RESEARCH ARTICLE

Nash Social Distancing Games with Equity Constraints: How Inequality Aversion Affects the Spread of Epidemics[†]

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Summary

In this paper, we present a game-theoretic model describing the voluntary social distancing during the spread of an epidemic. The payoffs of the agents depend on the social distancing they practice and on the probability of getting infected. We consider two types of agents, the non-vulnerable agents who have a small cost if they get infected, and the vulnerable agents who have a higher cost. For the modeling of the epidemic outbreak, we consider a variant of the SIR (Susceptible-Infected-Removed) model involving populations of susceptible, infected and removed persons of vulnerable and non-vulnerable types. The Nash equilibria of this social distancing game are studied. The main contribution of this work is the analysis of the case where the players, desiring to achieve a low social inequality, pose a bound on the variance of the payoffs. In this case, we introduce and characterize a notion of Generalized Nash Equilibrium (GNE) for games with a continuum of players. Through numerical studies, we show that inequality constraints result in a slower spread of the epidemic and an improved cost for the vulnerable players. Furthermore, it is possible that inequality constraints are beneficial for non-vulnerable players as well.

KEYWORDS:

COVID-19 pandemic, Nash games, inequality aversion, social distancing

1 | INTRODUCTION

Epidemics harass humanity for centuries, and people investigate several strategies to contain them. The development of medicines and vaccines and the evolution of healthcare systems with specialized personnel and equipped hospitals have significantly affected the spread of many epidemics and have even eliminated some contagious diseases. However, during the current COVID-19 pandemic, due to the lack or scarcity of appropriate medicines and vaccines, Non-Pharmaceutical Interventions (primarily social distancing) have been among the most effective strategies to reduce the disease spread. Due to the slow roll-out of the vaccines, their uneven distribution, the emergence of SARS-CoV-2 variants, age limitations, and people's resistance to vaccination, social distancing is likely to remain significant in a large part of the globe for the near future.

Epidemiological models are essential in designing measures and strategies to control epidemics¹. In the last century, epidemiologists have made significant progress in the mathematical modeling of the spread of epidemics. From the seminal works of Kermack and McKendrick² and Ross³, a prevalent approach in the mathematical modeling of epidemics is compartmental

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models. These models consider that each agent belongs in some compartment according to her infection state (e.g., Susceptible-Infected-Recovered) and study the evolution of each compartment's population. The literature on these models is extensive, so for a summary, we refer to Chapter 2 of Allen et al.(2008)⁴. There are also other elegant approaches to epidemics modeling, such as the ones that take into consideration the heterogeneous networked structure of human interconnections⁵. However, the compartmental models remain a well-studied and fruitful approach, widely used in real-life applications.

The development of epidemiological models is a valuable tool in designing protective measures against the spread of an epidemic. Still, these measures will be adopted by agents who act in a self-interested manner, at least to some extent. Thus, game theory is an appropriate complementary mathematical tool to be used in this field. Indeed, many game-theoretic models have been developed to study voluntary vaccination ^{6,7,8,9,10,11,12,13} and behavioral changes of the agents ^{14,15,16,17,18,19,20,21}, such as social distancing, use of face masks, and better hygiene practice. Another closely related stream of research is the study of the adoption of decentralized protection strategies in engineered and social networks ^{22,23,24,25}. Recently, with the emergence of the COVID-19 pandemic, there is a renewed interest in modeling individual behaviors. Related tools include dynamic game analysis of social distancing ²⁶, evolutionary game theory ^{27,28,29,30} and network game models ³¹.

The majority of game-theoretic models are based on the assumption that the rational behavior for an agent is to maximize selfishly her own payoff ignoring the social impacts (externalities) of this choice. This can lead to 'free-riding' phenomena in vaccination games ^{9,12} or in disobedience of social distancing rules in social distancing games, which can both result in a higher prevalence of the spread of the epidemic ^{9,32} and to harmful consequences for the vulnerable members of the society. The phenomenon that the Nash equilibrium strategies result to a social welfare less than the optimal one is well known in game theory community as the *Tragedy of the Commons* ³³. There are some notable exceptions ^{34,35} analyzing epidemic games involving altruistic individuals. However, even in the cases considering that the agents prefer the strategies that maximize a social welfare function, there may still exist significant inequalities among their payoffs.

Epidemics may create vastly unequal outcomes in terms of health risks. For example, the severe illness or fatality risk for a person infected by SARS-COV-2 varies widely with age and underlying health conditions ^{36,37}. There is a lot of empirical evidence that people are often motivated by fairness considerations ^{38,39}. That is, people are often willing to sacrifice some of their own payoff to achieve a more equitable outcome. When it comes to health inequalities, people are often very inequality averse ⁴⁰. Especially if an agent has vulnerable relatives, it is rather natural for her to alter her behavior during an epidemic outbreak to protect them. In the context of the current COVID-19 crisis, it has been observed that communication strategies that aim to indicate the effects of social distancing behavior on others, especially on vulnerable persons (strategy of the identifiable victim), are very effective ⁴¹.

In this work, we employ a novel approach to model the agents' possible desire to keep the inequality among their payoffs below a certain threshold. Particularly, we consider that the players share a common constraint bounding inequality, modeled as their costs variance. This modeling approach can be useful for future waves of COVID-19 pandemic (probably involving variants of the virus) or for future epidemics.

Following the literature ^{42,43,44}, we consider a compartmental model (SIR) for the spread of an epidemic and a social distancing game among the agents. The payoffs of the agents consist of two terms: a cost for the social distancing and a cost proportional to the probability of getting infected. There are two types of agents, non-vulnerable agents, who have a small cost if they get infected and vulnerable agents, who have a higher cost. The size of the society is considered large, so the game has a continuum of players (it is a non-atomic game). In this game, the agents determine their actions to optimize their payoff and simultaneously respect a constraint concerning the variance of all the agents' payoffs. Due to this constraint, the game is, in fact, a generalized game with a non-convex constraint. The majority of the bibliography on generalized games ^{45,46} does not analyze generalized games with non-convex constraints. Furthermore, there are a few references on generalized non-atomic games ^{47,48,49}. However, in these papers, the convexity of the constraint set is built into the definition of the Generalized Nash Equilibrium (GNE). Another related paper by Singh and Wiszniewska-Matyszkiel ⁵⁰ examines a dynamic game with a continuum of players having state-dependent constraints. In this work, we give a new definition and characterization of GNE for constrained non-atomic games.

Numerical examples indicate that there may be many Nash equilibria inducing different costs for the players. Thus, even in the absence of social distancing regulations, it is beneficial for the players to coordinate and choose the 'best' equilibrium. In the variance constrained case, we numerically find that the inequality constraint (bounding the variance) is always beneficial for the vulnerable players. Sometimes, inequality constraints are beneficial for the group of non-vulnerable agents as well.

The rest of the paper is organized as follows. Section 2 describes the compartmental model for the epidemic outbreak and the social distancing game between the agents. In Section 3, we analyze the game and characterize its Nash equilibria. In Section 4, we introduce the constraint that concerns the variance of the payoffs and derive an appropriate definition of generalized

Nash equilibrium for variance constrained games. In section 5, we present a methodology for the computation of the Nash equilibrium strategies of section 3 and then give some numerical examples. Moreover, in the same section, we present an example of generalized Nash equilibrium computation for the variance constrained game. In the Appendix, we collocate the proofs of several propositions of the paper.

2 | MATHEMATICAL MODEL

This section presents a variation of a popular epidemics model which assumes a continuum of agents. The state of each agent could be Susceptible (S), Infected (I), Recovered (R) or Dead (D). A susceptible person may be infected at a rate proportional to the rate she meets with infected people. Infected persons either recover or die at a constant rate. We assume that an individual recovered from the infection is immune i.e., she could not be infected again.

We distinguish between two types of agents: non-vulnerable and vulnerable. We use the index j=1 for non-vulnerable agents and j=2 for vulnerable. The difference of the two types of agents is the severity of a possible infection (including the probability to survive). An infected agent recovers with a probability rate α'_j and dies with a probability rate $\alpha_j - \alpha'_j$. The evolution of the individual states is presented in Figure 1.

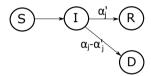


FIGURE 1 The Markov process describing the evolution of the state of each individual.

We analyze the behavior of the agents for a time interval [0, T]. Denote by $u_i \in [u_m, u_M]$ the action of player i, indicating the fraction of time this person spends in public places. The minimum value of the actions u_m describes the minimum contact a person needs for surviving and u_M describes a restriction placed by the government. In the absence of a restriction we consider $u_M = 1$.

Assumption 1: The actions u_i of all the players are *constant* during a time interval [0, T]. This interval represents a wave of the epidemic.

Remark 1. In a more general model u_i could be function of the state variables or time, but there are some arguments in favor of this choice. First, there may be a high uncertainty for the values of the state variables. Second, u_i 's reflect some everyday routine choices of the people and these choices may be difficult to adapt constantly.

Denote by μ_1 a (Borel) measure on $[u_m, u_M]$ describing the distribution of the actions of the players in category 1 i.e., for $A \subset [u_m, u_M]$, the value of $\mu_1(A)$ denotes the mass of the players using an action $u \in A$. Similarly, denote by μ_2 the distribution of actions of the players of type 2. The total mass of players of types 1 and 2 is n_1 and n_2 respectively i.e., $\mu_1([u_m, u_M]) = n_1$ and $\mu_2([u_m, u_M]) = n_2$. We first describe the evolution of the epidemic for general distributions μ_1, μ_2 .

