

# Overall error analysis for the training of deep neural networks via stochastic gradient descent with random initialisation

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## Abstract

In spite of the accomplishments of deep learning based algorithms in numerous applications and very broad corresponding research interest, at the moment there is still no rigorous understanding of the reasons why such algorithms produce useful results in certain situations. A thorough mathematical analysis of deep learning based algorithms seems to be crucial in order to improve our understanding and to make their implementation more effective and efficient. In this article we provide a mathematically rigorous full error analysis of deep learning based empirical risk minimisation with quadratic loss function in the probabilistically strong sense, where the underlying deep neural networks are trained using stochastic gradient descent with random initialisation. The convergence speed we obtain is presumably far from optimal and suffers under the curse of dimensionality. To the best of our knowledge, we establish, however, the first full error analysis in the scientific literature for a deep learning based algorithm in the probabilistically strong sense and, moreover, the first full error analysis in the scientific literature for a deep learning based algorithm where stochastic gradient descent with random initialisation is the employed optimisation method.

*Keywords:* deep learning, deep neural networks, empirical risk minimisation, full error analysis, approximation, generalisation, optimisation, strong convergence, stochastic gradient descent, random initialisation

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# 1 Introduction

Deep learning based algorithms have been applied extremely successfully to overcome fundamental challenges in many different areas, such as image recognition, natural language processing, game intelligence, autonomous driving, and computational advertising, just to name a few. In line with this, researchers from a wide range of different fields, including, for example, computer science, mathematics, chemistry, medicine, and finance, are investing significant efforts into studying such algorithms and employing them to tackle challenges arising in their fields. In spite of this broad research interest and the accomplishments of deep learning based algorithms in numerous applications, at the moment there is still no rigorous understanding of the reasons why such algorithms produce useful results in certain situations. Consequently, there is no rigorous way to predict, before actually implementing a deep learning based algorithm, in which situations it might perform reliably and in which situations it might fail. This necessitates in many cases a trial-and-error approach in order to move forward, which can cost a lot of time and resources. A thorough mathematical analysis of deep learning based algorithms (in scenarios where it is possible to formulate such an analysis) seems to be crucial in order to make progress on these issues. Moreover, such an analysis may lead to new insights that enable the design of more effective and efficient algorithms.

The aim of this article is to provide a mathematically rigorous full error analysis of deep learning based empirical risk minimisation with quadratic loss function in the probabilistically strong sense, where the underlying deep neural networks (DNNs) are trained using stochastic gradient descent (SGD) with random initialisation (cf. Theorem 1.1 below). For a brief illustration of deep learning based empirical risk minimisation with quadratic loss function, consider natural numbers  $d, \mathbf{d} \in \mathbb{N}$ , a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , random variables  $X: \Omega \rightarrow [0, 1]^d$  and  $Y: \Omega \rightarrow [0, 1]$ , and a measurable function  $\mathcal{E}: [0, 1]^d \rightarrow [0, 1]$  satisfying  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X) = \mathbb{E}[Y|X]$ . The goal is to find a DNN with appropriate architecture and appropriate parameter vector  $\theta \in \mathbb{R}^{\mathbf{d}}$  (collecting its weights and biases) such that its realisation  $\mathcal{N}_\theta: \mathbb{R}^d \rightarrow \mathbb{R}$  approximates the target function  $\mathcal{E}$  well in the sense that the error  $\mathbb{E}[|\mathcal{N}_\theta(X) - \mathcal{E}(X)|^p] = \int_{[0,1]^d} |\mathcal{N}_\theta(x) - \mathcal{E}(x)|^p \mathbb{P}_X(dx) \in [0, \infty)$  for some  $p \in [1, \infty)$  is as small as possible. In other words, given  $X$  we want  $\mathcal{N}_\theta(X)$  to predict  $Y$  as reliably as possible. Due to the well-known bias–variance decomposition (cf., e.g., Beck, Jentzen, & Kuckuck [10, Lemma 4.1]), for the case  $p = 2$  minimising the error function  $\mathbb{R}^{\mathbf{d}} \ni \theta \mapsto \mathbb{E}[|\mathcal{N}_\theta(X) - \mathcal{E}(X)|^2] \in [0, \infty)$  is equivalent to minimising the risk function  $\mathbb{R}^{\mathbf{d}} \ni \theta \mapsto \mathbb{E}[|\mathcal{N}_\theta(X) - Y|^2] \in [0, \infty)$  (corresponding to a quadratic loss function). Since in practice the joint distribution of  $X$  and  $Y$  is typically not known, the risk function is replaced by an empirical risk function based on i.i.d. training samples of  $(X, Y)$ . This empirical risk is then approximatively minimised using an optimisation method such as SGD. As is often the case for deep learning based algorithms, the overall error arising from this procedure consists of the following three different parts (cf. [10, Lemma 4.3] and Proposition 6.1 below): (i) the *approximation error* (cf., e.g., [5, 6, 14, 21, 24, 37, 39, 54–58, 66, 75] and the references in the introductory paragraph in Section 3), which arises from approximating the target function  $\mathcal{E}$  by the considered class of DNNs, (ii) the *generalisation error* (cf., e.g., [7, 10, 13, 23, 31–33, 52, 67, 87, 92]), which arises from replacing the true risk by the empirical risk, and (iii) the *optimisation error* (cf., e.g., [2, 4, 8, 10, 12, 18, 25, 26, 28, 29, 38, 60, 62, 63, 65, 88, 97, 98]), which arises from computing only an approximate minimiser using the selected optimisation method.

In this work we derive strong convergence rates for the approximation error, the generalisation error, and the optimisation error separately and combine these findings to

establish strong convergence results for the overall error (cf. Subsections 6.2 and 6.3), as illustrated in Theorem 1.1 below. The convergence speed we obtain (cf. (4) in Theorem 1.1) is presumably far from optimal, suffers under the curse of dimensionality (cf., e.g., Bellman [11] and Novak & Woźniakowski [73, Chapter 1; 74, Chapter 9]), and is, as a consequence, very slow. To the best of our knowledge, Theorem 1.1 is, however, the first full error result in the scientific literature for a deep learning based algorithm in the probabilistically strong sense and, moreover, the first full error result in the scientific literature for a deep learning based algorithm where SGD with random initialisation is the employed optimisation method. We now present Theorem 1.1, the statement of which is entirely self-contained, before we add further explanations and intuitions for the mathematical objects that are introduced.

**Theorem 1.1.** *Let  $d, \mathbf{d}, \mathbf{L}, \mathbf{J}, M, K, N \in \mathbb{N}$ ,  $\gamma, L \in \mathbb{R}$ ,  $c \in [\max\{2, L\}, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, \dots, N\}$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , for every  $m, n \in \mathbb{N}$ ,  $s \in \mathbb{N}_0$ ,  $\theta = (\theta_1, \dots, \theta_{\mathbf{d}}) \in \mathbb{R}^{\mathbf{d}}$  with  $\mathbf{d} \geq s + mn + m$  let  $\mathcal{A}_{m,n}^{\theta,s}: \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfy for all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  that*

$$\mathcal{A}_{m,n}^{\theta,s}(x) = \begin{pmatrix} \theta_{s+1} & \theta_{s+2} & \cdots & \theta_{s+n} \\ \theta_{s+n+1} & \theta_{s+n+2} & \cdots & \theta_{s+2n} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{s+(m-1)n+1} & \theta_{s+(m-1)n+2} & \cdots & \theta_{s+mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} \theta_{s+mn+1} \\ \theta_{s+mn+2} \\ \vdots \\ \theta_{s+mn+m} \end{pmatrix}, \quad (1)$$

let  $\mathbf{a}_i: \mathbb{R}^{\mathbf{l}_i} \rightarrow \mathbb{R}^{\mathbf{l}_i}$ ,  $i \in \{1, \dots, \mathbf{L}\}$ , satisfy for all  $i \in \mathbb{N} \cap [0, \mathbf{L})$ ,  $x = (x_1, \dots, x_{\mathbf{l}_i}) \in \mathbb{R}^{\mathbf{l}_i}$  that  $\mathbf{a}_i(x) = (\max\{x_1, 0\}, \dots, \max\{x_{\mathbf{l}_i}, 0\})$ , assume for all  $x \in \mathbb{R}$  that  $\mathbf{a}_{\mathbf{L}}(x) = \max\{\min\{x, 1\}, 0\}$ , for every  $\theta \in \mathbb{R}^{\mathbf{d}}$  let  $\mathcal{N}_{\theta}: \mathbb{R}^{\mathbf{d}} \rightarrow \mathbb{R}$  satisfy  $\mathcal{N}_{\theta} = \mathbf{a}_{\mathbf{L}} \circ \mathcal{A}_{\mathbf{l}_{\mathbf{L}}, \mathbf{l}_{\mathbf{L}-1}}^{\theta, \sum_{i=1}^{\mathbf{L}-1} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)} \circ \mathbf{a}_{\mathbf{L}-1} \circ \mathcal{A}_{\mathbf{l}_{\mathbf{L}-1}, \mathbf{l}_{\mathbf{L}-2}}^{\theta, \sum_{i=1}^{\mathbf{L}-2} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)} \circ \dots \circ \mathbf{a}_1 \circ \mathcal{A}_{\mathbf{l}_1, \mathbf{l}_0}^{\theta, 0}$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j^{k,n}: \Omega \rightarrow [0, 1]^d$ ,  $k, n, j \in \mathbb{N}_0$ , and  $Y_j^{k,n}: \Omega \rightarrow [0, 1]$ ,  $k, n, j \in \mathbb{N}_0$ , be functions, assume that  $(X_j^{0,0}, Y_j^{0,0})$ ,  $j \in \mathbb{N}$ , are i.i.d. random variables, let  $\mathcal{E}: [0, 1]^d \rightarrow [0, 1]$  satisfy  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1^{0,0}) = \mathbb{E}[Y_1^{0,0}|X_1^{0,0}]$ , assume for all  $x, y \in [0, 1]^d$  that  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq L\|x - y\|_1$ , let  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  be random variables, assume  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-c, c]^{\mathbf{d}}$ , assume that  $\Theta_{k,0}$ ,  $k \in \mathbb{N}$ , are i.i.d., assume that  $\Theta_{1,0}$  is continuous uniformly distributed on  $[-c, c]^{\mathbf{d}}$ , let  $\mathcal{R}_j^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$ ,  $k, n, J \in \mathbb{N}_0$ , and  $\mathcal{G}^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}$ , satisfy for all  $k, n \in \mathbb{N}$ ,  $\omega \in \Omega$ ,  $\theta \in \{\vartheta \in \mathbb{R}^{\mathbf{d}}: (\mathcal{R}_{\mathbf{J}}^{k,n}(\cdot, \omega)): \mathbb{R}^{\mathbf{d}} \rightarrow [0, \infty) \text{ is differentiable at } \vartheta\}$  that  $\mathcal{G}^{k,n}(\theta, \omega) = (\nabla_{\theta} \mathcal{R}_{\mathbf{J}}^{k,n})(\theta, \omega)$ , assume for all  $k, n \in \mathbb{N}$  that  $\Theta_{k,n} = \Theta_{k,n-1} - \gamma \mathcal{G}^{k,n}(\Theta_{k,n-1})$ , and assume for all  $k, n \in \mathbb{N}_0$ ,  $J \in \mathbb{N}$ ,  $\theta \in \mathbb{R}^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$\mathcal{R}_J^{k,n}(\theta, \omega) = \frac{1}{J} \left[ \sum_{j=1}^J |\mathcal{N}_{\theta}(X_j^{k,n}(\omega)) - Y_j^{k,n}(\omega)|^2 \right] \quad \text{and} \quad (2)$$

$$\mathbf{k}(\omega) \in \arg \min_{(l,m) \in \{1, \dots, K\} \times \mathbb{N}, \|\Theta_{l,m}(\omega)\|_{\infty} \leq c} \mathcal{R}_M^{0,0}(\Theta_{l,m}(\omega), \omega). \quad (3)$$

Then

$$\begin{aligned} & \mathbb{E} \left[ \int_{[0,1]^d} |\mathcal{N}_{\Theta_{\mathbf{k}}}(x) - \mathcal{E}(x)| \mathbb{P}_{X_1^{0,0}}(dx) \right] \\ & \leq \frac{dc^3}{[\min\{\mathbf{L}, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{c^3 \mathbf{L}(\|\mathbf{l}\|_{\infty} + 1) \ln(eM)}{M^{1/4}} + \frac{\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}} c^{\mathbf{L}+1}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_{\infty} + 1)^{-2}]}}. \end{aligned} \quad (4)$$

Recall that we denote for every  $p \in [1, \infty]$  by  $\|\cdot\|_p: (\bigcup_{n=1}^{\infty} \mathbb{R}^n) \rightarrow [0, \infty)$  the  $p$ -norm of vectors in  $\bigcup_{n=1}^{\infty} \mathbb{R}^n$  (cf. Definition 3.1). In addition, note that the function  $\Omega \times [0, 1]^d \ni$

$(\omega, x) \mapsto |\mathcal{N}_{\Theta_{\mathbf{k}}(\omega)}(\omega)(x) - \mathcal{E}(x)| \in [0, \infty)$  is measurable (cf. Lemma 6.2) and that the expression on the left hand side of (4) above is thus well-defined. Theorem 1.1 follows directly from Corollary 6.9 in Subsection 6.3, which, in turn, is a consequence of the main result of this article, Theorem 6.5 in Subsection 6.2.

In the following we provide additional explanations and intuitions for Theorem 1.1. For every  $\theta \in \mathbb{R}^d$  the functions  $\mathcal{N}_\theta: \mathbb{R}^d \rightarrow \mathbb{R}$  are realisations of fully connected feedforward artificial neural networks with  $\mathbf{L}+1$  layers consisting of an input layer of dimension  $\mathbf{l}_0 = d$ , of  $\mathbf{L}-1$  hidden layers of dimensions  $\mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}-1}$ , respectively, and of an output layer of dimension  $\mathbf{l}_{\mathbf{L}} = 1$  (cf. Definition 2.8). The weights and biases stored in the DNN parameter vector  $\theta \in \mathbb{R}^d$  determine the corresponding  $\mathbf{L}$  affine linear transformations (cf. (1) above). As activation functions we employ the multidimensional versions  $\mathbf{a}_1, \dots, \mathbf{a}_{\mathbf{L}-1}$  (cf. Definition 2.3) of the *rectifier function*  $\mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R}$  (cf. Definition 2.4) just in front of each of the hidden layers and the *clipping function*  $\mathbf{a}_{\mathbf{L}}$  (cf. Definition 2.6) just in front of the output layer. Furthermore, observe that we assume the target function  $\mathcal{E}: [0, 1]^d \rightarrow [0, 1]$ , the values of which we intend to approximately predict with the trained DNN, to be Lipschitz continuous with Lipschitz constant  $L$ . Moreover, for every  $k, n \in \mathbb{N}_0$ ,  $J \in \mathbb{N}$  the function  $\mathcal{R}_J^{k,n}: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$  is the empirical risk based on the  $J$  training samples  $(X_j^{k,n}, Y_j^{k,n})$ ,  $j \in \{1, \dots, J\}$  (cf. (2) above). Derived from the empirical risk, for every  $k, n \in \mathbb{N}$  the function  $\mathcal{G}^{k,n}: \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$  is a (generalised) gradient of the empirical risk  $\mathcal{R}_J^{k,n}$  with respect to its first argument, that is, with respect to the DNN parameter vector  $\theta \in \mathbb{R}^d$ . These gradients are required in order to formulate the training dynamics of the (random) DNN parameter vectors  $\Theta_{k,n} \in \mathbb{R}^d$ ,  $k \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$ , given by the SGD optimisation method with learning rate  $\gamma$ . Note that the subscript  $n \in \mathbb{N}_0$  of these SGD iterates (i.e., DNN parameter vectors) is the current training step number, whereas the subscript  $k \in \mathbb{N}$  counts the number of times the SGD iteration has been started from scratch so far. Such a new start entails the corresponding initial DNN parameter vector  $\Theta_{k,0} \in \mathbb{R}^d$  to be drawn continuous uniformly from the hypercube  $[-c, c]^d$ , in accordance with Xavier initialisation (cf. Glorot & Bengio [41]). The (random) double index  $\mathbf{k} \in \mathbb{N} \times \mathbb{N}_0$  represents the final choice made for the DNN parameter vector  $\Theta_{\mathbf{k}} \in \mathbb{R}^d$  (cf. (4) above), concluding the training procedure, and is selected as follows. During training the empirical risk  $\mathcal{R}_M^{0,0}$  has been calculated for the subset of the SGD iterates indexed by  $\mathbf{N} \subseteq \{0, \dots, N\}$  provided that they have not left the hypercube  $[-c, c]^d$  (cf. (3) above). After the SGD iteration has been started and finished  $K$  times (with maximally  $N$  training steps in each case) the final choice for the DNN parameter vector  $\Theta_{\mathbf{k}} \in \mathbb{R}^d$  is made among those SGD iterates for which the calculated empirical risk is minimal (cf. (3) above). Observe that we use mini-batches of size  $\mathbf{J}$  consisting, during SGD iteration number  $k \in \{1, \dots, K\}$  for training step number  $n \in \{1, \dots, N\}$ , of the training samples  $(X_j^{k,n}, Y_j^{k,n})$ ,  $j \in \{1, \dots, \mathbf{J}\}$ , and that we reserve the  $M$  training samples  $(X_j^{0,0}, Y_j^{0,0})$ ,  $j \in \{1, \dots, M\}$ , for checking the value of the empirical risk  $\mathcal{R}_M^{0,0}$ . Regarding the conclusion of Theorem 1.1, note that the left hand side of (4) is the expectation of the overall  $L^1$ -error, that is, the expected  $L^1$ -distance between the trained DNN  $\mathcal{N}_{\Theta_{\mathbf{k}}}$  and the target function  $\mathcal{E}$ . It is bounded from above by the right hand side of (4), which consists of following three summands: (i) the first summand corresponds to the *approximation error* and converges to zero as the number of hidden layers  $\mathbf{L}-1$  as well as the hidden layer dimensions  $\mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}-1}$  increase to infinity, (ii) the second summand corresponds to the *generalisation error* and converges to zero as number of training samples  $M$  used for calculating the empirical risk increases to infinity, and (iii) the third summand corresponds to the *optimisation error* and converges to zero as total number of times  $K$  the SGD iteration has been started from scratch increases to infinity. We would like to point

out that the the second summand (corresponding to the generalisation error) does not suffer under the curse of dimensionality with respect to any of the variables involved.

The main result of this article, Theorem 6.5 in Subsection 6.2, covers, in comparison with Theorem 1.1, the more general cases where  $L^p$ -norms of the overall  $L^2$ -error instead of the expectation of the overall  $L^1$ -error are considered (cf. (167) in Theorem 6.5), where the training samples are not restricted to unit hypercubes, and where a general stochastic approximation algorithm (cf., e.g., Robbins & Monro [83]) with random initialisation is used for optimisation. Our convergence proof for the optimisation error relies, in fact, on the convergence of the Minimum Monte Carlo method (cf. Proposition 5.6 in Section 5) and thus only exploits random initialisation but not the specific dynamics of the employed optimisation method (cf. (155) in the proof of Proposition 6.3). In this regard, note that Theorem 1.1 above also includes the application of deterministic gradient descent instead of SGD for optimisation since we do not assume the samples used for gradient iterations to be i.i.d. Parts of our derivation of Theorem 1.1 and Theorem 6.5, respectively, are inspired by Beck, Jentzen, & Kuckuck [10], Berner, Grohs, & Jentzen [13], and Cucker & Smale [23].

This article is structured in the following way. Section 2 recalls some basic definitions related to DNNs and thereby introduces the corresponding notation we use in the subsequent parts of this article. In Section 3 we examine the approximation error and, in particular, establish a convergence result for the approximation of Lipschitz continuous functions by DNNs. The following section, Section 4, contains our strong convergence analysis of the generalisation error. In Section 5, in turn, we address the optimisation error and derive in connection with this strong convergence rates for the Minimum Monte Carlo method. Finally, we combine in Section 6 a decomposition of the overall error (cf. Subsection 6.1) with our results for the different error sources from Sections 3, 4, and 5 to prove strong convergence results for the overall error. The employed optimisation method is initially allowed to be a general stochastic approximation algorithm with random initialisation (cf. Subsection 6.2) and is afterwards specialised to the setting of SGD with random initialisation (cf. Subsection 6.3).

## 2 Basics on deep neural networks (DNNs)

In this section we present the mathematical description of DNNs which we use throughout the remainder of this article. It is a vectorised description in the sense that all the weights and biases associated to the DNN under consideration are collected in a single parameter vector  $\theta \in \mathbb{R}^d$  with  $d \in \mathbb{N}$  sufficiently large (cf. Definitions 2.2 and 2.8). The content of this section is taken from Beck, Jentzen, & Kuckuck [10, Section 2.1] and is based on well-known material from the scientific literature, see, e.g., Beck et al. [8], Beck, E, & Jentzen [9], Berner, Grohs, & Jentzen [13], E, Han, & Jentzen [30], Goodfellow, Bengio, & Courville [43], and Grohs et al. [46]. In particular, Definition 2.1 is [10, Definition 2.1] (cf., e.g., (25) in [9]), Definition 2.2 is [10, Definition 2.2] (cf., e.g., (26) in [9]), Definition 2.3 is [10, Definition 2.3] (cf., e.g., [46, Definition 2.2]), and Definitions 2.4, 2.5, 2.6, 2.7, and 2.8 are [10, Definitions 2.4, 2.5, 2.6, 2.7, and 2.8] (cf., e.g., [13, Setting 2.5] and [43, Section 6.3]).

### 2.1 Vectorised description of DNNs

**Definition 2.1** (Affine function). *Let  $d, m, n \in \mathbb{N}$ ,  $s \in \mathbb{N}_0$ ,  $\theta = (\theta_1, \theta_2, \dots, \theta_d) \in \mathbb{R}^d$  satisfy  $d \geq s + mn + m$ . Then we denote by  $\mathcal{A}_{m,n}^{\theta,s} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  the function which satisfies*

for all  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  that

$$\begin{aligned}\mathcal{A}_{m,n}^{\theta,s}(x) &= \begin{pmatrix} \theta_{s+1} & \theta_{s+2} & \cdots & \theta_{s+n} \\ \theta_{s+n+1} & \theta_{s+n+2} & \cdots & \theta_{s+2n} \\ \theta_{s+2n+1} & \theta_{s+2n+2} & \cdots & \theta_{s+3n} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{s+(m-1)n+1} & \theta_{s+(m-1)n+2} & \cdots & \theta_{s+mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} \theta_{s+mn+1} \\ \theta_{s+mn+2} \\ \theta_{s+mn+3} \\ \vdots \\ \theta_{s+mn+m} \end{pmatrix} \\ &= \left( \left[ \sum_{i=1}^n \theta_{s+i} x_i \right] + \theta_{s+mn+1}, \left[ \sum_{i=1}^n \theta_{s+n+i} x_i \right] + \theta_{s+mn+2}, \dots, \left[ \sum_{i=1}^n \theta_{s+(m-1)n+i} x_i \right] + \theta_{s+mn+m} \right).\end{aligned}\quad (5)$$

**Definition 2.2** (Fully connected feedforward artificial neural network). Let  $\mathbf{d}, \mathbf{L}, \mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_L \in \mathbb{N}$ ,  $s \in \mathbb{N}_0$ ,  $\theta \in \mathbb{R}^{\mathbf{d}}$  satisfy  $\mathbf{d} \geq s + \sum_{i=1}^L \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$  and let  $\mathbf{a}_i: \mathbb{R}^{\mathbf{l}_i} \rightarrow \mathbb{R}^{\mathbf{l}_i}$ ,  $i \in \{1, 2, \dots, L\}$ , be functions. Then we denote by  $\mathcal{N}_{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_L}^{\theta, s, \mathbf{l}_0}: \mathbb{R}^{\mathbf{l}_0} \rightarrow \mathbb{R}^{\mathbf{l}_L}$  the function which satisfies for all  $x \in \mathbb{R}^{\mathbf{l}_0}$  that

$$\begin{aligned}(\mathcal{N}_{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_L}^{\theta, s, \mathbf{l}_0})(x) &= (\mathbf{a}_L \circ \mathcal{A}_{\mathbf{l}_L, \mathbf{l}_{L-1}}^{\theta, s + \sum_{i=1}^{L-1} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)} \circ \mathbf{a}_{L-1} \circ \mathcal{A}_{\mathbf{l}_{L-1}, \mathbf{l}_{L-2}}^{\theta, s + \sum_{i=1}^{L-2} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)} \circ \dots \\ &\quad \dots \circ \mathbf{a}_2 \circ \mathcal{A}_{\mathbf{l}_2, \mathbf{l}_1}^{\theta, s + \mathbf{l}_1 (\mathbf{l}_0 + 1)} \circ \mathbf{a}_1 \circ \mathcal{A}_{\mathbf{l}_1, \mathbf{l}_0}^{\theta, s})(x)\end{aligned}\quad (6)$$

(cf. Definition 2.1).

## 2.2 Activation functions

**Definition 2.3** (Multidimensional version). Let  $d \in \mathbb{N}$  and let  $\mathbf{a}: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Then we denote by  $\mathfrak{M}_{\mathbf{a}, d}: \mathbb{R}^d \rightarrow \mathbb{R}^d$  the function which satisfies for all  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$  that

$$\mathfrak{M}_{\mathbf{a}, d}(x) = (\mathbf{a}(x_1), \mathbf{a}(x_2), \dots, \mathbf{a}(x_d)). \quad (7)$$

**Definition 2.4** (Rectifier function). We denote by  $\mathfrak{r}: \mathbb{R} \rightarrow \mathbb{R}$  the function which satisfies for all  $x \in \mathbb{R}$  that

$$\mathfrak{r}(x) = \max\{x, 0\}. \quad (8)$$

**Definition 2.5** (Multidimensional rectifier function). Let  $d \in \mathbb{N}$ . Then we denote by  $\mathfrak{R}_d: \mathbb{R}^d \rightarrow \mathbb{R}^d$  the function given by

$$\mathfrak{R}_d = \mathfrak{M}_{\mathfrak{r}, d} \quad (9)$$

(cf. Definitions 2.3 and 2.4).

