BSDEs generated by fractional space-time noise and related SPDEs

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Abstract. This paper is concerned with the backward stochastic differential equations whose generator is a weighted fractional Brownian field: $Y_t = \xi + \int_t^T Y_s W(ds, B_s) - \int_t^T Z_s dB_s$, $0 \leq t \leq T$, where W is a $(d+1)$-parameter weighted fractional Brownian field of Hurst parameter $H = (H_0, H_1, \cdots, H_d)$, which provide probabilistic interpretations (Feynman-Kac formulas) for certain linear stochastic partial differential equations with colored space-time noise. Conditions on the Hurst parameter H and on the decay rate of the weight are given to ensure the existence and uniqueness of the solution pair. Moreover, the explicit expression for both components Y and Z of the solution pair are given.

Keywords. Backward stochastic differential equations; stochastic partial differential equations; Feynman-Kac formulas; fractional space-time noise; explicit solution, Malliavin calculus.

1 Introduction and main result

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. Let $W = (W(t,x), t \geq 0, x \in \mathbb{R}^d)$ be a weighted fractional Brownian field. Namely, W is a mean-zero Gaussian random field with the following covariance structure:

$$\mathbb{E}\left[W(t,x)W(s,y)\right] = R_{H_0}(s,t)\rho(x)\rho(y)\prod_{i=1}^{d} R_{H_i}(x_i,y_i)\,, \tag{1.1}$$

where and throughout the paper, we assume $H_i \in (1/2, 1)$ for all $i = 0, 1, \cdots, d$, and $R_H(\xi,\eta) = \left[|\xi|^{2H} + |\eta|^{2H} - |\xi-\eta|^{2H}\right]/2$, for all $\xi, \eta \in \mathbb{R}$ and $\rho(x)$ is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}$ satisfying some properties which will be specified later. We consider the following (one dimensional) linear backward stochastic differential equation (BSDE for short) with fractional noise generator:

$$Y_t = \xi + \int_t^T Y_s W(ds, B_s) - \int_t^T Z_s dB_s, \quad t \in [0,T]\,, \tag{1.2}$$

where B is a d-dimensional standard Brownian motion. Our interest in this equation is motivated from the following three aspects.

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(a) The first aspect is the nonlinear Feynman-Kac formula (in our special case) which relates the following two stochastic differential equations: the first one is the backward doubly stochastic differential equation (BDSDE for short)

$$
\begin{aligned}
Y_s^{t,x} &= \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\
&\quad + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})W(dr, X_r^{t,x}) - \int_s^t Z_r^{t,x} dB_r \,,
\end{aligned}
$$

where $X_s^{t,x}$ is the solution to the following stochastic differential equation

$$
dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s \,, \quad s \in [t, T] \,, \quad X_t^{t,x} = x \in \mathbb{R}^d \,.
$$

The second one is the stochastic partial differential equation (SPDE for short)

$$
\begin{cases}
-du(t,x) = [\mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(x))]\, dt \\
\qquad\qquad +g(t,x,u(t,x),\nabla u(t,x)\sigma(x))W(dt,x) \,, \quad (t,x) \in [0,T] \times \mathbb{R}^d \,, \\
u(T,x) = \phi(x) \,,
\end{cases}
\tag{1.3}
$$

where $\mathcal{L}$ is the generator associated with the Markov process $X_s^{t,x}$. There are many articles along this direction since the work of [10]. Most scholars studied the BDSDEs under various conditions, whose solution can be used as the nonlinear Feynman–Kac formula to represent the solution to the correlated semi-linear SPDEs driven by white noise. We refer to [7, Theorem 5.1] and the references therein for the exact relation between the solutions of these two equations. It is worth noting that, BDSDEs and probabilistic interpretation (nonlinear Feynman-KAC formula) of SPDEs driven only by temporal white noise have been studied extensively in several directions, see e.g. [4], [5], [6] and [8]. Although Feynman-Kac formulas of (linear or non-linear) SPDE with spatial-temporal noise is obtained in [1], [12]and [16] for example, there are limited works to characterize the solution of SPDEs by using the solution of BSDEs. To the best of our knowledge only [7] and [9] dealt with such problems.

(b) If $b = 0$ and $\sigma = 1$, then $\mathcal{L} = \frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the half of the Laplacian. If further $g(r,x,u,p) = u$, then the above SPDE (1.3) becomes

$$
-du(t,x) = \frac{1}{2}\Delta u dt + u(t,x)W(dt,x), \; u(T,x) = \phi(x) \,. \tag{1.4}
$$

This equation has enjoyed a great attention in recent decade (when the terminal condition is replaced by the initial condition and the noise $W(dt,x)$ is replaced by more singular one $\frac{\partial^d}{\partial x_1 \cdots \partial x_d} W(dt,x)$), often in the name of parabolic Anderson model. We refer to a survey work [11] and references therein for further study. Let us only point out that many works do not require that the noise is white in time in their study: For the SPDE in the above case (b), the associated BDSDEs becomes

$$
Y_s^{t,x} = \phi(B_T^{t,x}) + \int_s^T Y_r^{t,x} W(dr, B_{r-t}^{t,x}) - \int_s^t Z_r^{t,x} dB_r \,, \tag{1.5}
$$

where $B_s^{t,x} = x + (B_s - B_t)$ is a d-dimensional Brownian motion starting at time t from the point x. This equation is of the form (1.2). Its probabilistic interpretation, the explicit form and some sharp properties of solution will be the main focus of this paper.

To illustrate our main results of finding the explicit representation of the solution pair using partial Malliavin derivatives we shall follow the idea of [1]. Define (we shall justify it in the next section)

$$\alpha_s^t = \exp\left[\int_s^t W(dr, B_r)\right] \tag{1.6}$$

and denote by $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t\,; W(t,x), t \ge 0, x \in \mathbb{R}^d)$ the σ-algebra generated by Brownian motion up to time instant $t \ge 0$ and $W(t,x)$ for all t and $x \in \mathbb{R}^d$. Then we have formally the following candidate for the solution pair

$$\begin{cases} Y_t = (\alpha_0^t)^{-1}\mathbb{E}\left[\xi\alpha_0^T \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\xi \exp\left(\int_t^T W(dr, B_r)\right) \,\middle|\, \mathcal{F}_t\right] & \text{by [13, Equation (2.11)]}\ , \\ \\ Z_t = D_t^B Y_t = D_t^B \mathbb{E}\left[\xi \exp\left(\int_t^T W(dr, B_r)\right) \,\middle|\, \mathcal{F}_t\right] & \text{by [13, Equation (2.23)]}\ , \end{cases} \tag{1.7}$$

where D_t^B is the Malliavin gradient with respect to the Brownian motion B (see next section for the definition and properties), and $\mathbb{E}^B$ is the expectation with respect to B (explained in detail in the proof of Proposition 2.1). Here is the main result of this paper.