Denote by $S_{1u}(t)$ the probability a non-vulnerable player who plays $u \in [u_m, u_M]$ to be susceptible at time t and by $I_{1u}(t)$ the probability to be infected. Similarly define $S_{2u}(t)$, $I_{2u}(t)$. The rate at which this person gets infected is given by: ruI^f , where r is a positive constant and I^f denotes the density of infected people in 'public places'. Each player contributes to $I^f(t)$ proportionally to her probability of being infected at time t. The dynamics is given by:

$$\begin{split} \dot{S}_{ju} &= -ruS_{ju}I^{f}(t) \\ \dot{I}_{ju} &= ruS_{ju}I^{f}(t) - \alpha_{1}I_{ju}, \\ \dot{z} &= I^{f}(t) \end{split} \tag{1}$$

where z is an auxiliary variable, j = 1, 2 and:

$$I^{f}(t) = \int_{[u_{m}, u_{M}]} I_{1u'}(t)u' \cdot \mu_{1}(du') + \int_{[u_{m}, u_{M}]} I_{2u'}(t)u' \cdot \mu_{2}(du'). \tag{2}$$

The initial conditions are

$$S_{1u}(0) = S_{2u}(0) = (1 - I_0), \ I_{1u}(0) = I_{2u}(0) = I_0,$$
 (3)

where I_0 is the percentage of infected persons at time 0. Here we assume, without loss of generality, that at the beginning of the time interval [0, T], the agents of both types are infected with the same probability I_0 .

Before showing the existence of a solution for the initial value problem (1)–(3), we introduce some function spaces. Let:

$$X = C([u_m, u_M], \mathbb{R}^4) \times \mathbb{R},$$

where $C([u_m, u_M], \mathbb{R}^4)$ be the space of continuous functions defined on $[u_m, u_M]$ and values on \mathbb{R}^4 . The space X equipped with the norm:

$$||x|| = \max\{|x_{1u}|, |x_{2u}|, |x_{3u}|, |x_{4u}|, |x_5| : u \in [u_m, u_M]\},$$

is a Banach space. We also consider the Banach space Y of signed measures on $[u_m, u_M]$ with the total variation norm.

Proposition 1. The initial value problem (1)–(3) has a unique solution. Furthermore, this solution is continuous on u.

The cost of an agent *i* of type *j* consists of two terms. The first term is proportional to the probability of getting infected and the vulnerability of the player. The second term, represents the benefits earned from social interactions. The cost is given by:

$$J_{i} = G_{i} P_{i} - Q_{i}(u_{i}, \bar{u}_{1}, \bar{u}_{2}), \tag{4}$$

where G_j corresponds to the expected severity of a possible infection, P_i is the probability that i gets infected within the time interval [0,T]. Note that $G_2 > G_1$. The quantity $Q_j(u_i,\bar{u}_1,\bar{u}_2)$ represents the utility derived from the interaction with others where \bar{u}_1,\bar{u}_2 are the mean actions of the players of types j=1 and j=2 respectively. For simplicity we assume that Q_j has the form:

$$Q_i(u_i, \bar{u}_1, \bar{u}_2) = s_{i1}u_i\bar{u}_1 + s_{i2}u_i\bar{u}_2,$$

where s_{j1} , s_{j2} are non-negative constants.

Remark 2. In the computation of the second term in (4), we assume that the number of people in these types is approximately constant with time. This is a good approximation in epidemics with a low mortality rate and duration small compared to the average human life.

Let us then compute the probability of getting infected P_i . The probability $S_{ju_i}^i(t)$ that an agent i of either type (j = 1 or j = 2) is not infected up to time t evolves according to:

$$\dot{S}^i_{ju_i} = -rS^i_{ju_i}u_iI^f.$$

Thus, we have:

$$P_{i} = 1 - S_{ju_{i}}^{i}(T) = I_{0} + (1 - I_{0}) \left[1 - \exp\left(-ru_{i} \int_{0}^{T} I^{f}(t)dt\right) \right].$$

Here we assume that at the beginning of the time interval [0, T], all the agents are infected with a small probability I_0 . Denoting by $F(\mu_1, \mu_2) = r \int_0^T I^f(t) dt$ the cost is written as:

$$J_{i}(u_{i}, \mu_{1}, \mu_{2}) = G_{i} \left[I_{0} + (1 - I_{0}) \left[1 - \exp \left(-u_{i} F(\mu_{1}, \mu_{2}) \right) \right] \right] - s_{i1} u_{i} \bar{u}_{1} - s_{i2} u_{i} \bar{u}_{2}.$$

Since we assume a very large population of players, each one of them is not able to affect the distributions μ_1 , μ_2 . It is interesting to observe that the individual cost J_i is concave in u_i . To see this take the second derivative of J_i with respect to u_i :

$$\frac{\partial^2 J_i}{\partial u_i^2} = -G_j(1 - I_0) \exp\left(-ru_i \int_0^T I^f(t)dt\right) \left(-\int_0^T I^f(t)dt\right)^2 < 0.$$

Therefore, the possible actions minimizing the individual cost are $u_i = u_m$ and $u_i = u_M$. Thus, to compute the Nash equilibria, we focus on distributions assigning the entire mass on $\{u_m, u_M\}$.

Remark 3. The fact that the cost function J_i is concave in u_i simplifies the analysis a lot, implying that players choose either $u = u_m$ or $u = u_M$. It further allows us to describe dynamics using a finite-dimensional model.

3 | NASH EQUILIBRIUM

To analyze the Nash equilibrium we focus on distributions having all the mass on $\{u_m, u_M\}$. The dynamics is given by:

$$\dot{S}_{ju_{m}} = -ru_{m}I^{f}S_{ju_{m}}, \qquad \dot{S}_{1u_{M}} = -ru_{M}I^{f}S_{ju_{M}},
\dot{I}_{ju_{m}} = ru_{m}I^{f}S_{ju_{m}} - \alpha_{j}I_{ju_{m}}, \qquad \dot{I}_{ju_{M}} = ru_{M}I^{f}S_{ju_{M}} - \alpha_{j}I_{ju_{M}},
\dot{z} = I^{f},$$
(5)

where j = 1, 2, the total mass of 'free infected people' I^f is given by:

$$I^{f}(t) = \sum_{i=1}^{2} n_{j}((1 - \tilde{u}_{j})u_{m}I_{ju_{m}}(t) + \tilde{u}_{j}u_{M}I_{ju_{M}}(t)),$$

and $\tilde{u}_i = \mu_i(\{u_M\})$ is the percentage of players of type j using u_M . The initial conditions are given by:

$$S_{iu_{m}}(0) = S_{iu_{m}}(0) = (1 - I_{0}), \quad I_{iu_{m}}(0) = I_{iu_{m}}(0) = I_{0}, \quad z(0) = 0$$
 (6)

Denote by $\phi_{\tilde{u}.\tilde{u}_2}^z(t)$ the z-part of the solution of the differential equation (5) with initial conditions (6). Then it holds:

$$F(\mu_1, \mu_2) = F(\tilde{u}_1, \tilde{u}_2) = r\phi^z_{\tilde{u}_1, \tilde{u}_2}(T).$$

Remark 4. We view the equilibria where some players of type j play u_m and some u_M as equilibria in symmetric mixed strategies. Particularly, each player of type j plays u_M with probability \tilde{u}_j .

Proposition 2. Consider a set of strategies characterized by \tilde{u}_1 , \tilde{u}_2 , where a fraction $1 - \tilde{u}_j$ of the players of type j use $u = u_m$ and a fraction \tilde{u}_j of the players of type j use $u = u_M$. This set of strategies is a Nash equilibrium if and only if, for each j = 1, 2, one of the following holds:

(i) $0 < \tilde{u}_i < 1$ and:

$$G_{j}(1-I_{0})(e^{-u_{m}F(\tilde{u}_{1},\tilde{u}_{2})}-e^{-u_{M}F(\tilde{u}_{1},\tilde{u}_{2})})=(u_{M}-u_{m})(s_{j1}\bar{u}_{1}+s_{j2}\bar{u}_{2}),$$

where $\bar{u}_j = u_m + (u_M - u_m)\tilde{u}_j$.

(ii) $\tilde{u}_i = 0$ and:

$$G_j(1-I_0)(e^{-u_mF(\tilde{u}_1,\tilde{u}_2)}-e^{-u_MF(\tilde{u}_1,\tilde{u}_2)})\geq (u_M-u_m)(s_{j1}\bar{u}_1+s_{j2}\bar{u}_2).$$

(iii) $\tilde{u}_i = 1$ and:

$$G_i(1-I_0)(e^{-u_mF(\tilde{u}_1,\tilde{u}_2)}-e^{-u_MF(\tilde{u}_1,\tilde{u}_2)}) \le (u_M-u_m)(s_{i1}\bar{u}_1+s_{i2}\bar{u}_2).$$

Proof: The proof is immediate, observing that (i) corresponds to the case where the players of type j are indifferent between u_m and u_M and (ii), (iii) correspond to preference of u_m over u_M and u_M over u_m respectively.

The existence of a Nash equilibrium is a consequence of Theorem 1 of Mas-Colell(1984)⁵¹.

Corollary 1. Assume that $s_{11} = s_{21}$ and $s_{12} = s_{22}$. Then the possible Nash equilibria $(\tilde{u}_1, \tilde{u}_2)$ are in of one of the following forms $(0,0), (\tilde{u}_1,0), (1,0), (1,\tilde{u}_2), (1,1)$.