**Definition 2.6** (Clipping function). Let  $u \in [-\infty, \infty)$ ,  $v \in (u, \infty]$ . Then we denote by  $\mathfrak{c}_{u,v}: \mathbb{R} \rightarrow \mathbb{R}$  the function which satisfies for all  $x \in \mathbb{R}$  that

$$\mathfrak{c}_{u,v}(x) = \max\{u, \min\{x, v\}\}. \quad (10)$$

**Definition 2.7** (Multidimensional clipping function). Let  $d \in \mathbb{N}$ ,  $u \in [-\infty, \infty)$ ,  $v \in (u, \infty]$ . Then we denote by  $\mathfrak{C}_{u,v,d}: \mathbb{R}^d \rightarrow \mathbb{R}^d$  the function given by

$$\mathfrak{C}_{u,v,d} = \mathfrak{M}_{\mathfrak{c}_{u,v}, d} \quad (11)$$

(cf. Definitions 2.3 and 2.6).

## 2.3 Rectified DNNs

**Definition 2.8** (Rectified clipped DNN). Let  $\mathbf{d}, \mathbf{L} \in \mathbb{N}$ ,  $u \in [-\infty, \infty)$ ,  $v \in (u, \infty]$ ,  $\mathbf{l} = (l_0, l_1, \dots, l_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\theta \in \mathbb{R}^{\mathbf{d}}$  satisfy  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} l_i(l_{i-1} + 1)$ . Then we denote by  $\mathcal{N}_{u,v}^{\theta,\mathbf{l}}: \mathbb{R}^{l_0} \rightarrow \mathbb{R}^{l_{\mathbf{L}}}$  the function which satisfies for all  $x \in \mathbb{R}^{l_0}$  that

$$\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) = \begin{cases} (\mathcal{N}_{\mathfrak{C}_{u,v,l_{\mathbf{L}}}}^{\theta,0,l_0})(x) & : \mathbf{L} = 1 \\ (\mathcal{N}_{\mathfrak{R}_{l_1}, \mathfrak{R}_{l_2}, \dots, \mathfrak{R}_{l_{\mathbf{L}-1}}, \mathfrak{C}_{u,v,l_{\mathbf{L}}}}^{\theta,0,l_0})(x) & : \mathbf{L} > 1 \end{cases} \quad (12)$$

(cf. Definitions 2.2, 2.5, and 2.7).

## 3 Analysis of the approximation error

This section is devoted to establishing a convergence result for the approximation of Lipschitz continuous functions by DNNs (cf. Proposition 3.5). More precisely, Proposition 3.5 establishes that a Lipschitz continuous function defined on a  $d$ -dimensional hypercube for  $d \in \mathbb{N}$  can be approximated by DNNs with convergence rate  $1/d$  with respect to a parameter  $A \in (0, \infty)$  that bounds the architecture size (that is, depth and width) of the approximating DNN from below. Key ingredients of the proof of Proposition 3.5 are Beck, Jentzen, & Kuckuck [10, Corollary 3.8] as well as the elementary covering number estimate in Lemma 3.3. In order to improve the accessibility of Lemma 3.3, we recall the definition of covering numbers associated to a metric space in Definition 3.2, which is [10, Definition 3.11]. Lemma 3.3 provides upper bounds for the covering numbers of hypercubes equipped with the metric induced by the  $p$ -norm (cf. Definition 3.1) for  $p \in [1, \infty]$  and is a generalisation of Berner, Grohs, & Jentzen [13, Lemma 2.7] (cf. Cucker & Smale [23, Proposition 5] and [10, Proposition 3.12]). Furthermore, we present in Lemma 3.4 an elementary upper bound for the error arising when Lipschitz continuous functions defined on a hypercube are approximated by certain DNNs. Additional DNN approximation results can be found, e.g., in [3, 5, 6, 14–17, 19–21, 24, 27, 34–37, 39, 44–51, 53–59, 61, 64, 66, 68–72, 75–80, 82, 84–86, 89–91, 93, 95, 96] and the references therein.

### 3.1 A covering number estimate

**Definition 3.1** ( $p$ -norm). We denote by  $\|\cdot\|_p: (\bigcup_{d=1}^{\infty} \mathbb{R}^d) \rightarrow [0, \infty]$ ,  $p \in [1, \infty]$ , the functions which satisfy for all  $p \in [1, \infty)$ ,  $d \in \mathbb{N}$ ,  $\theta = (\theta_1, \theta_2, \dots, \theta_d) \in \mathbb{R}^d$  that

$$\|\theta\|_p = \left( \sum_{i=1}^d |\theta_i|^p \right)^{1/p} \quad \text{and} \quad \|\theta\|_{\infty} = \max_{i \in \{1, 2, \dots, d\}} |\theta_i|. \quad (13)$$

**Definition 3.2** (Covering number). Let  $(E, \delta)$  be a metric space and let  $r \in [0, \infty]$ . Then we denote by  $\mathcal{C}_{(E, \delta), r} \in \mathbb{N}_0 \cup \{\infty\}$  (we denote by  $\mathcal{C}_{E, r} \in \mathbb{N}_0 \cup \{\infty\}$ ) the extended real number given by

$$\mathcal{C}_{(E, \delta), r} = \inf \left( \left\{ n \in \mathbb{N}_0: \left[ \exists A \subseteq E: \left( \begin{array}{l} (|A| \leq n) \wedge (\forall x \in E: \\ \exists a \in A: \delta(a, x) \leq r) \end{array} \right) \right] \right\} \cup \{\infty\} \right). \quad (14)$$

**Lemma 3.3.** Let  $d \in \mathbb{N}$ ,  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $r \in (0, \infty)$ , for every  $p \in [1, \infty]$  let  $\delta_p: ([a, b]^d) \times ([a, b]^d) \rightarrow [0, \infty)$  satisfy for all  $x, y \in [a, b]^d$  that  $\delta_p(x, y) = \|x - y\|_p$ , and let  $\lceil \cdot \rceil: [0, \infty) \rightarrow \mathbb{N}_0$  satisfy for all  $x \in [0, \infty)$  that  $\lceil x \rceil = \min([x, \infty) \cap \mathbb{N}_0)$  (cf. Definition 3.1). Then

(i) it holds for all  $p \in [1, \infty)$  that

$$\mathcal{C}_{([a,b]^d, \delta_p), r} \leq \left( \left\lceil \frac{d^{1/p}(b-a)}{2r} \right\rceil \right)^d \leq \begin{cases} 1 & : r \geq d(b-a)/2 \\ \left( \frac{d(b-a)}{r} \right)^d & : r < d(b-a)/2 \end{cases} \quad (15)$$

and

(ii) it holds that

$$\mathcal{C}_{([a,b]^d, \delta_\infty), r} \leq \left( \left\lceil \frac{b-a}{2r} \right\rceil \right)^d \leq \begin{cases} 1 & : r \geq (b-a)/2 \\ \left( \frac{b-a}{r} \right)^d & : r < (b-a)/2 \end{cases} \quad (16)$$

(cf. Definition 3.2).

*Proof of Lemma 3.3.* Throughout this proof let  $(\mathfrak{N}_p)_{p \in [1, \infty]} \subseteq \mathbb{N}$  satisfy for all  $p \in [1, \infty)$  that

$$\mathfrak{N}_p = \left\lceil \frac{d^{1/p}(b-a)}{2r} \right\rceil \quad \text{and} \quad \mathfrak{N}_\infty = \left\lceil \frac{b-a}{2r} \right\rceil, \quad (17)$$

for every  $N \in \mathbb{N}$ ,  $i \in \{1, 2, \dots, N\}$  let  $g_{N,i} \in [a, b]$  be given by  $g_{N,i} = a + (i-1/2)(b-a)/N$ , and for every  $p \in [1, \infty]$  let  $A_p \subseteq [a, b]^d$  be given by  $A_p = \{g_{\mathfrak{N}_p,1}, g_{\mathfrak{N}_p,2}, \dots, g_{\mathfrak{N}_p,\mathfrak{N}_p}\}^d$ . Observe that it holds for all  $N \in \mathbb{N}$ ,  $i \in \{1, 2, \dots, N\}$ ,  $x \in [a + (i-1)(b-a)/N, g_{N,i}]$  that

$$|x - g_{N,i}| = a + \frac{(i-1/2)(b-a)}{N} - x \leq a + \frac{(i-1/2)(b-a)}{N} - \left( a + \frac{(i-1)(b-a)}{N} \right) = \frac{b-a}{2N}. \quad (18)$$

In addition, note that it holds for all  $N \in \mathbb{N}$ ,  $i \in \{1, 2, \dots, N\}$ ,  $x \in [g_{N,i}, a + i(b-a)/N]$  that

$$|x - g_{N,i}| = x - \left( a + \frac{(i-1/2)(b-a)}{N} \right) \leq a + \frac{i(b-a)}{N} - \left( a + \frac{(i-1/2)(b-a)}{N} \right) = \frac{b-a}{2N}. \quad (19)$$

Combining (18) and (19) implies for all  $N \in \mathbb{N}$ ,  $i \in \{1, 2, \dots, N\}$ ,  $x \in [a + (i-1)(b-a)/N, a + i(b-a)/N]$  that  $|x - g_{N,i}| \leq (b-a)/(2N)$ . This proves that for every  $N \in \mathbb{N}$ ,  $x \in [a, b]$  there exists  $y \in \{g_{N,1}, g_{N,2}, \dots, g_{N,N}\}$  such that

$$|x - y| \leq \frac{b-a}{2N}. \quad (20)$$

This, in turn, establishes that for every  $p \in [1, \infty)$ ,  $x = (x_1, x_2, \dots, x_d) \in [a, b]^d$  there exists  $y = (y_1, y_2, \dots, y_d) \in A_p$  such that

$$\delta_p(x, y) = \|x - y\|_p = \left( \sum_{i=1}^d |x_i - y_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^d \frac{(b-a)^p}{(2\mathfrak{N}_p)^p} \right)^{1/p} = \frac{d^{1/p}(b-a)}{2\mathfrak{N}_p} \leq \frac{d^{1/p}(b-a)2r}{2d^{1/p}(b-a)} = r. \quad (21)$$

Furthermore, again (20) shows that for every  $x = (x_1, x_2, \dots, x_d) \in [a, b]^d$  there exists  $y = (y_1, y_2, \dots, y_d) \in A_\infty$  such that

$$\delta_\infty(x, y) = \|x - y\|_\infty = \max_{i \in \{1, 2, \dots, d\}} |x_i - y_i| \leq \frac{b-a}{2\mathfrak{N}_\infty} \leq \frac{(b-a)2r}{2(b-a)} = r. \quad (22)$$

Note that (21), (17), and the fact that  $\forall x \in [0, \infty) : \lceil x \rceil \leq \mathbb{1}_{(0,1]}(x) + 2x\mathbb{1}_{(1,\infty)}(x) = \mathbb{1}_{(0,r]}(rx) + 2x\mathbb{1}_{(r,\infty)}(rx)$  yield for all  $p \in [1, \infty)$  that

$$\begin{aligned} \mathcal{C}_{([a,b]^d, \delta_p), r} &\leq |A_p| = (\mathfrak{N}_p)^d = \left( \left\lceil \frac{d^{1/p}(b-a)}{2r} \right\rceil \right)^d \leq \left( \left\lceil \frac{d(b-a)}{2r} \right\rceil \right)^d \\ &\leq \left( \mathbb{1}_{(0,r]} \left( \frac{d(b-a)}{2} \right) + \frac{2d(b-a)}{2r} \mathbb{1}_{(r,\infty)} \left( \frac{d(b-a)}{2} \right) \right)^d \\ &= \mathbb{1}_{(0,r]} \left( \frac{d(b-a)}{2} \right) + \left( \frac{d(b-a)}{r} \right)^d \mathbb{1}_{(r,\infty)} \left( \frac{d(b-a)}{2} \right). \end{aligned} \quad (23)$$

This proves (i). In addition, (22), (17), and again the fact that  $\forall x \in [0, \infty) : \lceil x \rceil \leq \mathbb{1}_{(0,r]}(rx) + 2x\mathbb{1}_{(r,\infty)}(rx)$  demonstrate that

$$\mathcal{C}_{([a,b]^d, \delta_\infty), r} \leq |A_\infty| = (\mathfrak{N}_\infty)^d = \left( \left\lceil \frac{b-a}{2r} \right\rceil \right)^d \leq \mathbb{1}_{(0,r]} \left( \frac{b-a}{2} \right) + \left( \frac{b-a}{r} \right)^d \mathbb{1}_{(r,\infty)} \left( \frac{b-a}{2} \right). \quad (24)$$

This implies (ii) and thus completes the proof of Lemma 3.3.  $\square$

### 3.2 Convergence rates for the approximation error

**Lemma 3.4.** Let  $d, \mathbf{d}, \mathbf{L} \in \mathbb{N}$ ,  $L, a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $u \in [-\infty, \infty)$ ,  $v \in (u, \infty]$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ , assume  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , and let  $f: [a, b]^d \rightarrow ([u, v] \cap \mathbb{R})$  satisfy for all  $x, y \in [a, b]^d$  that  $|f(x) - f(y)| \leq L\|x - y\|_1$  (cf. Definition 3.1). Then there exists  $\vartheta \in \mathbb{R}^{\mathbf{d}}$  such that  $\|\vartheta\|_\infty \leq \sup_{x \in [a, b]^d} |f(x)|$  and

$$\sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\vartheta, \mathbf{l}}(x) - f(x)| \leq \frac{dL(b-a)}{2} \quad (25)$$

(cf. Definition 2.8).

*Proof of Lemma 3.4.* Throughout this proof let  $\mathfrak{d} \in \mathbb{N}$  be given by  $\mathfrak{d} = \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , let  $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_d) \in [a, b]^d$  satisfy for all  $i \in \{1, 2, \dots, d\}$  that  $\mathbf{m}_i = (a+b)/2$ , and let  $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_{\mathfrak{d}}) \in \mathbb{R}^{\mathfrak{d}}$  satisfy for all  $i \in \{1, 2, \dots, \mathfrak{d}\} \setminus \{\mathfrak{d}\}$  that  $\vartheta_i = 0$  and  $\vartheta_{\mathfrak{d}} = f(\mathbf{m})$ . Observe that the assumption that  $\mathbf{l}_{\mathbf{L}} = 1$  and the fact that  $\forall i \in \{1, 2, \dots, \mathfrak{d}-1\}: \vartheta_i = 0$  show for all  $x = (x_1, x_2, \dots, x_{\mathbf{l}_{\mathbf{L}-1}}) \in \mathbb{R}^{\mathbf{l}_{\mathbf{L}-1}}$  that

$$\begin{aligned} \mathcal{A}_{1, \mathbf{l}_{\mathbf{L}-1}}^{\vartheta, \sum_{i=1}^{\mathbf{L}-1} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)}(x) &= \left[ \sum_{i=1}^{\mathbf{L}-1} \vartheta_{[\sum_{i=1}^{\mathbf{L}-1} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)] + i} x_i \right] + \vartheta_{[\sum_{i=1}^{\mathbf{L}-1} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)] + \mathbf{l}_{\mathbf{L}-1} + 1} \\ &= \left[ \sum_{i=1}^{\mathbf{L}-1} \vartheta_{[\sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)] - (\mathbf{l}_{\mathbf{L}-1} - i + 1)} x_i \right] + \vartheta_{\sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)} \\ &= \left[ \sum_{i=1}^{\mathbf{L}-1} \vartheta_{\mathfrak{d} - (\mathbf{l}_{\mathbf{L}-1} - i + 1)} x_i \right] + \vartheta_{\mathfrak{d}} = \vartheta_{\mathfrak{d}} = f(\mathbf{m}) \end{aligned} \quad (26)$$

(cf. Definition 2.1). Combining this with the fact that  $f(\mathbf{m}) \in [u, v]$  ensures for all  $x \in \mathbb{R}^{\mathbf{l}_{\mathbf{L}-1}}$  that

$$\begin{aligned} (\mathfrak{C}_{u, v, \mathbf{l}_{\mathbf{L}}} \circ \mathcal{A}_{\mathbf{l}_{\mathbf{L}}, \mathbf{l}_{\mathbf{L}-1}}^{\vartheta, \sum_{i=1}^{\mathbf{L}-1} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)})(x) &= (\mathfrak{C}_{u, v, 1} \circ \mathcal{A}_{1, \mathbf{l}_{\mathbf{L}-1}}^{\vartheta, \sum_{i=1}^{\mathbf{L}-1} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)})(x) = \mathfrak{c}_{u, v}(f(\mathbf{m})) \\ &= \max\{u, \min\{f(\mathbf{m}), v\}\} = \max\{u, f(\mathbf{m})\} = f(\mathbf{m}) \end{aligned} \quad (27)$$

(cf. Definitions 2.6 and 2.7). This proves for all  $x \in \mathbb{R}^d$  that

$$\mathcal{N}_{u, v}^{\vartheta, \mathbf{l}}(x) = f(\mathbf{m}). \quad (28)$$

In addition, note that it holds for all  $x \in [a, \mathbf{m}_1]$ ,  $\mathfrak{x} \in [\mathbf{m}_1, b]$  that  $|\mathbf{m}_1 - x| = \mathbf{m}_1 - x = (a+b)/2 - x \leq (a+b)/2 - a = (b-a)/2$  and  $|\mathbf{m}_1 - \mathfrak{x}| = \mathfrak{x} - \mathbf{m}_1 = \mathfrak{x} - (a+b)/2 \leq b - (a+b)/2 = (b-a)/2$ . The assumption that  $\forall x, y \in [a, b]^d: |f(x) - f(y)| \leq L\|x - y\|_1$  and (28) hence demonstrate for all  $x = (x_1, x_2, \dots, x_d) \in [a, b]^d$  that

$$\begin{aligned} |\mathcal{N}_{u, v}^{\vartheta, \mathbf{l}}(x) - f(x)| &= |f(\mathbf{m}) - f(x)| \leq L\|\mathbf{m} - x\|_1 = L \sum_{i=1}^d |\mathbf{m}_i - x_i| \\ &= L \sum_{i=1}^d |\mathbf{m}_1 - x_i| \leq \sum_{i=1}^d \frac{L(b-a)}{2} = \frac{dL(b-a)}{2}. \end{aligned} \quad (29)$$

This and the fact that  $\|\vartheta\|_\infty = \max_{i \in \{1, 2, \dots, \mathfrak{d}\}} |\vartheta_i| = |f(\mathbf{m})| \leq \sup_{x \in [a, b]^d} |f(x)|$  complete the proof of Lemma 3.4.  $\square$

**Proposition 3.5.** Let  $d, \mathbf{d}, \mathbf{L} \in \mathbb{N}$ ,  $A \in (0, \infty)$ ,  $L, a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $u \in [-\infty, \infty)$ ,  $v \in (u, \infty]$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ , assume  $\mathbf{L} \geq A\mathbb{1}_{(6^d, \infty)}(A)/(2d) + 1$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_1 \geq A\mathbb{1}_{(6^d, \infty)}(A)$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , assume for all  $i \in \{2, 3, \dots\} \cap [0, \mathbf{L}]$

that  $\mathbf{l}_i \geq \mathbb{1}_{(6^d, \infty)}(A) \max\{A/d - 2i + 3, 2\}$ , and let  $f: [a, b]^d \rightarrow ([u, v] \cap \mathbb{R})$  satisfy for all  $x, y \in [a, b]^d$  that  $|f(x) - f(y)| \leq L\|x - y\|_1$  (cf. Definition 3.1). Then there exists  $\vartheta \in \mathbb{R}^d$  such that  $\|\vartheta\|_\infty \leq \max\{1, L, |a|, |b|, 2[\sup_{x \in [a, b]^d} |f(x)|]\}$  and

$$\sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\vartheta, \mathbf{l}}(x) - f(x)| \leq \frac{3dL(b - a)}{A^{1/d}} \quad (30)$$

(cf. Definition 2.8).

*Proof of Proposition 3.5.* Throughout this proof assume w.l.o.g. that  $A > 6^d$  (cf. Lemma 3.4), let  $\mathfrak{N} \in \mathbb{N}$  be given by

$$\mathfrak{N} = \max \left\{ \mathfrak{n} \in \mathbb{N}: \mathfrak{n} \leq \left(\frac{A}{2d}\right)^{1/d} \right\}, \quad (31)$$

let  $r \in (0, \infty)$  be given by  $r = d(b-a)/(2\mathfrak{N})$ , let  $\delta: ([a, b]^d) \times ([a, b]^d) \rightarrow [0, \infty)$  satisfy for all  $x, y \in [a, b]^d$  that  $\delta(x, y) = \|x - y\|_1$ , let  $\mathcal{D} \subseteq [a, b]^d$  satisfy  $|\mathcal{D}| = \max\{2, \mathcal{C}_{([a, b]^d, \delta), r}\}$  and

$$\sup_{x \in [a, b]^d} \inf_{y \in \mathcal{D}} \delta(x, y) \leq r \quad (32)$$

(cf. Definition 3.2), and let  $\lceil \cdot \rceil: [0, \infty) \rightarrow \mathbb{N}_0$  satisfy for all  $x \in [0, \infty)$  that  $\lceil x \rceil = \min([x, \infty) \cap \mathbb{N}_0)$ . Note that it holds for all  $\mathfrak{d} \in \mathbb{N}$  that

$$2\mathfrak{d} \leq 2 \cdot 2^{\mathfrak{d}-1} = 2^{\mathfrak{d}}. \quad (33)$$

This implies that  $3^d = 6^d/2^d \leq A/(2d)$ . Equation (31) hence demonstrates that

$$2 \leq \frac{2}{3} \left(\frac{A}{2d}\right)^{1/d} = \left(\frac{A}{2d}\right)^{1/d} - \frac{1}{3} \left(\frac{A}{2d}\right)^{1/d} \leq \left(\frac{A}{2d}\right)^{1/d} - 1 < \mathfrak{N}. \quad (34)$$

This and (i) in Lemma 3.3 (with  $\delta_1 \leftarrow \delta$ ,  $p \leftarrow 1$  in the notation of (i) in Lemma 3.3) establish that

$$|\mathcal{D}| = \max\{2, \mathcal{C}_{([a, b]^d, \delta), r}\} \leq \max \left\{ 2, \left( \left\lceil \frac{d(b-a)}{2r} \right\rceil \right)^d \right\} = \max\{2, (\lceil \mathfrak{N} \rceil)^d\} = \mathfrak{N}^d. \quad (35)$$

Combining this with (31) proves that

$$4 \leq 2d|\mathcal{D}| \leq 2d\mathfrak{N}^d \leq \frac{2dA}{2d} = A. \quad (36)$$

The fact that  $\mathbf{L} \geq A\mathbb{1}_{(6^d, \infty)}(A)/(2d) + 1 = A/(2d) + 1$  hence yields that  $|\mathcal{D}| \leq A/(2d) \leq \mathbf{L} - 1$ . This, (36), and the facts that  $\mathbf{l}_1 \geq A\mathbb{1}_{(6^d, \infty)}(A) = A$  and  $\forall i \in \{2, 3, \dots\} \cap [0, \mathbf{L}] = \{2, 3, \dots, \mathbf{L} - 1\}: \mathbf{l}_i \geq \mathbb{1}_{(6^d, \infty)}(A) \max\{A/d - 2i + 3, 2\} = \max\{A/d - 2i + 3, 2\}$  imply for all  $i \in \{2, 3, \dots, |\mathcal{D}|\}$  that

$$\mathbf{L} \geq |\mathcal{D}| + 1, \quad \mathbf{l}_1 \geq A \geq 2d|\mathcal{D}|, \quad \text{and} \quad \mathbf{l}_i \geq A/d - 2i + 3 \geq 2|\mathcal{D}| - 2i + 3. \quad (37)$$

In addition, the fact that  $\forall i \in \{2, 3, \dots\} \cap [0, \mathbf{L}]: \mathbf{l}_i \geq \max\{A/d - 2i + 3, 2\}$  ensures for all  $i \in \mathbb{N} \cap (|\mathcal{D}|, \mathbf{L})$  that

$$\mathbf{l}_i \geq 2. \quad (38)$$

Furthermore, observe that it holds for all  $x = (x_1, x_2, \dots, x_d), y = (y_1, y_2, \dots, y_d) \in [a, b]^d$  that

$$|f(x) - f(y)| \leq L\|x - y\|_1 = L \left[ \sum_{i=1}^d |x_i - y_i| \right]. \quad (39)$$

This, the assumptions that  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , (37)–(38), and Beck, Jentzen, & Kuckuck [10, Corollary 3.8] (with  $d \leftarrow d$ ,  $\mathfrak{d} \leftarrow \mathbf{d}$ ,  $\mathfrak{L} \leftarrow \mathbf{L}$ ,  $L \leftarrow L$ ,  $u \leftarrow u$ ,  $v \leftarrow v$ ,  $D \leftarrow [a, b]^d$ ,  $f \leftarrow f$ ,  $\mathcal{M} \leftarrow \mathcal{D}$ ,  $l \leftarrow 1$  in the notation of [10, Corollary 3.8]) show that there exists  $\vartheta \in \mathbb{R}^{\mathbf{d}}$  such that  $\|\vartheta\|_\infty \leq \max\{1, L, \sup_{x \in \mathcal{D}} \|x\|_\infty, 2[\sup_{x \in \mathcal{D}} |f(x)|]\}$  and

$$\begin{aligned} \sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\vartheta, 1}(x) - f(x)| &\leq 2L \left[ \sup_{x=(x_1, x_2, \dots, x_d) \in [a, b]^d} \left( \inf_{y=(y_1, y_2, \dots, y_d) \in \mathcal{D}} \sum_{i=1}^d |x_i - y_i| \right) \right] \\ &= 2L \left[ \sup_{x \in [a, b]^d} \inf_{y \in \mathcal{D}} \|x - y\|_1 \right] = 2L \left[ \sup_{x \in [a, b]^d} \inf_{y \in \mathcal{D}} \delta(x, y) \right]. \end{aligned} \quad (40)$$

Note that this demonstrates that

$$\|\vartheta\|_\infty \leq \max\{1, L, |a|, |b|, 2[\sup_{x \in [a, b]^d} |f(x)|]\}. \quad (41)$$

Moreover, (40) and (32)–(34) prove that

$$\begin{aligned} \sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\vartheta, 1}(x) - f(x)| &\leq 2L \left[ \sup_{x \in [a, b]^d} \inf_{y \in \mathcal{D}} \delta(x, y) \right] \leq 2Lr \\ &= \frac{dL(b-a)}{\mathfrak{N}} \leq \frac{dL(b-a)}{\frac{2}{3} \left( \frac{A}{2d} \right)^{1/d}} = \frac{(2d)^{1/d} 3dL(b-a)}{2A^{1/d}} \leq \frac{3dL(b-a)}{A^{1/d}}. \end{aligned} \quad (42)$$

Combining this with (41) completes the proof of Proposition 3.5.  $\square$

## 4 Analysis of the generalisation error

In this section we consider the *worst-case* generalisation error arising in deep learning based empirical risk minimisation with quadratic loss function for DNNs with a fixed architecture and weights and biases bounded in size by a fixed constant (cf. Corollary 4.15 in Subsection 4.3). We prove that this worst-case generalisation error converges in the probabilistically strong sense with rate  $1/2$  (up to a logarithmic factor) with respect to the number of samples used for calculating the empirical risk and that the constant in the corresponding upper bound for the worst-case generalisation error scales favourably (i.e., only very moderately) in terms of depth and width of the DNNs employed, cf. (ii) in Corollary 4.15. Corollary 4.15 is a consequence of the main result of this section, Proposition 4.14 in Subsection 4.3, which provides a similar conclusion in a more general setting. The proofs of Proposition 4.14 and Corollary 4.15, respectively, rely on the tools developed in the two preceding subsections, Subsections 4.1 and 4.2.

