An extension of Thomassen's result on choosability

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Abstract

Thomassen proved that all planar graphs are 5-choosable. Škrekovski strengthened the result by showing that all K_5-minor-free graphs are 5-choosable. Dvořák and Postle pointed out that all planar graphs are DP-5-colorable. In this note, we first improve these results by showing that every K_5-minor-free or $K_{3,3}$-minor-free graph is DP-5-colorable. In the final section, we further improve these results under the term strictly f-degenerate transversal.

1 Introduction

Thomassen [6] proved that all planar graphs are 5-choosable. Škrekovski [9] (see also [3, 11]) extended the result to the class of K_5-minor-free graphs. Dvořák and Postle [2] gave a generalization of list coloring, under the name correspondence coloring, which was called DP-coloring by Bernshteyn, Kostochka, and Pron [1].

Let G be a graph and L be a list assignment for G. For each vertex $v \in V(G)$, we associate it with a set $L_v = \{v\} \times L(v)$; for each edge $uv \in E(G)$, we associate it with a matching $\mathscr{M}_{uv}$ between L_u and L_v. Let $\mathscr{M} = \bigcup_{uv \in E(G)} \mathscr{M}_{uv}$, and we call $\mathscr{M}$ the **matching assignment** over L. The matching assignment $\mathscr{M}$ is called a k**-matching assignment** if $L(v) = \{1, 2, \dots, k\}$ for every $v \in V(G)$. A **cover** of G is a graph $H_{L,\mathscr{M}}$ (simply write H) meeting two conditions:

- the vertex set of H is the disjoint union of L_v for all $v \in V(G)$; and
- the edge set of H is the matching assignment $\mathscr{M}$.

Let G be a graph and H be a cover of G over a list assignment L. An $(L, \mathscr{M})$**-coloring** of G is an independent set $\mathcal{I}$ of H such that $|\mathcal{I} \cap L_v| = 1$ for each $v \in V(G)$. A graph G is **DP-k-colorable** if for any list assignment $L(v) \supseteq \{1, 2, \dots, k\}$ and any matching assignment $\mathscr{M}$, it admits an $(L, \mathscr{M})$-coloring. Note that every DP-k-colorable graph is k-choosable.

Dvořák and Postle [2] have pointed out that all planar graphs are DP-5-colorable. We improve the result to the following Theorem 1.1, and we also extend the result for planar graphs to the class of $K_{3,3}$-minor-free graphs.

Theorem 1.1. All K_5-minor-free graphs are DP-5-colorable.

Theorem 1.2. All $K_{3,3}$-minor-free graphs are DP-5-colorable.

Let H be a cover of G, and let f be a function from $V(H)$ to $\{0, 1, 2, \dots\}$. A subset $T \subseteq V(H)$ is called a **transversal** if $|T \cap L_v| = 1$ for each $v \in V(G)$. A transversal T of a cover H is **strictly f-degenerate** if every nonempty subgraph Γ in $H[T]$ contains a vertex x with $\deg_{\Gamma}(x) < f(x)$. In other words, all the vertices of $H[T]$ can be ordered as $x_1, x_2, \dots, x_n$ such that each vertex x_i has less than $f(x_i)$ neighbors on the right

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hand side. Such an order is an f**-removing order**, and the reverse order $x_n, x_{n-1}, \dots, x_1$ is an f**-coloring order**.

By definition, a vertex x can never be chosen in a strictly f-degenerate transversal if $f(x) = 0$. Hence, we can add some vertices into L_v and define the value of f to be zero on these new vertices, so that all the L_v have the same cardinality. On the other hand, it doesn't matter what the labels of the vertices are, so we may assume that $L_v = \{v\} \times [s]$, where s is an integer. A cover H together with a function f is called a valued-cover.

In Section 3, we strengthen Theorems 1.1 and 1.2 to Theorem 1.3. In order to demonstrate how Thomassen's technique in [6] is extended, we first give a proof for Theorem 1.1 in Section 2, and then give one for Theorem 1.3, even though Theorems 1.1 and 1.2 are special cases of Theorem 1.3. For a function f, we use R_f to denote the range of f.