Proof: Let $(\tilde{u}_1, \tilde{u}_2)$ be a Nash equilibrium. Then, if $\tilde{u}_1 < 1$ it holds:

$$G_1(1-I_0)(e^{-u_mF(\tilde{u}_1,\tilde{u}_2)}-e^{-u_MF(\tilde{u}_1,\tilde{u}_2)})\geq (u_M-u_m)(s_{11}\bar{u}_1+s_{12}\bar{u}_2).$$

Since, $s_{11} = s_{21}$, $s_{12} = s_{22}$ and $G_2 > G_1$ we have:

$$G_2(1-I_0)(e^{-u_mF(\tilde{u}_1,\tilde{u}_2)}-e^{-u_MF(\tilde{u}_1,\tilde{u}_2)})>(u_M-u_m)(s_{21}\bar{u}_1+s_{22}\bar{u}_2).$$

But, since $(\tilde{u}_1, \tilde{u}_2)$ is a Nash equilibrium Proposition (2) implies that $\tilde{u}_2 = 0$

Corollary 2. If $u_m = 0$, then $(\tilde{u}_1, \tilde{u}_2) = (0, 0)$ is always a Nash equilibrium.

4 | THE VARIANCE CONSTRAINED GAME

This section analyzes a game situation, where the players pose a shared bound on the variance of their costs. To do so, we first introduce a notion of equilibrium with a shared constraint, for non-atomic games, and then characterize it in terms of small

variations. We assume that the strategies of the players are symmetric (that is all the players of the same type use the same strategy), allowing for randomization. Consider a pair of distributions (μ_1 , μ_2) for the actions of the players. Then, the players of type j randomize according to $\bar{\mu}_i(\cdot) = \mu_i(\cdot)/n_i$.

The variance of the costs is given by:

$$V(\mu_{1}, \mu_{2}) = \frac{\int (J_{1}(u', \mu_{1}, \mu_{2}) - \bar{J})^{2} \mu_{1}(du') + \int (J_{2}(u', \mu_{1}, \mu_{2}) - \bar{J})^{2} \mu_{2}(du')}{n_{1} + n_{2}},$$

$$= \frac{n_{1} \int (J_{1}(u', \mu_{1}, \mu_{2}) - \bar{J})^{2} \bar{\mu}_{1}(du') + n_{2} \int (J_{2}(u', \mu_{1}, \mu_{2}) - \bar{J})^{2} \bar{\mu}_{2}(du')}{n_{1} + n_{2}},$$
(7)

where $\bar{J} = (n_1 \bar{J}_1 + n_2 \bar{J}_2)/(n_1 + n_2)$, and:

$$\bar{J}_j = \frac{1}{n_i} \int J_j(u',\mu_1,\mu_2) \mu_j(du') = \int J_j(u',\mu_1,\mu_2) \bar{\mu}_j(du').$$

We then describe a notion of equilibrium for the generalized game with variance constraint. Ideally, to have an equilibrium, the actions of each player should minimize the cost subject to the variance constraint. The difficult point here is that, since we have a continuum of players, the variance does not depend on the actions of individual players. To define a meaningful notion of equilibrium, instead of analyzing the effect of a deviation of a single player, we consider the deviation of a small fraction of players of type j and see how a variation from the nominal mixed strategy $\bar{\mu}_j$ affects the cost of this small group of players and the total variance. Then, we take the limit as the total mass of the group of players tends to zero.

Denote by j the type of players containing the deviating group and by -j the other type of players. Assume that the total mass of deviating players is ε and that the deviating players use a mixed strategy $\bar{\mu}'_j$ (note that it holds $\bar{\mu}'_j([u_m, u_M]) = 1$). Then, the distribution of the actions of the players of type j is given by:

$$\mu_j + \varepsilon(\bar{\mu}'_j - \bar{\mu}_j) = \mu_j + \varepsilon \delta \mu_j.$$

The mean cost of the deviating players after the deviation is:

$$\bar{J}_{j}^{\text{dev}} = \int J_{j}(u', \mu_{j} + \varepsilon(\bar{\mu}'_{j} - \bar{\mu}_{j}), \mu_{-j})\mu_{j}'(du'),$$

while before the deviation is \bar{J}_j . The following lemma expresses this the limit of this deviation, as well as the directional (Gateaux) derivative of the variance, in terms of linear bounded operators. This result will be used to define the Generalized Nash equilibrium.

Lemma 1. For all μ_{-i} it holds:

(i) The limit of the variation $\bar{J}_i^{\text{dev}} - \bar{J}_j$, as $\varepsilon \to 0$, is a linear function of $\delta \mu_j$. Particularly, it is written as:

$$\lim_{\varepsilon \to 0} (\bar{J}_j^{\text{dev}} - \bar{J}_j) = \mathcal{K}_{\mu_1, \mu_2}^j \delta \mu_j = \int J_j(u', \mu_j, \mu_{-j}) \delta \mu_j(du'), \tag{8}$$

where $\mathcal{K}^j_{\mu_1,\mu_2} \in Y^\star$ and Y^\star is the space of bounded linear functionals on Y .

(ii) The directional derivative of the variance $V(\mu_i, \mu_{-i})$ in the direction $\delta \mu_i$ is expressed as:

$$\lim_{\varepsilon \to 0} \frac{V(\mu_j + \varepsilon \delta \mu_j, \mu_{-j}) - V(\mu_j, \mu_{-j})}{\varepsilon} = \mathcal{L}^j_{\mu_1, \mu_2} \delta \mu_j,$$

where $\mathscr{L}^{j}_{\mu_{1},\mu_{2}} \in Y^{\star}$. Furthermore, $\mathscr{L}^{j}_{\mu_{1},\mu_{2}}$ can be written as:

$$\mathscr{L}_{\mu_1,\mu_2}^j \delta \mu_j = \int f_{j,V,\mu_1,\mu_2}^{\text{var}}(u') \delta \mu_j(du'),$$

with $f_{j,V,\mu_1,\mu_2}^{\text{var}}(u')$ continuous.

Proof: See Appendix B. □

Definition 1 (Generalized Nash Equilibrium). A distribution of actions described by (μ_1, μ_2) is a Generalized Nash Equilibrium (GNE) with variance constraint $V \le C$ if either:

(i) $V(\mu_1, \mu_2) < C$ and for any j = 1, 2 and any probability measure $\bar{\mu}'_i$, it holds:

$$\mathcal{K}_{\mu_1,\mu_2}^j \delta \mu_j \geq 0,$$

where $\delta \mu_i = \bar{\mu}'_i - \bar{\mu}_i$, or

(ii) $V(\mu_1, \mu_2) = C$ and for any j = 1, 2 and any probability measure $\bar{\mu}'_j$, it holds:

$$\mathcal{X}_{\mu_1,\mu_2}^j \delta \mu_j < 0 \implies \mathcal{L}_{\mu_1,\mu_2}^j \delta \mu_j > 0,$$

where $\delta \mu_i = \bar{\mu}'_i - \bar{\mu}_i$.

Remark 5. In the first case of the definition, any small group of players is not sufficient to increase the variance above C. Thus, if μ_1, μ_2 is an equilibrium, there is no profitable deviation, and the definition coincides with the equilibrium of Section 3. In the second case, μ_1, μ_2 is an equilibrium, if any profitable deviation for a small group of players, increases the variance above C.

Remark 6. Some notions of GNE for games with a continuum of players were already introduced in the literature ^{47,48,49}. However, these definitions assume a convex constraint set. Let us note that Definition 1 is neither a generalization nor a special case of the these definitions.

We then introduce a refinement of GNE, called non-singular GNE. It turns out that non-singular GNE are easier to compute.

Definition 2 (non-singular GNE). A pair (μ_1, μ_2) is variance stationary if either:

(i) for all $\delta \mu_1 = \bar{\mu}_1' - \bar{\mu}_1$, with $\bar{\mu}_1'$ probability measure, it holds:

$$\mathcal{L}^1_{\mu_1,\mu_2}\delta\mu_1\geq 0,$$

or

(ii) for all $\delta \mu_2 = \bar{\mu}_2' - \bar{\mu}_2$, with $\bar{\mu}_2'$ probability measure, it holds:

$$\mathcal{L}^2_{\mu_1,\mu_2}\delta\mu_2 \geq 0.$$

We call a GNE (μ_1, μ_2) non-singular if it is not variance stationary.

Lemma 2. Assume that (μ_1, μ_2) is not variance stationary. Then, (μ_1, μ_2) is a Generalized Nash equilibrium with variance constraint $V \le C$ if and only if either it satisfies (i) of Definition 1 or $V(\mu_1, \mu_2) = C$ and for any probability measure $\bar{\mu}'$, it holds:

$$\mathcal{X}_{\mu_1,\mu_2}^j \delta \mu_j < 0 \implies \mathcal{L}_{\mu_1,\mu_2}^j \delta \mu_j \ge 0, \tag{9}$$

where $\delta \mu_j = \bar{\mu}'_j - \bar{\mu}_j$.

Proof: The direct part is immediate. Assume that the converse is not true, that is, there is a probability measure $\bar{\mu}'_i$ such that:

$$\mathcal{K}^{j}_{\mu_{1},\mu_{2}}(\bar{\mu}'_{j}-\bar{\mu}_{j})<0,\;\mathcal{L}^{j}_{\mu_{1},\mu_{2}}(\bar{\mu}'_{j}-\bar{\mu}_{j})=0.$$

Then, since (μ_1, μ_2) is not variance stationary, there is a probability measure $\bar{\mu}_j''$ such that $\mathcal{L}_{\mu_1, \mu_2}^j(\bar{\mu}_j'' - \bar{\mu}_j) < 0$. Hence, there is a $\theta \in (0, 1)$ such that:

$$\mathcal{K}^{j}_{\mu_{1},\mu_{2}}(\theta\bar{\mu}'_{j}+(1-\theta)\bar{\mu}''_{j}-\bar{\mu}_{j})<0,\ \mathcal{L}^{j}_{\mu_{1},\mu_{2}}(\theta\bar{\mu}'_{j}+(1-\theta)\bar{\mu}''_{j}-\bar{\mu}_{j})<0.$$

But this contradicts (9).