Theorem 1.1. *Suppose* $\sum_{i=1}^d (2H_i - \beta_i) < 2$ *and* $\xi \in \mathbb{D}_B^{1,q}$ *is measurable w.r.t.* σ*-field* $\mathcal{F}_T^B$*, for* $q > \dfrac{2}{2\underline{H} - 1}$*, where* $\underline{H} = \min\{H_0, \ldots, H_d\}$*. Then we have the following results:*

(1) The processes $\{(Y_t, Z_t), 0 \le t \le T\}$ *formally defined by* (1.7) *are well-defined and square integrable, and they are the solution pair to the BSDE* (1.2)*. Moreover,* Z *has the following alternative expression:*

$$Z_t = \mathbb{E}\left[e^{\int_t^T W(d\tau, B_\tau)} D_t^B \xi + \int_t^T e^{\int_t^s W(d\tau, B_\tau)} Y_s (\nabla_x W)(ds, B_s) \Big| \mathcal{F}_t\right]. \tag{1.8}$$

(2) If for all $q > 2$, $\mathbb{E}|D_t^B\xi - D_s^B\xi|^q \le C|t-s|^{\kappa q/2}$ *for some* $\kappa \in (0,2)$*, then for any* $a > 1$ *and for any* $\varepsilon > 0$*, we have the following Hölder continuity for* Y *and* Z*:*

$$\mathbb{E}|Y_t - Y_s|^a \le C_a\,|t-s|^{a/2}, \qquad \mathbb{E}\big|Z_t - Z_s\big|^2 \le C_\varepsilon |t-s|^{(2H_0 + \underline{H} - 1 - \varepsilon)\wedge\kappa}\,, \quad \forall\ s,t \in [0,T]. \tag{1.9}$$

(3) If a pair (Y, Z) *satisfies (2) for some* $a, \kappa > 0$*, then* (Y, Z) *is represented by* (1.7) *and hence the BSDEs* (1.2) *has a unique solution.*

(4) If $(Y, Z) \in \mathcal{S}_{\mathbb{F}}^2(0,T;\mathbb{R}) \times \mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$ *is the solution pair of BSDEs* (1.2) *so that* Y, D^BY *are* $\mathbb{D}^{1,2}$ *then the solution also has the explicit expression* (1.7) *and hence the BSDEs* (1.2) *has a unique solution.*

Remark 1.2. *Since we assume* $H_0 > 1/2$*, we see* $2H_0 + \underline{H} - 1 > 0$*. We can only obtain the Hölder continuity of* Z *in the mean square sense. We encounter the difficulty to deal with high moments for* Z*.*

Now let us point out the novelty compared to two relevant works. In the work [9], the generator W is a fractional Brownian motion (the generator W does not depend on x). In the work [7] W can depend on the space x, but it is assumed that that it is a backward martingale with to time variable t so that the backward martingale technology can be used. In our above theorem, neither the assumption that W is independent of x, nor the assumption that W is a backward martingale is assumed. In particular, we can obtain the explicit solution (for linear equation) and use this expression to obtain the some kind sharp Hölder continuity for the solution pair which, to our best knowledge, are new.

Here is the organization of this work. In next section, we shall show that the quantity $\int_s^t W(dr, B_r)$ in (1.7) is well-defined and is exponentially integrable so that Y_t is well-defined. In Section 3, we obtain some properties of the process Y_t and show that it is Malliavin differentiable and Z_t is well-defined. We show that the pair (Y_t, Z_t) is the solution to the linear BSDE (1.2). A great difficulty is that we need to show that the process Y is in $\mathcal{S}_{\mathbb{F}}^p(0,T;\mathbb{R})$ and Z is in $\mathcal{M}_{\mathbb{F}}^p(0,T;\mathbb{R})$ due to the singularity of the noise W in the generator. We overcome this difficulty by Talagrand theorem 3.2, Borell theorem 3.3, and a new Lemma 3.7. In Section 4, we use the explicit expression to obtain Hölder continuity of the solution pair. The Hölder continuity of the process Z_t is always a difficult problem (see e.g. [13, 18, 19]) however plays a critical role in numerical method. In Section 6 we discuss the relation between the linear BSDE (1.5) and the stochastic PDE (1.4).

2 Exponential integrability of $\int_t^T W(ds, B_s)$

Let $T > 0$ be a fixed time horizon and let $(\Omega, \mathcal{F}, P)$ be a complete probability space, on which the expectation is denoted by $\mathbb{E}$. Let $\{B_t, 0 \le t \le T\}$ be a d-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$. Suppose $W = \{W(t,x), t \ge 0,\ x \in \mathbb{R}^d\}$ is a weighted fractional Brownian space-time field whose covariance is given by (1.1). The stochastic integral with respect to W is well-defined in many references, and we refer to [11] and references therein for more details. We shall use this concept freely. For example, we denote $W(\phi) = \int_{\mathbb{R}_+\times\mathbb{R}^d} \phi(t,x) W(dt,x)dx$ for any $\phi \in \mathcal{D} = \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$, where $\mathcal{D}$ is the set of all smooth functions with compact support from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}$. We denote the spatial covariance as

$$q(x,y) = \rho(x)\rho(y) \prod_{i=1}^d R_{H_i}(x_i, y_i)\,, \quad \forall\, x = (x_1, \cdots, x_d)^T\,, \quad y = (y_1, \cdots, y_d)^T \in \mathbb{R}^d\,, \tag{2.1}$$

where $\rho : \mathbb{R}^d \to \mathbb{R}$ is a continuous function of power decay, and we will specify the conditions that ρ are satisfied later. It is known that

$$\mathbb{E}\big[W(h)W(g)\big] = \int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}} h(t,x)g(s,y)|s-t|^{2H_0-2}\rho(x)\rho(y) \prod_{i=1}^d R_{H_i}(x_i,y_i) ds dt dx dy \tag{2.2}$$

for all $h, g \in \mathcal{D}$. It is clear that $\langle h, g\rangle_{\mathcal{H}}$ is a scalar product on $\mathcal{D}$. We denote $\mathcal{H}$ the Hilbert space by completing $\mathcal{D}$ with respect to this scalar product.