Theorem 1.3. Assume that G is a K_5-minor-free or $K_{3,3}$-minor-free graph, and (H, f) is a valued-cover with $R_f \subseteq \{0, 1, 2\}$. Then H contains a strictly f-degenerate transversal.

Assume that G is a plane graph and C is a cycle in it. We will use $\mathrm{Int}(C)$ (resp. $\mathrm{Ext}(C)$) to denote the subgraph induced by $V(C)$ and the vertices inside (resp. outside) of C. The cycle C is a **separating** cycle of G if both the interior and the exterior of C have at least one vertex.

2 DP-5-coloring

A **plane triangulation** is an embedded plane graph such that each of its faces is bounded by a cycle of length three. A **near-triangulation** is an embedded plane graph such that each bounded face is bounded by a triangle and the unbounded face (outer face) is bounded by a cycle. An ℓ**-sum** of two graphs G' and G'' is the graph G such that $G = G' \cup G''$ and $G' \cap G'' = K_\ell$.

The **Wagner graph** is a 3-regular graph with 8 vertices and 12 edges, see Fig. 1. Note that the Wagner graph is non-planar, thus the Wagner graph cannot be a subgraph of a planar graph.

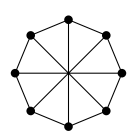

Fig. 1: Wagner graph.

Wagner [10] gave the following characterization of planar graphs in terms of graph minors.

Theorem 2.1 (Wagner [10])**.** A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

By Wagner's Theorem, the class of K_5-minor-free graphs and the class of $K_{3,3}$-minor-free graphs are two superclasses of planar graphs.

A graph G is **maximal** K_5**-minor-free** if it does not contain K_5 as a minor, but $G + xy$ contains a K_5-minor for every pair nonadjacent vertices x and y in G. Wagner [10] also gave the following characterization of maximal K_5-minor-free graphs.

Theorem 2.2 (Wagner [10])**.** Every maximal K_5-minor-free graph can be obtained from the Wagner graph and plane triangulations by recursively 2-sums or 3-sums.

The following theorem and its proof are very similar to that in [6], but for completeness we give a complete proof here.

Theorem 2.3. Assume that G is a near-triangulation such that the outer face is bounded by a cycle $\mathcal{O} = v_1 v_2 \dots v_p v_1$. Let L be a list assignment of G such that $|L(v)| \geq 3$ for each $v \in V(\mathcal{O})$ and $|L(v)| \geq 5$ for each $v \notin V(\mathcal{O})$. If $\mathscr{M}$ is a matching assignment for G and R_0 is an $(L, \mathscr{M})$-coloring of $G[\{v_1, v_2\}]$, then G admits an $(L, \mathscr{M})$-coloring such that its restriction on $G[\{v_1, v_2\}]$ is R_0.

Proof. The assertion is proved by induction on $|V(G)|$. When G has only three vertices, $G = \mathcal{O} = K_3$ and the assertion is obvious. So we can assume that $|V(G)| \geq 4$ and the assertion is true for smaller graphs. Suppose that $\mathcal{O}$ has a chord v_iv_j. It follows that v_iv_j lies in two cycles C_1 and C_2 of $\mathcal{O} + v_iv_j$. Let v_1v_2 lie in C_1. Applying the induction hypothesis to $\mathrm{Int}(C_1)$, R_0 can be extended to an $(L, \mathscr{M})$-coloring of $\mathrm{Int}(C_1)$. After v_i and v_j are colored, it can be further extended to an $(L, \mathscr{M})$-coloring of $\mathrm{Int}(C_2)$. This yields a desired $(L, \mathscr{M})$-coloring of G.