Assume that $V_A(\mu_1, \mu_2) = C$ and (μ_1, μ_2) is not variance stationary. Then, (μ_1, μ_2) is an equilibrium if and only if there is no $\delta \mu_j = \bar{\mu}_j' - \bar{\mu}_j$ such that:

$$\mathcal{K}^{j}_{\mu_1,\mu_2}\delta\mu_j < 0 \text{ and } \mathcal{L}^{j}_{\mu_1,\mu_2}\delta\mu_j < 0. \tag{10}$$

The following proposition characterizes the non-singular GNE in terms of measures supported on at most two points.

Proposition 3. If there is a probability measure $\bar{\mu}'_j$ such that (10) holds true, then there is another probability measure $\bar{\mu}''_j$ supported on at most two points which also satisfies (10) with $\delta \mu = \bar{\mu}'' - \bar{\mu}$.

Let us introduce the following quantities:

$$g_{j,\mu_1,\mu_2}^{\mathcal{X}}(u) = \mathcal{X}_{\mu_1,\mu_2}^{j}(d_u - \bar{\mu}_j), \quad g_{j,\mu_1,\mu_2}^{\mathcal{L}}(u) = \mathcal{L}_{\mu_1,\mu_2}^{j}(d_u - \bar{\mu}_j),$$

where d_u is a Dirac measure supported on u. Using these quantities we have the following necessary (Corollary 3), and necessary and sufficient conditions (Corollary 4).

Corollary 3. If (μ_1, μ_2) is a GNE then for all $u \in [u_m, u_M]$, j = 1, 2 if $g_{j,\mu_1,\mu_2}^{\mathcal{X}}(u) < 0$ then $g_{j,\mu_1,\mu_2}^{\mathcal{L}}(u) \ge 0$.

Corollary 4. A non variance stationary pair (μ_1, μ_2) is a GNE if and only if for all $u', u'' \in [u_m, u_M], \rho \in [0, 1], j = 1, 2$ if $\rho g_{j,\mu_1,\mu_2}^{\mathcal{X}}(u') + (1 - \rho) g_{j,\mu_1,\mu_2}^{\mathcal{X}}(u'') < 0$ then $\rho g_{j,\mu_1,\mu_2}^{\mathcal{L}}(u') + (1 - \rho) g_{j,\mu_1,\mu_2}^{\mathcal{L}}(u'') \geq 0$.

Remark 7. The proposed formulation describes pro-social behaviors in terms of bounding the variance of the costs. There are various alternative formulations. For example, people may bound the maximum number of infected individuals, reflecting the bounded capacity of the healthcare systems. Another alternative would be to consider altruistic players ⁵². Finally, pro-social behavior can be modeled as Kantian behavior ⁵³. We chose to model pro-social behavior as bounding the variance, because of the vast health inequities created by the current COVID-19 pandemic.

5 | COMPUTATIONAL STUDY

We then present some numerical results. In Subsection 5.1, we compute numerically the Nash equilibria of the unconstrained game providing two illustrative examples and in Subsection 5.2, we study an example for the variance constrained game.

5.1 | Computing Unconstrained Nash Equilibria

The computation of the value of $F(\bar{u}_1, \bar{u}_2)$ corresponds to the numerical integration of (5). The search for pure Nash equilibria needs just the computation of F(0, 1), F(1, 0), and F(1, 1) and checking the corresponding inequalities.

Let us then describe the procedure to find equilibria in the form of the $(\tilde{u}_1, 0)$ or $(\tilde{u}_1, 1)$. We have first to find the solutions of:

$$H_{\tilde{u}_2}(\tilde{u}_1) = G_1(1 - I_0)(e^{-u_m F(\tilde{u}_1, \tilde{u}_2)} - e^{-u_M F(\tilde{u}_1, \tilde{u}_2)}) - (u_M - u_m)(s_{i1}(u_m + (u_M - u_m)\tilde{u}_1) + s_{i2}\bar{u}_2) = 0,$$

with respect to \tilde{u}_1 , for a fixed value of $\tilde{u}_2 = 0$, 1. To do so we use line search (an alternative, would be to use a multi-start Newton algorithm). Having found a solution of $H_{\tilde{u}_2}(\tilde{u}_1) = 0$ for $\tilde{u}_2 = 0$ or $\tilde{u}_2 = 1$ we need also to check the corresponding inequality. The computation of possible equilibria in the form $(0, \tilde{u}_2)$ or $(1, \tilde{u}_2)$ is similar.

Let us compute any possible Nash equilibrium where both types use mixed strategies (internal Nash equilibria). Then we should have:

$$G_{1}(1-I_{0})(e^{-u_{m}F(\bar{u}_{1},\bar{u}_{2})}-e^{-u_{M}F(\bar{u}_{1},\bar{u}_{2})}) = (u_{M}-u_{m})(s_{11}\bar{u}_{1}+s_{12}\bar{u}_{2})$$

$$G_{2}(1-I_{0})(e^{-u_{m}F(\bar{u}_{1},\bar{u}_{2})}-e^{-u_{M}F(\bar{u}_{1},\bar{u}_{2})}) = (u_{M}-u_{m})(s_{21}\bar{u}_{1}+s_{22}\bar{u}_{2})$$

$$(11)$$

Any solution of this equation should belong to the line:

$$G_2(s_{11}\bar{u}_1 + s_{12}\bar{u}_2) = G_1(s_{21}\bar{u}_1 + s_{22}\bar{u}_2)$$

or equivalently:

$$(G_2s_{11} - G_1s_{21})\tilde{u}_1 + (G_2s_{12} - G_1s_{22})\tilde{u}_2 = \frac{G_1(s_{21} + s_{22}) - G_2(s_{11} + s_{12})}{u_M - u_m}u_m. \tag{12}$$

Therefore, to find any internal Nash equilibria, we examine using line search if there are solutions of (11) on the line (12). We then present some examples with concrete values for the parameters.

Example 1: The parameters are T = 100, r = 5/16, $I_0 = 0.01$, $n_1 = 0.8$, $n_2 = 0.2$, $\alpha_1 = \alpha_2 = 1/8$. These parameters correspond to an epidemic with basic reproduction number $R_0 = 2.5$, where people remain infectious for a mean time of 8 days 30 . The s parameters are $s_{11} = s_{21} = 2$ and $s_{12} = s_{22} = 0.5$. We assume that $u_M = 0.8$ and $u_m = 0.5$. We compute the equilibria for different values of G_1 and G_2 , assuming that $G_2/G_1 = 10$. This choice roughly corresponds to the infection fatality risks of the older people compared with the infection fatality risks of younger people 36 . The variation described corresponds to varied ways that people may weight health, money and well being.

The equilibria of the game are presented in Figure 2. We observe for all the values of G_1 , G_2 with $G_2/G_1=10$ there is a unique Nash equilibrium. When G_1 , G_2 are small, the equilibrium strategies are $\tilde{u}_1=\tilde{u}_2=1$. Then, as G_1 , G_2 become larger, there is a mixed Nash equilibrium $(1,\tilde{u}_2)$. For intermediate values of G_1 , G_2 , there is a unique equilibrium with $\tilde{u}_1=1$ and $\tilde{u}_2=0$. For larger values of G_1 , G_2 the equilibrium has the form $\tilde{u}_1=\tilde{u}_2=0$. Finally for large G_1 , G_2 there is a unique equilibrium $\tilde{u}_1=0$ and $\tilde{u}_2=0$.

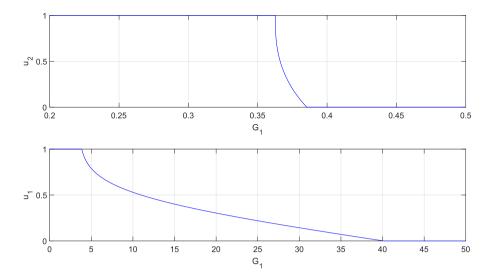


FIGURE 2 The upper part of the image presents the equilibria when $G_1 \in [0.2, 0.5]$. In this region, players of type 1 (non-vulnerable players) play $u = u_M$ and players of type 2 randomize between u_m and u_M . The probability to play u_M denoted by \tilde{u}_2 is illustrated in the upper part. Similarly the lower part presents the equilibria for $G_1 \in [0.5, 50]$. In this region, all the players of type 2 play $u = u_m$, while the players of type 1 randomize with probability \tilde{u}_1 .

Example 2: In this example there is a strong homophily. Particularly, $s_{11} = s_{22} = 2$ and $s_{21} = s_{12} = 0.5$. The rest of the parameters are as in Example 1, including the fact that $G_2/G_1 = 10$. The equilibria for various values of are presented in Figure 3. For low values of G_1 there is a unique pure Nash equilibrium, where all the players play u_M . Then around $G_1 = 0.292$, in addition to the pure equilibrium (1,1), a pair of equilibria appears. One of the new equilibra is pure and the other is mixed. In the new pure equilibrium all the players of type 1 play u_M and all the players of type 2 play u_m . In the mixed equilibrium, players of type 2 randomize, that is the mixed equilibrium has the form $(1,\tilde{u}_2)$. As G_1 becomes larger the value of \tilde{u}_2 increases and eventually, around $G_1 = 0.363$, the mixed equilibrium $(1,\tilde{u}_2)$ meets with the pure equilibrium (1,1) and they both disappear. Then, for $G_1 \in [0,363,3.8]$ there is a unique Nash equilibrium where all the players of type 1 play u_M and players of type 2 play u_m . On the interval $G_1 \in [3.8,40.1]$, there is a unique mixed Nash equilibrium where all the players of type 2 play u_m and the players of type 1 randomize. For larger values of G_1 all the players play u_m .