Let F be a cylindrical random variable of the form

$$F = f\left(W\left(\phi^1\right), \ldots, W\left(\phi^n\right)\right),$$

where $\phi^i \in \mathcal{D}\,, i = 1, \cdots, n$ and $f \in C_p^\infty(\mathbb{R}^n)$, i.e., f and all its partial derivatives have polynomial growth. The set of all such cylindrical random variables is denoted by $\mathcal{P}$. If $F \in \mathcal{P}$ has the above form, then $D^W F$ is the $\mathcal{H}$-valued random variable defined by

$$D^W F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \left(W\left(\phi_1\right), \ldots, W\left(\phi_n\right)\right) \phi_j.$$

The operator D^W is closable from $L^2(\Omega)$ into $L^2(\Omega, \mathcal{H})$, namely D^W is the Malliavin derivative operator with respect to the fractional Brownian motion W. We define the Sobolev space $\mathbb{D}_W^{1,p}$ as the closure of $\mathcal{P}$ under the following norm :

$$\|D^W F\|_{1,p} = \left(\mathbb{E}\,|F|^p + \mathbb{E}\|D^W F\|_{\mathcal{H}}^p\right)^{1/p}.$$

Let us denote by δ the adjoint of the derivative operator given by duality formula

$$\mathbb{E}(\delta(u)F) = \mathbb{E}\left(\langle D^W F, u\rangle_{\mathcal{H}}\right) \qquad \text{for any } F \in \mathbb{D}_W^{1,2},$$

where $\delta(u)$ is also called the Skorohod integral of u. We refer to [3] and [2] for a detailed account on the Malliavin calculus. For any random variable $F \in \mathbb{D}_W^{1,2}$ and $\phi \in \mathcal{H}$, we will often use the following formula in the text:

$$FW(\phi) = \delta(F\phi) + \langle D^W F, \phi\rangle_{\mathcal{H}}.$$

Accordingly, we can define D^B the Malliavin derivative operator with respect to the standard Brownian motion B and $\mathbb{D}_B^{1,p}$ the Sobolev space in the same way. We say a random field $F \in \mathbb{D}^{1,p}$ if F is an element both in $\mathbb{D}_B^{1,p}$ and $\mathbb{D}_W^{1,p}$.

The stochastic integral studied earlier is useful in this paper but is not sufficient for our purpose. We also need to introduce a new kind of nonlinear stochastic integral similar to that of Kunita ([20]). To this end, we introduce the approximation of $\dot{W}$ as follows.

$$\dot{W}_{\varepsilon,\eta}(s, B_s) = \int_0^s \int_{R^d} \varphi_\eta(s-r) p_\varepsilon(B_s - y) W(dr, y) dy\,, \tag{2.3}$$

where φ_η and p_ε are the approximation of the Dirac delta functions:

$$\varphi_\eta(t) = \frac{1}{\eta}\mathbf{1}_{[0,\eta]}(t),\ p_\varepsilon(x) = (2\pi\epsilon)^{-d/2} e^{-|x|^2/2\varepsilon}, \quad \text{for all } \eta,\ \varepsilon > 0.$$

Proposition 2.1. *Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a continuous function of power decay, i.e., ρ satisfying $0 \le \rho(x) \le C \prod_{i=1}^d (1 + |x_i|)^{-\beta_i}$, where $\beta_i \in (0, 2)$ and $2H_i > \beta_i$ for all $i = 1, 2, \ldots d$ and suppose*

$$\alpha := \sum_{i=1}^{d} (2H_i - \beta_i) < 2\,. \tag{2.4}$$

Then the stochastic integral $V_t^{\varepsilon,\eta} := \int_t^T \dot{W}_{\varepsilon,\eta}(s, B_s) ds$ converges in $L^2(\Omega)$ to a limit denoted by

$$V_t = \int_t^T W(ds, B_s). \tag{2.5}$$

Moreover, conditioning on $\mathcal{F}^B$, V_t is a mean-zero Gaussian random variable with variance

$$\mathrm{Var}^W(V_t) = \int_t^T \int_t^T |s - s'|^{2H_0 - 2} \rho(B_s) \rho(B_{s'}) \prod_{i=1}^{d} R_{H_i}(B_s^i, B_{s'}^i) ds ds'\,. \tag{2.6}$$

Proof. Suppose $\varepsilon, \varepsilon', \eta, \eta' \in (0,1)$. Throughout the paper denote by $\mathbb{E}^W$ the expectation with respect to the random field W which considers other random elements as fixed "constant". For example, if $F(W,B)$ is a functional of W and B, then $\mathbb{E}^W(F(W,B)) = \mathbb{E}(F(W,B)|\mathcal{F}^B)$. By Fubini's theorem and (2.3), we have

$$
\begin{aligned}
&\mathbb{E}^W\left[\int_t^T \dot{W}_{\varepsilon,\eta}(s,B_s)ds \int_t^T \dot{W}_{\varepsilon',\eta'}(s,B_s)ds\right] \\
=&\alpha_{H_0}\int_t^T\int_t^T\int_t^s\int_t^{s'}\int_{\mathbb{R}^{2d}} \varphi_\eta(s-r)\varphi_{\eta'}(s'-r')p_\varepsilon(B_s-y)p_{\varepsilon'}(B_{s'}-y') \\
&\qquad\qquad |r-r'|^{2H_0-2}\rho(y)\rho(y')\prod_{i=1}^d R_{H_i}(y_i,y_i')dydy'drdr'dsds' \\
=&\alpha_{H_0}\int_t^T\int_t^T\int_t^s\int_t^{s'} \varphi_\eta(s-r)\varphi_{\eta'}(s'-r')|r-r'|^{2H_0-2} \\
&\quad \mathbb{E}^{X,X'}\left\{\rho(\sqrt{\varepsilon}X+B_s)\rho(\sqrt{\varepsilon'}X'+B_{s'})\prod_{i=1}^d\left[R_{H_i}(\sqrt{\varepsilon}X_i+B_s^i,\sqrt{\varepsilon}X_i'+B_{s'}^i)\right]\right\}drdr'dsds' \\
=:&I(\varepsilon,\varepsilon',\eta,\eta'),
\end{aligned} \tag{2.7}
$$

where $X=(X_1,\cdots,X_d), X'=(X_1',\cdots,X_d')$ are independent standard random variables, which are also independent of $\mathcal{F}^B$.