So we can assume that $\mathcal{O}$ has no chord. Let $v_1, u_1, u_2, \dots, u_m, v_{p-1}$ be the neighbors of v_p in a natural cyclic order around v_p. Since all the bounded faces of G are bounded by triangles and $\mathcal{O}$ has no chord, $P = v_1u_1u_2 \dots u_mv_{p-1}$ is a path and $\mathcal{O}' = P \cup (\mathcal{O} - v_p)$ is a cycle. Let j and ℓ be two distinct elements in $L(v_p)$ which do not conflict with the color of v_1 under the matching $\mathscr{M}_{v_1v_p}$. Now define $L'(v) = L(v)$ for every $v \notin \{u_1, u_2, \dots, u_m, v_p\}$, for $1 \leq i \leq m$, define $L'(u_i)$ from $L(u_i)$ by deleting the neighbors of $j, \ell \in L(v_p)$ under the matching $\mathscr{M}_{v_pu_i}$. It is easy to check that $|L'(v)| \geq 3$ for all $v \in V(\mathcal{O}')$ and $|L'(v)| \geq 5$ for all $V(G) - \{v_p\} - V(\mathcal{O}')$. Applying the induction hypothesis to $\mathcal{O}'$ and its interior and the new list L', we have an $(L', \mathscr{M})$-coloring for $G - v_p$. There is at least one color in $\{j, \ell\} \subset L(v_p)$ which do not conflict with the color of v_{p-1} under $\mathscr{M}_{v_{p-1}v_p}$, so we can assign it to the vertex v_p. This completes the proof. □

Theorem 2.4. Assume that G is a maximal K_5-minor-free graph. If K is a subgraph of G isomorphic to K_2 or K_3, then every DP-5-coloring φ of K can be extended to a DP-5-coloring of G.

Proof. Suppose to the contrary that G is a counterexample with $|V(G)|$ as small as possible.

Assume that G is a plane triangulation and K is a separating 3-cycle of G. Note that $\mathrm{Int}(K)$ and $\mathrm{Ext}(K)$ are both plane triangulations and maximal K_5-minor-free graphs. By minimality, every DP-5-coloring φ of K can be extended to a DP-5-coloring φ_1 of $\mathrm{Int}(K)$ and a DP-5-coloring φ_2 of $\mathrm{Ext}(K)$. Combining φ_1 and φ_2 yields a DP-5-coloring of G, a contradiction.

Assume that G is a plane triangulation and $K = [x_1x_2x_3]$ bounds a 3-face. Note that G has at least four vertices. We can redraw the plane triangulation such that K is the boundary of the outer face. Note that $G - x_3$ is a near-triangulation. Since x_3 on K is precolored, every uncolored vertex incident with the outer face of $G - x_3$ has at least four admissible colors other than $\varphi(x_3)$. Applying Theorem 2.3 to $G - x_3$, we obtain a DP-5-coloring of G whose restriction on K is the precoloring φ.

Assume that G is a plane triangulation and $K = y_1y_2$. We can further assume that y_1y_2 is incident with a 3-face $[y_1y_2y_3]$. Clearly, the precoloring of K can be extended to a DP-5-coloring of $G[y_1, y_2, y_3]$, and we can reduce the problem to the previous case.

If G is the Wagner graph, then we can greedily extend the precoloring of K to a DP-5-coloring of G since G is 3-regular.

By Theorem 2.2, we can assume that G is a 2-sum or 3-sum of two maximal K_5-minor-free graphs G_1 and G_2 with $K \subset G_1$. By minimality, the precoloring φ of K can be extended to a DP-5-coloring φ_1 of G_1. By minimality once again, we can extended the restriction of φ_1 on $G_1 \cap G_2$ to G_2. This yields a DP-5-coloring of G whose restriction on K is the precoloring φ. □

Now, we can easily prove Theorem 1.1.

Theorem 1.1. All K_5-minor-free graphs are DP-5-colorable.

Proof. Since every K_5-minor-free graph is a spanning subgraph of a maximal K_5-minor-free graph, it suffices to prove the result for maximal K_5-minor-free graphs. We can first color two adjacent vertices in G, and extend the coloring to the whole graph according to Theorem 2.4. □

Wagner [10] also gave a characterization of maximal $K_{3,3}$-minor-free graphs by 2-sums.

Theorem 2.5 (Wagner [10])**.** Every maximal $K_{3,3}$-minor-free graph can be obtained from the complete graph K_5 and plane triangulations by recursively 2-sums.

Since the proof of the following result is similar to that in Theorem 2.4, we leave it as an exercise to the readers.