5.2 | Computing Constrained Equilibria

We then search for generalized Nash equilibria with variance constraints. Let us first note that if the pair (μ_1, μ_2) satisfies Definition 1.(i) then it also is an unconstrained Nash equilibrium. Thus, it is sufficient to check if the Nash equilibria computed in the previous section satisfy the constraint $V(\mu_1, \mu_2) \le C$.

We then compute GNE, satisfying Definition 1.(ii), in the case where the policies (μ_1, μ_2) are of the form $\mu_1 = n_1 \mathcal{d}_{u_1}$, $\mu_2 = n_2 \mathcal{d}_{u_2}$, where \mathcal{d}_u is a Dirac measure concentrated on u. We use a grid to find the pairs (u_1, u_2) such that the equality $V(\mu_1, \mu_2) = C$ holds approximately. These points are candidates for GNE. For each of these points in the grid, we compute the functions $g_{j,\mu_1,\mu_2}^{\mathcal{H}}(u)$ and $g_{j,\mu_1,\mu_2}^{\mathcal{H}}(u)$ for j=1,2 and a grid of points u. The details of the computation of $g_{j,\mu_1,\mu_2}^{\mathcal{H}}(u)$ and $g_{j,\mu_1,\mu_2}^{\mathcal{H}}(u)$ are given in Appendix D. We then use Corollary 4 to check weather each of these points is a GNE.

Example 3: In this example the parameters are as in Example 1, and the vulnerability parameters $G_1 = 8$ and $G_2 = 80$. The unconstrained Nash equilibrium is $\tilde{u}_1 = 0.602$, $\tilde{u}_2 = 0$. The cost for the non-vulnerable and vulnerable players under the Nash equilibrium are $J_1 = 0.185$ and $J_2 = 9.12$ respectively.

The GNE under the constraint $V \le C$, for various values of C is illustrated in Figure 4. We observe that for some values of C there are multiple GNE. Figure 5 shows how costs of the non-vulnerable and vulnerable players vary as a function of C. We observe that, as the value of the constraint C becomes smaller the cost of the vulnerable players decreases monotonically. Furthermore, compared to the unconstrained case, the cost of the non-vulnerable players under the variance constrained is

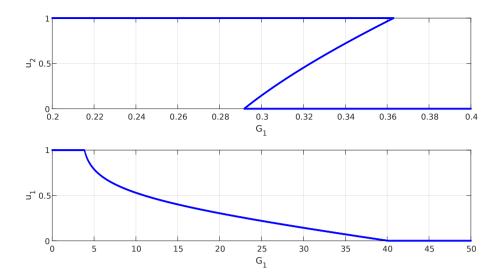


FIGURE 3 The figure presents the equilibria of the game. The upper part present \tilde{u}_2 , when $G_1 \in [0.2, 0.4]$. In this interval all the players of type 1 play u_M , that is $\tilde{u}_1 = 1$. There are at most three equilibria. The equilibria for $G_1 \in [0.4, 50]$ are presented in the lower part of the figure. In this region all the players of type 2 play $u = u_m$, while the players of type 1 randomize with probability \tilde{u}_1 .

improved as well. Figure 6 illustrates the evolution of the epidemic for various values of C. We observe that, as the constraint becomes more restrictive i.e., as C decreases the prevalence of the epidemic decreases as well.

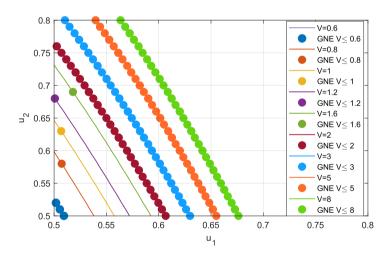


FIGURE 4 The contour line V = C, for various values of C and the corresponding GNE.

6 | CONCLUSION

We analyzed social distancing games, involving vulnerable and non-vulnerable populations of players, characterized the Nash equilibria and investigated how inequality constraints influence the epidemic spread and the costs of the players. We also defined

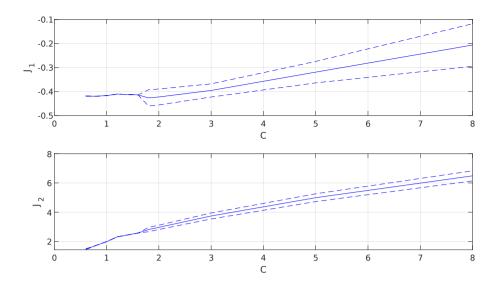


FIGURE 5 The costs for vulnerable and non-vulnerable players for various values of *C*. The dashed lines correspond to the minimum and maximum values for all the equilibria and the solid lines for an average value.

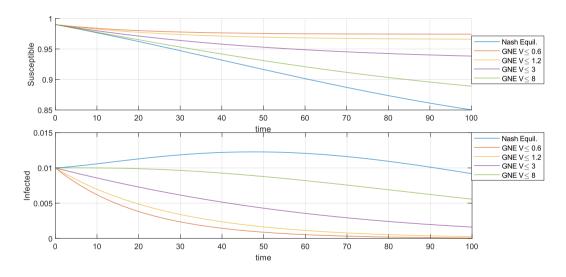


FIGURE 6 The evolution over time for the total number of susceptible and infected persons for the Nash equilibrium and the GNE for various values of *C*.

a Generalized Nash equilibrium concept for non-atomic games with variance constraints, and characterized it in terms of single-point-supported deviations. Inequality constraints are always beneficial for the vulnerable players, and in some cases they could be beneficial for the non-vulnerable players as well. Furthermore, inequality constraints delay the spread of epidemics and reduce its prevalence.

There are several directions for future research. First, the model can be generalized, including many classes of players having different vulnerabilities, minimum actions, degrees, etc. Another direction is to use real data to check the predictions of the model, the modeling of the dynamic response of players and the study of other applications of the variance constrained games defined.

APPENDIX

A PROOF OF PROPOSITION 1

Equation (1) can be written as:

$$\dot{x}_{1u} = -rux_{1u} \cdot (\mathcal{M}x)
\dot{x}_{2u} = -rux_{2u} \cdot (\mathcal{M}x)
\dot{x}_{3u} = rux_{1u} \cdot (\mathcal{M}x) - \alpha_1 x_{3u},
\dot{x}_{4u} = rux_{2u} \cdot (\mathcal{M}x) - \alpha_1 x_{4u}
\dot{x}_5 = \mathcal{M}x$$
(A1)

where $\mathcal{M} \in X^*$ with:

$$\mathcal{M}x = \int x_{3u} u \mu_1(du) + \int x_{4u} u \mu_2(du).$$

Note that $\|\mathcal{M}\| \le n_1 + n_2$. The initial conditions are $x_{1u}(0) = x_{2u}(0) = 1 - I_0$, $x_{3u}(0) = x_{4u}(0) = I_0$, for all $u \in [u_m, u_M]$ and $x_5(0) = 0$. In compact form we write $\dot{x} = f_{\mu_1, \mu_2}(x)$.

Lemma 3. Any solution of (A1) with the given initial conditions satisfies $0 \le x_{1u}, \dots, x_{4u} \le 1$. Let us denote this set by X_0 i.e., $X_0 = \{x \in X : 0 \le x_{1u}, \dots, x_{4u} \le 1, \text{ for all } u\}$.

Proof: Consider such a solution. Observe that, $x_{1u}(t), x_{2u}(t) \ge 0$ for all u and t. Similarly, since $x \ge 0$ implies $\mathcal{M}x \ge 0$ we have $x_{3u}(t), x_{4u}(t) \ge 0$ for all t, u. Finally, $\dot{x}_{1u} + \dot{x}_{3u} \le 0$ and thus, $x_{3u}(t) \le x_{1u}(0) + x_{3u}(0) = 1$. Similarly, $x_{4u}(t) \le 1$.

Lemma 4. Let x(t) be a solution of (A1) with the given initial conditions. Then, $x_{iu}(t)$ is continuous on u, for $j = 1, \dots, 4$.

Proof: Let $h(t) = \mathcal{M}x(t)$. Then, $0 \le h(t) \le n_1 + n_2$. The solution of $\dot{x}_{1u} = -ruh(t)x_{1u}$ given by:

$$x_{1u}(t) = (1 - I_0) \exp\left(-ru \int_0^t h(s)ds\right)$$

depends continuously on u, t. A similar argument shows that x_{2u} is continuous on u, t. Then, observe that the third equation of (A1) can be written as:

$$\dot{x}_{3u} = b(u,t) - \alpha_1 x_{3u},$$

where $b(u,t) = rux_{1u} \cdot \mathcal{M}x$ is continuous on u and bounded by $u_M r(n_1 + n_2)(1 - I_0)$. The solution of this differential equation is:

$$x_{3u}(t) = x_{3u}(0)e^{-\alpha_1 t} + \int_0^t e^{\alpha_1(\tau - t)}b(u, \tau)dt,$$

which is again continuous on u, t. A similar argument shows that x_{4u} is continuous on u, t.

We then proceed to the proof of the proposition. Consider a saturated version of (A1):

$$\begin{split} \dot{x}_{1u} &= -ru \operatorname{sat}_1(x_{1u}) \cdot \operatorname{sat}_2(\mathcal{M}x) \\ \dot{x}_{2u} &= -ru \operatorname{sat}_1(x_{2u}) \cdot \operatorname{sat}_2(\mathcal{M}x) \\ \dot{x}_{3u} &= ru \operatorname{sat}_1(x_{1u}) \cdot \operatorname{sat}_2(\mathcal{M}x) - \alpha_1 x_{3u}, \\ \dot{x}_{4u} &= ru \operatorname{sat}_1(x_{2u}) \cdot \operatorname{sat}_2(\mathcal{M}x) - \alpha_2 x_{4u} \\ \dot{x}_5 &= \mathcal{M}x \end{split} \tag{A2}$$

where $\operatorname{sat}_1(z) = \max\{\min\{z,1\},0\}$ and $\operatorname{sat}_2(z) = \max\{\min\{z,n_1+n_2\},0\}$. Due to Lemma 3, any solution of (A1), with the given initial conditions, is also a solution of the modified system. Denote this system in compact form as $\dot{x} = \tilde{f}(x)$. It is not difficult to see that $\tilde{f}: X \to X$ is Lipschitz with constant:

$$L = 2ru_M(n_1 + n_2) + \alpha_1 + \alpha_2.$$

Thus, Theorem 7.3 of Brezis $(2010)^{54}$ implies that there exists a unique solution within the space C([0,T],X) of continuous functions with values in X. Lemma 4, along with the compactness of $[u_m,u_M]$ show that there is no solution of (A1) not belonging to C([0,T],X).