To study the limit of above $I(\varepsilon,\varepsilon',\eta,\eta')$ as $\varepsilon,\varepsilon',\eta,\eta'\to 0$, we observe that, firstly [1, Lemma A.3] directly yields

$$
\int_t^s\int_t^{s'} \varphi_\eta(s-r)\varphi_{\eta'}(s'-r')|r-r'|^{2H_0-2}drdr' \le |s-s'|^{2H_0-2}. \tag{2.8}
$$

Moreover,

$$
\begin{aligned}
\left|q(y,y')\right| &= \frac{1}{2}\rho(y)\rho(y')\prod_{i=1}^d\left||y_i|^{2H_i}+|y_i'|^{2H_i}-|y_i-y_i'|^{2H_i}\right| \\
&\le C\rho(y)\rho(y')\prod_{i=1}^d(|y_i|^{2H_i}+|y_i'|^{2H_i}) \\
&\le C\prod_{i=1}^d(1+|y_i|^{2H_i})(1+|y_i'|^{2H_i})(1+|y_i|)^{-\beta_i}(1+|y_i'|)^{-\beta_i} \\
&\le C\prod_{i=1}^d(1+|y_i|)^{2H_i-\beta_i}(1+|y_i'|)^{2H_i-\beta_i},
\end{aligned} \tag{2.9}
$$

where and throughout this paper C is a generic constant depending only on $H_i,\ i=1,\dots,d$. This can be used to show that

$$
I_1(\varepsilon,\varepsilon',s,s') := \mathbb{E}^{X,X'}\Big[\rho(\sqrt{\varepsilon}X+B_s)\rho(\sqrt{\varepsilon'}X'+B_{s'})\prod_{i=1}^d R_{H_i}(\sqrt{\varepsilon}X_i+B_s^i,\sqrt{\varepsilon}X_i'+B_{s'}^i)\Big] \tag{2.10}
$$

is a pathwise bounded continuous function of $\varepsilon, \varepsilon', s, s'$ in the concerned domain (almost surely with respect to B). Thus, we have

$$
\begin{aligned}
&\mathbb{E}[I(\varepsilon,\varepsilon',\eta,\eta')] \\
&\quad = \alpha_{H_0}\mathbb{E}\int_t^T\int_t^T\int_t^s\int_t^{s'} \varphi_\eta(s-r)\varphi_{\eta'}(s'-r')|r-r'|^{2H_0-2}I_1(\varepsilon,\varepsilon',s,s')drdr'dsds' \\
&\quad \le \alpha_{H_0}\int_t^T\int_t^T |s-s'|^{2H_0-2}\prod_{i=1}^d \mathbb{E}\left(1+|B_s^i|\right)^{2H_i-\beta_i}(1+|B_{s'}^i|)^{2H_i-\beta_i}dsds' \\
&\quad \le C|T-t|^{2H_0} < \infty\,.
\end{aligned}
\tag{2.11}
$$

Moreover, for $s \neq s'$, as $\varepsilon, \varepsilon', \eta, \eta'$ tend to zero we have

$$
\begin{aligned}
&\lim_{\varepsilon,\varepsilon',\eta,\eta'\to 0} I(\varepsilon,\varepsilon',\eta,\eta') \\
&\quad = \alpha_{H_0}\lim_{\varepsilon,\varepsilon',\eta,\eta'\to 0}\int_t^T\int_t^T\int_t^s\int_t^{s'} \varphi_\eta(s-r)\varphi_{\eta'}(s'-r')|r-r'|^{2H_0-2}I_1(\varepsilon,\varepsilon',s,s')drdr'dsds' \\
&\quad = \alpha_{H_0}\int_t^T\int_t^T |s-s'|^{2H_0-2}\rho(B_s)\rho(B_{s'})\prod_{i=1}^d R_{H_i}(B_s^i,B_{s'}^i)dsds'\,.
\end{aligned}
$$

Therefore, if we put $\varepsilon=\varepsilon'$, $\eta=\eta'$ and use the estimates (2.9) and (2.10), and with the help of Lebesgue's convergence theorem we have

$$
\mathbb{E}\left(V_t^{\varepsilon,\eta}-V_t^{\varepsilon',\eta'}\right)^2 = \mathbb{E}\left(V_t^{\varepsilon,\eta}\right)^2 - 2\mathbb{E}\left(V_t^{\varepsilon,\eta}V_t^{\varepsilon',\eta'}\right) + \mathbb{E}\left(V_t^{\varepsilon',\eta'}\right)^2 \to 0, \text{ as } \varepsilon,\varepsilon',\eta,\eta'\to 0.
$$

As a consequence we have $V_t^{\varepsilon_n,\eta_n}$ is a Cauchy sequence in $L^2(\Omega)$. It has then a limit denoted by V_t, proving the proposition. □

Proposition 2.2. *Let $\rho:\mathbb{R}^d\to\mathbb{R}$ be a continuous function satisfying* (2.4). *Then, for all $\lambda\in\mathbb{R}$,*

$$
\mathbb{E}\left[\exp\left(\lambda\int_t^T W(dr,B_r)\right)\right] < \infty.
$$

Proof. From (2.6) and the first inequality in (2.9), it follows

$$
\begin{aligned}
I :&= \mathbb{E}\left[\mathbb{E}^W\exp\left(\lambda\int_t^T W(dr,B_r)\right)\right] \\
&= \mathbb{E}\left[\exp\left((\alpha_{H_0}\lambda^2)/2\int_t^T\int_t^T \left|s-r\right|^{2H_0-2}\rho(B_s)\rho(B_r)\prod_{i=1}^d R_{H_i}(B_s^i,B_r^i)dsdr\right)\right] \\
&\le \mathbb{E}\left[\exp\left(C(\alpha_{H_0}\lambda^2)/2\int_t^T\int_t^T \left|s-r\right|^{2H_0-2}\rho(B_s)\rho(B_r)\prod_{i=1}^d\left(\left|B_s^i\right|^{2H_i}+\left|B_r^i\right|^{2H_i}\right)dsdr\right)\right]\,.
\end{aligned}
\tag{2.12}
$$