Theorem 2.6. Assume that G is a maximal $K_{3,3}$-minor-free graph. If K is a subgraph of G isomorphic to K_2, then every DP-5-coloring of K can be extended to a DP-5-coloring of G.

Theorem 1.2. All $K_{3,3}$-minor-free graphs are DP-5-colorable.

Proof. Since each $K_{3,3}$-minor-free graph is a spanning subgraph of a maximal $K_{3,3}$-minor-free graph, it suffices to show the result for maximal $K_{3,3}$-minor-free graphs. We can first color two adjacent vertices in G, and further extend the precoloring to the whole graph according to Theorem 2.6. □

3 Strictly f-degnerate transversal

In this section, we extend the results on DP-5-coloring to particular strictly f-degenerate transversal. The following two lemmas were presented by Nakprasit and Nakprasit [5, Lemma 2.3] with a different term.

For a vertex subset K of $V(G)$, or a subgraph K of G, we use H_K to denote the cover restricted on K, i.e., $H_K \coloneqq H[\bigcup_{v\in K} L_v]$.

Lemma 3.1. Assume that G is a graph and K is a subgraph of G. Let (H, f) be a valued cover, and T be a transversal of H_K such that $H[T]$ has no edges and $f(x) = 1$ for each $x \in T$. If T can be extended to a strictly f-degenerate transversal T' of H, then there exists an f-removing order of T' such that the vertices in T are on the rightest of the order.

Proof. Let S' be an f-removing order of T'. Since $f(x) = 1$ for each $x \in T$, every vertex in T has no neighbor on the right of the order S', so we can move all the vertices in T to the rightest of the order. In other words, we can delete all the vertices in T from the order S' and put the vertices in T on the right side of all the other vertices of S'. Observe that the resulting order satisfies the desired condition. □

Lemma 3.2. Assume that $G = G_1 \cup G_2$, $V(G_1 \cap G_2) = K$ and G_1 is an induced subgraph of G. Let (H, f) be a valued cover of G, and H_i be the restriction of H on G_i for $i \in \{1, 2\}$. If R is a strictly f-degenerate transversal of H_1, and $R \cap H_K$ can be extended to a strictly f^*-degenerate transversal R^* of H^*, where H^* is obtained from H_2 by deleting all the edges in H_K, and f^* is obtained from f by defining $f^*(x) = 1$ for each $x \in R \cap H_K$, then $R \cup R^*$ must be a strictly f-degenerate transversal of H.

Proof. It suffices to give an f-removing order of $H[R \cup R^*]$. By Lemma 3.1, there exists an f^*-removing order of R^* such that the vertices in $R \cap H_K$ are on the rightest of the order. Then we list all the vertices of $R^* \setminus (R \cap H_K)$ according to the f^*-removing order and then list the vertices of R according to an f-removing order. It is easy to check that the resulting order is an f-removing order for $H[R \cup R^*]$. □

We first extend Theorem 2.3 to the following result. Note that Theorem 3.1 was first proved in [5, Theorem 1.6], but the following proof is a little bit different from that one.

Theorem 3.1. Assume that G is a near-triangulation such that the outer face is bounded by a cycle $\mathcal{O} = v_1v_2 \dots v_pv_1$. Let (H, f) be a valued cover of G with $R_f \subseteq \{0, 1, 2\}$ such that

$$f(v, 1) + \cdots + f(v, s) \geq 3 \text{ for every } v \in V(\mathcal{O}) \tag{1}$$

and

$$f(v, 1) + \cdots + f(v, s) \geq 5 \text{ for every } v \notin V(\mathcal{O}). \tag{2}$$

If R_0 is a strictly f-degenerate transversal of $H[L_{v_1} \cup L_{v_2}]$, then R_0 can be extended to a strictly f-degenerate transversal of H.