On the one hand, Subsection 4.1 provides an essentially well-known estimate for the  $L^p$ -error of Monte Carlo-type approximations, cf. Corollary 4.5. Corollary 4.5 is a consequence of the well-known result stated here as Proposition 4.4, which, in turn, follows directly from, e.g., Cox et al. [22, Corollary 5.11] (with  $M \leftarrow M$ ,  $q \leftarrow 2$ ,  $(E, \|\cdot\|_E) \leftarrow (\mathbb{R}^d, \|\cdot\|_2|_{\mathbb{R}^d})$ ,  $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$ ,  $(\xi_j)_{j \in \{1, 2, \dots, M\}} \leftarrow (X_j)_{j \in \{1, 2, \dots, M\}}$ ,  $p \leftarrow p$  in the notation of [22, Corollary 5.11] and Proposition 4.4, respectively). In the proof of Corollary 4.5 we also apply Lemma 4.3, which is Grohs et al. [45, Lemma 2.2]. In order to make the statements of Lemma 4.3 and Proposition 4.4 more accessible for the reader, we recall in Definition 4.1 (cf., e.g., [22, Definition 5.1]) the notion of a Rademacher family and in Definition 4.2 (cf., e.g., [22, Definition 5.4] or Gonon et al. [42, Definition 2.1]) the notion of the  $p$ -Kahane–Khintchine constant.

On the other hand, we derive in Subsection 4.2 uniform  $L^p$ -estimates for Lipschitz continuous random fields with a separable metric space as index set (cf. Lemmas 4.10

and 4.11 and Corollary 4.12). These estimates are uniform in the sense that the supremum over the index set is *inside* the expectation belonging to the  $L^p$ -norm, which is necessary since we intend to prove error bounds for the *worst-case* generalisation error, as illustrated above. One of the elementary but crucial arguments in our derivation of such uniform  $L^p$ -estimates is given in Lemma 4.9 (cf. Lemma 4.8). Roughly speaking, Lemma 4.9 illustrates how the  $L^p$ -norm of a supremum can be bounded from above by the supremum of certain  $L^p$ -norms, where the  $L^p$ -norms are integrating over a general measure space and where the suprema are taken over a general (bounded) separable metric space. Furthermore, the elementary and well-known Lemmas 4.6 and 4.7, respectively, follow immediately from Beck, Jentzen, & Kuckuck [10, (ii) in Lemma 3.13 and (ii) in Lemma 3.14] and ensure that the mathematical statements of Lemmas 4.8, 4.9, and 4.10 do indeed make sense.

The results in Subsections 4.2 and 4.3 are in parts inspired by [10, Subsection 3.2] and we refer, e.g., to [7, 13, 23, 31–33, 52, 67, 87, 92] and the references therein for further results on the generalisation error.

## 4.1 Monte Carlo estimates

**Definition 4.1** (Rademacher family). *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $J$  be a set. Then we say that  $(r_j)_{j \in J}$  is a  $\mathbb{P}$ -Rademacher family if and only if it holds that  $r_j: \Omega \rightarrow \{-1, 1\}$ ,  $j \in J$ , are independent random variables with  $\forall j \in J: \mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1)$ .*

**Definition 4.2** ( $p$ -Kahane–Khintchine constant). *Let  $p \in (0, \infty)$ . Then we denote by  $\mathfrak{K}_p \in (0, \infty]$  the extended real number given by*

$$\mathfrak{K}_p = \sup \left\{ c \in [0, \infty): \begin{bmatrix} \exists \mathbb{R}\text{-Banach space } (E, \|\cdot\|_E): \\ \exists \text{probability space } (\Omega, \mathcal{F}, \mathbb{P}): \\ \exists \mathbb{P}\text{-Rademacher family } (r_j)_{j \in \mathbb{N}}: \\ \exists k \in \mathbb{N}: \exists x_1, x_2, \dots, x_k \in E \setminus \{0\}: \\ \left( \mathbb{E} \left[ \left\| \sum_{j=1}^k r_j x_j \right\|_E^p \right] \right)^{1/p} = c \left( \mathbb{E} \left[ \left\| \sum_{j=1}^k r_j x_j \right\|_E^2 \right] \right)^{1/2} \end{bmatrix} \right\} \quad (43)$$

(cf. Definition 4.1).

**Lemma 4.3.** *It holds for all  $p \in [2, \infty)$  that  $\mathfrak{K}_p \leq \sqrt{p-1} < \infty$  (cf. Definition 4.2).*

**Proposition 4.4.** *Let  $d, M \in \mathbb{N}$ ,  $p \in [2, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow \mathbb{R}^d$ ,  $j \in \{1, 2, \dots, M\}$ , be independent random variables, and assume  $\max_{j \in \{1, 2, \dots, M\}} \mathbb{E}[\|X_j\|_2] < \infty$  (cf. Definition 3.1). Then*

$$\left( \mathbb{E} \left[ \left\| \left[ \sum_{j=1}^M X_j \right] - \mathbb{E} \left[ \sum_{j=1}^M X_j \right] \right\|_2^p \right] \right)^{1/p} \leq 2\mathfrak{K}_p \left[ \sum_{j=1}^M \left( \mathbb{E} [\|X_j - \mathbb{E}[X_j]\|_2^p] \right)^{2/p} \right]^{1/2} \quad (44)$$

(cf. Definition 4.2 and Lemma 4.3).

**Corollary 4.5.** *Let  $d, M \in \mathbb{N}$ ,  $p \in [2, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow \mathbb{R}^d$ ,  $j \in \{1, 2, \dots, M\}$ , be independent random variables, and assume  $\max_{j \in \{1, 2, \dots, M\}} \mathbb{E}[\|X_j\|_2] < \infty$  (cf. Definition 3.1). Then*

$$\left( \mathbb{E} \left[ \left\| \frac{1}{M} \left[ \sum_{j=1}^M X_j \right] - \mathbb{E} \left[ \frac{1}{M} \sum_{j=1}^M X_j \right] \right\|_2^p \right] \right)^{1/p} \leq \frac{2\sqrt{p-1}}{\sqrt{M}} \left[ \max_{j \in \{1, 2, \dots, M\}} \left( \mathbb{E} [\|X_j - \mathbb{E}[X_j]\|_2^p] \right)^{1/p} \right]. \quad (45)$$

*Proof of Corollary 4.5.* Observe that Proposition 4.4 and Lemma 4.3 imply that

$$\begin{aligned}
& \left( \mathbb{E} \left[ \left\| \frac{1}{M} \left[ \sum_{j=1}^M X_j \right] - \mathbb{E} \left[ \frac{1}{M} \sum_{j=1}^M X_j \right] \right\|_2^p \right] \right)^{1/p} \\
&= \frac{1}{M} \left( \mathbb{E} \left[ \left\| \sum_{j=1}^M X_j \right\|_2^p \right] - \mathbb{E} \left[ \left\| \sum_{j=1}^M X_j \right\|_2^p \right] \right)^{1/p} \\
&\leq \frac{2\mathfrak{K}_p}{M} \left[ \sum_{j=1}^M (\mathbb{E} [\|X_j - \mathbb{E}[X_j]\|_2^p])^{2/p} \right]^{1/2} \\
&\leq \frac{2\mathfrak{K}_p}{M} \left[ M \left( \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E} [\|X_j - \mathbb{E}[X_j]\|_2^p])^{2/p} \right) \right]^{1/2} \\
&= \frac{2\mathfrak{K}_p}{\sqrt{M}} \left[ \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E} [\|X_j - \mathbb{E}[X_j]\|_2^p])^{1/p} \right] \\
&\leq \frac{2\sqrt{p-1}}{\sqrt{M}} \left[ \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E} [\|X_j - \mathbb{E}[X_j]\|_2^p])^{1/p} \right]
\end{aligned} \tag{46}$$

(cf. Definition 4.2). The proof of Corollary 4.5 is thus complete.  $\square$

## 4.2 Uniform strong error estimates for random fields

**Lemma 4.6.** *Let  $(E, \mathcal{E})$  be a separable topological space, assume  $E \neq \emptyset$ , let  $(\Omega, \mathcal{F})$  be a measurable space, let  $f_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , be  $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable functions, and assume for all  $\omega \in \Omega$  that  $E \ni x \mapsto f_x(\omega) \in \mathbb{R}$  is a continuous function. Then it holds that the function*

$$\Omega \ni \omega \mapsto \sup_{x \in E} f_x(\omega) \in \mathbb{R} \cup \{\infty\} \tag{47}$$

*is  $\mathcal{F}/\mathcal{B}(\mathbb{R} \cup \{\infty\})$ -measurable.*

**Lemma 4.7.** *Let  $(E, \delta)$  be a separable metric space, assume  $E \neq \emptyset$ , let  $L \in \mathbb{R}$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $Z_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , be random variables, and assume for all  $x, y \in E$  that  $\mathbb{E}[|Z_x|] < \infty$  and  $|Z_x - Z_y| \leq L\delta(x, y)$ . Then it holds that the function*

$$\Omega \ni \omega \mapsto \sup_{x \in E} |Z_x(\omega) - \mathbb{E}[Z_x]| \in [0, \infty] \tag{48}$$

*is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable.*

**Lemma 4.8.** *Let  $(E, \delta)$  be a separable metric space, let  $N \in \mathbb{N}$ ,  $p, L, r_1, r_2, \dots, r_N \in [0, \infty)$ ,  $z_1, z_2, \dots, z_N \in E$  satisfy  $E \subseteq \bigcup_{i=1}^N \{x \in E : \delta(x, z_i) \leq r_i\}$ , let  $(\Omega, \mathcal{F}, \mu)$  be a measure space, let  $Z_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , be  $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable functions, and assume for all  $\omega \in \Omega$ ,  $x, y \in E$  that  $|Z_x(\omega) - Z_y(\omega)| \leq L\delta(x, y)$ . Then*

$$\int_{\Omega} \sup_{x \in E} |Z_x(\omega)|^p \mu(d\omega) \leq \sum_{i=1}^N \int_{\Omega} (Lr_i + |Z_{z_i}(\omega)|)^p \mu(d\omega) \tag{49}$$

(cf. Lemma 4.6).

*Proof of Lemma 4.8.* Throughout this proof let  $B_1, B_2, \dots, B_N \subseteq E$  satisfy for all  $i \in \{1, 2, \dots, N\}$  that  $B_i = \{x \in E : \delta(x, z_i) \leq r_i\}$ . Note that the fact that  $E = \bigcup_{i=1}^N B_i$  shows for all  $\omega \in \Omega$  that

$$\sup_{x \in E} |Z_x(\omega)| = \sup_{x \in (\bigcup_{i=1}^N B_i)} |Z_x(\omega)| = \max_{i \in \{1, 2, \dots, N\}} \sup_{x \in B_i} |Z_x(\omega)|. \tag{50}$$

This establishes that

$$\begin{aligned} \int_{\Omega} \sup_{x \in E} |Z_x(\omega)|^p \mu(d\omega) &= \int_{\Omega} \max_{i \in \{1, 2, \dots, N\}} \sup_{x \in B_i} |Z_x(\omega)|^p \mu(d\omega) \\ &\leq \int_{\Omega} \sum_{i=1}^N \sup_{x \in B_i} |Z_x(\omega)|^p \mu(d\omega) = \sum_{i=1}^N \int_{\Omega} \sup_{x \in B_i} |Z_x(\omega)|^p \mu(d\omega). \end{aligned} \quad (51)$$

Furthermore, the assumption that  $\forall \omega \in \Omega, x, y \in E: |Z_x(\omega) - Z_y(\omega)| \leq L\delta(x, y)$  implies for all  $\omega \in \Omega, i \in \{1, 2, \dots, N\}, x \in B_i$  that

$$\begin{aligned} |Z_x(\omega)| &= |Z_x(\omega) - Z_{z_i}(\omega) + Z_{z_i}(\omega)| \leq |Z_x(\omega) - Z_{z_i}(\omega)| + |Z_{z_i}(\omega)| \\ &\leq L\delta(x, z_i) + |Z_{z_i}(\omega)| \leq Lr_i + |Z_{z_i}(\omega)|. \end{aligned} \quad (52)$$

Combining this with (51) proves that

$$\int_{\Omega} \sup_{x \in E} |Z_x(\omega)|^p \mu(d\omega) \leq \sum_{i=1}^N \int_{\Omega} (Lr_i + |Z_{z_i}(\omega)|)^p \mu(d\omega). \quad (53)$$

The proof of Lemma 4.8 is thus complete.  $\square$

**Lemma 4.9.** Let  $p, L, r \in (0, \infty)$ , let  $(E, \delta)$  be a separable metric space, let  $(\Omega, \mathcal{F}, \mu)$  be a measure space, assume  $E \neq \emptyset$  and  $\mu(\Omega) \neq 0$ , let  $Z_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , be  $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable functions, and assume for all  $\omega \in \Omega, x, y \in E$  that  $|Z_x(\omega) - Z_y(\omega)| \leq L\delta(x, y)$ . Then

$$\int_{\Omega} \sup_{x \in E} |Z_x(\omega)|^p \mu(d\omega) \leq \mathcal{C}_{(E, \delta), r} \left[ \sup_{x \in E} \int_{\Omega} (Lr + |Z_x(\omega)|)^p \mu(d\omega) \right] \quad (54)$$

(cf. Definition 3.2 and Lemma 4.6).

*Proof of Lemma 4.9.* Throughout this proof assume w.l.o.g. that  $\mathcal{C}_{(E, \delta), r} < \infty$ , let  $N \in \mathbb{N}$  be given by  $N = \mathcal{C}_{(E, \delta), r}$ , and let  $z_1, z_2, \dots, z_N \in E$  satisfy  $E \subseteq \bigcup_{i=1}^N \{x \in E: \delta(x, z_i) \leq r\}$ . Note that Lemma 4.8 (with  $r_1 \leftarrow r, r_2 \leftarrow r, \dots, r_N \leftarrow r$  in the notation of Lemma 4.8) establishes that

$$\begin{aligned} \int_{\Omega} \sup_{x \in E} |Z_x(\omega)|^p \mu(d\omega) &\leq \sum_{i=1}^N \int_{\Omega} (Lr + |Z_{z_i}(\omega)|)^p \mu(d\omega) \\ &\leq \sum_{i=1}^N \left[ \sup_{x \in E} \int_{\Omega} (Lr + |Z_x(\omega)|)^p \mu(d\omega) \right] = N \left[ \sup_{x \in E} \int_{\Omega} (Lr + |Z_x(\omega)|)^p \mu(d\omega) \right]. \end{aligned} \quad (55)$$

The proof of Lemma 4.9 is thus complete.  $\square$

**Lemma 4.10.** Let  $p \in [1, \infty)$ ,  $L, r \in (0, \infty)$ , let  $(E, \delta)$  be a separable metric space, assume  $E \neq \emptyset$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $Z_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , be random variables, and assume for all  $x, y \in E$  that  $\mathbb{E}[|Z_x|] < \infty$  and  $|Z_x - Z_y| \leq L\delta(x, y)$ . Then

$$(\mathbb{E}[\sup_{x \in E} |Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \leq (\mathcal{C}_{(E, \delta), r})^{1/p} \left[ 2Lr + \sup_{x \in E} (\mathbb{E}[|Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \right] \quad (56)$$

(cf. Definition 3.2 and Lemma 4.7).

*Proof of Lemma 4.10.* Throughout this proof let  $Y_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , satisfy for all  $x \in E, \omega \in \Omega$  that  $Y_x(\omega) = Z_x(\omega) - \mathbb{E}[Z_x]$ . Note that it holds for all  $\omega \in \Omega, x, y \in E$  that

$$\begin{aligned} |Y_x(\omega) - Y_y(\omega)| &= |(Z_x(\omega) - \mathbb{E}[Z_x]) - (Z_y(\omega) - \mathbb{E}[Z_y])| \\ &\leq |Z_x(\omega) - Z_y(\omega)| + |\mathbb{E}[Z_x] - \mathbb{E}[Z_y]| \leq L\delta(x, y) + \mathbb{E}[|Z_x - Z_y|] \\ &\leq 2L\delta(x, y). \end{aligned} \quad (57)$$

Combining this with Lemma 4.9 (with  $L \leftarrow 2L$ ,  $(\Omega, \mathcal{F}, \mu) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$ ,  $(Z_x)_{x \in E} \leftarrow (Y_x)_{x \in E}$  in the notation of Lemma 4.9) implies that

$$\begin{aligned} & (\mathbb{E}[\sup_{x \in E}|Z_x - \mathbb{E}[Z_x]|^p])^{1/p} = (\mathbb{E}[\sup_{x \in E}|Y_x|^p])^{1/p} \\ & \leq (\mathcal{C}_{(E, \delta), r})^{1/p} \left[ \sup_{x \in E} (\mathbb{E}[(2Lr + |Y_x|)^p])^{1/p} \right] \\ & \leq (\mathcal{C}_{(E, \delta), r})^{1/p} \left[ 2Lr + \sup_{x \in E} (\mathbb{E}[|Y_x|^p])^{1/p} \right] \\ & = (\mathcal{C}_{(E, \delta), r})^{1/p} \left[ 2Lr + \sup_{x \in E} (\mathbb{E}[|Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \right]. \end{aligned} \quad (58)$$

The proof of Lemma 4.10 is thus complete.  $\square$

**Lemma 4.11.** *Let  $M \in \mathbb{N}$ ,  $p \in [2, \infty)$ ,  $L, r \in (0, \infty)$ , let  $(E, \delta)$  be a separable metric space, assume  $E \neq \emptyset$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, for every  $x \in E$  let  $Y_{x,j}: \Omega \rightarrow \mathbb{R}$ ,  $j \in \{1, 2, \dots, M\}$ , be independent random variables, assume for all  $x, y \in E$ ,  $j \in \{1, 2, \dots, M\}$  that  $\mathbb{E}[|Y_{x,j}|] < \infty$  and  $|Y_{x,j} - Y_{y,j}| \leq L\delta(x, y)$ , and let  $Z_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , satisfy for all  $x \in E$  that*

$$Z_x = \frac{1}{M} \left[ \sum_{j=1}^M Y_{x,j} \right]. \quad (59)$$

Then

- (i) it holds for all  $x \in E$  that  $\mathbb{E}[|Z_x|] < \infty$ ,
- (ii) it holds that the function  $\Omega \ni \omega \mapsto \sup_{x \in E}|Z_x(\omega) - \mathbb{E}[Z_x]| \in [0, \infty]$  is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable, and
- (iii) it holds that

$$\begin{aligned} & (\mathbb{E}[\sup_{x \in E}|Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \\ & \leq 2(\mathcal{C}_{(E, \delta), r})^{1/p} \left[ Lr + \frac{\sqrt{p-1}}{\sqrt{M}} \left( \sup_{x \in E} \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|Y_{x,j} - \mathbb{E}[Y_{x,j}]|^p])^{1/p} \right) \right] \end{aligned} \quad (60)$$

(cf. Definition 3.2).

*Proof of Lemma 4.11.* Note that the assumption that  $\forall x \in E, j \in \{1, 2, \dots, M\}$ :  $\mathbb{E}[|Y_{x,j}|] < \infty$  implies for all  $x \in E$  that

$$\mathbb{E}[|Z_x|] = \mathbb{E}\left[\frac{1}{M} \left| \sum_{j=1}^M Y_{x,j} \right|\right] \leq \frac{1}{M} \left[ \sum_{j=1}^M \mathbb{E}[|Y_{x,j}|] \right] \leq \max_{j \in \{1, 2, \dots, M\}} \mathbb{E}[|Y_{x,j}|] < \infty. \quad (61)$$

This proves (i). Next observe that the assumption that  $\forall x, y \in E, j \in \{1, 2, \dots, M\}$ :  $|Y_{x,j} - Y_{y,j}| \leq L\delta(x, y)$  demonstrates for all  $x, y \in E$  that

$$|Z_x - Z_y| = \frac{1}{M} \left| \left[ \sum_{j=1}^M Y_{x,j} \right] - \left[ \sum_{j=1}^M Y_{y,j} \right] \right| \leq \frac{1}{M} \left[ \sum_{j=1}^M |Y_{x,j} - Y_{y,j}| \right] \leq L\delta(x, y). \quad (62)$$

Combining this with (i) and Lemma 4.7 establishes (ii). It thus remains to show (iii). For this note that (i), (62), and Lemma 4.10 yield that

$$(\mathbb{E}[\sup_{x \in E}|Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \leq (\mathcal{C}_{(E, \delta), r})^{1/p} \left[ 2Lr + \sup_{x \in E} (\mathbb{E}[|Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \right]. \quad (63)$$

Moreover, (61) and Corollary 4.5 (with  $d \leftarrow 1$ ,  $(X_j)_{j \in \{1, 2, \dots, M\}} \leftarrow (Y_{x,j})_{j \in \{1, 2, \dots, M\}}$  for  $x \in E$  in the notation of Corollary 4.5) prove for all  $x \in E$  that

$$\begin{aligned} (\mathbb{E}[|Z_x - \mathbb{E}[Z_x]|^p])^{1/p} &= \left( \mathbb{E}\left[ \left| \frac{1}{M} \left[ \sum_{j=1}^M Y_{x,j} \right] - \mathbb{E}\left[ \frac{1}{M} \sum_{j=1}^M Y_{x,j} \right] \right|^p \right] \right)^{1/p} \\ &\leq \frac{2\sqrt{p-1}}{\sqrt{M}} \left[ \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|Y_{x,j} - \mathbb{E}[Y_{x,j}]|^p])^{1/p} \right]. \end{aligned} \quad (64)$$

This and (63) imply that

$$\begin{aligned} &(\mathbb{E}[\sup_{x \in E} |Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \\ &\leq (\mathcal{C}_{(E,\delta),r})^{1/p} \left[ 2Lr + \frac{2\sqrt{p-1}}{\sqrt{M}} \left( \sup_{x \in E} \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|Y_{x,j} - \mathbb{E}[Y_{x,j}]|^p])^{1/p} \right) \right] \\ &= 2(\mathcal{C}_{(E,\delta),r})^{1/p} \left[ Lr + \frac{\sqrt{p-1}}{\sqrt{M}} \left( \sup_{x \in E} \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|Y_{x,j} - \mathbb{E}[Y_{x,j}]|^p])^{1/p} \right) \right]. \end{aligned} \quad (65)$$

The proof of Lemma 4.11 is thus complete.  $\square$

**Corollary 4.12.** Let  $M \in \mathbb{N}$ ,  $p \in [2, \infty)$ ,  $L, C \in (0, \infty)$ , let  $(E, \delta)$  be a separable metric space, assume  $E \neq \emptyset$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, for every  $x \in E$  let  $Y_{x,j}: \Omega \rightarrow \mathbb{R}$ ,  $j \in \{1, 2, \dots, M\}$ , be independent random variables, assume for all  $x, y \in E$ ,  $j \in \{1, 2, \dots, M\}$  that  $\mathbb{E}[|Y_{x,j}|] < \infty$  and  $|Y_{x,j} - Y_{y,j}| \leq L\delta(x, y)$ , and let  $Z_x: \Omega \rightarrow \mathbb{R}$ ,  $x \in E$ , satisfy for all  $x \in E$  that

$$Z_x = \frac{1}{M} \left[ \sum_{j=1}^M Y_{x,j} \right]. \quad (66)$$

Then

- (i) it holds for all  $x \in E$  that  $\mathbb{E}[|Z_x|] < \infty$ ,
- (ii) it holds that the function  $\Omega \ni \omega \mapsto \sup_{x \in E} |Z_x(\omega) - \mathbb{E}[Z_x]| \in [0, \infty]$  is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable, and
- (iii) it holds that

$$\begin{aligned} &(\mathbb{E}[\sup_{x \in E} |Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \\ &\leq \frac{2\sqrt{p-1}}{\sqrt{M}} \left( \mathcal{C}_{(E,\delta), \frac{C\sqrt{p-1}}{L\sqrt{M}}} \right)^{1/p} \left[ C + \sup_{x \in E} \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|Y_{x,j} - \mathbb{E}[Y_{x,j}]|^p])^{1/p} \right] \end{aligned} \quad (67)$$

(cf. Definition 3.2).