Note that

$$
\left|\rho(B_s)\rho(B_r)\prod_{i=1}^d\left(\left|B_s^i\right|^{2H_i}+\left|B_r^i\right|^{2H_i}\right)\right| \le 2^d\rho(B_s)\prod_{i=1}^d\sup_{s\in[t,T]}\left|B_s^i\right|^{2H_i}
$$

$$\le 2^d \prod_{i=1}^d \left(1 + \sup_{s\in[t,T]} |B_s^i|\right)^{2H_i-\beta_i} \le 2^d \left(1 + \sup_{s\in[t,T]} \sum_{i=1}^d |B_s^i|\right)^{\sum_{i=1}^d 2H_i-\beta_i} \tag{2.13}$$
$$=: C_d\left(1 + \sup_{s\in[t,T]} \sum_{i=1}^d |B_s^i|\right)^{\alpha}.$$

We have

$$I \le \mathbb{E}\left[\exp\left(C\int_t^T\int_t^T \left|s-r\right|^{2H_0-2} dsdr \cdot \left(1+\sup_{s\in[t,T]}\sum_{i=1}^d |B_s^i|\right)^{\alpha}\right)\right],$$

which is finite thanks to Fernique's theorem (e.g. [3, Theorem 4.14]) since $\alpha < 2$, completing the proof of the proposition. $\square$

3 Linear backward stochastic differential equation

Now we consider the backward stochastic differential equation (1.2). In order to study the regularity of (Y, Z), we approximate it by (2.3) and obtain the following approximation of (1.2):

$$Y_t^{\varepsilon,\eta} = \xi + \int_t^T Y_s^{\varepsilon,\eta}\dot{W}_{\varepsilon,\eta}(s, B_s)ds - \int_t^T Z_s^{\varepsilon,\eta} dB_s, \quad t \in [0, T]. \tag{3.1}$$

Due to the regularity of the approximated noise $\dot{W}_{\varepsilon,\eta}$ and Proposition 2.2, we can explicitly express its solution as follows (see e.g. [1] and references therein):

$$\begin{cases} Y_t^{\varepsilon,\eta} = \mathbb{E}\left[\xi \exp\left(\int_t^T \dot{W}_{\varepsilon,\eta}(r, B_r)dr\right) \,\middle|\, \mathcal{F}_t\right] & \text{by [13, Equation (2.11)]}, \\ Z_t^{\varepsilon,\eta} = D_t^B \mathbb{E}\left[\xi \exp\left(\int_t^T \dot{W}_{\varepsilon,\eta}(r, B_r)dr\right) \,\middle|\, \mathcal{F}_t\right] & \text{by [13, Equation (2.23)]}, \end{cases} \tag{3.2}$$

where $D^B = (D^{B^1}, \cdots, D^{B^d})^T$ is the Malliavin gradient operator with respect to the Brownian motion B, so that $Z_t^{\varepsilon,\eta}$ is a d-dimensional vector.

We have proved that $\int_t^T W(ds, B_s)$ is exponentially integrable in Proposition 2.2. Then we can define

$$Y_t := \mathbb{E}\left[\xi \exp\left(\int_t^T W(dr, B_r)\right) \,\middle|\, \mathcal{F}_t\right]. \tag{3.3}$$

Lemma 3.1. *Assume $\xi \in L^q(\Omega)$ for some $q > 2$. Then for any $t \in [0, T]$, we have $Y_t^{\varepsilon,\eta}$ converges to Y_t in $L^p(\Omega)$ for all $p \in [1, q)$.*

Proof. Denote $V_t^{\varepsilon,\eta} = \int_t^T \dot{W}_{\varepsilon,\eta}(s, B_s)ds$. Let $q'p = q$ and $1/p' + 1/q' = 1$. From (3.2), (3.3), Jensen's inequality and Hölder's inequality it follows

$$\begin{aligned} \mathbb{E}\left[\left|Y_t^{\varepsilon,\eta} - Y_t\right|^p\right] &= \mathbb{E}\left|\mathbb{E}\left[\xi\left(\exp\left(V_t^{\varepsilon,\eta}\right) - \exp\left(V_t\right)\right)\middle|\mathcal{F}_t\right]\right|^p \\ &\le \mathbb{E}\left[|\xi|^p \left|\exp\left(V_t^{\varepsilon,\eta}\right) - \exp\left(V_t\right)\right|^p\right] \\ &\le \|\xi\|_q^{1/q'} \left[\mathbb{E}\left|\exp\left(V_t^{\varepsilon,\eta}\right) - \exp\left(V_t\right)\right|^{pp'}\right]^{1/p'}. \end{aligned} \tag{3.4}$$

Similar to Proposition 2.2 we can prove

$$\sup_{\varepsilon,\eta\in(0,1]} \mathbb{E}\Big|\exp(\lambda V_t^{\varepsilon,\eta})\Big| < \infty, \quad \forall\, \lambda \in \mathbb{R}\,.$$

Proposition 2.1 implies that $V_t^{\varepsilon,\eta} \to V_t$ in probability. Thus, we prove the proposition by Lebesgue's convergence theorem. $\square$

Let us denote

$$\mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R}) := \Big\{\psi = (\psi_s)_{s\in[0,T]} : \psi \text{ is a real-valued } \mathcal{F}\text{-adapted} \\ \text{continuous process; } \mathbb{E}\left[\sup_{0\le s\le T} |\psi_s|^p\right] < \infty\Big\}.$$

To prove $Y = \{Y_t, t \in [0,T]\} \in \mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R})$ for all $p \in [1,q)$, we shall first recall Talagrand's majorizing measure theorem.