Proof. We prove the assertion by induction on $|V(G)|$. When G has exactly three vertices, $G = \mathcal{O} = K_3$ and the assertion is obvious. Then $|V(G)| \geq 4$ and the assertion is true for smaller graphs. Suppose that $\mathcal{O}$ has a chord uw. It follows that uw lies in two cycles C_1 and C_2 of $\mathcal{O} + uw$ with v_1v_2 in C_1. Let $G_1 := \mathrm{Int}(C_1)$ and $G_2 := \mathrm{Int}(C_2)$. Applying the induction hypothesis to G_1, R_0 can be extended to a strictly f-degenerate transversal R of H_1, and then $R \cap H[L_u \cup L_w]$ can be extended to a strictly f^*-degenerate transversal R^* of H^* as in Lemma 3.2. Therefore, $R^* \cup R$ is a desired strictly f-degenerate transversal of H.

The other case is that $\mathcal{O}$ has no chord. Let $v_1, u_1, u_2, \dots, u_m, v_{p-1}$ be the neighbors of v_p in a natural cyclic order around v_p, and let $U = \{u_1, u_2, \dots, u_m\}$. Since all the bounded faces of G are bounded by triangles and $\mathcal{O}$ has no chord, we have $P = v_1u_1u_2 \dots u_mv_{p-1}$ is a path and $\mathcal{O}' = P \cup (\mathcal{O} - v_p)$ is a cycle. For each $x \in \{v_p\} \times [s]$, let

$$f'(x) = \begin{cases} \max\{0, f(x) - 1\}, & \text{if } x \text{ is adjacent to } R_0 \cap L_{v_1} \text{ under } \mathscr{M}_{v_1v_p}; \\ \\ f(x), & \text{otherwise.} \end{cases}$$

Since $R_0 \cap L_{v_1}$ has at most one neighbor in L_{v_p}, we have $f'(v_p, 1) + \cdots + f'(v_p, s) \geq 2$. Let

$$X' = \{\, x \in \{v_p\} \times [s] : f'(x) > 0 \,\}.$$

Case 1. $|X'| \geq 2$.

Let X^* be a subset of X' with $|X^*| = 2$. A new function $f^\dagger$ on $H - L_{v_p}$ is defined as

$$f^\dagger(x) = \begin{cases} \max\{0, f(x) - 1\}, & \text{if } x \in U \times [s] \text{ and } x \text{ is connected to a vertex in } X^*; \\ \\ f(x), & \text{otherwise.} \end{cases}$$

It follows that, for each $u \in \mathcal{O}'$, we have

$$\sum_{z \in L_u} f^\dagger(z) \geq 3.$$

By induction hypothesis and Lemma 3.1, $(H - L_{v_p}, f^\dagger)$ contains a strictly $f^\dagger$-degenerate transversal $R^\dagger$ with an $f^\dagger$-removing order $S^\dagger$ such that the vertices in R_0 are on the rightest of the order. Let (v_p, c_p) be a vertex in X^* which is not adjacent to $R^\dagger \cap L_{v_{p-1}}$. Therefore, we insert (v_p, c_p) into $S^\dagger$ such that it is the reciprocal third element to obtain an f-removing order of a strictly f-degenerate transversal of H.

Case 2. $|X'| = 1$.

Without loss of generality, assume that $X' = \{(v_p, 1)\}$. Since $f'(v_p, 1) + \cdots + f'(v_p, s) \geq 2$ and $R_f \subseteq \{0, 1, 2\}$, we have $f'(v_p, 1) = 2$. Define a function $f^\dagger$ on $H - L_{v_p}$ by

$$f^\dagger(x) = \begin{cases} 0, & \text{if } x \in U \times [s] \text{ and } x \text{ is adjacent to } (v_p, 1) \text{ in } H; \\ \\ f(x), & \text{otherwise.} \end{cases}$$

Note that the range of f is a subset of $\{0, 1, 2\}$, for each $u \in \mathcal{O}'$,

$$\sum_{z \in L_u} f^\dagger(z) \geq 3.$$

By induction hypothesis, $(H - L_{v_p}, f^\dagger)$ admits a strictly $f^\dagger$-degenerate transversal $R^\dagger$ with an $f^\dagger$-removing order $S^\dagger$ such that the vertices in R_0 are on the rightest of the order. Let S be a sequence obtained from $S^\dagger$ by inserting $(v_p, 1)$ into $S^\dagger$ such that $(v_p, 1)$ is the immediate predecessor of (v_{p-1}, c_{p-1}), where $(v_{p-1}, c_{p-1}) \in L_{v_{p-1}} \cap R^\dagger$. Recall that $f^\dagger(v_p, 1) = 2$, it is not hard to check that S is an f-removing order of a strictly f-degenerate transversal of H. $\square$

Instead of proving Theorem 1.3, we prove the following stronger theorem for K_5-minor-free graphs, and leave the corresponding result for $K_{3,3}$-minor-free graphs to the readers.