*Proof of Corollary 4.12.* Note that Lemma 4.11 shows (i) and (ii). In addition, Lemma 4.11 (with  $r \leftarrow C\sqrt{p-1}/(L\sqrt{M})$  in the notation of Lemma 4.11) ensures that

$$\begin{aligned} &(\mathbb{E}[\sup_{x \in E} |Z_x - \mathbb{E}[Z_x]|^p])^{1/p} \\ &\leq 2 \left( \mathcal{C}_{(E,\delta), \frac{C\sqrt{p-1}}{L\sqrt{M}}} \right)^{1/p} \left[ L \frac{C\sqrt{p-1}}{L\sqrt{M}} + \frac{\sqrt{p-1}}{\sqrt{M}} \left( \sup_{x \in E} \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|Y_{x,j} - \mathbb{E}[Y_{x,j}]|^p])^{1/p} \right) \right] \\ &= \frac{2\sqrt{p-1}}{\sqrt{M}} \left( \mathcal{C}_{(E,\delta), \frac{C\sqrt{p-1}}{L\sqrt{M}}} \right)^{1/p} \left[ C + \sup_{x \in E} \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|Y_{x,j} - \mathbb{E}[Y_{x,j}]|^p])^{1/p} \right]. \end{aligned} \quad (68)$$

This establishes (iii) and thus completes the proof of Corollary 4.12.  $\square$

### 4.3 Strong convergence rates for the generalisation error

**Lemma 4.13.** Let  $M \in \mathbb{N}$ ,  $p \in [2, \infty)$ ,  $L, C, b \in (0, \infty)$ , let  $(E, \delta)$  be a separable metric space, assume  $E \neq \emptyset$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_{x,j}: \Omega \rightarrow \mathbb{R}$ ,  $j \in \{1, 2, \dots, M\}$ ,  $x \in E$ , and  $Y_j: \Omega \rightarrow \mathbb{R}$ ,  $j \in \{1, 2, \dots, M\}$ , be functions, assume for every  $x \in E$  that  $(X_{x,j}, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, assume for all  $x, y \in E$ ,  $j \in \{1, 2, \dots, M\}$  that  $|X_{x,j} - Y_j| \leq b$  and  $|X_{x,j} - X_{y,j}| \leq L\delta(x, y)$ , let  $\mathbf{R}: E \rightarrow [0, \infty)$  satisfy for all  $x \in E$  that  $\mathbf{R}(x) = \mathbb{E}[|X_{x,1} - Y_1|^2]$ , and let  $\mathcal{R}: E \times \Omega \rightarrow [0, \infty)$  satisfy for all  $x \in E$ ,  $\omega \in \Omega$  that

$$\mathcal{R}(x, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |X_{x,j}(\omega) - Y_j(\omega)|^2 \right]. \quad (69)$$

Then

(i) it holds that the function  $\Omega \ni \omega \mapsto \sup_{x \in E} |\mathcal{R}(x, \omega) - \mathbf{R}(x)| \in [0, \infty]$  is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable and

(ii) it holds that

$$(\mathbb{E}[\sup_{x \in E} |\mathcal{R}(x) - \mathbf{R}(x)|^p])^{1/p} \leq \left( \mathcal{C}_{(E, \delta), \frac{Cb\sqrt{p-1}}{2L\sqrt{M}}} \right)^{1/p} \left[ \frac{2(C+1)b^2\sqrt{p-1}}{\sqrt{M}} \right] \quad (70)$$

(cf. Definition 3.2).

*Proof of Lemma 4.13.* Throughout this proof let  $\mathcal{Y}_{x,j}: \Omega \rightarrow \mathbb{R}$ ,  $j \in \{1, 2, \dots, M\}$ ,  $x \in E$ , satisfy for all  $x \in E$ ,  $j \in \{1, 2, \dots, M\}$  that  $\mathcal{Y}_{x,j} = |X_{x,j} - Y_j|^2$ . Note that the assumption that for every  $x \in E$  it holds that  $(X_{x,j}, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables ensures for all  $x \in E$  that

$$\mathbb{E}[\mathcal{R}(x)] = \frac{1}{M} \left[ \sum_{j=1}^M \mathbb{E}[|X_{x,j} - Y_j|^2] \right] = \frac{M \mathbb{E}[|X_{x,1} - Y_1|^2]}{M} = \mathbf{R}(x). \quad (71)$$

Furthermore, the assumption that  $\forall x \in E$ ,  $j \in \{1, 2, \dots, M\}$ :  $|X_{x,j} - Y_j| \leq b$  shows for all  $x \in E$ ,  $j \in \{1, 2, \dots, M\}$  that

$$\mathbb{E}[|\mathcal{Y}_{x,j}|] = \mathbb{E}[|X_{x,j} - Y_j|^2] \leq b^2 < \infty, \quad (72)$$

$$\mathcal{Y}_{x,j} - \mathbb{E}[\mathcal{Y}_{x,j}] = |X_{x,j} - Y_j|^2 - \mathbb{E}[|X_{x,j} - Y_j|^2] \leq |X_{x,j} - Y_j|^2 \leq b^2, \quad (73)$$

and

$$\mathbb{E}[\mathcal{Y}_{x,j}] - \mathcal{Y}_{x,j} = \mathbb{E}[|X_{x,j} - Y_j|^2] - |X_{x,j} - Y_j|^2 \leq \mathbb{E}[|X_{x,j} - Y_j|^2] \leq b^2. \quad (74)$$

Combining (72)–(74) implies for all  $x \in E$ ,  $j \in \{1, 2, \dots, M\}$  that

$$(\mathbb{E}[|\mathcal{Y}_{x,j} - \mathbb{E}[\mathcal{Y}_{x,j}]|^p])^{1/p} \leq (\mathbb{E}[b^{2p}])^{1/p} = b^2. \quad (75)$$

Moreover, note that the assumptions that  $\forall x, y \in E$ ,  $j \in \{1, 2, \dots, M\}$ :  $[|X_{x,j} - Y_j| \leq b$  and  $|X_{x,j} - X_{y,j}| \leq L\delta(x, y)]$  and the fact that  $\forall x_1, x_2, y \in \mathbb{R}$ :  $(x_1 - y)^2 - (x_2 - y)^2 = (x_1 - x_2)((x_1 - y) + (x_2 - y))$  establish for all  $x, y \in E$ ,  $j \in \{1, 2, \dots, M\}$  that

$$\begin{aligned} |\mathcal{Y}_{x,j} - \mathcal{Y}_{y,j}| &= |(X_{x,j} - Y_j)^2 - (X_{y,j} - Y_j)^2| \\ &\leq |X_{x,j} - X_{y,j}|(|X_{x,j} - Y_j| + |X_{y,j} - Y_j|) \\ &\leq 2b|X_{x,j} - X_{y,j}| \leq 2bL\delta(x, y). \end{aligned} \quad (76)$$

Combining this, (71), (72), and the fact that for every  $x \in E$  it holds that  $\mathcal{Y}_{x,j}$ ,  $j \in \{1, 2, \dots, M\}$ , are independent random variables with Corollary 4.12 (with  $L \leftarrow 2bL$ ,  $C \leftarrow Cb^2$ ,  $(Y_{x,j})_{x \in E, j \in \{1, 2, \dots, M\}} \leftarrow (\mathcal{Y}_{x,j})_{x \in E, j \in \{1, 2, \dots, M\}}$ ,  $(Z_x)_{x \in E} \leftarrow (\Omega \ni \omega \mapsto \mathcal{R}(x, \omega) \in \mathbb{R})_{x \in E}$  in the notation of Corollary 4.12) and (75) proves (i) and

$$\begin{aligned} & (\mathbb{E}[\sup_{x \in E} |\mathcal{R}(x) - \mathbf{R}(x)|^p])^{1/p} = (\mathbb{E}[\sup_{x \in E} |\mathcal{R}(x) - \mathbb{E}[\mathcal{R}(x)]|^p])^{1/p} \\ & \leq \frac{2\sqrt{p-1}}{\sqrt{M}} \left( \mathcal{C}_{(E, \delta), \frac{Cb^2\sqrt{p-1}}{2bL\sqrt{M}}} \right)^{1/p} \left[ Cb^2 + \sup_{x \in E} \max_{j \in \{1, 2, \dots, M\}} (\mathbb{E}[|\mathcal{Y}_{x,j} - \mathbb{E}[\mathcal{Y}_{x,j}]|^p])^{1/p} \right] \quad (77) \\ & \leq \frac{2\sqrt{p-1}}{\sqrt{M}} \left( \mathcal{C}_{(E, \delta), \frac{Cb\sqrt{p-1}}{2L\sqrt{M}}} \right)^{1/p} [Cb^2 + b^2] = \left( \mathcal{C}_{(E, \delta), \frac{Cb\sqrt{p-1}}{2L\sqrt{M}}} \right)^{1/p} \left[ \frac{2(C+1)b^2\sqrt{p-1}}{\sqrt{M}} \right]. \end{aligned}$$

This shows (ii) and thus completes the proof of Lemma 4.13.  $\square$

**Proposition 4.14.** Let  $d, \mathbf{d}, M \in \mathbb{N}$ ,  $L, b \in (0, \infty)$ ,  $\alpha \in \mathbb{R}$ ,  $\beta \in (\alpha, \infty)$ ,  $D \subseteq \mathbb{R}^d$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow D$ ,  $j \in \{1, 2, \dots, M\}$ , and  $Y_j: \Omega \rightarrow \mathbb{R}$ ,  $j \in \{1, 2, \dots, M\}$ , be functions, assume that  $(X_j, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $f = (f_\theta)_{\theta \in [\alpha, \beta]^d}: [\alpha, \beta]^d \rightarrow C(D, \mathbb{R})$  be a function, assume for all  $\theta, \vartheta \in [\alpha, \beta]^d$ ,  $j \in \{1, 2, \dots, M\}$ ,  $x \in D$  that  $|f_\theta(X_j) - Y_j| \leq b$  and  $|f_\theta(x) - f_\vartheta(x)| \leq L\|\theta - \vartheta\|_\infty$ , let  $\mathbf{R}: [\alpha, \beta]^d \rightarrow [0, \infty)$  satisfy for all  $\theta \in [\alpha, \beta]^d$  that  $\mathbf{R}(\theta) = \mathbb{E}[|f_\theta(X_1) - Y_1|^2]$ , and let  $\mathcal{R}: [\alpha, \beta]^d \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in [\alpha, \beta]^d$ ,  $\omega \in \Omega$  that

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |f_\theta(X_j(\omega)) - Y_j(\omega)|^2 \right] \quad (78)$$

(cf. Definition 3.1). Then

- (i) it holds that the function  $\Omega \ni \omega \mapsto \sup_{\theta \in [\alpha, \beta]^d} |\mathcal{R}(\theta, \omega) - \mathbf{R}(\theta)| \in [0, \infty]$  is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable and
- (ii) it holds for all  $p \in (0, \infty)$  that

$$\begin{aligned} & (\mathbb{E}[\sup_{\theta \in [\alpha, \beta]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} \\ & \leq \inf_{C, \varepsilon \in (0, \infty)} \left[ \frac{2(C+1)b^2 \max\{1, [2\sqrt{M}L(\beta-\alpha)(Cb)^{-1}]^\varepsilon\} \sqrt{\max\{1, p, \mathbf{d}/\varepsilon\}}}{\sqrt{M}} \right] \quad (79) \\ & \leq \inf_{C \in (0, \infty)} \left[ \frac{2(C+1)b^2 \sqrt{e \max\{1, p, \mathbf{d} \ln(4ML^2(\beta-\alpha)^2(Cb)^{-2})\}}}{\sqrt{M}} \right]. \end{aligned}$$

*Proof of Proposition 4.14.* Throughout this proof let  $p \in (0, \infty)$ , let  $(\kappa_C)_{C \in (0, \infty)} \subseteq (0, \infty)$  satisfy for all  $C \in (0, \infty)$  that  $2\sqrt{ML(\beta-\alpha)/(Cb)}$ , let  $\mathcal{X}_{\theta,j}: \Omega \rightarrow \mathbb{R}$ ,  $j \in \{1, 2, \dots, M\}$ ,  $\theta \in [\alpha, \beta]^d$ , satisfy for all  $\theta \in [\alpha, \beta]^d$ ,  $j \in \{1, 2, \dots, M\}$  that  $\mathcal{X}_{\theta,j} = f_\theta(X_j)$ , and let  $\delta: ([\alpha, \beta]^d) \times ([\alpha, \beta]^d) \rightarrow [0, \infty)$  satisfy for all  $\theta, \vartheta \in [\alpha, \beta]^d$  that  $\delta(\theta, \vartheta) = \|\theta - \vartheta\|_\infty$ . First of all, note that the assumption that  $\forall \theta \in [\alpha, \beta]^d$ ,  $j \in \{1, 2, \dots, M\}$ :  $|f_\theta(X_j) - Y_j| \leq b$  implies for all  $\theta \in [\alpha, \beta]^d$ ,  $j \in \{1, 2, \dots, M\}$  that

$$|\mathcal{X}_{\theta,j} - Y_j| = |f_\theta(X_j) - Y_j| \leq b. \quad (80)$$

In addition, the assumption that  $\forall \theta, \vartheta \in [\alpha, \beta]^d$ ,  $x \in D$ :  $|f_\theta(x) - f_\vartheta(x)| \leq L\|\theta - \vartheta\|_\infty$  ensures for all  $\theta, \vartheta \in [\alpha, \beta]^d$ ,  $j \in \{1, 2, \dots, M\}$  that

$$|\mathcal{X}_{\theta,j} - \mathcal{X}_{\vartheta,j}| = |f_\theta(X_j) - f_\vartheta(X_j)| \leq \sup_{x \in D} |f_\theta(x) - f_\vartheta(x)| \leq L\|\theta - \vartheta\|_\infty = L\delta(\theta, \vartheta). \quad (81)$$

Combining this, (80), and the fact that for every  $\theta \in [\alpha, \beta]^{\mathbf{d}}$  it holds that  $(\mathcal{X}_{\theta,j}, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables with Lemma 4.13 (with  $p \leftarrow q$ ,  $C \leftarrow C$ ,  $(E, \delta) \leftarrow ([\alpha, \beta]^{\mathbf{d}}, \delta)$ ,  $(X_{x,j})_{x \in E, j \in \{1, 2, \dots, M\}} \leftarrow (\mathcal{X}_{\theta,j})_{\theta \in [\alpha, \beta]^{\mathbf{d}}, j \in \{1, 2, \dots, M\}}$  for  $q \in [2, \infty)$ ,  $C \in (0, \infty)$  in the notation of Lemma 4.13) demonstrates for all  $C \in (0, \infty)$ ,  $q \in [2, \infty)$  that the function  $\Omega \ni \omega \mapsto \sup_{\theta \in [\alpha, \beta]^{\mathbf{d}}} |\mathcal{R}(\theta, \omega) - \mathbf{R}(\theta)| \in [0, \infty]$  is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable and

$$(\mathbb{E}[\sup_{\theta \in [\alpha, \beta]^{\mathbf{d}}} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^q])^{1/q} \leq \left( \mathcal{C}_{([\alpha, \beta]^{\mathbf{d}}, \delta), \frac{Cb\sqrt{q-1}}{2L\sqrt{M}}} \right)^{1/q} \left[ \frac{2(C+1)b^2\sqrt{q-1}}{\sqrt{M}} \right] \quad (82)$$

(cf. Definition 3.2). This finishes the proof of (i). Next observe that (ii) in Lemma 3.3 (with  $d \leftarrow \mathbf{d}$ ,  $a \leftarrow \alpha$ ,  $b \leftarrow \beta$ ,  $r \leftarrow r$  for  $r \in (0, \infty)$  in the notation of Lemma 3.3) shows for all  $r \in (0, \infty)$  that

$$\begin{aligned} \mathcal{C}_{([\alpha, \beta]^{\mathbf{d}}, \delta), r} &\leq \mathbb{1}_{[0, r]} \left( \frac{\beta-\alpha}{2} \right) + \left( \frac{\beta-\alpha}{r} \right)^{\mathbf{d}} \mathbb{1}_{(r, \infty)} \left( \frac{\beta-\alpha}{2} \right) \\ &\leq \max \left\{ 1, \left( \frac{\beta-\alpha}{r} \right)^{\mathbf{d}} \right\} \left( \mathbb{1}_{[0, r]} \left( \frac{\beta-\alpha}{2} \right) + \mathbb{1}_{(r, \infty)} \left( \frac{\beta-\alpha}{2} \right) \right) \\ &= \max \left\{ 1, \left( \frac{\beta-\alpha}{r} \right)^{\mathbf{d}} \right\}. \end{aligned} \quad (83)$$

This yields for all  $C \in (0, \infty)$ ,  $q \in [2, \infty)$  that

$$\begin{aligned} \left( \mathcal{C}_{([\alpha, \beta]^{\mathbf{d}}, \delta), \frac{Cb\sqrt{q-1}}{2L\sqrt{M}}} \right)^{1/q} &\leq \max \left\{ 1, \left( \frac{2(\beta-\alpha)L\sqrt{M}}{Cb\sqrt{q-1}} \right)^{\frac{\mathbf{d}}{q}} \right\} \\ &\leq \max \left\{ 1, \left( \frac{2(\beta-\alpha)L\sqrt{M}}{Cb} \right)^{\frac{\mathbf{d}}{q}} \right\} = \max \left\{ 1, (\kappa_C)^{\frac{\mathbf{d}}{q}} \right\}. \end{aligned} \quad (84)$$

Jensen's inequality and (82) hence prove for all  $C, \varepsilon \in (0, \infty)$  that

$$\begin{aligned} &(\mathbb{E}[\sup_{\theta \in [\alpha, \beta]^{\mathbf{d}}} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} \\ &\leq (\mathbb{E}[\sup_{\theta \in [\alpha, \beta]^{\mathbf{d}}} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^{\max\{2, p, \mathbf{d}/\varepsilon\}}])^{\frac{1}{\max\{2, p, \mathbf{d}/\varepsilon\}}} \\ &\leq \max \left\{ 1, (\kappa_C)^{\frac{\mathbf{d}}{\max\{2, p, \mathbf{d}/\varepsilon\}}} \right\} \frac{2(C+1)b^2\sqrt{\max\{2, p, \mathbf{d}/\varepsilon\}-1}}{\sqrt{M}} \\ &= \max \left\{ 1, (\kappa_C)^{\min\{\mathbf{d}/2, \mathbf{d}/p, \varepsilon\}} \right\} \frac{2(C+1)b^2\sqrt{\max\{1, p-1, \mathbf{d}/\varepsilon-1\}}}{\sqrt{M}} \\ &\leq \frac{2(C+1)b^2 \max\{1, (\kappa_C)^{\varepsilon}\} \sqrt{\max\{1, p, \mathbf{d}/\varepsilon\}}}{\sqrt{M}}. \end{aligned} \quad (85)$$

Next note that the fact that  $\forall a \in (1, \infty) : a^{1/(2 \ln(a))} = e^{\ln(a)/(2 \ln(a))} = e^{1/2} = \sqrt{e} \geq 1$  ensures for all  $C \in (0, \infty)$  with  $\kappa_C > 1$  that

$$\begin{aligned} &\inf_{\varepsilon \in (0, \infty)} \left[ \frac{2(C+1)b^2 \max\{1, (\kappa_C)^{\varepsilon}\} \sqrt{\max\{1, p, \mathbf{d}/\varepsilon\}}}{\sqrt{M}} \right] \\ &\leq \frac{2(C+1)b^2 \max\{1, (\kappa_C)^{1/(2 \ln(\kappa_C))}\} \sqrt{\max\{1, p, 2\mathbf{d} \ln(\kappa_C)\}}}{\sqrt{M}} \\ &= \frac{2(C+1)b^2 \sqrt{e \max\{1, p, \mathbf{d} \ln([\kappa_C]^2)\}}}{\sqrt{M}}. \end{aligned} \quad (86)$$

In addition, observe that it holds for all  $C \in (0, \infty)$  with  $\kappa_C \leq 1$  that

$$\begin{aligned} & \inf_{\varepsilon \in (0, \infty)} \left[ \frac{2(C+1)b^2 \max\{1, (\kappa_C)^\varepsilon\} \sqrt{\max\{1, p, \mathbf{d}/\varepsilon\}}}{\sqrt{M}} \right] \\ & \leq \inf_{\varepsilon \in (0, \infty)} \left[ \frac{2(C+1)b^2 \sqrt{\max\{1, p, \mathbf{d}/\varepsilon\}}}{\sqrt{M}} \right] \leq \frac{2(C+1)b^2 \sqrt{\max\{1, p\}}}{\sqrt{M}} \\ & \leq \frac{2(C+1)b^2 \sqrt{e \max\{1, p, \mathbf{d} \ln([\kappa_C]^2)\}}}{\sqrt{M}}. \end{aligned} \quad (87)$$

Combining (85) with (86) and (87) demonstrates that

$$\begin{aligned} & (\mathbb{E}[\sup_{\theta \in [\alpha, \beta]^\mathbf{d}} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} \\ & \leq \inf_{C, \varepsilon \in (0, \infty)} \left[ \frac{2(C+1)b^2 \max\{1, (\kappa_C)^\varepsilon\} \sqrt{\max\{1, p, \mathbf{d}/\varepsilon\}}}{\sqrt{M}} \right] \\ & = \inf_{C, \varepsilon \in (0, \infty)} \left[ \frac{2(C+1)b^2 \max\{1, [2\sqrt{ML}(\beta-\alpha)(Cb)^{-1}]^\varepsilon\} \sqrt{\max\{1, p, \mathbf{d}/\varepsilon\}}}{\sqrt{M}} \right] \\ & \leq \inf_{C \in (0, \infty)} \left[ \frac{2(C+1)b^2 \sqrt{e \max\{1, p, \mathbf{d} \ln([\kappa_C]^2)\}}}{\sqrt{M}} \right] \\ & = \inf_{C \in (0, \infty)} \left[ \frac{2(C+1)b^2 \sqrt{e \max\{1, p, \mathbf{d} \ln(4ML^2(\beta-\alpha)^2(Cb)^{-2})\}}}{\sqrt{M}} \right]. \end{aligned} \quad (88)$$

This establishes (ii) and thus completes the proof of Proposition 4.14.  $\square$

**Corollary 4.15.** Let  $d, \mathbf{d}, \mathbf{L}, M \in \mathbb{N}$ ,  $B, b \in [1, \infty)$ ,  $u \in \mathbb{R}$ ,  $v \in [u+1, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $D \subseteq [-b, b]^d$ , assume  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow D$ ,  $j \in \{1, 2, \dots, M\}$ , and  $Y_j: \Omega \rightarrow [u, v]$ ,  $j \in \{1, 2, \dots, M\}$ , be functions, assume that  $(X_j, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $\mathbf{R}: [-B, B]^{\mathbf{d}} \rightarrow [0, \infty)$  satisfy for all  $\theta \in [-B, B]^{\mathbf{d}}$  that  $\mathbf{R}(\theta) = \mathbb{E}[|\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(X_1) - Y_1|^2]$ , and let  $\mathcal{R}: [-B, B]^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in [-B, B]^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \quad (89)$$

(cf. Definition 2.8). Then

- (i) it holds that the function  $\Omega \ni \omega \mapsto \sup_{\theta \in [-B, B]^{\mathbf{d}}} |\mathcal{R}(\theta, \omega) - \mathbf{R}(\theta)| \in [0, \infty]$  is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable and
- (ii) it holds for all  $p \in (0, \infty)$  that

$$\begin{aligned} & (\mathbb{E}[\sup_{\theta \in [-B, B]^{\mathbf{d}}} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} \\ & \leq \frac{9(v-u)^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1) \sqrt{\max\{p, \ln(4(Mb)^{1/\mathbf{L}}(\|\mathbf{l}\|_\infty + 1)B)\}}}{\sqrt{M}} \\ & \leq \frac{9(v-u)^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBb)\}}{\sqrt{M}} \end{aligned} \quad (90)$$

(cf. Definition 3.1).