Lemma 3.2. *(Majorizing Measure Theorem, see e.g. [[15, Theorem 2.4.2]]. Let T be a given set and let $\{X_t, t\in T\}$ be a centred Gaussian process indexed by T. Denote by $d(t,s) = \left(\mathbb{E}\,|X_t - X_s|^2\right)^{\frac{1}{2}}$ the associated natural metric on T. Then*

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \asymp \gamma_2(T,d) := \inf_{\mathcal{A}} \sup_{t\in T} \sum_{n\ge 0} 2^{n/2}\, \mathrm{diam}\,(A_n(t))\,,$$

where "$\asymp$" indicates the asymptotic notation. Note that the infinimum is taken over all increasing sequence $\mathcal{A} := \{\mathcal{A}_n, n = 1,2,\cdots\}$ of partitions of T such that $\#\mathcal{A}_n \le 2^{2^n}$ ($\#A$ denotes the number of elements in the set A), $A_n(t)$ denotes the unique element of $\mathcal{A}_n$ that contains t, and diam $(A_n(t))$ *is the diameter (with respect to the natural distance $d(\cdot,\cdot)$) of $A_n(t)$.*

We shall apply the above majorizing measure theorem to $V_t = \int_t^T W(ds, B_s)$ as a random variable of W (which is Gaussian under the conditional law knowing B). The associated natural metric (which is a random variable of B) is (assuming $t > s$)

$$\begin{aligned} d(t,s) &:= \sqrt{(\mathbb{E}^W|V_t - V_s|^2)} = \sqrt{(\mathbb{E}^W|\int_s^t W(dr, B_r)|^2)} \\ &= \sqrt{\int_s^t\int_s^t \alpha_{H_0}|u-v|^{2H_0-2}\rho(B_u)\rho(B_r)\prod_{i=1}^d R_{H_i}(B_u^i, B_r^i)dudv} \\ &\le C_H \mathsf{v}(B)|t-s|^{H_0}, \end{aligned} \tag{3.5}$$

where C_H is a constant depending only $H_i,\ i = 1,\ldots,d$ and

$$\mathsf{v}(B) := C_d(1 + \sup_{u\in[0,T]} \sum_{i=1}^d |B_u^i|)^{\alpha}. \tag{3.6}$$

Next, we choose the admissible sequences $(\mathcal{A}_n)$ as uniform partition of $[0,T]$ such that $\#(\mathcal{A}_n) \le 2^{2^n}$:

$$[0,T] = \bigcup_{j=0}^{2^{2^{n-1}}-1} \Big[j\cdot 2^{-2^{n-1}}T, (j+1)\cdot 2^{-2^{n-1}}T\Big)\,.$$

Thus, we can deduce that, by Lemma 3.2,

$$\mathbb{E}^W\big[\sup_{t\in[0,T]} V_t\big] \le C \sup_{t\in[0,T]} \sum_{n\ge 0} 2^{n/2}\,\mathrm{diam}\,(A_n(t)) \tag{3.7}$$

where $A_n(t)$ is the element of uniform partition $\mathcal{A}_n$ that contains (t), i.e.,

$$A_n(t) = \Big[j\cdot 2^{-2^{n-1}}T, (j+1)\cdot 2^{-2^{n-1}}T\Big)$$

such that $j\cdot 2^{-2^{n-1}}T \le t < (j+1)\cdot 2^{-2^{n-1}}T$.
Since $(\mathcal{A}_n)$ is a uniform partition, and by using the bound (3.5) we see the diameter of $A_n(t)$ with respect to $d(t,s)$ can be estimated by

$$\mathrm{diam}\big(A_n(t)\big) \le C_H \nu(B) 2^{-H_0 2^{n-1}} T^{H_0}.$$

Inserting this result into (3.7), we have

$$\begin{aligned}\mathbb{E}^W\big[\sup_{t\in[0,T]} V_t\big] &\le C \sup_{t\in[0,T]} \sum_{n\ge 0} 2^{n/2}\,\mathrm{diam}\,(A_n(t)) \\ &\le C_H \nu(B) T^{H_0} \sum_{n\ge 0} 2^{n/2} 2^{-H_0 2^{n-1}} \le C_H \nu(B) T^{H_0}.\end{aligned} \tag{3.8}$$

We also need the following two results to show $Y \in \mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R})$.

Lemma 3.3. *(Borell-TIS inequality, see e.g. [14, Theorem 2.1]). Let $\{X_t, t\in T\}$ be a centered separable Gaussian process on some topological index set T with almost surely bounded sample paths. Then $\mathbb{E}\left(\sup_{t\in T} X_t\right) < \infty$, and for all $\lambda > 0$,*

$$\mathbf{P}^W\left\{\left|\sup_{t\in T} X_t - \mathbb{E}\left(\sup_{t\in T} X_t\right)\right| > \lambda\right\} \le 2\exp\left(-\frac{\lambda^2}{2\sigma_T^2}\right),$$

where $\sigma_T^2 := \sup_{t\in T}\mathbb{E}\left(X_t^2\right)$.

Lemma 3.4. *If the process $\{X_t, t\in T\}$ is symmetric, then we have*

$$\mathbb{E}\left[\sup_{t\in T}|X_t|\right] \leqslant 2\mathbb{E}\left[\sup_{t\in T} X_t\right] + \inf_{t_0\in T}\mathbb{E}\left[|X_{t_0}|\right]. \tag{3.9}$$

Now we can state and prove one of the main results of this work.

Theorem 3.5. *Suppose $\xi \in L^q(\Omega)$ for some $q > 2$ and suppose that (2.4) holds. Then we have $Y^{\varepsilon,\eta}$ converges to $Y = \{Y_t, t\in[0,T]\} \in \mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R})$ for all $p\in[1,q)$.*

Proof. We just need to verify $Y_t \in \mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R})$. Let $q'p = q$ and $1/p' + 1/q' = 1$. By (3.2) and Jessen's inequality and Doob's martingale inequality we see

$$\begin{aligned}\mathbb{E}\big|\sup_{t\in[0,T]} Y_t\big|^p &= \mathbb{E}\Big|\sup_{t\in[0,T]} \mathbb{E}^B\Big[\xi\exp\big(\int_t^T W(ds,B_s)\big)\,\big|\,\mathcal{F}_t^B\Big]\Big|^p \\ &\le \mathbb{E}\left|\sup_{t\in[0,T]}\mathbb{E}^B\left[|\xi|^p\exp\{p\sup_{t\in[0,T]}|V_t|\}\,\big|\,\mathcal{F}_t^B\right]\right|\end{aligned}$$

$$\leq \left(\frac{p}{p-1}\right)^p \|\xi\|_q^{1/q'} \Big[\mathbb{E} \exp\big(pp' \sup_{t\in[0,T]} |V_t|\big)\Big]^{1/p'}.$$