Theorem 3.2. Assume that G is a K_5-minor-free graph, and (H, f) is a valued-cover with $R_f \subseteq \{0, 1, 2\}$. If K is a subgraph isomorphic to K_2 or K_3, and $f(v, 1) + \cdots + f(v, s) \geq 5$ for each $v \in V(G)$, then every strictly f-degenerate transversal of H_K can be extended to a strictly f-degenerate transversal of H.

Proof. Suppose to the contrary that (G, H, f, R_0) is a counterexample with $|V(G)|$ as small as possible, where R_0 is a strictly f-degenerate transversal of H_K. Similar to the previous results, we only need to consider the case that G is a maximal K_5-minor-free graph.

Assume that G is a plane triangulation and K is a separating triangle of G. Note that $\mathrm{Ext}(K)$ and $\mathrm{Int}(K)$ are both plane triangulations and maximal K_5-minor-free graphs. By minimality and Lemma 3.2, R_0 can be extended to a strictly f-degenerate transversal of H.

Assume that G is a plane triangulation and $K = [x_1x_2x_3]$ bounds a 3-face. We can redraw the plane triangulation such that K bounds the outer face. Let (x_3, c_3) be in R_0, define a function f' on $H - L_{x_3}$ by

$$f'(x) = \begin{cases} 0, & \text{if } x \in \{u\} \times [s] \text{ with } u \notin \{x_1, x_2\} \text{ and } x \text{ is connected to } (x_3, c_3) \text{ in } H; \\ \\ f(x), & \text{otherwise.} \end{cases}$$

Note that the graph $G - x_3$ is a near-triangulation. Since the range of f is a subset of $\{0, 1, 2\}$, we have that, for each w on the outer face of $G - x_3$,

$$\sum_{x \in \{w\} \times [s]} f'(x) \geq 3.$$

By Theorem 3.1, $R_0 \setminus \{(x_3, c_3)\}$ can be extended to a strictly f'-degenerate transversal of $H \setminus L_{x_3}$ with an f'-removing order S' such that the two vertices in $R_0 \setminus \{(x_3, c_3)\}$ are on the rightest of the order. According to an f-removing order of R_0, we can insert (x_3, c_3) into S' such that the three vertices in R_0 are the three rightest elements in the order to obtain an f-removing order of a strictly f-degenerate transversal of H.

Assume that G is a plane triangulation and $K = x_1x_2$. We may assume that x_1x_2 is incident with a 3-face $[x_1x_2x_3]$. Clearly, R_0 can be extended to a strictly f-degenerate transversal of $H_{[x_1,x_2,x_3]}$, and we can reduce the problem to the previous case.

If G is the Wagner graph, then we can greedily extend R_0 to a strictly f-degenerate transversal of H since G is 3-regular.

By Theorem 2.2, assume that G is a 2-sum or 3-sum of two maximal K_5-minor-free graphs G_1 and G_2 with $K \subset G_1$. By minimality and Lemma 3.2, R_0 can be extended to a strictly f-degenerate transversal of H. $\square$

In Theorems 3.1 and 3.2, there is a restriction on f, i.e., the range of f is a subset of $\{0, 1, 2\}$. If the restriction can be dropped, the results can imply two theorems due to Thomassen. Thomassen proved that every planar graph can be partitioned into a 3-degenerate graph and an independent set [8], and every planar graph can be partitioned into a 2-degenerate graph and a forest [7]. So the second author and some others made the following conjecture in [4].

Conjecture. Assume that G is a planar graph and (H, f) is a positive-valued cover. If $s \geq 2$ and $f(v, 1) + \cdots + f(v, s) \geq 5$ for each $v \in V(G)$, then H admits a strictly f-degenerate transversal.

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