*Proof of Corollary 4.15.* Throughout this proof let  $\mathfrak{d} \in \mathbb{N}$  be given by  $\mathfrak{d} = \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$ , let  $L \in (0, \infty)$  be given by  $L = b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} B^{\mathbf{L}-1}$ , let  $f = (f_\theta)_{\theta \in [-B, B]^\mathfrak{d}} : [-B, B]^\mathfrak{d} \rightarrow C(D, \mathbb{R})$  satisfy for all  $\theta \in [-B, B]^\mathfrak{d}$ ,  $x \in D$  that  $f_\theta(x) = \mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x)$ , let  $\mathcal{R} : [-B, B]^\mathfrak{d} \rightarrow [0, \infty)$  satisfy for all  $\theta \in [-B, B]^\mathfrak{d}$  that  $\mathcal{R}(\theta) = \mathbb{E}[|f_\theta(X_1) - Y_1|^2] = \mathbb{E}[|\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_1) - Y_1|^2]$ , and let  $R : [-B, B]^\mathfrak{d} \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in [-B, B]^\mathfrak{d}$ ,  $\omega \in \Omega$  that

$$R(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |f_\theta(X_j(\omega)) - Y_j(\omega)|^2 \right] = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right]. \quad (91)$$

Note that the fact that  $\forall \theta \in \mathbb{R}^\mathfrak{d}, x \in \mathbb{R}^d : \mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) \in [u, v]$  and the assumption that  $\forall j \in \{1, 2, \dots, M\} : Y_j(\Omega) \subseteq [u, v]$  imply for all  $\theta \in [-B, B]^\mathfrak{d}, j \in \{1, 2, \dots, M\}$  that

$$|f_\theta(X_j) - Y_j| = |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j) - Y_j| \leq \sup_{y_1, y_2 \in [u, v]} |y_1 - y_2| = v - u. \quad (92)$$

Moreover, the assumptions that  $D \subseteq [-b, b]^d$ ,  $\mathbf{l}_0 = d$ , and  $\mathbf{l}_{\mathbf{L}} = 1$ , Beck, Jentzen, & Kuckuck [10, Corollary 2.37] (with  $a \leftarrow -b$ ,  $b \leftarrow b$ ,  $u \leftarrow u$ ,  $v \leftarrow v$ ,  $d \leftarrow \mathfrak{d}$ ,  $L \leftarrow \mathbf{L}$ ,  $l \leftarrow 1$  in the notation of [10, Corollary 2.37]), and the assumptions that  $b \geq 1$  and  $B \geq 1$  ensure for all  $\theta, \vartheta \in [-B, B]^\mathfrak{d}, x \in D$  that

$$\begin{aligned} |f_\theta(x) - f_\vartheta(x)| &\leq \sup_{y \in [-b, b]^d} |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(y) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(y)| \\ &\leq \mathbf{L} \max\{1, b\} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} (\max\{1, \|\theta\|_\infty, \|\vartheta\|_\infty\})^{\mathbf{L}-1} \|\theta - \vartheta\|_\infty \\ &\leq b\mathbf{L} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} B^{\mathbf{L}-1} \|\theta - \vartheta\|_\infty = L \|\theta - \vartheta\|_\infty. \end{aligned} \quad (93)$$

Furthermore, the facts that  $\mathbf{d} \geq \mathfrak{d}$  and  $\forall \theta = (\theta_1, \theta_2, \dots, \theta_{\mathbf{d}}) \in \mathbb{R}^{\mathbf{d}} : \mathcal{N}_{u,v}^{\theta,\mathbf{l}} = \mathcal{N}_{u,v}^{(\theta_1, \theta_2, \dots, \theta_{\mathfrak{d}}), \mathbf{l}}$  prove for all  $\omega \in \Omega$  that

$$\sup_{\theta \in [-B, B]^{\mathbf{d}}} |\mathcal{R}(\theta, \omega) - \mathbf{R}(\theta)| = \sup_{\theta \in [-B, B]^\mathfrak{d}} |R(\theta, \omega) - \mathcal{R}(\theta)|. \quad (94)$$

Next observe that (92), (93), Proposition 4.14 (with  $\mathbf{d} \leftarrow \mathfrak{d}$ ,  $b \leftarrow v - u$ ,  $\alpha \leftarrow -B$ ,  $\beta \leftarrow B$ ,  $\mathbf{R} \leftarrow \mathcal{R}$ ,  $\mathcal{R} \leftarrow R$  in the notation of Proposition 4.14), and the facts that  $v - u \geq (u + 1) - u = 1$  and  $\mathfrak{d} \leq \mathbf{L}\|\mathbf{l}\|_\infty(\|\mathbf{l}\|_\infty + 1) \leq \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2$  demonstrate for all  $p \in (0, \infty)$  that the function  $\Omega \ni \omega \mapsto \sup_{\theta \in [-B, B]^\mathfrak{d}} |R(\theta, \omega) - \mathcal{R}(\theta)| \in [0, \infty]$  is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable and

$$\begin{aligned} &\left( \mathbb{E} [\sup_{\theta \in [-B, B]^\mathfrak{d}} |R(\theta) - \mathcal{R}(\theta)|^p] \right)^{1/p} \\ &\leq \inf_{C \in (0, \infty)} \left[ \frac{2(C+1)(v-u)^2 \sqrt{e \max\{1, p, \mathfrak{d} \ln(4ML^2(2B)^2(C[v-u])^{-2})\}}}{\sqrt{M}} \right] \\ &\leq \inf_{C \in (0, \infty)} \left[ \frac{2(C+1)(v-u)^2 \sqrt{e \max\{1, p, \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \ln(2^4ML^2B^2C^{-2})\}}}{\sqrt{M}} \right]. \end{aligned} \quad (95)$$

This and (94) establish (i). In addition, combining (94)–(95) with the fact that  $2^6\mathbf{L}^2 \leq 2^6 \cdot 2^{2(\mathbf{L}-1)} = 2^{4+2\mathbf{L}} \leq 2^{4\mathbf{L}+2\mathbf{L}} = 2^{6\mathbf{L}}$  and the facts that  $3 \geq e$ ,  $B \geq 1$ ,  $\mathbf{L} \geq 1$ ,  $M \geq 1$ , and

$b \geq 1$  shows for all  $p \in (0, \infty)$  that

$$\begin{aligned}
& (\mathbb{E}[\sup_{\theta \in [-B, B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} = (\mathbb{E}[\sup_{\theta \in [-B, B]^d} |R(\theta) - \mathcal{R}(\theta)|^p])^{1/p} \\
& \leq \frac{2(1/2 + 1)(v - u)^2 \sqrt{e \max\{1, p, \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \ln(2^4 M L^2 B^2 2^2)\}}}{\sqrt{M}} \\
& = \frac{3(v - u)^2 \sqrt{e \max\{p, \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \ln(2^6 M b^2 \mathbf{L}^2 (\|\mathbf{l}\|_\infty + 1)^{2\mathbf{L}} B^{2\mathbf{L}})\}}}{\sqrt{M}} \\
& \leq \frac{3(v - u)^2 \sqrt{e \max\{p, 3\mathbf{L}^2(\|\mathbf{l}\|_\infty + 1)^2 \ln([2^{6\mathbf{L}} M b^2 (\|\mathbf{l}\|_\infty + 1)^{2\mathbf{L}} B^{2\mathbf{L}}]^{1/(3\mathbf{L})})\}}}{\sqrt{M}} \\
& \leq \frac{3(v - u)^2 \sqrt{3 \max\{p, 3\mathbf{L}^2(\|\mathbf{l}\|_\infty + 1)^2 \ln(2^2(Mb^2)^{1/(3\mathbf{L})}(\|\mathbf{l}\|_\infty + 1)B)\}}}{\sqrt{M}} \\
& \leq \frac{9(v - u)^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1) \sqrt{\max\{p, \ln(4(Mb)^{1/\mathbf{L}}(\|\mathbf{l}\|_\infty + 1)B)\}}}{\sqrt{M}}.
\end{aligned} \tag{96}$$

Furthermore, note that the fact that  $\forall n \in \mathbb{N}: n \leq 2^{n-1}$  and the fact that  $\|\mathbf{l}\|_\infty \geq 1$  imply that

$$4(\|\mathbf{l}\|_\infty + 1) \leq 2^2 \cdot 2^{(\|\mathbf{l}\|_\infty + 1) - 1} = 2^3 \cdot 2^{(\|\mathbf{l}\|_\infty + 1) - 2} \leq 3^2 \cdot 3^{(\|\mathbf{l}\|_\infty + 1) - 2} = 3^{(\|\mathbf{l}\|_\infty + 1)}. \tag{97}$$

This demonstrates for all  $p \in (0, \infty)$  that

$$\begin{aligned}
& \frac{9(v - u)^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1) \sqrt{\max\{p, \ln(4(Mb)^{1/\mathbf{L}}(\|\mathbf{l}\|_\infty + 1)B)\}}}{\sqrt{M}} \\
& \leq \frac{9(v - u)^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1) \sqrt{\max\{p, (\|\mathbf{l}\|_\infty + 1) \ln([3^{(\|\mathbf{l}\|_\infty + 1)}(Mb)^{1/\mathbf{L}} B]^{1/(\|\mathbf{l}\|_\infty + 1)})\}}}{\sqrt{M}} \\
& \leq \frac{9(v - u)^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBb)\}}{\sqrt{M}}.
\end{aligned} \tag{98}$$

Combining this with (96) shows (ii). The proof of Corollary 4.15 is thus complete.  $\square$

## 5 Analysis of the optimisation error

The main result of this section, Proposition 5.6, establishes that the optimisation error of the Minimum Monte Carlo method applied to a Lipschitz continuous random field with a  $\mathbf{d}$ -dimensional hypercube as index set, where  $\mathbf{d} \in \mathbb{N}$ , converges in the probabilistically strong sense with rate  $1/\mathbf{d}$  with respect to the number of samples used, provided that the sample indices are continuous uniformly drawn from the index hypercube (cf. (ii) in Proposition 5.6). We refer to Beck, Jentzen, & Kuckuck [10, Lemmas 3.22–3.23] for analogous results for convergence in probability instead of strong convergence and to Beck et al. [8, Lemma 3.5] for a related result. Corollary 5.8 below specialises Proposition 5.6 to the case where the empirical risk from deep learning based empirical risk minimisation with quadratic loss function indexed by a hypercube of DNN parameter vectors plays the role of the random field under consideration. In the proof of Corollary 5.8 we make use of the elementary and well-known fact that this choice for the random field is indeed Lipschitz continuous, which is the assertion of Lemma 5.7. Further results on the optimisation error in the context of stochastic approximation can be found, e.g., in [2, 4, 12, 18, 25, 26, 28, 29, 38, 60, 62, 63, 65, 88, 97, 98] and the references therein.

The proof of the main result of this section, Proposition 5.6, crucially relies (cf. Lemma 5.5) on the complementary distribution function formula (cf., e.g., Elbrächter et al. [35, Lemma 2.2]) and the elementary estimate for the beta function given in Corollary 5.4. In order to prove Corollary 5.4, we first collect a few basic facts about the gamma and the beta function in the elementary and well-known Lemma 5.1 and derive from these in Proposition 5.3 further elementary and essentially well-known properties of the gamma function. In particular, the inequalities in (100) in Proposition 5.3 below are slightly reformulated versions of the well-known inequalities called *Wendel's double inequality* (cf. Wendel [94]) or *Gautschi's double inequality* (cf. Gautschi [40]); cf., e.g., Qi [81, Subsection 2.1 and Subsection 2.4].

## 5.1 Properties of the gamma and the beta function

**Lemma 5.1.** *Let  $\Gamma: (0, \infty) \rightarrow (0, \infty)$  satisfy for all  $x \in (0, \infty)$  that  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$  and let  $\mathbb{B}: (0, \infty)^2 \rightarrow (0, \infty)$  satisfy for all  $x, y \in (0, \infty)$  that  $\mathbb{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ . Then*

- (i) *it holds for all  $x \in (0, \infty)$  that  $\Gamma(x+1) = x\Gamma(x)$ ,*
- (ii) *it holds that  $\Gamma(1) = \Gamma(2) = 1$ , and*
- (iii) *it holds for all  $x, y \in (0, \infty)$  that  $\mathbb{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ .*

**Lemma 5.2.** *It holds for all  $\alpha, x \in [0, 1]$  that  $(1-x)^\alpha \leq 1 - \alpha x$ .*

*Proof of Lemma 5.2.* Note that the fact that for every  $y \in [0, \infty)$  it holds that the function  $[0, \infty) \ni z \mapsto y^z \in [0, \infty)$  is a convex function implies for all  $\alpha, x \in [0, 1]$  that

$$\begin{aligned} (1-x)^\alpha &= (1-x)^{\alpha \cdot 1 + (1-\alpha) \cdot 0} \\ &\leq \alpha(1-x)^1 + (1-\alpha)(1-x)^0 \\ &= \alpha - \alpha x + 1 - \alpha = 1 - \alpha x. \end{aligned} \tag{99}$$

The proof of Lemma 5.2 is thus complete.  $\square$

**Proposition 5.3.** *Let  $\Gamma: (0, \infty) \rightarrow (0, \infty)$  satisfy for all  $x \in (0, \infty)$  that  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$  and let  $\lfloor \cdot \rfloor: (0, \infty) \rightarrow \mathbb{N}_0$  satisfy for all  $x \in (0, \infty)$  that  $\lfloor x \rfloor = \max([0, x] \cap \mathbb{N}_0)$ . Then*

- (i) *it holds that  $\Gamma: (0, \infty) \rightarrow (0, \infty)$  is a convex function,*
- (ii) *it holds for all  $x \in (0, \infty)$  that  $\Gamma(x+1) = x\Gamma(x) \leq x^{\lfloor x \rfloor} \leq \max\{1, x^x\}$ ,*
- (iii) *it holds for all  $x \in (0, \infty)$ ,  $\alpha \in [0, 1]$  that*

$$(\max\{x+\alpha-1, 0\})^\alpha \leq \frac{x}{(x+\alpha)^{1-\alpha}} \leq \frac{\Gamma(x+\alpha)}{\Gamma(x)} \leq x^\alpha, \tag{100}$$

and

- (iv) *it holds for all  $x \in (0, \infty)$ ,  $\alpha \in [0, \infty)$  that*

$$(\max\{x + \min\{\alpha-1, 0\}, 0\})^\alpha \leq \frac{\Gamma(x+\alpha)}{\Gamma(x)} \leq (x + \max\{\alpha-1, 0\})^\alpha. \tag{101}$$

*Proof of Proposition 5.3.* First, observe that the fact that for every  $t \in (0, \infty)$  it holds that the function  $\mathbb{R} \ni x \mapsto t^x \in (0, \infty)$  is a convex function implies for all  $x, y \in (0, \infty)$ ,  $\alpha \in [0, 1]$  that

$$\begin{aligned}\Gamma(\alpha x + (1 - \alpha)y) &= \int_0^\infty t^{\alpha x + (1 - \alpha)y - 1} e^{-t} dt = \int_0^\infty t^{\alpha x + (1 - \alpha)y} t^{-1} e^{-t} dt \\ &\leq \int_0^\infty (\alpha t^x + (1 - \alpha)t^y) t^{-1} e^{-t} dt \\ &= \alpha \int_0^\infty t^{x-1} e^{-t} dt + (1 - \alpha) \int_0^\infty t^{y-1} e^{-t} dt \\ &= \alpha \Gamma(x) + (1 - \alpha) \Gamma(y).\end{aligned}\tag{102}$$

This shows (i).

Second, note that (ii) in Lemma 5.1 and (i) establish for all  $\alpha \in [0, 1]$  that

$$\Gamma(\alpha + 1) = \Gamma(\alpha \cdot 2 + (1 - \alpha) \cdot 1) \leq \alpha \Gamma(2) + (1 - \alpha) \Gamma(1) = \alpha + (1 - \alpha) = 1.\tag{103}$$

This yields for all  $x \in (0, 1]$  that

$$\Gamma(x + 1) \leq 1 = x^x = \max\{1, x^x\}.\tag{104}$$

Induction, (i) in Lemma 5.1, and the fact that  $\forall x \in (0, \infty) : x - \lfloor x \rfloor \in (0, 1]$  hence ensure for all  $x \in [1, \infty)$  that

$$\Gamma(x + 1) = \left[ \prod_{i=1}^{\lfloor x \rfloor} (x - i + 1) \right] \Gamma(x - \lfloor x \rfloor + 1) \leq x^{\lfloor x \rfloor} \Gamma(x - \lfloor x \rfloor + 1) \leq x^{\lfloor x \rfloor} \leq x^x = \max\{1, x^x\}.\tag{105}$$

Combining this with again (i) in Lemma 5.1 and (104) establishes (ii).

Third, note that Hölder's inequality and (i) in Lemma 5.1 prove for all  $x \in (0, \infty)$ ,  $\alpha \in [0, 1]$  that

$$\begin{aligned}\Gamma(x + \alpha) &= \int_0^\infty t^{x+\alpha-1} e^{-t} dt = \int_0^\infty t^{\alpha x} e^{-\alpha t} t^{(1-\alpha)x-(1-\alpha)} e^{-(1-\alpha)t} dt \\ &= \int_0^\infty [t^x e^{-t}]^\alpha [t^{x-1} e^{-t}]^{1-\alpha} dt \\ &\leq \left( \int_0^\infty t^x e^{-t} dt \right)^\alpha \left( \int_0^\infty t^{x-1} e^{-t} dt \right)^{1-\alpha} \\ &= [\Gamma(x + 1)]^\alpha [\Gamma(x)]^{1-\alpha} = x^\alpha [\Gamma(x)]^\alpha [\Gamma(x)]^{1-\alpha} \\ &= x^\alpha \Gamma(x).\end{aligned}\tag{106}$$

This and again (i) in Lemma 5.1 demonstrate for all  $x \in (0, \infty)$ ,  $\alpha \in [0, 1]$  that

$$x \Gamma(x) = \Gamma(x + 1) = \Gamma(x + \alpha + (1 - \alpha)) \leq (x + \alpha)^{1-\alpha} \Gamma(x + \alpha).\tag{107}$$

Combining (106) and (107) yields for all  $x \in (0, \infty)$ ,  $\alpha \in [0, 1]$  that

$$\frac{x}{(x + \alpha)^{1-\alpha}} \leq \frac{\Gamma(x + \alpha)}{\Gamma(x)} \leq x^\alpha.\tag{108}$$

Furthermore, observe that (i) in Lemma 5.1 and (108) imply for all  $x \in (0, \infty)$ ,  $\alpha \in [0, 1]$  that

$$\frac{\Gamma(x + \alpha)}{\Gamma(x + 1)} = \frac{\Gamma(x + \alpha)}{x \Gamma(x)} \leq x^{\alpha-1}.\tag{109}$$

This shows for all  $\alpha \in [0, 1]$ ,  $x \in (\alpha, \infty)$  that

$$\frac{\Gamma(x)}{\Gamma(x + (1 - \alpha))} = \frac{\Gamma((x - \alpha) + \alpha)}{\Gamma((x - \alpha) + 1)} \leq (x - \alpha)^{\alpha-1} = \frac{1}{(x - \alpha)^{1-\alpha}}. \quad (110)$$

This, in turn, ensures for all  $\alpha \in [0, 1]$ ,  $x \in (1 - \alpha, \infty)$  that

$$(x + \alpha - 1)^\alpha = (x - (1 - \alpha))^\alpha \leq \frac{\Gamma(x + \alpha)}{\Gamma(x)}. \quad (111)$$

Next note that Lemma 5.2 proves for all  $x \in (0, \infty)$ ,  $\alpha \in [0, 1]$  that

$$\begin{aligned} (\max\{x + \alpha - 1, 0\})^\alpha &= (x + \alpha)^\alpha \left( \frac{\max\{x + \alpha - 1, 0\}}{x + \alpha} \right)^\alpha \\ &= (x + \alpha)^\alpha \left( \max\left\{1 - \frac{1}{x + \alpha}, 0\right\}\right)^\alpha \\ &\leq (x + \alpha)^\alpha \left(1 - \frac{\alpha}{x + \alpha}\right) = (x + \alpha)^\alpha \left(\frac{x}{x + \alpha}\right) \\ &= \frac{x}{(x + \alpha)^{1-\alpha}}. \end{aligned} \quad (112)$$

This and (108) establish (iii).

Fourth, we show (iv). For this let  $\lfloor \cdot \rfloor : [0, \infty) \rightarrow \mathbb{N}_0$  satisfy for all  $x \in [0, \infty)$  that  $\lfloor x \rfloor = \max([0, x] \cap \mathbb{N}_0)$ . Observe that induction, (i) in Lemma 5.1, the fact that  $\forall \alpha \in [0, \infty) : \alpha - \lfloor \alpha \rfloor \in [0, 1)$ , and (iii) demonstrate for all  $x \in (0, \infty)$ ,  $\alpha \in [0, \infty)$  that

$$\begin{aligned} \frac{\Gamma(x + \alpha)}{\Gamma(x)} &= \left[ \prod_{i=1}^{\lfloor \alpha \rfloor} (x + \alpha - i) \right] \frac{\Gamma(x + \alpha - \lfloor \alpha \rfloor)}{\Gamma(x)} \leq \left[ \prod_{i=1}^{\lfloor \alpha \rfloor} (x + \alpha - i) \right] x^{\alpha - \lfloor \alpha \rfloor} \\ &\leq (x + \alpha - 1)^{\lfloor \alpha \rfloor} x^{\alpha - \lfloor \alpha \rfloor} \\ &\leq (x + \max\{\alpha - 1, 0\})^{\lfloor \alpha \rfloor} (x + \max\{\alpha - 1, 0\})^{\alpha - \lfloor \alpha \rfloor} \\ &= (x + \max\{\alpha - 1, 0\})^\alpha. \end{aligned} \quad (113)$$

Furthermore, again the fact that  $\forall \alpha \in [0, \infty) : \alpha - \lfloor \alpha \rfloor \in [0, 1)$ , (iii), induction, and (i) in Lemma 5.1 imply for all  $x \in (0, \infty)$ ,  $\alpha \in [0, \infty)$  that

$$\begin{aligned} \frac{\Gamma(x + \alpha)}{\Gamma(x)} &= \frac{\Gamma(x + \lfloor \alpha \rfloor + \alpha - \lfloor \alpha \rfloor)}{\Gamma(x)} \\ &\geq (\max\{x + \lfloor \alpha \rfloor + \alpha - \lfloor \alpha \rfloor - 1, 0\})^{\alpha - \lfloor \alpha \rfloor} \left[ \frac{\Gamma(x + \lfloor \alpha \rfloor)}{\Gamma(x)} \right] \\ &= (\max\{x + \alpha - 1, 0\})^{\alpha - \lfloor \alpha \rfloor} \left[ \prod_{i=1}^{\lfloor \alpha \rfloor} (x + \lfloor \alpha \rfloor - i) \right] \frac{\Gamma(x)}{\Gamma(x)} \\ &\geq (\max\{x + \alpha - 1, 0\})^{\alpha - \lfloor \alpha \rfloor} x^{\lfloor \alpha \rfloor} \\ &= (\max\{x + \alpha - 1, 0\})^{\alpha - \lfloor \alpha \rfloor} (\max\{x, 0\})^{\lfloor \alpha \rfloor} \\ &\geq (\max\{x + \min\{\alpha - 1, 0\}, 0\})^{\alpha - \lfloor \alpha \rfloor} (\max\{x + \min\{\alpha - 1, 0\}, 0\})^{\lfloor \alpha \rfloor} \\ &= (\max\{x + \min\{\alpha - 1, 0\}, 0\})^\alpha. \end{aligned} \quad (114)$$

Combining this with (113) shows (iv). The proof of Proposition 5.3 is thus complete.  $\square$

**Corollary 5.4.** Let  $\mathbb{B}: (0, \infty)^2 \rightarrow (0, \infty)$  satisfy for all  $x, y \in (0, \infty)$  that  $\mathbb{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$  and let  $\Gamma: (0, \infty) \rightarrow (0, \infty)$  satisfy for all  $x \in (0, \infty)$  that  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ . Then it holds for all  $x, y \in (0, \infty)$  with  $x + y > 1$  that

$$\frac{\Gamma(x)}{(y + \max\{x - 1, 0\})^x} \leq \mathbb{B}(x, y) \leq \frac{\Gamma(x)}{(y + \min\{x - 1, 0\})^x} \leq \frac{\max\{1, x^x\}}{x(y + \min\{x - 1, 0\})^x}. \quad (115)$$

*Proof of Corollary 5.4.* Note that (iii) in Lemma 5.1 ensures for all  $x, y \in (0, \infty)$  that

$$\mathbb{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(y+x)}. \quad (116)$$

In addition, observe that it holds for all  $x, y \in (0, \infty)$  with  $x + y > 1$  that  $y + \min\{x - 1, 0\} > 0$ . This and (iv) in Proposition 5.3 demonstrate for all  $x, y \in (0, \infty)$  with  $x + y > 1$  that

$$0 < (y + \min\{x - 1, 0\})^x \leq \frac{\Gamma(y+x)}{\Gamma(y)} \leq (y + \max\{x - 1, 0\})^x. \quad (117)$$

Combining this with (116) and (ii) in Proposition 5.3 shows for all  $x, y \in (0, \infty)$  with  $x + y > 1$  that

$$\frac{\Gamma(x)}{(y + \max\{x - 1, 0\})^x} \leq \mathbb{B}(x, y) \leq \frac{\Gamma(x)}{(y + \min\{x - 1, 0\})^x} \leq \frac{\max\{1, x^x\}}{x(y + \min\{x - 1, 0\})^x}. \quad (118)$$

The proof of Corollary 5.4 is thus complete.  $\square$

## 5.2 Strong convergence rates for the optimisation error

**Lemma 5.5.** Let  $K \in \mathbb{N}$ ,  $p, L \in (0, \infty)$ , let  $(E, \delta)$  be a metric space, let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $\mathcal{R}: E \times \Omega \rightarrow \mathbb{R}$  be a  $(\mathcal{B}(E) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$ -measurable function, assume for all  $x, y \in E$ ,  $\omega \in \Omega$  that  $|\mathcal{R}(x, \omega) - \mathcal{R}(y, \omega)| \leq L\delta(x, y)$ , and let  $X_k: \Omega \rightarrow E$ ,  $k \in \{1, 2, \dots, K\}$ , be i.i.d. random variables. Then it holds for all  $x \in E$  that

$$\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(X_k) - \mathcal{R}(x)|^p] \leq L^p \int_0^\infty [\mathbb{P}(\delta(X_1, x) > \varepsilon^{1/p})]^K d\varepsilon. \quad (119)$$

*Proof of Lemma 5.5.* Throughout this proof let  $x \in E$  and let  $Y: \Omega \rightarrow [0, \infty)$  be the function which satisfies for all  $\omega \in \Omega$  that  $Y(\omega) = \min_{k \in \{1, 2, \dots, K\}} [\delta(X_k(\omega), x)]^p$ . Observe that the fact that  $Y$  is a random variable, the assumption that  $\forall x, y \in E, \omega \in \Omega: |\mathcal{R}(x, \omega) - \mathcal{R}(y, \omega)| \leq L\delta(x, y)$ , and the complementary distribution function formula (see, e.g., Elbrächter et al. [35, Lemma 2.2]) demonstrate that

$$\begin{aligned} \mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(X_k) - \mathcal{R}(x)|^p] &\leq L^p \mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} [\delta(X_k, x)]^p] \\ &= L^p \mathbb{E}[Y] = L^p \int_0^\infty y \mathbb{P}_Y(dy) = L^p \int_0^\infty \mathbb{P}_Y((\varepsilon, \infty)) d\varepsilon \\ &= L^p \int_0^\infty \mathbb{P}(Y > \varepsilon) d\varepsilon = L^p \int_0^\infty \mathbb{P}(\min_{k \in \{1, 2, \dots, K\}} [\delta(X_k, x)]^p > \varepsilon) d\varepsilon. \end{aligned} \quad (120)$$

Moreover, the assumption that  $\Theta_k$ ,  $k \in \{1, 2, \dots, K\}$ , are i.i.d. random variables shows for all  $\varepsilon \in (0, \infty)$  that

$$\begin{aligned} \mathbb{P}(\min_{k \in \{1, 2, \dots, K\}} [\delta(X_k, x)]^p > \varepsilon) &= \mathbb{P}(\forall k \in \{1, 2, \dots, K\}: [\delta(X_k, x)]^p > \varepsilon) \\ &= \prod_{k=1}^K \mathbb{P}([\delta(X_k, x)]^p > \varepsilon) = [\mathbb{P}([\delta(X_1, x)]^p > \varepsilon)]^K = [\mathbb{P}(\delta(X_1, x) > \varepsilon^{1/p})]^K. \end{aligned} \quad (121)$$

Combining (120) with (121) proves (119). The proof of Lemma 5.5 is thus complete.  $\square$

**Proposition 5.6.** Let  $\mathbf{d}, K \in \mathbb{N}$ ,  $L, \alpha \in \mathbb{R}$ ,  $\beta \in (\alpha, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $\mathcal{R}: [\alpha, \beta]^{\mathbf{d}} \times \Omega \rightarrow \mathbb{R}$  be a random field, assume for all  $\theta, \vartheta \in [\alpha, \beta]^{\mathbf{d}}$ ,  $\omega \in \Omega$  that  $|\mathcal{R}(\theta, \omega) - \mathcal{R}(\vartheta, \omega)| \leq L\|\theta - \vartheta\|_{\infty}$ , let  $\Theta_k: \Omega \rightarrow [\alpha, \beta]^{\mathbf{d}}$ ,  $k \in \{1, 2, \dots, K\}$ , be i.i.d. random variables, and assume that  $\Theta_1$  is continuous uniformly distributed on  $[\alpha, \beta]^{\mathbf{d}}$  (cf. Definition 3.1). Then

- (i) it holds that  $\mathcal{R}$  is a  $(\mathcal{B}([\alpha, \beta]^{\mathbf{d}}) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$ -measurable function and
- (ii) it holds for all  $\theta \in [\alpha, \beta]^{\mathbf{d}}$ ,  $p \in (0, \infty)$  that

$$\begin{aligned} & (\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_k) - \mathcal{R}(\theta)|^p])^{1/p} \\ & \leq \frac{L(\beta - \alpha) \max\{1, (p/\mathbf{d})^{1/\mathbf{d}}\}}{K^{1/\mathbf{d}}} \leq \frac{L(\beta - \alpha) \max\{1, p\}}{K^{1/\mathbf{d}}}. \end{aligned} \quad (122)$$