Remark 8. Note that \tilde{f} is uniformly Lipschitz. The constant is uniform in μ_1, μ_2 , for positive measures of total mass n_1, n_2 .

B PROOF OF LEMMA 1

We then proceed to the proof Lemma 1 and the computation of a formula for the directional derivative of the variance. To this end, we first consider the sensitivity of the solution of (1) with respect to the deviation $\varepsilon \delta \mu_i$.

We first compute the directional derivative of $f_{\mu_1,\mu_2}(x)$ in the direction $\delta\mu$. The value of $f_{\mu_1,\mu_2}(x)$ depends on μ_j through the quantity $I^f(t)$. This quantity can be written as:

$$I^{f}(t) = \mathcal{T}_{\mu_{i}}I_{j\cdot}(t) + \mathcal{T}_{\mu_{i}}I_{-j\cdot}(t),$$

where $I_{j.}(t):[u_m,u_M]\to\mathbb{R}$ with $u\mapsto I_{ju}(t)$ and $\mathcal{T}_{\mu}\in (C([u_m,u_M],\mathbb{R}))^{\star}$ with:

$$\mathcal{T}_{\mu}I_{j.}(t) = \int I_{ju'}(t)u' \cdot \mu(du'). \tag{B3}$$

The directional derivative of $f_{\mu_1,\mu_2}(x)$ in the direction $\delta \mu_i$ is:

$$\begin{split} \mathcal{D}_{\delta\mu_{j}}[f_{\mu_{1},\mu_{2}}(x)] = \\ [-ruS_{ju}\mathcal{T}_{\delta\mu_{i}}I_{j.}, & -ruS_{(-j)u}\mathcal{T}_{\delta\mu_{i}}I_{j.}, & ruS_{ju}\mathcal{T}_{\delta\mu_{i}}I_{j.}, & ruS_{(-j)u}\mathcal{T}_{\delta\mu_{i}}I_{j.}, & \mathcal{T}_{\delta\mu_{i}}I_{j.}]^{T}. \end{split}$$

Observe that, due to the form of $\mathcal{T}_{\delta\mu_i}$ in (B3), there is a bounded linear operator $\mathcal{B}_{\mu_1,\mu_2}(t): Y \to X$ such that:

$$\mathcal{D}_{\delta\mu_i}[f_{\mu_1,\mu_2}(x(t))] = \mathcal{B}_{\mu_1,\mu_2}(t)\delta\mu_i.$$

The linearized version of (1), around the trajectories $(S_{1u}, S_{2u}, I_{1u}, I_{2u})$, is given by:

$$\begin{split} \dot{x}'_{1u} &= -(ruI^f) x'_{1u} - ruS_{1u} (\mathcal{T}_{\mu_1} x'_{3.} + \mathcal{T}_{\mu_2} x'_{4.}) - ruS_{1u} \mathcal{T}_{\delta \mu_j} I_j. \\ \dot{x}'_{2u} &= -(ruI^f) x'_{2u} - ruS_{2u} (\mathcal{T}_{\mu_1} x'_{3.} + \mathcal{T}_{\mu_2} x'_{4.}) - ruS_{2u} \mathcal{T}_{\delta \mu_j} I_j. \\ \dot{x}'_{3u} &= (ruI^f) x'_{1u} + ruS_{1u} (\mathcal{T}_{\mu_1} x'_{3.} + \mathcal{T}_{\mu_2} x'_{4.}) - \alpha_1 x'_{3u} + ruS_{1u} \mathcal{T}_{\delta \mu_j} I_j. \\ \dot{x}'_{4u} &= (ruI^f) x'_{2u} + ruS_{2u} (\mathcal{T}_{\mu_1} x'_{3.} + \mathcal{T}_{\mu_2} x'_{4.}) - \alpha_2 x'_{4u} + ruS_{2u} \mathcal{T}_{\delta \mu_j} I_j. \end{split}$$

$$\dot{x}'_5 &= \mathcal{T}_{\mu_1} x'_{3u} + \mathcal{T}_{\mu_2} x'_{4u} + \mathcal{T}_{\delta \mu_j} I_j.$$
(B4)

where $I^f = I^f_{\mu_1,\mu_2}$ as in (2). Thus, (B4) is a LTV system in the form:

$$\dot{x}' = \mathcal{A}_{\mu_1, \mu_2}(t)x' + \mathcal{B}_{\mu_1, \mu_2}(t)\delta\mu_j.$$
(B5)

Lemma 5. Denote by $\phi_{\mu_1,\mu_2}(t)$ the solution of (1). Then, the directional derivatives $x'_{lu} = \mathcal{D}_{\delta\mu_j}\phi^{lu}_{\mu_1,\mu_2}(t)$, for $l=1,\ldots,4$ and $x'_5 = \mathcal{D}_{\delta\mu_l}\phi^5_{\mu_1,\mu_2}(t)$ satisfy (B5), with zero initial conditions.

Proof: Consider the function: $\bar{f}: Y \times X \to X$ with:

$$\bar{f}(\bar{\mu},x)=f_{\bar{\mu},\mu_{-i}}(x)$$

and the set $D=X_0\times\{\mu\in Y\,:\,\|\mu\|=n_j\}$

The function $\bar{f}(\bar{\mu}, x)$ is continuously Fréchet differentiable with respect to x. Its derivative is the operator $\mathcal{A}_{\mu_1, \mu_2}(t)$ given in (B4)–(B5), which is continuous and bounded in D. The directional derivative:

$$\mathcal{D}_{\delta\mu_j}[\bar{f}(\bar{\mu},x)] = \mathcal{B}_{\mu_1,\mu_2}(t)\delta\mu_j,$$

is also continuous in x and bounded in D. Furthermore, X_0 is positively invariant. Thus, Theorem 1 of Banks et al. $(2006)^{55}$ applies and the proof is complete.

Therefore, F is continuous in μ_i and $J_i(u', \mu_i, \mu_{-i})$ is continuous in μ_i , uniformly in u'. This proves (8).

Lemma 6. There is a continuous function $f_{j,F,\mu_1,\mu_2}^{\text{var}}(u)$ such that the directional derivative of $F(\mu_1,\mu_2)$ on the $\delta\mu_j$ direction is given by:

$$\mathcal{D}_{\delta\mu_{j}}F(\mu_{1},\mu_{2}) = rx'_{5}(T) = \int f_{j,F,\mu_{1},\mu_{2}}^{\text{var}}(u')\delta\mu_{j}(du').$$

Proof: Observe that $\mathscr{B}_{\mu_1,\mu_2}(t): Y \to X$ can be written as a composition of two linear operators $\mathscr{B}^1_{\mu_1,\mu_2}(t): Y \to \mathbb{R}$ and $\mathscr{B}^2_{\mu_1,\mu_2}(t): \mathbb{R} \to X$ with: $\mathscr{B}^1_{\mu_1,\mu_2}(\delta\mu) = \mathscr{T}_{\delta\mu_i}I_{j\cdot}$ and

$$\mathcal{B}_{\mu_1,\mu_2}^2(l) = \begin{bmatrix} -rux_{1u}l & -rux_{2u}l & rux_{1u}l & rux_{2u}l & l \end{bmatrix}^T.$$

The solution of the LTV system (B5) with zero initial conditions is written in the form (e.g. paragraph 1.4 of Deimling(2006)⁵⁶):

$$x'(T) = \int_{0}^{T} \Psi(T, t) \mathcal{B}_{\mu_1, \mu_2}(t) \delta \mu_j dt$$

where $\Psi(t, s) : X \to X$ is the state transition operator.