Denote by $\|V\|_T := \sup_{t\in[0,T]} V_t$. From Lemma 3.3 and Lemma 3.4 it follows for all $\lambda > 0$,

$$\mathbf{P}^W \left\{\big|\|V\|_T - \mathbb{E}^W \|V\|_T\big| \geq \lambda\right\} \leq 2\exp\left(-\frac{\lambda^2}{2\sigma_T^2}\right), \tag{3.10}$$

The above term σ_T^2 is defined and bounded by

$$\sigma_T^2 = \sup_{t\in[0,T]} \mathbb{E}^W[|V_t|^2] \leq C_{T,H_0} \sup_{u\in[0,T]} \rho(B_u) \prod_{i=1}^d |B_u^i|^{2H_i} \leq C_{T,H_0,d}\big(1 + \sup_{s\in[t,T]} \sum_{i=1}^d |B_s^i|\big)^\alpha, \tag{3.11}$$

by (2.6) and (2.13). From (3.10) we have for any $m > 0$,

$$\begin{aligned}
\mathbb{E}^W\Big[\exp\big(m\|V\|_T\big)\Big] &= \mathbb{E}^W\left[\exp\left\{m(\|V\|_T - \mathbb{E}^W[\|V\|_T])\right\}\right] \cdot \exp\left\{m\mathbb{E}^W[\|V\|_T]\right\} \\
&\leq m\exp\left\{m\mathbb{E}^W[\|V\|_T]\right\} \int_0^\infty e^{m\lambda} \mathbf{P}\left(\big|\|V\|_T - \mathbb{E}^W[\|V\|_T]\big| \geq \lambda\right) d\lambda \\
&\leq 2m\exp\left\{m\mathbb{E}^W[\|V\|_T]\right\} \int_0^\infty e^{m\lambda} \cdot e^{-\frac{\lambda^2}{2\sigma_T^2}} d\lambda \\
&\leq 2\sqrt{2\pi} m \cdot \sigma_T \exp\left\{mC_H T^{H_0} \nu(B) + \frac{m^2}{2}\sigma_T^2\right\}.
\end{aligned} \tag{3.12}$$

Since for all $x > 0$, we have $x \cdot e^{\frac{x^2}{2}} \leq 2e^{x^2}$. Therefore, taking account (3.6) and (3.11) it yields that, there is a constant $C_{T,H,m,d}$ which only depends on $T, m, d, H_i,\ i = 0, 1, \ldots, d$ such that:

$$\mathbb{E}^W\Big[\exp\big(m\|V\|_T\big)\Big] \leq 4\sqrt{2\pi}\exp\left\{C_{T,H,m,d}(1 + \sup_{u\in[0,T]} \sum_{i=1}^d |B_u^i|)^\alpha\right\}.$$

By the Fernique's theorem we obtain $\mathbb{E}\left[\mathbb{E}^W\left[\exp\big(m\|V\|_T\big)\right]\right] < \infty$, which implies $\mathbb{E}\big|\sup_{t\in[0,T]} Y_t\big|^p < \infty$. That is to say $Y = \{Y_t, t \in [0,T]\} \in \mathcal{S}_{\mathbb{F}}^p(0,T;\mathbb{R})$ for all $p \in [1,q)$. The convergence of $Y^{\varepsilon,\eta}$ to $Y = \{Y_t, t \in [0,T]\} \in \mathcal{S}_{\mathbb{F}}^p(0,T;\mathbb{R})$ for all $p \in [1,q)$ is routine and a little bit more complicated. But the essential estimates are the same as above. □

Now we want to study the second component of the solution pair of (3.1), i.e. $Z^{\varepsilon,\eta} = \{Z_s^{\varepsilon,\eta},\ s \in [0,T]\}$ defined by (3.2). Introduce the space

$$\mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d) := \Big\{\phi = (\phi_s)_{s\in[0,T]} : \mathbb{R}^d\text{-valued } \mathbb{F}\text{-progressively measurable and } \mathbb{E}\big[\int_0^T |\phi_s|^2 ds\big] < \infty\Big\}.$$

Theorem 3.6. *Denote*

$$\bar{H} = \max\{H_0, H_1, \cdots, H_d\} \quad \text{and} \quad \underline{H} = \min\{H_0, H_1, \cdots, H_d\}.$$

Suppose $\sum_{i=1}^d (2H_i - \beta_i) < 2$, *terminal condition* $\xi \in \mathbb{D}_B^{1,q}$ *is measurable w.r.t.* σ*-field* $\mathcal{F}_T^B$, *for* $q > \dfrac{2}{2\underline{H} - 1}$. *Then* $Z^{\varepsilon,\eta} \in \mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$ *and* $Z^{\varepsilon,\eta}$ *has a limit* $Z = \{Z_s,\ s \in [0,T]\}$ *in* $\mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$. *This limit can be written as*

$$Z_t \;=\; D_t^B Y_t = D_t^B \mathbb{E}\left[\xi \exp\big(\int_t^T W(dr, B_r)\big) \,\big|\, \mathcal{F}_t\right] \tag{3.13}$$

$$= \quad \mathbb{E}\Big[e^{\int_t^T W(d\tau,B_\tau)}D_t^B\xi + \int_t^T e^{\int_t^s W(d\tau,B_\tau)}Y_s(\nabla_x W)(ds,B_s)\big|\mathcal{F}_t\Big]\,. \tag{3.14}$$

Proof. Presumably we may apply D_r^B to $Y_t^{\varepsilon,\eta}$ given by (3.2). But it is inconvenient to deal with the Malliavin derivative of the conditional expectation. We find that it is more convenient to find $D_r^B Y_t^{\varepsilon,\eta}$ by working on (3.1) directly. In fact applying D_r^B to (3.1) yields

$$\begin{aligned} D_r^B Y_t^{\varepsilon,\eta} &= D_r^B\xi + \int_t^T \dot W_{\varepsilon,\eta}(s,B_s)D_r^B Y_s^{\varepsilon,\eta}ds \\ &\quad + \int_t^T Y_s^{\varepsilon,\eta}\nabla_x \dot W_{\varepsilon,\eta}(s,B_s)I_{[0,s]}(r)ds - \int_t^T D_r^B Z_s^{\varepsilon,\eta}dB_s. \end{aligned}$$