*Proof of Proposition 5.6.* Throughout this proof assume w.l.o.g. that  $L > 0$ , let  $\delta: ([\alpha, \beta]^{\mathbf{d}}) \times ([\alpha, \beta]^{\mathbf{d}}) \rightarrow [0, \infty)$  satisfy for all  $\theta, \vartheta \in [\alpha, \beta]^{\mathbf{d}}$  that  $\delta(\theta, \vartheta) = \|\theta - \vartheta\|_{\infty}$ , let  $\mathbb{B}: (0, \infty)^2 \rightarrow (0, \infty)$  satisfy for all  $x, y \in (0, \infty)$  that  $\mathbb{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ , and let  $\Theta_{1,1}, \Theta_{1,2}, \dots, \Theta_{1,\mathbf{d}}: \Omega \rightarrow [\alpha, \beta]$  satisfy  $\Theta_1 = (\Theta_{1,1}, \Theta_{1,2}, \dots, \Theta_{1,\mathbf{d}})$ . First of all, note that the assumption that  $\forall \theta, \vartheta \in [\alpha, \beta]^{\mathbf{d}}, \omega \in \Omega: |\mathcal{R}(\theta, \omega) - \mathcal{R}(\vartheta, \omega)| \leq L\|\theta - \vartheta\|_{\infty}$  ensures for all  $\omega \in \Omega$  that the function  $[\alpha, \beta]^{\mathbf{d}} \ni \theta \mapsto \mathcal{R}(\theta, \omega) \in \mathbb{R}$  is continuous. Combining this with the fact that  $([\alpha, \beta]^{\mathbf{d}}, \delta)$  is a separable metric space, the fact that for every  $\theta \in [\alpha, \beta]^{\mathbf{d}}$  it holds that the function  $\Omega \ni \omega \mapsto \mathcal{R}(\theta, \omega) \in \mathbb{R}$  is  $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable, and, e.g., Aliprantis & Border [1, Lemma 4.51] (see also, e.g., Beck et al. [8, Lemma 2.4]) proves (i). Next observe that it holds for all  $\theta \in [\alpha, \beta], \varepsilon \in [0, \infty)$  that

$$\begin{aligned} & \min\{\theta + \varepsilon, \beta\} - \max\{\theta - \varepsilon, \alpha\} = \min\{\theta + \varepsilon, \beta\} + \min\{\varepsilon - \theta, -\alpha\} \\ & = \min\{\theta + \varepsilon + \min\{\varepsilon - \theta, -\alpha\}, \beta + \min\{\varepsilon - \theta, -\alpha\}\} \\ & = \min\{\min\{2\varepsilon, \theta - \alpha + \varepsilon\}, \min\{\beta - \theta + \varepsilon, \beta - \alpha\}\} \\ & \geq \min\{\min\{2\varepsilon, \alpha - \alpha + \varepsilon\}, \min\{\beta - \beta + \varepsilon, \beta - \alpha\}\} \\ & = \min\{2\varepsilon, \varepsilon, \varepsilon, \beta - \alpha\} = \min\{\varepsilon, \beta - \alpha\}. \end{aligned} \quad (123)$$

The assumption that  $\Theta_1$  is continuous uniformly distributed on  $[\alpha, \beta]^{\mathbf{d}}$  hence shows for all  $\theta = (\theta_1, \theta_2, \dots, \theta_{\mathbf{d}}) \in [\alpha, \beta]^{\mathbf{d}}, \varepsilon \in [0, \infty)$  that

$$\begin{aligned} & \mathbb{P}(\|\Theta_1 - \theta\|_{\infty} \leq \varepsilon) = \mathbb{P}(\max_{i \in \{1, 2, \dots, \mathbf{d}\}} |\Theta_{1,i} - \theta_i| \leq \varepsilon) \\ & = \mathbb{P}(\forall i \in \{1, 2, \dots, \mathbf{d}\}: -\varepsilon \leq \Theta_{1,i} - \theta_i \leq \varepsilon) \\ & = \mathbb{P}(\forall i \in \{1, 2, \dots, \mathbf{d}\}: \theta_i - \varepsilon \leq \Theta_{1,i} \leq \theta_i + \varepsilon) \\ & = \mathbb{P}(\forall i \in \{1, 2, \dots, \mathbf{d}\}: \max\{\theta_i - \varepsilon, \alpha\} \leq \Theta_{1,i} \leq \min\{\theta_i + \varepsilon, \beta\}) \\ & = \mathbb{P}(\Theta_1 \in [\times_{i=1}^{\mathbf{d}} [\max\{\theta_i - \varepsilon, \alpha\}, \min\{\theta_i + \varepsilon, \beta\}]]]) \\ & = \frac{1}{(\beta - \alpha)^{\mathbf{d}}} \prod_{i=1}^{\mathbf{d}} (\min\{\theta_i + \varepsilon, \beta\} - \max\{\theta_i - \varepsilon, \alpha\}) \\ & \geq \frac{1}{(\beta - \alpha)^{\mathbf{d}}} [\min\{\varepsilon, \beta - \alpha\}]^{\mathbf{d}} = \min\left\{1, \frac{\varepsilon^{\mathbf{d}}}{(\beta - \alpha)^{\mathbf{d}}}\right\}. \end{aligned} \quad (124)$$

Therefore, we obtain for all  $\theta \in [\alpha, \beta]^{\mathbf{d}}, p \in (0, \infty), \varepsilon \in [0, \infty)$  that

$$\begin{aligned} & \mathbb{P}(\|\Theta_1 - \theta\|_{\infty} > \varepsilon^{1/p}) = 1 - \mathbb{P}(\|\Theta_1 - \theta\|_{\infty} \leq \varepsilon^{1/p}) \\ & \leq 1 - \min\left\{1, \frac{\varepsilon^{\mathbf{d}/p}}{(\beta - \alpha)^{\mathbf{d}}}\right\} = \max\left\{0, 1 - \frac{\varepsilon^{\mathbf{d}/p}}{(\beta - \alpha)^{\mathbf{d}}}\right\}. \end{aligned} \quad (125)$$

This, (i), the assumption that  $\forall \theta, \vartheta \in [\alpha, \beta]^{\mathbf{d}}, \omega \in \Omega: |\mathcal{R}(\theta, \omega) - \mathcal{R}(\vartheta, \omega)| \leq L\|\theta - \vartheta\|_{\infty}$ , the assumption that  $\Theta_k, k \in \{1, 2, \dots, K\}$ , are i.i.d. random variables, and Lemma 5.5 (with  $(E, \delta) \leftarrow ([\alpha, \beta]^{\mathbf{d}}, \delta)$ ,  $(X_k)_{k \in \{1, 2, \dots, K\}} \leftarrow (\Theta_k)_{k \in \{1, 2, \dots, K\}}$  in the notation of Lemma 5.5) establish for all  $\theta \in [\alpha, \beta]^{\mathbf{d}}, p \in (0, \infty)$  that

$$\begin{aligned} \mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_k) - \mathcal{R}(\theta)|^p] &\leq L^p \int_0^\infty [\mathbb{P}(\|\Theta_1 - \theta\|_{\infty} > \varepsilon^{1/p})]^K d\varepsilon \\ &\leq L^p \int_0^\infty \left[ \max \left\{ 0, 1 - \frac{\varepsilon^{\mathbf{d}/p}}{(\beta-\alpha)^{\mathbf{d}}} \right\} \right]^K d\varepsilon = L^p \int_0^{(\beta-\alpha)^p} \left( 1 - \frac{\varepsilon^{\mathbf{d}/p}}{(\beta-\alpha)^{\mathbf{d}}} \right)^K d\varepsilon \\ &= \frac{p}{\mathbf{d}} L^p (\beta - \alpha)^p \int_0^1 t^{p/\mathbf{d}-1} (1-t)^K dt = \frac{p}{\mathbf{d}} L^p (\beta - \alpha)^p \int_0^1 t^{p/\mathbf{d}-1} (1-t)^{K+1-1} dt \\ &= \frac{p}{\mathbf{d}} L^p (\beta - \alpha)^p \mathbb{B}(p/\mathbf{d}, K+1). \end{aligned} \quad (126)$$

Corollary 5.4 (with  $x \leftarrow p/\mathbf{d}$ ,  $y \leftarrow K+1$  for  $p \in (0, \infty)$  in the notation of (115) in Corollary 5.4) hence demonstrates for all  $\theta \in [\alpha, \beta]^{\mathbf{d}}, p \in (0, \infty)$  that

$$\begin{aligned} \mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_k) - \mathcal{R}(\theta)|^p] &\leq \frac{\frac{p}{\mathbf{d}} L^p (\beta - \alpha)^p \max\{1, (p/\mathbf{d})^{p/\mathbf{d}}\}}{\frac{p}{\mathbf{d}} (K+1 + \min\{p/\mathbf{d} - 1, 0\})^{p/\mathbf{d}}} \leq \frac{L^p (\beta - \alpha)^p \max\{1, (p/\mathbf{d})^{p/\mathbf{d}}\}}{K^{p/\mathbf{d}}}. \end{aligned} \quad (127)$$

This implies for all  $\theta \in [\alpha, \beta]^{\mathbf{d}}, p \in (0, \infty)$  that

$$\begin{aligned} &(\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_k) - \mathcal{R}(\theta)|^p])^{1/p} \\ &\leq \frac{L(\beta - \alpha) \max\{1, (p/\mathbf{d})^{1/\mathbf{d}}\}}{K^{1/\mathbf{d}}} \leq \frac{L(\beta - \alpha) \max\{1, p\}}{K^{1/\mathbf{d}}}. \end{aligned} \quad (128)$$

This shows (ii) and thus completes the proof of Proposition 5.6.  $\square$

**Lemma 5.7.** Let  $d, \mathbf{d}, \mathbf{L}, M \in \mathbb{N}, B, b \in [1, \infty), u \in \mathbb{R}, v \in (u, \infty), \mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}, D \subseteq [-b, b]^d$ , assume  $\mathbf{l}_0 = d, \mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$ , let  $\Omega$  be a set, let  $X_j: \Omega \rightarrow D, j \in \{1, 2, \dots, M\}$ , and  $Y_j: \Omega \rightarrow [u, v], j \in \{1, 2, \dots, M\}$ , be functions, and let  $\mathcal{R}: [-B, B]^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in [-B, B]^{\mathbf{d}}, \omega \in \Omega$  that

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \quad (129)$$

(cf. Definition 2.8). Then it holds for all  $\theta, \vartheta \in [-B, B]^{\mathbf{d}}, \omega \in \Omega$  that

$$|\mathcal{R}(\theta, \omega) - \mathcal{R}(\vartheta, \omega)| \leq 2(v-u)b\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}} B^{\mathbf{L}-1} \|\theta - \vartheta\|_{\infty} \quad (130)$$

(cf. Definition 3.1).

*Proof of Lemma 5.7.* Observe that the fact that  $\forall x_1, x_2, y \in \mathbb{R}: (x_1 - y)^2 - (x_2 - y)^2 = (x_1 - x_2)((x_1 - y) + (x_2 - y))$ , the fact that  $\forall \theta \in \mathbb{R}^{\mathbf{d}}, x \in \mathbb{R}^d: \mathcal{N}_{u,v}^{\theta, \mathbf{l}}(x) \in [u, v]$ , and the assumption that  $\forall j \in \{1, 2, \dots, M\}, \omega \in \Omega: Y_j(\omega) \in [u, v]$  prove for all  $\theta, \vartheta \in [-B, B]^{\mathbf{d}}$ ,

$\omega \in \Omega$  that

$$\begin{aligned}
& |\mathcal{R}(\theta, \omega) - \mathcal{R}(\vartheta, \omega)| \\
&= \frac{1}{M} \left| \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] - \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \right| \\
&\leq \frac{1}{M} \left[ \sum_{j=1}^M |[\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)]^2 - [\mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)]^2| \right] \\
&= \frac{1}{M} \left[ \sum_{j=1}^M (|\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(X_j(\omega))| \right. \\
&\quad \cdot \left. |[\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)] + [\mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)]|) \right] \\
&\leq \frac{2}{M} \left[ \sum_{j=1}^M ([\sup_{x \in D} |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(x)|] [\sup_{y_1, y_2 \in [u,v]} |y_1 - y_2|]) \right] \\
&= 2(v-u) [\sup_{x \in D} |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(x)|].
\end{aligned} \tag{131}$$

In addition, combining the assumptions that  $D \subseteq [-b, b]^d$ ,  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ ,  $b \geq 1$ , and  $B \geq 1$  with Beck, Jentzen, & Kuckuck [10, Corollary 2.37] (with  $a \leftarrow -b$ ,  $b \leftarrow b$ ,  $u \leftarrow u$ ,  $v \leftarrow v$ ,  $d \leftarrow \mathbf{d}$ ,  $L \leftarrow \mathbf{L}$ ,  $l \leftarrow \mathbf{l}$  in the notation of [10, Corollary 2.37]) shows for all  $\theta, \vartheta \in [-B, B]^{\mathbf{d}}$  that

$$\begin{aligned}
& \sup_{x \in D} |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(x)| \leq \sup_{x \in [-b,b]^d} |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(x)| \\
&\leq \mathbf{L} \max\{1, b\} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} (\max\{1, \|\theta\|_\infty, \|\vartheta\|_\infty\})^{\mathbf{L}-1} \|\theta - \vartheta\|_\infty \\
&\leq b \mathbf{L} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} B^{\mathbf{L}-1} \|\theta - \vartheta\|_\infty.
\end{aligned} \tag{132}$$

This and (131) imply for all  $\theta, \vartheta \in [-B, B]^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$|\mathcal{R}(\theta, \omega) - \mathcal{R}(\vartheta, \omega)| \leq 2(v-u) b \mathbf{L} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} B^{\mathbf{L}-1} \|\theta - \vartheta\|_\infty. \tag{133}$$

The proof of Lemma 5.7 is thus complete.  $\square$

**Corollary 5.8.** Let  $d, \mathbf{d}, \mathbf{d}, \mathbf{L}, M, K \in \mathbb{N}$ ,  $B, b \in [1, \infty)$ ,  $u \in \mathbb{R}$ ,  $v \in (u, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $D \subseteq [-b, b]^d$ , assume  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \mathbf{d} = \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $\Theta_k: \Omega \rightarrow [-B, B]^{\mathbf{d}}$ ,  $k \in \{1, 2, \dots, K\}$ , be i.i.d. random variables, assume that  $\Theta_1$  is continuous uniformly distributed on  $[-B, B]^{\mathbf{d}}$ , let  $X_j: \Omega \rightarrow D$ ,  $j \in \{1, 2, \dots, M\}$ , and  $Y_j: \Omega \rightarrow [u, v]$ ,  $j \in \{1, 2, \dots, M\}$ , be random variables, and let  $\mathcal{R}: [-B, B]^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in [-B, B]^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \tag{134}$$

(cf. Definition 2.8). Then

- (i) it holds that  $\mathcal{R}$  is a  $(\mathcal{B}([-B, B]^{\mathbf{d}}) \otimes \mathcal{F})/\mathcal{B}([0, \infty))$ -measurable function and
- (ii) it holds for all  $\theta \in [-B, B]^{\mathbf{d}}$ ,  $p \in (0, \infty)$  that

$$\begin{aligned}
& (\mathbb{E} [\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_k) - \mathcal{R}(\theta)|^p])^{1/p} \\
&\leq \frac{4(v-u) b \mathbf{L} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} B^{\mathbf{L}} \sqrt{\max\{1, p/\mathbf{d}\}}}{K^{1/\mathbf{d}}} \leq \frac{4(v-u) b \mathbf{L} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} B^{\mathbf{L}} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}}
\end{aligned} \tag{135}$$

(cf. Definition 3.1).

*Proof of Corollary 5.8.* Throughout this proof let  $L \in \mathbb{R}$  be given by  $L = 2(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}B^{\mathbf{L}-1}$ , let  $P: [-B, B]^\mathbf{d} \rightarrow [-B, B]^\mathfrak{d}$  satisfy for all  $\theta = (\theta_1, \theta_2, \dots, \theta_\mathbf{d}) \in [-B, B]^\mathbf{d}$  that  $P(\theta) = (\theta_1, \theta_2, \dots, \theta_\mathfrak{d})$ , and let  $R: [-B, B]^\mathfrak{d} \times \Omega \rightarrow \mathbb{R}$  satisfy for all  $\theta \in [-B, B]^\mathfrak{d}$ ,  $\omega \in \Omega$  that

$$R(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right]. \quad (136)$$

Note that the fact that  $\forall \theta \in [-B, B]^\mathbf{d}: \mathcal{N}_{u,v}^{\theta,\mathbf{l}} = \mathcal{N}_{u,v}^{P(\theta),\mathbf{l}}$  implies for all  $\theta \in [-B, B]^\mathbf{d}$ ,  $\omega \in \Omega$  that

$$\begin{aligned} \mathcal{R}(\theta, \omega) &= \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \\ &= \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{P(\theta),\mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] = R(P(\theta), \omega). \end{aligned} \quad (137)$$

Furthermore, Lemma 5.7 (with  $\mathbf{d} \leftarrow \mathfrak{d}$ ,  $\mathcal{R} \leftarrow ([-B, B]^\mathfrak{d} \times \Omega \ni (\theta, \omega) \mapsto R(\theta, \omega) \in [0, \infty))$  in the notation of Lemma 5.7) demonstrates for all  $\theta, \vartheta \in [-B, B]^\mathfrak{d}$ ,  $\omega \in \Omega$  that

$$|R(\theta, \omega) - R(\vartheta, \omega)| \leq 2(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}B^{\mathbf{L}-1}\|\theta - \vartheta\|_\infty = L\|\theta - \vartheta\|_\infty. \quad (138)$$

Moreover, observe that the assumption that  $X_j$ ,  $j \in \{1, 2, \dots, M\}$ , and  $Y_j$ ,  $j \in \{1, 2, \dots, M\}$ , are random variables ensures that  $R: [-B, B]^\mathfrak{d} \times \Omega \rightarrow \mathbb{R}$  is a random field. This, (138), the fact that  $P \circ \Theta_k: \Omega \rightarrow [-B, B]^\mathfrak{d}$ ,  $k \in \{1, 2, \dots, K\}$ , are i.i.d. random variables, the fact that  $P \circ \Theta_1$  is continuous uniformly distributed on  $[-B, B]^\mathfrak{d}$ , and Proposition 5.6 (with  $\mathbf{d} \leftarrow \mathfrak{d}$ ,  $\alpha \leftarrow -B$ ,  $\beta \leftarrow B$ ,  $\mathcal{R} \leftarrow R$ ,  $(\Theta_k)_{k \in \{1, 2, \dots, K\}} \leftarrow (P \circ \Theta_k)_{k \in \{1, 2, \dots, K\}}$  in the notation of Proposition 5.6) prove for all  $\theta \in [-B, B]^\mathbf{d}$ ,  $p \in (0, \infty)$  that  $R$  is a  $(\mathcal{B}([-B, B]^\mathfrak{d}) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$ -measurable function and

$$\begin{aligned} &(\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |R(P(\Theta_k)) - R(P(\theta))|^p])^{1/p} \\ &\leq \frac{L(2B) \max\{1, (p/\mathfrak{d})^{1/\mathfrak{d}}\}}{K^{1/\mathfrak{d}}} = \frac{4(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}B^{\mathbf{L}} \max\{1, (p/\mathfrak{d})^{1/\mathfrak{d}}\}}{K^{1/\mathfrak{d}}}. \end{aligned} \quad (139)$$

The fact that  $P$  is a  $\mathcal{B}([-B, B]^\mathbf{d})/\mathcal{B}([-B, B]^\mathfrak{d})$ -measurable function and (137) hence show (i). In addition, (137), (139), and the fact that  $2 \leq \mathfrak{d} = \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1) \leq \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2$  yield for all  $\theta \in [-B, B]^\mathbf{d}$ ,  $p \in (0, \infty)$  that

$$\begin{aligned} &(\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_k) - \mathcal{R}(\theta)|^p])^{1/p} \\ &= (\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |R(P(\Theta_k)) - R(P(\theta))|^p])^{1/p} \\ &\leq \frac{4(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}B^{\mathbf{L}} \sqrt{\max\{1, p/\mathfrak{d}\}}}{K^{1/\mathfrak{d}}} \leq \frac{4(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}B^{\mathbf{L}} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}}. \end{aligned} \quad (140)$$

This establishes (ii). The proof of Corollary 5.8 is thus complete.  $\square$

## 6 Analysis of the overall error

In Subsection 6.2 below we present the main result of this article, Theorem 6.5, that provides an estimate for the overall  $L^2$ -error arising in deep learning based empirical risk minimisation with quadratic loss function in the probabilistically strong sense and that covers the case where the underlying DNNs are trained using a general stochastic optimisation algorithm with random initialisation.

In order to prove Theorem 6.5, we require a link to combine the results from Sections 3, 4, and 5, which is given in Subsection 6.1 below. More specifically, Proposition 6.1 in Subsection 6.1 shows that the overall error can be decomposed into three different error sources: the *approximation error* (cf. Section 3), the *worst-case generalisation error* (cf. Section 4), and the *optimisation error* (cf. Section 5). Proposition 6.1 is a consequence of the well-known bias-variance decomposition (cf., e.g., Beck, Jentzen, & Kuckuck [10, Lemma 4.1] or Berner, Grohs, & Jentzen [13, Lemma 2.2]) and is very similar to [10, Lemma 4.3].

Thereafter, Subsection 6.2 is devoted to strong convergence results for deep learning based empirical risk minimisation with quadratic loss function where a general stochastic approximation algorithm with random initialisation is allowed to be the employed optimisation method. Apart from the main result (cf. Theorem 6.5), Subsection 6.2 also includes on the one hand Proposition 6.3, which combines the overall error decomposition (cf. Proposition 6.1) with our convergence result for the generalisation error (cf. Corollary 4.15 in Section 4) and our convergence result for the optimisation error (cf. Corollary 5.8 in Section 5), and on the other hand Corollary 6.6, which replaces the architecture parameter  $A \in (0, \infty)$  in Theorem 6.5 (cf. Proposition 3.5) by the minimum of the depth parameter  $\mathbf{L} \in \mathbb{N}$  and the hidden layer sizes  $\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1} \in \mathbb{N}$  of the trained DNN (cf. (178) below).

Finally, in Subsection 6.3 we present three more strong convergence results for the special case where SGD with random initialisation is the employed optimisation method. In particular, Corollary 6.7 specifies Corollary 6.6 to this special case, Corollary 6.8 provides a convergence estimate for the expectation of the  $L^1$ -distance between the trained DNN and the target function, and Corollary 6.9 reaches an analogous conclusion in a simplified setting.