Denote by $\mathscr{C}: X \to \mathbb{R}$ the operator picking the last component i.e., $\mathscr{C}(x') = x'_5$. Then, $x'_5(T)$ can be written as:

$$x_{5}'(T) = \int_{0}^{T} [\mathscr{C}\Psi(T, t)\mathscr{B}_{\mu_{1}, \mu_{2}}^{2}(t)] \mathscr{B}_{\mu_{1}, \mu_{2}}^{1}(t) \delta \mu_{j} dt.$$

Note that $\psi_t = \mathscr{C}\Psi(T, t)\mathscr{B}^2_{\mu_1, \mu_2}$ is a linear function $\psi_t : \mathbb{R} \to \mathbb{R}$. Write this function as $\psi_t(l) = \varphi(t) \cdot l$. Thus:

$$x_5'(T) = \int_0^T \varphi(t) \int u x_{j+2,u}(t) \delta \mu_j(du) dt = \int \left[u \int_0^T \varphi(t) x_{j+2,u}(t) dt \right] \delta \mu_j(du)$$

Define:

$$f_{j,F,\mu_1,\mu_2}^{\text{var}}(u) = ru \int_{0}^{T} \varphi(t) x_{j+2,u}(t) dt$$

Observe that, since $x_{j+2,u}(t)$ is continuous with respect to u, the function $f_{j,F,\mu_1,\mu_2}^{\text{var}}(u)$ is also continuous in u. Thus:

$$\mathcal{D}_{\delta\mu_j}F(\mu_1,\mu_2) = rx_5'(T) = \int f_{j,F,\mu_1,\mu_2}^{\text{var}}(u)\delta\mu_j(du),$$

and the proof is complete

The directional derivative of \bar{u}_i is given by:

$$\mathcal{D}_{\delta\mu_j}\bar{u}_j=\int u'\delta\mu_j(du')/n_j.$$

The directional derivative of the cost of a player of type j' who uses an action u is given by:

$$\mathcal{D}_{\delta\mu_{j}}J_{j'}(u,\mu_{j},\mu_{-j}) = G_{j'}(1-I_{0})ue^{-uF}\mathcal{D}_{\delta\mu_{j}}F - s_{j'j}u\mathcal{D}_{\delta\mu_{j}}\bar{u}_{j}$$

$$= \int f_{j,J_{j'},\mu_{1},\mu_{2}}^{\text{var}}(u',u)\delta\mu_{j}(du'), \tag{B6}$$

where:

$$f_{j,J_{j'},\mu_1,\mu_2}^{\text{var}}(u',u) = G_{j'}(1-I_0)ue^{-uF}f_{j,F,\mu_1,\mu_2}^{\text{var}}(u') - s_{j'j}uu'/n_j,$$

is a continuous function of u', u. The directional derivative of the mean cost of the players of type j (the same with the type of the deviating players) is given by:

$$\begin{split} \mathcal{D}_{\delta\mu_{j}}\bar{J}_{j} &= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon n_{j}} \left[\int J_{j}(u,\mu_{j} + \varepsilon \delta \mu_{j},\mu_{-j})(\mu_{j} + \varepsilon \delta \mu_{j})(du) - \int J_{j}(u,\mu_{j},\mu_{-j})\mu_{j}(du) \right] \right] \\ &= \int \mathcal{D}_{\delta\mu_{j}}J_{j}(u,\mu_{j},\mu_{-j})\bar{\mu}_{j}(du) + \lim_{\varepsilon \to 0} \left[\frac{1}{n_{j}} \int J_{j}(u,\mu_{j} + \varepsilon \delta \mu_{j},\mu_{-j})\delta \mu_{j}(du) \right] \\ &= \int \mathcal{D}_{\delta\mu_{j}}J_{j}(u,\mu_{j},\mu_{-j})\bar{\mu}_{j}(du) + \frac{1}{n_{j}} \int J_{j}(u,\mu_{j},\mu_{-j})\delta \mu_{j}(du). \end{split}$$

Substituting (B6) into the last equation and using Fubini's theorem, we get:

$$\mathcal{D}_{\delta\mu_{j}}\bar{J}_{j} = \int \left[\frac{1}{n_{j}}J_{j}(u,\mu_{j},\mu_{-j}) + \int f_{J_{j},\mu_{1},\mu_{2}}^{\text{var}}(u',u)\bar{\mu}_{j}(du)\right]\delta\mu_{j}(du')$$

$$= \int f_{j,\bar{J}_{j},\mu_{1},\mu_{2}}^{\text{var}}(u')\delta\mu_{j}(du')$$
(B7)

Similarly:

$$\mathcal{D}_{\delta\mu_{j}}\bar{J}_{-j} = \int \left[\int f_{j,\bar{J}_{-j},\mu_{1},\mu_{2}}^{\text{var}}(u',u)\bar{\mu}_{-j}(du) \right] \delta\mu_{j}(du') = \int f_{j,\bar{J}_{-j},\mu_{1},\mu_{2}}^{\text{var}}(u')\delta\mu_{j}(du')$$

Combining the last two equations we compute the variation of the mean cost:

$$\mathcal{D}_{\delta\mu_{j}}\bar{J} = (n_{1}\mathcal{D}_{\delta\mu_{j}}\bar{J}_{1} + n_{2}\mathcal{D}_{\delta\mu_{j}}\bar{J}_{2})/(n_{1} + n_{2}) = \int f_{j,\bar{J},\mu_{1},\mu_{2}}^{\mathrm{var}}(u')\delta\mu_{j}(du').$$

We then proceed to the computation of the directional derivative of the variance in two steps. The first part is:

$$\begin{split} \mathcal{D}_{\delta\mu_{j}} \left[\int (J_{j}(u,\mu_{j},\mu_{-j}) - \bar{J})^{2} \mu_{j}(du) \right] &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\int (J_{j}(u,\mu_{j} + \varepsilon \delta \mu_{j},\mu_{-j}) - \bar{J}(\mu_{j} + \varepsilon \delta \mu_{j},\mu_{-j}))^{2} (\mu_{j} + \varepsilon \delta \mu_{j})(du) - \\ &- \int (J_{j}(u,\mu_{j},\mu_{-j}) - \bar{J}(\mu_{j},\mu_{-j}))^{2} \mu_{j}(du) \right] \\ &= \int (J_{j}(u,\mu_{j},\mu_{-j}) - \bar{J})^{2} \delta \mu_{j}(du) + 2 \int (J_{j}(u,\mu_{j},\mu_{-j}) - \bar{J}(\mu_{j},\mu_{-j})) \int [f_{j,J_{j},\mu_{1},\mu_{2}}^{\text{var}}(u',u) - f_{j,\bar{J},\mu_{1},\mu_{2}}^{\text{var}}(u')] \delta \mu_{j}(du') \mu_{j}(du) \\ &= \int f_{j,V_{j},\mu_{1},\mu_{2}}^{\text{var}}(u') \delta \mu_{j}(du'). \end{split}$$

In the second equality the interchange of the integral with the limit is possible, because the convergence $\lim_{\epsilon \to 0} [J_j(u, \mu_j + \epsilon \delta \mu_j, \mu_{-j}) - J_j(u, \mu_j, \mu_{-j})]/\epsilon$ is uniform in u. In the last equality we use again Fubini's theorem. Note that $f_{j,V_j,\mu_1,\mu_2}^{\text{var}}(u')$ is continuous in u'.

Similarly:

$$\begin{split} \mathscr{D}_{\delta\mu_{j}} \left[\int (J_{-j}(u,\mu_{j},\mu_{-j}) - \bar{J})^{2} \mu_{j}(du) \right] &= \\ &= 2 \int (J_{-j}(u,\mu_{j},\mu_{-j}) - \bar{J}(\mu_{j},\mu_{-j})) \int [f_{j,J_{-j},\mu_{1},\mu_{2}}^{\text{var}}(u',u) - f_{j,\bar{J},\mu_{1},\mu_{2}}^{\text{var}}(u')] \delta\mu_{j}(du') \\ &= \int f_{j,V_{-j},\mu_{1},\mu_{2}}^{\text{var}}(u') \delta\mu_{j}(du') \end{split}$$

Therefore, the directional derivative of the variance is written as:

$$\mathcal{D}_{\delta\mu_j}V(\mu_j,\mu_{-j}) = \int f_{j,V,\mu_1,\mu_2}^{\text{var}}(u')\delta\mu_j(du'),$$

for a continuous function $f_{j,V,\mu_1,\mu_2}^{\text{var}}(u')$.

C PROOF OF PROPOSITION 3

Let us drop the dependence on j. Inequalities (10), recalling that $\bar{\mu}'$ is a probability measure, can be written as:

$$\int \bar{f}_{\mathcal{X}}(u')\bar{\mu}'(du') < 0,$$

$$\int \bar{f}_{\mathcal{L}}(u')\bar{\mu}'(du') < 0,$$

where $\bar{f}_{\mathscr{R}}(u') = J_j(u', \mu_1, \mu_2) - \int J_j(u', \mu_1, \mu_2) \bar{\mu}(du')$, $\bar{f}_{\mathscr{L}}(u') = f_{j,V_j,\mu_1,\mu_2}^{\mathrm{var}}(u') - \int f_{j,V_j,\mu_1,\mu_2}^{\mathrm{var}}(u') \bar{\mu}(du')$. Using the (uniform) continuity of $\bar{f}_{\mathscr{R}}$, $\bar{f}_{\mathscr{R}}$ we have that for a set of u_1, \ldots, u_N and $c_1, \ldots, c_N > 0$ it holds:

$$c_1 \bar{f}_{\mathcal{H}}(u_1) + \dots + c_N \bar{f}_{\mathcal{H}}(u_N) < 0,$$

$$c_1 \bar{f}_{\mathcal{L}}(u_1) + \dots + c_N \bar{f}_{\mathcal{L}}(u_N) < 0,$$

If there is a point u_k where $\bar{f}_{\mathscr{R}}(u_k) < 0$ and $\bar{f}_{\mathscr{L}}(u_k) \leq 0$ or $\bar{f}_{\mathscr{R}}(u_k) \leq 0$ and $\bar{f}_{\mathscr{L}}(u_k) < 0$ we are done. Thus, assume that there is no such point. Excluding from the summation the terms where both $\bar{f}_{\mathscr{L}}(u_k)$ and $\bar{f}_{\mathscr{L}}(u_k)$ are non-negative the inequalities still hold true. Thus, assume that for any point u_k either $\bar{f}_{\mathscr{R}}(u_k) < 0 < \bar{f}_{\mathscr{L}}(u_k)$ or $\bar{f}_{\mathscr{L}}(u_k) < 0 < \bar{f}_{\mathscr{L}}(u_k)$. Reordering the terms, the inequalities can be written as:

$$x_1 + \dots + x_n - y_1 - \dots - y_m < 0$$

-z_1 - \dots - z_n + \omega_1 + \dots + \omega_m < 0 (C8)

where

$$\begin{split} & x_i = c_{k_i} \bar{f}_{\mathcal{X}}(u_{k_i}), \quad k_i \in \{k : c_k \bar{f}_{\mathcal{X}}(u_k) > 0\}, \\ & y_i = |c_{k_i} \bar{f}_{\mathcal{X}}(u_{k_i})|, \quad k_i \in \{k : c_k \bar{f}_{\mathcal{X}}(u_k) < 0\}, \\ & w_i = c_{k_i} \bar{f}_{\mathcal{L}}(u_{k_i}), \quad k_i \in \{k : c_k \bar{f}_{\mathcal{L}}(u_k) > 0\}, \\ & z_i = |c_{k_i} \bar{f}_{\mathcal{L}}(u_{k_i})|, \quad k_i \in \{k : c_k \bar{f}_{\mathcal{L}}(u_k) < 0\}. \end{split}$$

Let i_0 be such that $y_{i_0}/w_{i_0} \ge y_i/w_i$ for all i and denote $\alpha = y_{i_0}/w_{i_0}$.