## 6.1 Overall error decomposition

**Proposition 6.1.** *Let  $d, \mathbf{d}, \mathbf{L}, M, K, N \in \mathbb{N}$ ,  $B \in [0, \infty)$ ,  $u \in \mathbb{R}$ ,  $v \in (u, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, 1, \dots, N\}$ ,  $D \subseteq \mathbb{R}^d$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow D$ ,  $j \in \{1, 2, \dots, M\}$ , and  $Y_j: \Omega \rightarrow [u, v]$ ,  $j \in \{1, 2, \dots, M\}$ , be random variables, let  $\mathcal{E}: D \rightarrow [u, v]$  be a  $\mathcal{B}(D)/\mathcal{B}([u, v])$ -measurable function, assume that it holds  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1) = \mathbb{E}[Y_1|X_1]$ , let  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , satisfy  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-B, B]^{\mathbf{d}}$ , let  $\mathbf{R}: \mathbb{R}^{\mathbf{d}} \rightarrow [0, \infty)$  satisfy for all  $\theta \in \mathbb{R}^{\mathbf{d}}$  that  $\mathbf{R}(\theta) = \mathbb{E}[|\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(X_1) - Y_1|^2]$ , and let  $\mathcal{R}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$  and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  satisfy for all  $\theta \in \mathbb{R}^{\mathbf{d}}$ ,  $\omega \in \Omega$  that*

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \quad \text{and} \quad (141)$$

$$\mathbf{k}(\omega) \in \arg \min_{(k,n) \in \{1, 2, \dots, K\} \times \mathbf{N}, \|\Theta_{k,n}(\omega)\|_{\infty} \leq B} \mathcal{R}(\Theta_{k,n}(\omega), \omega) \quad (142)$$

(cf. Definitions 2.8 and 3.1). Then it holds for all  $\vartheta \in [-B, B]^{\mathbf{d}}$  that

$$\begin{aligned} & \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}}, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \\ & \leq [\sup_{x \in D} |\mathcal{N}_{u,v}^{\vartheta, \mathbf{l}}(x) - \mathcal{E}(x)|^2] + 2[\sup_{\theta \in [-B, B]^{\mathbf{d}}} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|] \\ & \quad + \min_{(k,n) \in \{1, 2, \dots, K\} \times \mathbf{N}, \|\Theta_{k,n}\|_{\infty} \leq B} |\mathcal{R}(\Theta_{k,n}) - \mathcal{R}(\vartheta)|. \end{aligned} \quad (143)$$

*Proof of Proposition 6.1.* Throughout this proof let  $\mathcal{R}: \mathcal{L}^2(\mathbb{P}_{X_1}; \mathbb{R}) \rightarrow [0, \infty)$  satisfy for all  $f \in \mathcal{L}^2(\mathbb{P}_{X_1}; \mathbb{R})$  that  $\mathcal{R}(f) = \mathbb{E}[|f(X_1) - Y_1|^2]$ . Observe that the assumption that

$\forall \omega \in \Omega: Y_1(\omega) \in [u, v]$  and the fact that  $\forall \theta \in \mathbb{R}^d, x \in \mathbb{R}^d: \mathcal{N}_{u,v}^{\theta,1}(x) \in [u, v]$  ensure for all  $\theta \in \mathbb{R}^d$  that  $\mathbb{E}[|Y_1|^2] \leq v^2 < \infty$  and

$$\int_D |\mathcal{N}_{u,v}^{\theta,1}(x)|^2 \mathbb{P}_{X_1}(dx) = \mathbb{E}[|\mathcal{N}_{u,v}^{\theta,1}(X_1)|^2] \leq v^2 < \infty. \quad (144)$$

The bias-variance decomposition (cf., e.g., Beck, Jentzen, & Kuckuck [10, (iii) in Lemma 4.1] with  $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$ ,  $(S, \mathcal{S}) \leftarrow (D, \mathcal{B}(D))$ ,  $X \leftarrow X_1$ ,  $Y \leftarrow (\Omega \ni \omega \mapsto Y_1(\omega) \in \mathbb{R})$ ,  $\mathcal{E} \leftarrow \mathcal{R}$ ,  $f \leftarrow \mathcal{N}_{u,v}^{\theta,1}|_D$ ,  $g \leftarrow \mathcal{N}_{u,v}^{\vartheta,1}|_D$  for  $\theta, \vartheta \in \mathbb{R}^d$  in the notation of [10, (iii) in Lemma 4.1]) hence proves for all  $\theta, \vartheta \in \mathbb{R}^d$  that

$$\begin{aligned} & \int_D |\mathcal{N}_{u,v}^{\theta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \\ &= \mathbb{E}[|\mathcal{N}_{u,v}^{\theta,1}(X_1) - \mathcal{E}(X_1)|^2] = \mathbb{E}[|\mathcal{N}_{u,v}^{\theta,1}(X_1) - \mathbb{E}[Y_1|X_1]|^2] \\ &= \mathbb{E}[|\mathcal{N}_{u,v}^{\vartheta,1}(X_1) - \mathbb{E}[Y_1|X_1]|^2] + \mathcal{R}(\mathcal{N}_{u,v}^{\theta,1}|_D) - \mathcal{R}(\mathcal{N}_{u,v}^{\vartheta,1}|_D) \\ &= \mathbb{E}[|\mathcal{N}_{u,v}^{\vartheta,1}(X_1) - \mathcal{E}(X_1)|^2] + \mathbb{E}[|\mathcal{N}_{u,v}^{\theta,1}(X_1) - Y_1|^2] - \mathbb{E}[|\mathcal{N}_{u,v}^{\vartheta,1}(X_1) - Y_1|^2] \\ &= \int_D |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) + \mathbf{R}(\theta) - \mathbf{R}(\vartheta). \end{aligned} \quad (145)$$

This implies for all  $\theta, \vartheta \in \mathbb{R}^d$  that

$$\begin{aligned} & \int_D |\mathcal{N}_{u,v}^{\theta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \\ &= \int_D |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) - [\mathcal{R}(\theta) - \mathbf{R}(\theta)] + \mathcal{R}(\vartheta) - \mathbf{R}(\vartheta) + \mathcal{R}(\theta) - \mathcal{R}(\vartheta) \\ &\leq \int_D |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) + |\mathcal{R}(\theta) - \mathbf{R}(\theta)| + |\mathcal{R}(\vartheta) - \mathbf{R}(\vartheta)| + \mathcal{R}(\theta) - \mathcal{R}(\vartheta) \\ &\leq \int_D |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) + 2[\max_{\eta \in \{\theta, \vartheta\}} |\mathcal{R}(\eta) - \mathbf{R}(\eta)|] + \mathcal{R}(\theta) - \mathcal{R}(\vartheta). \end{aligned} \quad (146)$$

Next note that the fact that  $\forall \omega \in \Omega: \|\Theta_{\mathbf{k}(\omega)}(\omega)\|_\infty \leq B$  ensures for all  $\omega \in \Omega$  that  $\Theta_{\mathbf{k}(\omega)}(\omega) \in [-B, B]^d$ . Combining (146) with (142) hence establishes for all  $\vartheta \in [-B, B]^d$  that

$$\begin{aligned} & \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \\ &\leq \int_D |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) + 2[\sup_{\theta \in [-B, B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|] + \mathcal{R}(\Theta_{\mathbf{k}}) - \mathcal{R}(\vartheta) \\ &= \int_D |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) + 2[\sup_{\theta \in [-B, B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|] \\ &\quad + \min_{(k,n) \in \{1, 2, \dots, K\} \times \mathbb{N}, \|\Theta_{k,n}\|_\infty \leq B} |\mathcal{R}(\Theta_{k,n}) - \mathcal{R}(\vartheta)| \\ &\leq [\sup_{x \in D} |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2] + 2[\sup_{\theta \in [-B, B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|] \\ &\quad + \min_{(k,n) \in \{1, 2, \dots, K\} \times \mathbb{N}, \|\Theta_{k,n}\|_\infty \leq B} |\mathcal{R}(\Theta_{k,n}) - \mathcal{R}(\vartheta)|. \end{aligned} \quad (147)$$

The proof of Proposition 6.1 is thus complete.  $\square$

## 6.2 Full strong error analysis for the training of DNNs

**Lemma 6.2.** *Let  $d, \mathbf{d}, \mathbf{L} \in \mathbb{N}$ ,  $p \in [0, \infty)$ ,  $u \in [-\infty, \infty)$ ,  $v \in (u, \infty]$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $D \subseteq \mathbb{R}^d$ , assume  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , let  $\mathcal{E}: D \rightarrow \mathbb{R}$  be a  $\mathcal{B}(D)/\mathcal{B}(\mathbb{R})$ -measurable function, let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and let  $X: \Omega \rightarrow D$ ,  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$ , and  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , be random variables. Then*

- (i) it holds that the function  $\mathbb{R}^d \times \mathbb{R}^d \ni (\theta, x) \mapsto \mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) \in \mathbb{R}$  is  $(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)) / \mathcal{B}(\mathbb{R})$ -measurable,
- (ii) it holds that the function  $\Omega \ni \omega \mapsto \Theta_{\mathbf{k}(\omega)}(\omega) \in \mathbb{R}^d$  is  $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ -measurable, and
- (iii) it holds that the function

$$\Omega \ni \omega \mapsto \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}(\omega)}(\omega),\mathbf{l}}(x) - \mathcal{E}(x)|^p \mathbb{P}_X(dx) \in [0, \infty] \quad (148)$$

is  $\mathcal{F}/\mathcal{B}([0, \infty])$ -measurable

(cf. Definition 2.8).

*Proof of Lemma 6.2.* First, observe that Beck, Jentzen, & Kuckuck [10, Corollary 2.37] (with  $a \leftarrow -\|x\|_\infty$ ,  $b \leftarrow \|x\|_\infty$ ,  $u \leftarrow u$ ,  $v \leftarrow v$ ,  $d \leftarrow \mathbf{d}$ ,  $L \leftarrow \mathbf{L}$ ,  $l \leftarrow 1$  for  $x \in \mathbb{R}^d$  in the notation of [10, Corollary 2.37]) demonstrates for all  $x \in \mathbb{R}^d$ ,  $\theta, \vartheta \in \mathbb{R}^d$  that

$$\begin{aligned} |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(x)| &\leq \sup_{y \in [-\|x\|_\infty, \|x\|_\infty]^L} |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(y) - \mathcal{N}_{u,v}^{\vartheta,\mathbf{l}}(y)| \\ &\leq \mathbf{L} \max\{1, \|x\|_\infty\} (\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} (\max\{1, \|\theta\|_\infty, \|\vartheta\|_\infty\})^{\mathbf{L}-1} \|\theta - \vartheta\|_\infty \end{aligned} \quad (149)$$

(cf. Definition 3.1). This implies for all  $x \in \mathbb{R}^d$  that the function

$$\mathbb{R}^d \ni \theta \mapsto \mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) \in \mathbb{R} \quad (150)$$

is continuous. In addition, the fact that  $\forall \theta \in \mathbb{R}^d: \mathcal{N}_{u,v}^{\theta,\mathbf{l}} \in C(\mathbb{R}^d, \mathbb{R})$  ensures for all  $\theta \in \mathbb{R}^d$  that the function  $\mathbb{R}^d \ni x \mapsto \mathcal{N}_{u,v}^{\theta,\mathbf{l}}(x) \in \mathbb{R}$  is  $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ -measurable. This, (150), the fact that  $(\mathbb{R}^d, \|\cdot\|_\infty|_{\mathbb{R}^d})$  is a separable normed  $\mathbb{R}$ -vector space, and, e.g., Aliprantis & Border [1, Lemma 4.51] (see also, e.g., Beck et al. [8, Lemma 2.4]) show (i).

Second, we prove (ii). For this let  $\Xi: \Omega \rightarrow \mathbb{R}^d$  satisfy for all  $\omega \in \Omega$  that  $\Xi(\omega) = \Theta_{\mathbf{k}(\omega)}(\omega)$ . Observe that the assumption that  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^d$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  are random variables establishes for all  $U \in \mathcal{B}(\mathbb{R}^d)$  that

$$\begin{aligned} \Xi^{-1}(U) &= \{\omega \in \Omega: \Xi(\omega) \in U\} = \{\omega \in \Omega: \Theta_{\mathbf{k}(\omega)}(\omega) \in U\} \\ &= \{\omega \in \Omega: [\exists k, n \in \mathbb{N}_0: ([\Theta_{k,n}(\omega) \in U] \wedge [\mathbf{k}(\omega) = (k, n)])]\} \\ &= \bigcup_{k=0}^{\infty} \bigcup_{n=0}^{\infty} (\{\omega \in \Omega: \Theta_{k,n}(\omega) \in U\} \cap \{\omega \in \Omega: \mathbf{k}(\omega) = (k, n)\}) \\ &= \bigcup_{k=0}^{\infty} \bigcup_{n=0}^{\infty} ([(\Theta_{k,n})^{-1}(U)] \cap [\mathbf{k}^{-1}(\{(k, n)\})]) \in \mathcal{F}. \end{aligned} \quad (151)$$

This implies (ii).

Third, note that (i)–(ii) yield that the function  $\Omega \times \mathbb{R}^d \ni (\omega, x) \mapsto \mathcal{N}_{u,v}^{\Theta_{\mathbf{k}(\omega)}(\omega),\mathbf{l}}(x) \in \mathbb{R}$  is  $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))/\mathcal{B}(\mathbb{R})$ -measurable. This and the assumption that  $\mathcal{E}: D \rightarrow \mathbb{R}$  is  $\mathcal{B}(D)/\mathcal{B}(\mathbb{R})$ -measurable demonstrate that the function  $\Omega \times D \ni (\omega, x) \mapsto |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}(\omega)}(\omega),\mathbf{l}}(x) - \mathcal{E}(x)|^p \in [0, \infty)$  is  $(\mathcal{F} \otimes \mathcal{B}(D))/\mathcal{B}([0, \infty))$ -measurable. Tonelli's theorem hence establishes (iii). The proof of Lemma 6.2 is thus complete.  $\square$

**Proposition 6.3.** Let  $d, \mathbf{d}, \mathbf{L}, M, N \in \mathbb{N}$ ,  $b, c \in [1, \infty)$ ,  $B \in [c, \infty)$ ,  $u \in \mathbb{R}$ ,  $v \in (u, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, 1, \dots, N\}$ ,  $D \subseteq [-b, b]^d$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow D$ ,  $j \in \mathbb{N}$ , and  $Y_j: \Omega \rightarrow [u, v]$ ,  $j \in \mathbb{N}$ , be functions, assume that  $(X_j, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $\mathcal{E}: D \rightarrow [u, v]$  be a  $\mathcal{B}(D)/\mathcal{B}([u, v])$ -measurable function,

assume that it holds  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1) = \mathbb{E}[Y_1|X_1]$ , let  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^d$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  be random variables, assume  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-B, B]^d$ , assume that  $\Theta_{k,0}$ ,  $k \in \{1, 2, \dots, K\}$ , are i.i.d., assume that  $\Theta_{1,0}$  is continuous uniformly distributed on  $[-c, c]^d$ , and let  $\mathcal{R}: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in \mathbb{R}^d$ ,  $\omega \in \Omega$  that

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta,1}(X_j(\omega)) - Y_j(\omega)|^2 \right] \quad \text{and} \quad (152)$$

$$\mathbf{k}(\omega) \in \arg \min_{(k,n) \in \{1,2,\dots,K\} \times \mathbf{N}, \|\Theta_{k,n}(\omega)\|_{\infty} \leq B} \mathcal{R}(\Theta_{k,n}(\omega), \omega) \quad (153)$$

(cf. Definitions 2.8 and 3.1). Then it holds for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^p \right] \right)^{1/p} \\ & \leq [\inf_{\theta \in [-c,c]^d} \sup_{x \in D} |\mathcal{N}_{u,v}^{\theta,1}(x) - \mathcal{E}(x)|^2] + \frac{4(v-u)b\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}}c^{\mathbf{L}} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_{\infty} + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^2 \max\{p, \ln(3MBb)\}}{\sqrt{M}} \\ & \leq [\inf_{\theta \in [-c,c]^d} \sup_{x \in D} |\mathcal{N}_{u,v}^{\theta,1}(x) - \mathcal{E}(x)|^2] \\ & \quad + \frac{20 \max\{1, (v-u)^2\} b\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}+1} B^{\mathbf{L}} \max\{p, \ln(3M)\}}{\min\{\sqrt{M}, K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_{\infty} + 1)^{-2}]}\}} \end{aligned} \quad (154)$$

(cf. (iii) in Lemma 6.2).

*Proof of Proposition 6.3.* Throughout this proof let  $\mathbf{R}: \mathbb{R}^d \rightarrow [0, \infty)$  satisfy for all  $\theta \in \mathbb{R}^d$  that  $\mathbf{R}(\theta) = \mathbb{E}[|\mathcal{N}_{u,v}^{\theta,1}(X_1) - Y_1|^2]$ . First of all, observe that the assumption that  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-B, B]^d$ , the assumption that  $0 \in \mathbf{N}$ , and Proposition 6.1 show for all  $\vartheta \in [-B, B]^d$  that

$$\begin{aligned} & \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \\ & \leq [\sup_{x \in D} |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2] + 2[\sup_{\theta \in [-B,B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|] \\ & \quad + \min_{(k,n) \in \{1,2,\dots,K\} \times \mathbf{N}, \|\Theta_{k,n}\|_{\infty} \leq B} |\mathcal{R}(\Theta_{k,n}) - \mathcal{R}(\vartheta)| \\ & \leq [\sup_{x \in D} |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2] + 2[\sup_{\theta \in [-B,B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|] \\ & \quad + \min_{k \in \{1,2,\dots,K\}, \|\Theta_{k,0}\|_{\infty} \leq B} |\mathcal{R}(\Theta_{k,0}) - \mathcal{R}(\vartheta)| \\ & = [\sup_{x \in D} |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2] + 2[\sup_{\theta \in [-B,B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|] \\ & \quad + \min_{k \in \{1,2,\dots,K\}} |\mathcal{R}(\Theta_{k,0}) - \mathcal{R}(\vartheta)|. \end{aligned} \quad (155)$$

Minkowski's inequality hence establishes for all  $p \in [1, \infty)$ ,  $\vartheta \in [-c, c]^d \subseteq [-B, B]^d$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^p \right] \right)^{1/p} \\ & \leq (\mathbb{E} [\sup_{x \in D} |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^{2p}])^{1/p} + 2(\mathbb{E} [\sup_{\theta \in [-B,B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} \\ & \quad + (\mathbb{E} [\min_{k \in \{1,2,\dots,K\}} |\mathcal{R}(\Theta_{k,0}) - \mathcal{R}(\vartheta)|^p])^{1/p} \\ & \leq [\sup_{x \in D} |\mathcal{N}_{u,v}^{\vartheta,1}(x) - \mathcal{E}(x)|^2] + 2(\mathbb{E} [\sup_{\theta \in [-B,B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} \\ & \quad + \sup_{\theta \in [-c,c]^d} (\mathbb{E} [\min_{k \in \{1,2,\dots,K\}} |\mathcal{R}(\Theta_{k,0}) - \mathcal{R}(\theta)|^p])^{1/p} \end{aligned} \quad (156)$$

(cf. (i) in Corollary 4.15 and (i) in Corollary 5.8). Next note that Corollary 4.15 (with  $v \leftarrow \max\{u+1, v\}$ ,  $\mathbf{R} \leftarrow \mathbf{R}|_{[-B,B]^d}$ ,  $\mathcal{R} \leftarrow \mathcal{R}|_{[-B,B]^d \times \Omega}$  in the notation of Corollary 4.15)

proves for all  $p \in (0, \infty)$  that

$$\begin{aligned} & (\mathbb{E}[\sup_{\theta \in [-B, B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^p])^{1/p} \\ & \leq \frac{9(\max\{u+1, v\} - u)^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBb)\}}{\sqrt{M}} \\ & = \frac{9 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBb)\}}{\sqrt{M}}. \end{aligned} \quad (157)$$

In addition, observe that Corollary 5.8 (with  $\mathfrak{d} \leftarrow \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ ,  $B \leftarrow c$ ,  $(\Theta_k)_{k \in \{1, 2, \dots, K\}} \leftarrow (\Omega \ni \omega \mapsto \mathbb{1}_{\{\Theta_{k,0} \in [-c, c]^d\}}(\omega) \Theta_{k,0}(\omega) \in [-c, c]^d)_{k \in \{1, 2, \dots, K\}}$ ,  $\mathcal{R} \leftarrow \mathcal{R}|_{[-c, c]^d \times \Omega}$  in the notation of Corollary 5.8) implies for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \sup_{\theta \in [-c, c]^d} (\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_{k,0}) - \mathcal{R}(\theta)|^p])^{1/p} \\ & = \sup_{\theta \in [-c, c]^d} (\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\mathbb{1}_{\{\Theta_{k,0} \in [-c, c]^d\}} \Theta_{k,0}) - \mathcal{R}(\theta)|^p])^{1/p} \\ & \leq \frac{4(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}}. \end{aligned} \quad (158)$$

Combining this, (156), (157), and the fact that  $\ln(3MBb) \geq 1$  with Jensen's inequality demonstrates for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}}, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^p \right] \right)^{1/p} \\ & \leq \left( \mathbb{E} \left[ \left( \int_D |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}}, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^{\max\{1, p\}} \right] \right)^{\frac{1}{\max\{1, p\}}} \\ & \leq [\inf_{\theta \in [-c, c]^d} \sup_{x \in D} |\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(x) - \mathcal{E}(x)|^2] \\ & \quad + \sup_{\theta \in [-c, c]^d} (\mathbb{E}[\min_{k \in \{1, 2, \dots, K\}} |\mathcal{R}(\Theta_{k,0}) - \mathcal{R}(\theta)|^{\max\{1, p\}}])^{\frac{1}{\max\{1, p\}}} \\ & \quad + 2(\mathbb{E}[\sup_{\theta \in [-B, B]^d} |\mathcal{R}(\theta) - \mathbf{R}(\theta)|^{\max\{1, p\}}])^{\frac{1}{\max\{1, p\}}} \\ & \leq [\inf_{\theta \in [-c, c]^d} \sup_{x \in D} |\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(x) - \mathcal{E}(x)|^2] + \frac{4(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBb)\}}{\sqrt{M}}. \end{aligned} \quad (159)$$

Moreover, note that the fact that  $\forall x \in [0, \infty): x+1 \leq e^x \leq 3^x$  and the facts that  $Bb \geq 1$  and  $M \geq 1$  ensure that

$$\ln(3MBb) \leq \ln(3M3^{Bb-1}) = \ln(3^{Bb}M) = Bb \ln([3^{Bb}M]^{1/(Bb)}) \leq Bb \ln(3M). \quad (160)$$

The facts that  $\|\mathbf{l}\|_\infty + 1 \geq 2$ ,  $B \geq c \geq 1$ ,  $\ln(3M) \geq 1$ ,  $b \geq 1$ , and  $\mathbf{L} \geq 1$  hence show for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \frac{4(v-u)b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBb)\}}{\sqrt{M}} \\ & \leq \frac{2(\|\mathbf{l}\|_\infty + 1) \max\{1, (v-u)^2\} b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} B^{\mathbf{L}} \max\{p, \ln(3M)\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 B \max\{p, \ln(3M)\}}{\sqrt{M}} \\ & \leq \frac{20 \max\{1, (v-u)^2\} b\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}+1} B^{\mathbf{L}} \max\{p, \ln(3M)\}}{\min\{\sqrt{M}, K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}\}}. \end{aligned} \quad (161)$$

This and (159) complete the proof of Proposition 6.3.  $\square$

**Lemma 6.4.** *Let  $a, x, p \in (0, \infty)$ ,  $M, c \in [1, \infty)$ ,  $B \in [c, \infty)$ . Then*

- (i) *it holds that  $ax^p \leq \exp(a^{1/p} \frac{px}{e})$  and*
- (ii) *it holds that  $\ln(3MBc) \leq \frac{23B}{18} \ln(eM)$ .*

*Proof of Lemma 6.4.* First, note that the fact that  $\forall y \in \mathbb{R}: y + 1 \leq e^y$  demonstrates that

$$ax^p = (a^{1/p}x)^p = [e(a^{1/p} \frac{x}{e} - 1 + 1)]^p \leq [e \exp(a^{1/p} \frac{x}{e} - 1)]^p = \exp(a^{1/p} \frac{px}{e}). \quad (162)$$

This proves (i).

Second, observe that (i) and the fact that  $2\sqrt{3}/e \leq 23/18$  ensure that

$$3B^2 \leq \exp(\sqrt{3} \frac{2B}{e}) = \exp(\frac{2\sqrt{3}B}{e}) \leq \exp(\frac{23B}{18}). \quad (163)$$

The facts that  $B \geq c \geq 1$  and  $M \geq 1$  hence imply that

$$\ln(3MBc) \leq \ln(3B^2M) \leq \ln([eM]^{23B/18}) = \frac{23B}{18} \ln(eM). \quad (164)$$

This establishes (ii). The proof of Lemma 6.4 is thus complete.  $\square$

**Theorem 6.5.** *Let  $d, \mathbf{d}, \mathbf{L}, M, K, N \in \mathbb{N}$ ,  $A \in (0, \infty)$ ,  $L, a, u \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $v \in (u, \infty)$ ,  $c \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$ ,  $B \in [c, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, 1, \dots, N\}$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{L} \geq A\mathbf{1}_{(6^d, \infty)}(A)/(2d) + 1$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_1 \geq A\mathbf{1}_{(6^d, \infty)}(A)$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , assume for all  $i \in \{2, 3, \dots\} \cap [0, \mathbf{L}]$  that  $\mathbf{l}_i \geq \mathbf{1}_{(6^d, \infty)}(A) \max\{A/d - 2i + 3, 2\}$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow [a, b]^d$ ,  $j \in \mathbb{N}$ , and  $Y_j: \Omega \rightarrow [u, v]$ ,  $j \in \mathbb{N}$ , be functions, assume that  $(X_j, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $\mathcal{E}: [a, b]^d \rightarrow [u, v]$  satisfy  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1) = \mathbb{E}[Y_1|X_1]$ , assume for all  $x, y \in [a, b]^d$  that  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq L\|x - y\|_1$ , let  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  be random variables, assume  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-B, B]^{\mathbf{d}}$ , assume that  $\Theta_{k,0}$ ,  $k \in \{1, 2, \dots, K\}$ , are i.i.d., assume that  $\Theta_{1,0}$  is continuous uniformly distributed on  $[-c, c]^{\mathbf{d}}$ , and let  $\mathcal{R}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in \mathbb{R}^{\mathbf{d}}$ ,  $\omega \in \Omega$  that*

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \quad \text{and} \quad (165)$$

$$\mathbf{k}(\omega) \in \arg \min_{(k,n) \in \{1, 2, \dots, K\} \times \mathbf{N}, \|\Theta_{k,n}(\omega)\|_{\infty} \leq B} \mathcal{R}(\Theta_{k,n}(\omega), \omega) \quad (166)$$

(cf. Definitions 2.8 and 3.1). Then it holds for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_{[a,b]^d} |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}}, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(\mathrm{d}x) \right)^p \right] \right)^{1/p} \\ & \leq \frac{9d^2 L^2 (b-a)^2}{A^{2/d}} + \frac{4(v-u)\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_{\infty} + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^2 \max\{p, \ln(3MBc)\}}{\sqrt{M}} \\ & \leq \frac{36d^2 c^4}{A^{2/d}} + \frac{4\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}} c^{\mathbf{L}+2} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_{\infty} + 1)^{-2}]}} + \frac{23B^3 \mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^2 \max\{p, \ln(eM)\}}{\sqrt{M}} \end{aligned} \quad (167)$$

(cf. (iii) in Lemma 6.2).

*Proof of Theorem 6.5.* First of all, note that the assumption that  $\forall x, y \in [a, b]^d: |\mathcal{E}(x) - \mathcal{E}(y)| \leq L\|x - y\|_1$  ensures that  $\mathcal{E}: [a, b]^d \rightarrow [u, v]$  is a  $\mathcal{B}([a, b]^d)/\mathcal{B}([u, v])$ -measurable function. The fact that  $\max\{1, |a|, |b|\} \leq c$  and Proposition 6.3 (with  $b \leftarrow \max\{1, |a|, |b|\}$ ,  $D \leftarrow [a, b]^d$  in the notation of Proposition 6.3) hence show for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_{[a, b]^d} |\mathcal{N}_{u, v}^{\Theta_k, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^p \right] \right)^{1/p} \\ & \leq [\inf_{\theta \in [-c, c]^d} \sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\theta, \mathbf{l}}(x) - \mathcal{E}(x)|^2] \\ & \quad + \frac{4(v-u) \max\{1, |a|, |b|\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MB \max\{1, |a|, |b|\})\}}{\sqrt{M}} \quad (168) \\ & \leq [\inf_{\theta \in [-c, c]^d} \sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\theta, \mathbf{l}}(x) - \mathcal{E}(x)|^2] + \frac{4(v-u) \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBc)\}}{\sqrt{M}}. \end{aligned}$$

Furthermore, observe that Proposition 3.5 (with  $f \leftarrow \mathcal{E}$  in the notation of Proposition 3.5) proves that there exists  $\vartheta \in \mathbb{R}^d$  such that  $\|\vartheta\|_\infty \leq \max\{1, L, |a|, |b|, 2[\sup_{x \in [a, b]^d} |\mathcal{E}(x)|]\}$  and

$$\sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\vartheta, \mathbf{l}}(x) - \mathcal{E}(x)| \leq \frac{3dL(b-a)}{A^{1/d}}. \quad (169)$$

The fact that  $\forall x \in [a, b]^d: \mathcal{E}(x) \in [u, v]$  hence implies that

$$\|\vartheta\|_\infty \leq \max\{1, L, |a|, |b|, 2|u|, 2|v|\} \leq c. \quad (170)$$

This and (169) demonstrate that

$$\begin{aligned} & \inf_{\theta \in [-c, c]^d} \sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\theta, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \\ & \leq \sup_{x \in [a, b]^d} |\mathcal{N}_{u, v}^{\vartheta, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \\ & \leq \left[ \frac{3dL(b-a)}{A^{1/d}} \right]^2 = \frac{9d^2L^2(b-a)^2}{A^{2/d}}. \quad (171) \end{aligned}$$

Combining this with (168) establishes for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_{[a, b]^d} |\mathcal{N}_{u, v}^{\Theta_k, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^p \right] \right)^{1/p} \\ & \leq \frac{9d^2L^2(b-a)^2}{A^{2/d}} + \frac{4(v-u) \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & \quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p, \ln(3MBc)\}}{\sqrt{M}}. \quad (172) \end{aligned}$$

Moreover, note that the facts that  $\max\{1, L, |a|, |b|\} \leq c$  and  $(b-a)^2 \leq (|a| + |b|)^2 \leq 2(a^2 + b^2)$  yield that

$$9L^2(b-a)^2 \leq 18c^2(a^2 + b^2) \leq 18c^2(c^2 + c^2) = 36c^4. \quad (173)$$

In addition, the fact that  $B \geq c \geq 1$ , the fact that  $M \geq 1$ , and (ii) in Lemma 6.4 ensure that  $\ln(3MBc) \leq \frac{23B}{18} \ln(eM)$ . This, (173), the fact that  $(v-u) \leq 2 \max\{|u|, |v|\} =$

$\max\{2|u|, 2|v|\} \leq c \leq B$ , and the fact that  $B \geq 1$  prove for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \frac{9d^2L^2(b-a)^2}{A^{2/d}} + \frac{4(v-u)\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}c^{\mathbf{L}+1}\max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & + \frac{18\max\{1, (v-u)^2\}\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2\max\{p, \ln(3MBc)\}}{\sqrt{M}} \\ & \leq \frac{36d^2c^4}{A^{2/d}} + \frac{4\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}c^{\mathbf{L}+2}\max\{1, p\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} + \frac{23B^3\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2\max\{p, \ln(eM)\}}{\sqrt{M}}. \end{aligned} \quad (174)$$

Combining this with (172) shows (167). The proof of Theorem 6.5 is thus complete.  $\square$

**Corollary 6.6.** Let  $d, \mathbf{d}, \mathbf{L}, M, K, N \in \mathbb{N}$ ,  $L, a, u \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $v \in (u, \infty)$ ,  $c \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$ ,  $B \in [c, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, 1, \dots, N\}$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j: \Omega \rightarrow [a, b]^d$ ,  $j \in \mathbb{N}$ , and  $Y_j: \Omega \rightarrow [u, v]$ ,  $j \in \mathbb{N}$ , be functions, assume that  $(X_j, Y_j)$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $\mathcal{E}: [a, b]^d \rightarrow [u, v]$  satisfy  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1) = \mathbb{E}[Y_1|X_1]$ , assume for all  $x, y \in [a, b]^d$  that  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq L\|x - y\|_1$ , let  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  be random variables, assume  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-B, B]^{\mathbf{d}}$ , assume that  $\Theta_{k,0}$ ,  $k \in \{1, 2, \dots, K\}$ , are i.i.d., assume that  $\Theta_{1,0}$  is continuous uniformly distributed on  $[-c, c]^{\mathbf{d}}$ , and let  $\mathcal{R}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$  satisfy for all  $\theta \in \mathbb{R}^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$\mathcal{R}(\theta, \omega) = \frac{1}{M} \left[ \sum_{j=1}^M |\mathcal{N}_{u,v}^{\theta, \mathbf{l}}(X_j(\omega)) - Y_j(\omega)|^2 \right] \quad \text{and} \quad (175)$$

$$\mathbf{k}(\omega) \in \arg \min_{(k,n) \in \{1, 2, \dots, K\} \times \mathbf{N}, \|\Theta_{k,n}(\omega)\|_\infty \leq B} \mathcal{R}(\Theta_{k,n}(\omega), \omega) \quad (176)$$

(cf. Definitions 2.8 and 3.1). Then it holds for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_{[a,b]^d} |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}}, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^{p/2} \right] \right)^{1/p} \\ & \leq \frac{3dL(b-a)}{[\min(\{\mathbf{L}\} \cup \{\mathbf{l}_i: i \in \mathbb{N} \cap [0, \mathbf{L}]\})]^{1/d}} + \frac{2[(v-u)\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}c^{\mathbf{L}+1}\max\{1, p/2\}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & + \frac{3\max\{1, v-u\}(\|\mathbf{l}\|_\infty + 1)[\mathbf{L}\max\{p, 2\ln(3MBc)\}]^{1/2}}{M^{1/4}} \\ & \leq \frac{6dc^2}{[\min(\{\mathbf{L}\} \cup \{\mathbf{l}_i: i \in \mathbb{N} \cap [0, \mathbf{L}]\})]^{1/d}} + \frac{2\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^\mathbf{L}c^{\mathbf{L}+1}\max\{1, p\}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ & + \frac{5B^2\mathbf{L}(\|\mathbf{l}\|_\infty + 1)\max\{p, \ln(eM)\}}{M^{1/4}} \end{aligned} \quad (177)$$

(cf. (iii) in Lemma 6.2).

*Proof of Corollary 6.6.* Throughout this proof let  $A \in (0, \infty)$  be given by

$$A = \min(\{\mathbf{L}\} \cup \{\mathbf{l}_i: i \in \mathbb{N} \cap [0, \mathbf{L}]\}). \quad (178)$$

Note that (178) ensures that

$$\begin{aligned} \mathbf{L} & \geq A = A - 1 + 1 \geq (A - 1)\mathbb{1}_{[2,\infty)}(A) + 1 \\ & \geq (A - \frac{A}{2})\mathbb{1}_{[2,\infty)}(A) + 1 = \frac{A\mathbb{1}_{[2,\infty)}(A)}{2} + 1 \geq \frac{A\mathbb{1}_{(6^d,\infty)}(A)}{2d} + 1. \end{aligned} \quad (179)$$

Moreover, the assumption that  $\mathbf{l}_{\mathbf{L}} = 1$  and (178) imply that

$$\mathbf{l}_1 = \mathbf{l}_1 \mathbb{1}_{\{1\}}(\mathbf{L}) + \mathbf{l}_1 \mathbb{1}_{[2,\infty)}(\mathbf{L}) \geq \mathbb{1}_{\{1\}}(\mathbf{L}) + A \mathbb{1}_{[2,\infty)}(\mathbf{L}) = A \geq A \mathbb{1}_{(6^d,\infty)}(A). \quad (180)$$

Moreover, again (178) shows for all  $i \in \{2, 3, \dots\} \cap [0, \mathbf{L}]$  that

$$\begin{aligned} \mathbf{l}_i &\geq A \geq A \mathbb{1}_{[2, \infty)}(A) \geq \mathbb{1}_{[2, \infty)}(A) \max\{A - 1, 2\} = \mathbb{1}_{[2, \infty)}(A) \max\{A - 4 + 3, 2\} \\ &\geq \mathbb{1}_{[2, \infty)}(A) \max\{A - 2i + 3, 2\} \geq \mathbb{1}_{(6^d, \infty)}(A) \max\{A/d - 2i + 3, 2\}. \end{aligned} \quad (181)$$

Combining (179)–(181) and Theorem 6.5 (with  $p \leftarrow p/2$  for  $p \in (0, \infty)$  in the notation of Theorem 6.5) establishes for all  $p \in (0, \infty)$  that

$$\begin{aligned} &\left( \mathbb{E} \left[ \left( \int_{[a, b]^d} |\mathcal{N}_{u, v}^{\Theta_{\mathbf{k}}, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^{p/2} \right] \right)^{2/p} \\ &\leq \frac{9d^2 L^2(b-a)^2}{A^{2/d}} + \frac{4(v-u)\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p/2\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ &\quad + \frac{18 \max\{1, (v-u)^2\} \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p/2, \ln(3MBc)\}}{\sqrt{M}} \\ &\leq \frac{36d^2 c^4}{A^{2/d}} + \frac{4\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+2} \max\{1, p/2\}}{K^{[\mathbf{L}^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} + \frac{23B^3 \mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p/2, \ln(eM)\}}{\sqrt{M}}. \end{aligned} \quad (182)$$

This, (178), and the facts that  $\mathbf{L} \geq 1$ ,  $c \geq 1$ ,  $B \geq 1$ , and  $\ln(eM) \geq 1$  demonstrate for all  $p \in (0, \infty)$  that

$$\begin{aligned} &\left( \mathbb{E} \left[ \left( \int_{[a, b]^d} |\mathcal{N}_{u, v}^{\Theta_{\mathbf{k}}, \mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1}(dx) \right)^{p/2} \right] \right)^{1/p} \\ &\leq \frac{3dL(b-a)}{[\min(\{\mathbf{L}\} \cup \{\mathbf{l}_i : i \in \mathbb{N} \cap [0, \mathbf{L}]\})]^{1/d}} + \frac{2[(v-u)\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p/2\}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ &\quad + \frac{3 \max\{1, v-u\} (\|\mathbf{l}\|_\infty + 1) [\mathbf{L} \max\{p, 2 \ln(3MBc)\}]^{1/2}}{M^{1/4}} \\ &\leq \frac{6dc^2}{[\min(\{\mathbf{L}\} \cup \{\mathbf{l}_i : i \in \mathbb{N} \cap [0, \mathbf{L}]\})]^{1/d}} + \frac{2[\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+2} \max\{1, p/2\}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ &\quad + \frac{5B^3 [\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^2 \max\{p/2, \ln(eM)\}]^{1/2}}{M^{1/4}} \\ &\leq \frac{6dc^2}{[\min(\{\mathbf{L}\} \cup \{\mathbf{l}_i : i \in \mathbb{N} \cap [0, \mathbf{L}]\})]^{1/d}} + \frac{2\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p\}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ &\quad + \frac{5B^2 \mathbf{L}(\|\mathbf{l}\|_\infty + 1) \max\{p, \ln(eM)\}}{M^{1/4}}. \end{aligned} \quad (183)$$

The proof of Corollary 6.6 is thus complete.  $\square$

### 6.3 Full strong error analysis for the training of DNNs with optimisation via stochastic gradient descent with random initialisation

**Corollary 6.7.** *Let  $d, \mathbf{d}, \mathbf{L}, M, K, N \in \mathbb{N}$ ,  $L, a, u \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $v \in (u, \infty)$ ,  $c \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$ ,  $B \in [c, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, 1, \dots, N\}$ ,  $(\mathbf{J}_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ ,  $(\gamma_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i (\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j^{k,n} : \Omega \rightarrow [a, b]^d$ ,  $k, n, j \in \mathbb{N}_0$ , and  $Y_j^{k,n} : \Omega \rightarrow [u, v]$ ,  $k, n, j \in \mathbb{N}_0$ , be functions, assume that  $(X_j^{0,0}, Y_j^{0,0})$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $\mathcal{E} : [a, b]^d \rightarrow [u, v]$  satisfy  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1^{0,0}) = \mathbb{E}[Y_1^{0,0}|X_1^{0,0}]$ , assume for all  $x, y \in [a, b]^d$  that  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq L \|x - y\|_1$ , let  $\Theta_{k,n} : \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k} : \Omega \rightarrow (\mathbb{N}_0)^2$  be random variables, assume  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-B, B]^{\mathbf{d}}$ , assume that  $\Theta_{k,0}$ ,  $k \in$*

$\{1, 2, \dots, K\}$ , are i.i.d., assume that  $\Theta_{1,0}$  is continuous uniformly distributed on  $[-c, c]^{\mathbf{d}}$ , let  $\mathcal{R}_{J^n}^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$ ,  $k, n, J \in \mathbb{N}_0$ , and  $\mathcal{G}^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}$ , satisfy for all  $k, n \in \mathbb{N}$ ,  $\omega \in \Omega$ ,  $\theta \in \{\vartheta \in \mathbb{R}^{\mathbf{d}}: (\mathcal{R}_{J^n}^{k,n}(\cdot, \omega)): \mathbb{R}^{\mathbf{d}} \rightarrow [0, \infty) \text{ is differentiable at } \vartheta\}$  that  $\mathcal{G}^{k,n}(\theta, \omega) = (\nabla_{\theta} \mathcal{R}_{J^n}^{k,n})(\theta, \omega)$ , assume for all  $k, n \in \mathbb{N}$  that  $\Theta_{k,n} = \Theta_{k,n-1} - \gamma_n \mathcal{G}^{k,n}(\Theta_{k,n-1})$ , and assume for all  $k, n \in \mathbb{N}_0$ ,  $J \in \mathbb{N}$ ,  $\theta \in \mathbb{R}^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$\mathcal{R}_{J^n}^{k,n}(\theta, \omega) = \frac{1}{J} \left[ \sum_{j=1}^J |\mathcal{N}_{u,v}^{\theta,1}(X_j^{k,n}(\omega)) - Y_j^{k,n}(\omega)|^2 \right] \quad \text{and} \quad (184)$$

$$\mathbf{k}(\omega) \in \arg \min_{(l,m) \in \{1,2,\dots,K\} \times \mathbf{N}, \|\Theta_{l,m}(\omega)\|_{\infty} \leq B} \mathcal{R}_M^{0,0}(\Theta_{l,m}(\omega), \omega) \quad (185)$$

(cf. Definitions 2.8 and 3.1). Then it holds for all  $p \in (0, \infty)$  that

$$\begin{aligned} & \left( \mathbb{E} \left[ \left( \int_{[a,b]^d} |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},1}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1^{0,0}}(dx) \right)^{p/2} \right] \right)^{1/p} \\ & \leq \frac{3dL(b-a)}{[\min(\{\mathbf{L}\} \cup \{\mathbf{l}_i: i \in \mathbb{N} \cap [0, \mathbf{L}]\})]^{1/d}} + \frac{2[(v-u)\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p/2\}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_{\infty} + 1)^{-2}]}} \\ & \quad + \frac{3 \max\{1, v-u\}(\|\mathbf{l}\|_{\infty} + 1)[\mathbf{L} \max\{p, 2 \ln(3MBc)\}]^{1/2}}{M^{1/4}} \quad (186) \\ & \leq \frac{6dc^2}{[\min(\{\mathbf{L}\} \cup \{\mathbf{l}_i: i \in \mathbb{N} \cap [0, \mathbf{L}]\})]^{1/d}} + \frac{2\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1)^{\mathbf{L}} c^{\mathbf{L}+1} \max\{1, p\}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_{\infty} + 1)^{-2}]}} \\ & \quad + \frac{5B^2\mathbf{L}(\|\mathbf{l}\|_{\infty} + 1) \max\{p, \ln(eM)\}}{M^{1/4}} \end{aligned}$$

(cf. (iii) in Lemma 6.2).

*Proof of Corollary 6.7.* Observe that Corollary 6.6 (with  $(X_j)_{j \in \mathbb{N}} \leftarrow (X_j^{0,0})_{j \in \mathbb{N}}$ ,  $(Y_j)_{j \in \mathbb{N}} \leftarrow (Y_j^{0,0})_{j \in \mathbb{N}}$ ,  $\mathcal{R} \leftarrow \mathcal{R}_M^{0,0}$  in the notation of Corollary 6.6) shows (186). The proof of Corollary 6.7 is thus complete.  $\square$

**Corollary 6.8.** Let  $d, \mathbf{d}, \mathbf{L}, M, K, N \in \mathbb{N}$ ,  $L, a, u \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $v \in (u, \infty)$ ,  $c \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$ ,  $B \in [c, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, 1, \dots, N\}$ ,  $(\mathbf{J}_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ ,  $(\gamma_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j^{k,n}: \Omega \rightarrow [a, b]^d$ ,  $k, n, j \in \mathbb{N}_0$ , and  $Y_j^{k,n}: \Omega \rightarrow [u, v]$ ,  $k, n, j \in \mathbb{N}_0$ , be functions, assume that  $(X_j^{0,0}, Y_j^{0,0})$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $\mathcal{E}: [a, b]^d \rightarrow [u, v]$  satisfy  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1^{0,0}) = \mathbb{E}[Y_1^{0,0}|X_1^{0,0}]$ , assume for all  $x, y \in [a, b]^d$  that  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq L\|x - y\|_1$ , let  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  be random variables, assume  $(\bigcup_{k=1}^{\infty} \Theta_{k,0}(\Omega)) \subseteq [-B, B]^{\mathbf{d}}$ , assume that  $\Theta_{k,0}$ ,  $k \in \{1, 2, \dots, K\}$ , are i.i.d., assume that  $\Theta_{1,0}$  is continuous uniformly distributed on  $[-c, c]^{\mathbf{d}}$ , let  $\mathcal{R}_{J^n}^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$ ,  $k, n, J \in \mathbb{N}_0$ , and  $\mathcal{G}^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}$ , satisfy for all  $k, n \in \mathbb{N}$ ,  $\omega \in \Omega$ ,  $\theta \in \{\vartheta \in \mathbb{R}^{\mathbf{d}}: (\mathcal{R}_{J^n}^{k,n}(\cdot, \omega)): \mathbb{R}^{\mathbf{d}} \rightarrow [0, \infty) \text{ is differentiable at } \vartheta\}$  that  $\mathcal{G}^{k,n}(\theta, \omega) = (\nabla_{\theta} \mathcal{R}_{J^n}^{k,n})(\theta, \omega)$ , assume for all  $k, n \in \mathbb{N}$  that  $\Theta_{k,n} = \Theta_{k,n-1} - \gamma_n \mathcal{G}^{k,n}(\Theta_{k,n-1})$ , and assume for all  $k, n \in \mathbb{N}_0$ ,  $J \in \mathbb{N}$ ,  $\theta \in \mathbb{R}^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$\mathcal{R}_{J^n}^{k,n}(\theta, \omega) = \frac{1}{J} \left[ \sum_{j=1}^J |\mathcal{N}_{u,v}^{\theta,1}(X_j^{k,n}(\omega)) - Y_j^{k,n}(\omega)|^2 \right] \quad \text{and} \quad (187)$$

$$\mathbf{k}(\omega) \in \arg \min_{(l,m) \in \{1,2,\dots,K\} \times \mathbf{N}, \|\Theta_{l,m}(\omega)\|_{\infty} \leq B} \mathcal{R}_M^{0,0}(\Theta_{l,m}(\omega), \omega) \quad (188)$$

(cf. Definitions 2.8 and 3.1). Then

$$\begin{aligned} \mathbb{E}\left[\int_{[a,b]^d} |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},\mathbf{l}}(x) - \mathcal{E}(x)| \mathbb{P}_{X_1^{0,0}}(dx)\right] &\leq \frac{2[(v-u)\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ &+ \frac{3dL(b-a)}{[\min\{\mathbf{L}, \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{3\max\{1, v-u\}(\|\mathbf{l}\|_\infty + 1)[2\mathbf{L} \ln(3MBc)]^{1/2}}{M^{1/4}} \quad (189) \\ &\leq \frac{6dc^2}{[\min\{\mathbf{L}, \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{5B^2\mathbf{L}(\|\mathbf{l}\|_\infty + 1) \ln(eM)}{M^{1/4}} + \frac{2\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \end{aligned}$$

(cf. (iii) in Lemma 6.2).

*Proof of Corollary 6.8.* Note that Jensen's inequality implies that

$$\mathbb{E}\left[\int_{[a,b]^d} |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},\mathbf{l}}(x) - \mathcal{E}(x)| \mathbb{P}_{X_1^{0,0}}(dx)\right] \leq \mathbb{E}\left[\left(\int_{[a,b]^d} |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},\mathbf{l}}(x) - \mathcal{E}(x)|^2 \mathbb{P}_{X_1^{0,0}}(dx)\right)^{1/2}\right]. \quad (190)$$

This and Corollary 6.7 (with  $p \leftarrow 1$  in the notation of Corollary 6.7) complete the proof of Corollary 6.8.  $\square$

**Corollary 6.9.** Let  $d, \mathbf{d}, \mathbf{L}, M, K, N \in \mathbb{N}$ ,  $L \in \mathbb{R}$ ,  $c \in [\max\{2, L\}, \infty)$ ,  $B \in [c, \infty)$ ,  $\mathbf{l} = (\mathbf{l}_0, \mathbf{l}_1, \dots, \mathbf{l}_{\mathbf{L}}) \in \mathbb{N}^{\mathbf{L}+1}$ ,  $\mathbf{N} \subseteq \{0, 1, \dots, N\}$ ,  $(\mathbf{J}_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ ,  $(\gamma_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ , assume  $0 \in \mathbf{N}$ ,  $\mathbf{l}_0 = d$ ,  $\mathbf{l}_{\mathbf{L}} = 1$ , and  $\mathbf{d} \geq \sum_{i=1}^{\mathbf{L}} \mathbf{l}_i(\mathbf{l}_{i-1} + 1)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X_j^{k,n}: \Omega \rightarrow [0, 1]^d$ ,  $k, n, j \in \mathbb{N}_0$ , and  $Y_j^{k,n}: \Omega \rightarrow [0, 1]$ ,  $k, n, j \in \mathbb{N}_0$ , be functions, assume that  $(X_j^{0,0}, Y_j^{0,0})$ ,  $j \in \{1, 2, \dots, M\}$ , are i.i.d. random variables, let  $\mathcal{E}: [0, 1]^d \rightarrow [0, 1]$  satisfy  $\mathbb{P}$ -a.s. that  $\mathcal{E}(X_1^{0,0}) = \mathbb{E}[Y_1^{0,0}|X_1^{0,0}]$ , assume for all  $x, y \in [0, 1]^d$  that  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq L\|x - y\|_1$ , let  $\Theta_{k,n}: \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}_0$ , and  $\mathbf{k}: \Omega \rightarrow (\mathbb{N}_0)^2$  be random variables, assume  $(\bigcup_{k=1}^\infty \Theta_{k,0}(\Omega)) \subseteq [-B, B]^{\mathbf{d}}$ , assume that  $\Theta_{k,0}$ ,  $k \in \{1, 2, \dots, K\}$ , are i.i.d., assume that  $\Theta_{1,0}$  is continuous uniformly distributed on  $[-c, c]^{\mathbf{d}}$ , let  $\mathcal{R}_{j,n}^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow [0, \infty)$ ,  $k, n, J \in \mathbb{N}_0$ , and  $\mathcal{G}^{k,n}: \mathbb{R}^{\mathbf{d}} \times \Omega \rightarrow \mathbb{R}^{\mathbf{d}}$ ,  $k, n \in \mathbb{N}$ , satisfy for all  $k, n \in \mathbb{N}$ ,  $\omega \in \Omega$ ,  $\theta \in \{\vartheta \in \mathbb{R}^{\mathbf{d}}: (\mathcal{R}_{j,n}^{k,n}(\cdot, \omega)): \mathbb{R}^{\mathbf{d}} \rightarrow [0, \infty) \text{ is differentiable at } \vartheta\}$  that  $\mathcal{G}^{k,n}(\theta, \omega) = (\nabla_\theta \mathcal{R}_{j,n}^{k,n})(\theta, \omega)$ , assume for all  $k, n \in \mathbb{N}$  that  $\Theta_{k,n} = \Theta_{k,n-1} - \gamma_n \mathcal{G}^{k,n}(\Theta_{k,n-1})$ , and assume for all  $k, n \in \mathbb{N}_0$ ,  $J \in \mathbb{N}$ ,  $\theta \in \mathbb{R}^{\mathbf{d}}$ ,  $\omega \in \Omega$  that

$$\mathcal{R}_{j,n}^{k,n}(\theta, \omega) = \frac{1}{J} \left[ \sum_{j=1}^J |\mathcal{N}_{u,v}^{\theta,\mathbf{l}}(X_j^{k,n}(\omega)) - Y_j^{k,n}(\omega)|^2 \right] \quad \text{and} \quad (191)$$

$$\mathbf{k}(\omega) \in \arg \min_{(l,m) \in \{1, 2, \dots, K\} \times \mathbf{N}, \|\Theta_{l,m}(\omega)\|_\infty \leq B} \mathcal{R}_M^{0,0}(\Theta_{l,m}(\omega), \omega) \quad (192)$$

(cf. Definitions 2.8 and 3.1). Then

$$\begin{aligned} \mathbb{E}\left[\int_{[0,1]^d} |\mathcal{N}_{u,v}^{\Theta_{\mathbf{k}},\mathbf{l}}(x) - \mathcal{E}(x)| \mathbb{P}_{X_1^{0,0}}(dx)\right] &\leq \frac{3dL}{[\min\{\mathbf{L}, \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{3(\|\mathbf{l}\|_\infty + 1)[2\mathbf{L} \ln(3MBc)]^{1/2}}{M^{1/4}} + \frac{2[\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\ &\leq \frac{dc^3}{[\min\{\mathbf{L}, \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{B^3\mathbf{L}(\|\mathbf{l}\|_\infty + 1) \ln(eM)}{M^{1/4}} + \frac{\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \quad (193) \end{aligned}$$

(cf. (iii) in Lemma 6.2).

*Proof of Corollary 6.9.* Observe that Corollary 6.8 (with  $a \leftarrow 0$ ,  $u \leftarrow 0$ ,  $b \leftarrow 1$ ,  $v \leftarrow 1$  in the notation of Corollary 6.8), the facts that  $B \geq c \geq \max\{2, L\}$  and  $M \geq 1$ , and (ii) in

Lemma 6.4 show that

$$\begin{aligned}
& \mathbb{E} \left[ \int_{[0,1]^d} |\mathcal{N}_{u,v}^{\Theta_k, l}(x) - \mathcal{E}(x)| \mathbb{P}_{X_1^{0,0}}(dx) \right] \\
& \leq \frac{3dL}{[\min\{\mathbf{L}, \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{3(\|\mathbf{l}\|_\infty + 1)[2\mathbf{L} \ln(3MBc)]^{1/2}}{M^{1/4}} + \frac{2[\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\
& \leq \frac{dc^3}{[\min\{\mathbf{L}, \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{(\|\mathbf{l}\|_\infty + 1)[23B\mathbf{L} \ln(eM)]^{1/2}}{M^{1/4}} + \frac{[\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{2\mathbf{L}+2}]^{1/2}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}} \\
& \leq \frac{dc^3}{[\min\{\mathbf{L}, \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_{\mathbf{L}-1}\}]^{1/d}} + \frac{B^3\mathbf{L}(\|\mathbf{l}\|_\infty + 1) \ln(eM)}{M^{1/4}} + \frac{\mathbf{L}(\|\mathbf{l}\|_\infty + 1)^{\mathbf{L}} c^{\mathbf{L}+1}}{K^{[(2\mathbf{L})^{-1}(\|\mathbf{l}\|_\infty + 1)^{-2}]}}. \tag{194}
\end{aligned}$$

The proof of Corollary 6.9 is thus complete.  $\square$

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