Claim 1: There is a j_0 such that $\alpha z_{j_0} > x_{j_0}$. Indeed if this in not true, and $\alpha z_j \le x_j$ for all j, then multiplying the second relationship with α we get:

$$-\alpha z_1 - \dots - \alpha z_n + \alpha w_1 + \dots + \alpha w_m \ge -x_1 - \dots - x_n + y_1 + \dots + y_m > 0.$$

This completes the proof of the claim.

Claim 2: We may choose a $\rho \in [0, 1]$ such that:

$$\rho x_{j_0} - (1 - \rho) y_{i_0} < 0$$
$$-\rho z_{j_0} + (1 - \rho) w_{i_0} < 0$$

Indeed for ρ in the interval:

$$\frac{y_{i_0}}{\alpha z_{j_0} + y_{i_0}} < \rho < \frac{y_{i_0}}{x_{j_0} + y_{i_0}},$$

both inequalities are valid. The interval is not empty, since $\alpha z_{i_0} > x_{i_0}$.

Choose such a ρ . Then, using the form of x, y, z, w, there are k_1, k_2 such that:

$$\begin{split} & \rho c_{k_1} \bar{f}_{\mathcal{X}}(u_{k_1}) + (1 - \rho) c_{k_2} \bar{f}_{\mathcal{X}}(u_{k_2}) < 0, \\ & \rho c_{k_1} \bar{f}_{\mathcal{Z}}(u_{k_1}) + (1 - \rho) c_{k_2} \bar{f}_{\mathcal{X}}(u_{k_2}) < 0, \end{split}$$

Thus, taking

$$\bar{\mu}'' = \frac{\rho c_{k_1}}{\rho c_{k_1} + (1 - \rho) c_{k_2}} \mathcal{A}_{u_{k_1}} + \frac{(1 - \rho) c_{k_2}}{\rho c_{k_1} + (1 - \rho) c_{k_2}} \mathcal{A}_{u_{k_2}},$$

where d_u is the Dirac measure at u, and the proof is complete.

D COMPUTATION OF GNE

Consider a discrete set $U_d \subset [u_m, u_M]$ and a grid $U_d \times U_d$ of pairs (u_1, u_2) . We first find the set of pairs (u_1, u_2) on the grid, such that $V(n_1 \mathcal{A}_{u_1}, n_2 \mathcal{A}_{u_2}) = C$ approximately holds. The dynamics becomes:

$$\begin{split} \dot{S}_{1u_{1}} &= -ru_{1}I^{f}S_{1u_{1}}, & \dot{S}_{2u_{2}} &= -ru_{2}I^{f}S_{2u_{2}}, \\ \dot{I}_{1u_{1}} &= ru_{1}I^{f}S_{1u_{1}} - \alpha_{1}I_{1u_{1}}, & \dot{I}_{2u_{2}} &= ru_{2}I^{f}S_{2u_{2}} - \alpha_{2}I_{2u_{2}}, \\ \dot{z} &= I^{f}. \end{split} \tag{D9}$$

where $I^f(t) = u_1 n_1 I_{1u_1}(t) + u_2 n_2 I_{2u_2}(t)$.

For every point on the grid that $V(n_1 d_{u_1}, n_2 d_{u_2}) = C$ approximately holds, we compute the function $g_{j,u_1,u_2}^{\mathcal{X}}(u)$, for $u \in U$, as:

$$g_{i,u_1,u_2}^{\mathcal{H}}(u) = J_j(u, d_{u_1}, d_{u_2}) - J_j(u_j, d_{u_1}, d_{u_2}).$$

For j = 1, 2 and a grid of values of $u \in U$, we solve the pair of differential equations:

$$\dot{S}_{1u} = -ruI^f S_{1u}, \qquad \qquad \dot{I}_{1u} = ruI^f S_{1u} - \alpha_i I_{1u}$$

where I^f is the quantity computed during the solution of (D9).

Consider the variation $\delta \mu_j = \mathcal{A}_u - \mathcal{A}_{u_i}$ The linearized system around the solution of (D9) consists of 5 differential equations:

$$\begin{split} \dot{x}_{1u_{1}}' &= -(ru_{1}I^{f})x_{1u_{1}}' - ru_{1}S_{1u_{1}}(n_{1}u_{1}x_{3u_{1}}' + n_{2}u_{2}x_{4u_{2}}') - ru_{1}S_{1u_{1}}(I_{ju} - I_{ju_{j}}) \\ \dot{x}_{2u_{2}}' &= -(ruI^{f})x_{2u_{2}}' - ru_{2}S_{2u_{2}}(n_{1}u_{1}x_{3u_{1}}' + n_{2}u_{2}x_{4u_{2}}') - ru_{2}S_{2u_{2}}(I_{ju} - I_{ju_{j}}) \\ \dot{x}_{3u_{1}}' &= (ru_{1}I^{f})x_{1u_{1}}' + ru_{1}S_{1u_{1}}(n_{1}u_{1}x_{3u_{1}}' + n_{2}u_{2}x_{4u_{2}}') - \alpha_{1}x_{3u_{1}}' + ru_{1}S_{1u_{1}}(I_{ju} - I_{ju_{j}}), \\ \dot{x}_{4u_{2}}' &= (ru_{2}I^{f})x_{2u_{2}}' + ru_{2}S_{2u_{2}}(n_{1}u_{1}x_{3u_{1}}' + n_{2}u_{2}x_{4u_{2}}') - \alpha_{2}x_{4u_{2}}' + ru_{2}S_{2u_{2}}(I_{ju} - I_{ju_{j}}), \\ \dot{x}_{5}' &= n_{1}u_{1}x_{3u_{1}}' + n_{2}u_{2}x_{4u_{2}}' + I_{ju} - I_{ju_{1}}. \end{split}$$

with zero initial conditions. It holds $\mathcal{D}_{\delta\mu_j}F(\mu_1,\mu_2)=x_5'(T)$, and $\mathcal{D}_{\delta\mu_j}\bar{u}_j=(u-u_j)/n_j$. The directional derivative $\mathcal{D}_{\delta\mu_i}J_{j'}(u,\mu_j,\mu_{-j})$ is given in (B6) and the directional derivative $\mathcal{D}_{\delta\mu_i}\bar{J}_j$ by:

$$\mathcal{D}_{\delta\mu_j}\bar{J}_j = \mathcal{D}_{\delta\mu_j}J_j(u_j,\mu_j,\mu_{-j}) + \frac{1}{n_j}(J_j(u,\mu_j,\mu_{-j}) - J_j(u_j,\mu_j,\mu_{-j})).$$

Furthermore $\mathcal{D}_{\delta\mu_j}\bar{J}_{-j}=\mathcal{D}_{\delta\mu_j}J_{-j}(u_{-j},\mu_j,\mu_{-j})$ and $\mathcal{D}_{\delta\mu_j}\bar{J}=(n_1\mathcal{D}_{\delta\mu_j}\bar{J}_1+n_2\mathcal{D}_{\delta\mu_j}\bar{J}_2)/(n_1+n_2)$. The directional derivative of the variance can be written as:

$$\mathcal{D}_{\delta\mu_j}V=(\mathcal{D}_{\delta\mu_j}V_1+\mathcal{D}_{\delta\mu_j}V_2)/(n_1+n_2),$$

where

$$\begin{split} \mathcal{D}_{\delta\mu_j}V_1 &= \mathcal{D}_{\delta\mu_j} \left[\int (J_j(u',\mu_j,\mu_{-j}) - \bar{J})^2 \mu_j(du') \right] \\ &= (J_j(u,\mu_j,\mu_{-j}) - \bar{J})^2 - (J_j(u_j,\mu_j,\mu_{-j}) - \bar{J})^2 + \\ &\quad + 2n_j(J_j(u_j,\mu_j,\mu_{-j}) - \bar{J}(\mu_j,\mu_{-j})) \left[\mathcal{D}_{\delta\mu_j} J_j(u_j,\mu_j,\mu_{-j}) - \mathcal{D}_{\delta\mu_j} \bar{J} \right], \end{split}$$

and:

$$\begin{split} \mathcal{D}_{\delta\mu_{j}}V_{2} &= \mathcal{D}_{\delta\mu_{j}} \left[\int (J_{-j}(u',\mu_{j},\mu_{-j}) - \bar{J})^{2} \mu_{-j}(du') \right] \\ &= 2n_{-j}(J_{-j}(u_{-j},\mu_{j},\mu_{-j}) - \bar{J}(\mu_{j},\mu_{-j})) \left[\mathcal{D}_{\delta\mu_{j}} J_{-j}(u_{-j},\mu_{j},\mu_{-j}) - \mathcal{D}_{\delta\mu_{j}} \bar{J} \right] \end{split}$$

We then compute the values of $g_{j,\mu_1,\mu_2}^{\mathcal{L}}(u)$, for $u \in U$ and use Corollary 4 to check if u_1, u_2 is a GNE.

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