Denote $\tilde Y_t = D_r^B Y_t^{\varepsilon,\eta}$, $\tilde Z_t = D_r^B Z_t^{\varepsilon,\eta}$ (we fix r) and we can rewrite the above equation as

$$\begin{cases} d\tilde Y_t = -\dot W_{\varepsilon,\eta}(t,B_t)\tilde Y_t dt - Y_t^{\varepsilon,\eta}\nabla_x \dot W_{\varepsilon,\eta}(t,B_t)I_{[0,t]}(r)dt + \tilde Z_t dB_t, \quad r\le t\le T \\ \tilde Y_T = D_r^B\xi\,. \end{cases}$$

This is another linear backward stochastic differential equation, whose solution has the following explicit form.

$$\begin{aligned} D_r^B Y_t^{\varepsilon,\eta} &= \mathbb{E}\Big[e^{\int_t^T \dot W_{\varepsilon,\eta}(\tau,B_\tau)d\tau}D_r^B\xi \\ &\quad + \int_r^T e^{\int_t^s \dot W_{\varepsilon,\eta}(\tau,B_\tau)d\tau}Y_s^{\varepsilon,\eta}\nabla_x \dot W_{\varepsilon,\eta}(s,B_s)ds\big|\mathcal{F}_t\Big]\,, \quad t\ge r. \end{aligned}$$

By [1, Equation (2.11)] and [1, Equation (2.23)] we have

$$\begin{aligned} Z_t^{\varepsilon,\eta} &= D_t^B Y_t^{\varepsilon,\eta} = \mathbb{E}\Big[e^{\int_t^T \dot W_{\varepsilon,\eta}(\tau,B_\tau)d\tau}D_t^B\xi \\ &\quad + \int_t^T e^{\int_t^s \dot W_{\varepsilon,\eta}(\tau,B_\tau)d\tau}Y_s^{\varepsilon,\eta}\nabla_x \dot W_{\varepsilon,\eta}(s,B_s)ds\big|\mathcal{F}_t\Big] \\ &:= Z_t^{0,\varepsilon,\eta} + Z_t^{1,\varepsilon,\eta}. \end{aligned}$$

Assuming $D_r^B\xi$ is nice, we $Z_t^{0,\varepsilon,\eta}$ can be treated in exactly the same way as $Y_s^{\varepsilon,\eta}$.

We shall focus our effort on showing $Z_t^{1,\varepsilon,\eta}\in\mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$. Substituting $Y_t^{\varepsilon,\eta}$ given by (3.2) into the above expression, we have

$$\begin{aligned} Z_t^{1,\varepsilon,\eta} &= \mathbb{E}\Big[\int_t^T e^{\int_t^s \dot W_{\varepsilon,\eta}(\tau,B_\tau)d\tau}\mathbb{E}\Big[\xi\exp\Big(\int_s^T \dot W_{\varepsilon,\eta}(u,B_u)du\Big)\,\big|\,\mathcal{F}_s\Big]\nabla_x \dot W_{\varepsilon,\eta}(s,B_s)ds\big|\mathcal{F}_t\Big] \\ &= \int_t^T \mathbb{E}\Big[\xi\,\nabla_x \dot W_{\varepsilon,\eta}(s,B_s)\exp\Big(\int_t^T \dot W_{\varepsilon,\eta}(u,B_u)du\Big)\,\big|\,\mathcal{F}_t\Big]ds\,. \end{aligned}$$

Since it involves the term $\nabla_x\dot W_{\varepsilon,\eta}(s,B_s)$, this term is much more difficult to deal with. We shall fully explore the normality of the Gaussian field W. Moreover, there is a conditional expectation in the expression of $Z_t^{1,\varepsilon,\eta}$ which seems to stop us carrying out any meaningful computations. We shall get around this difficulty by introducing two independent standard Brownian motions B^1, B^2 which are identical copies of the Brownian motion B. Denote $\mathcal{F}_t^{B^1,B^2} = \sigma\{B_s^1, B_r^2,\ 0\le s,r\le$

$t; W(t,x),\ t \geq 0, x \in \mathbb{R}^d\}$ by the σ-algebra generated by sBm B^1, B^2 up to time instant t and $W(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$.
Note that, $\mathbb{E}^W$ only denotes the expectation with respect to W, which consider other random variables as "fixed constant". Then, we have

$$\begin{aligned}
\mathbb{E}^W \left[Z_t^{1,\varepsilon,\eta} \right]^2 &= \mathbb{E}^W \int_t^T \int_t^T \mathbb{E}\Big[\xi(B^1)\xi(B^2) \big(\nabla_x \dot{W}_{\varepsilon,\eta}(s_1, B_{s_1}^1) \big)^T \nabla_x \dot{W}_{\varepsilon,\eta}(s_2, B_{s_2}^2) \\
&\qquad \exp\Big(\int_t^T \Big[\dot{W}_{\varepsilon,\eta}^H(u, B_u^1) + \dot{W}_{\varepsilon,\eta}^H(u, B_u^2) \Big] du \Big) \,\Big|\, \mathcal{F}_t^{B^1,B^2} \Big] \Big|_{B^1=B^2=B} ds_1 ds_2 \\
&= \int_t^T \int_t^T \mathbb{E}\Big[\xi(B^1)\xi(B^2) I^{\varepsilon,\eta}(s_1,s_2) \,\Big|\, \mathcal{F}_t^{B^1,B^2} \Big] \Big|_{B^1=B^2=B} ds_1 ds_2 \,,
\end{aligned} \tag{3.15}$$

where $I^{\varepsilon,\eta}(s_1,s_2)$ is defined by

$$I^{\varepsilon,\eta}(s_1,s_2) := \sum_{i=1}^d \mathbb{E}^W \Big\{ \nabla_{x_i} \dot{W}_{\varepsilon,\eta}(s_1, B_{s_1}^1) \nabla_{x_i} \dot{W}_{\varepsilon,\eta}(s_2, B_{s_2}^2) \exp\Big(\int_t^T \Big[\dot{W}_{\varepsilon,\eta}(u, B_u^1) + \dot{W}_{\varepsilon,\eta}(u, B_u^2) \Big] du \Big) \Big\} . \tag{3.16}$$

Denote

$$\begin{cases} Z_{1,i}^{\varepsilon,\eta} = \nabla_{x_i} \dot{W}_{\varepsilon,\eta}(s_1, B_{s_1}^1)\,; \quad Z_{2,i}^{\varepsilon,\eta} = \nabla_{x_i} \dot{W}_{\varepsilon,\eta}(s_2, B_{s_2}^2)\,; \\ Y^{\varepsilon,\eta} = \int_t^T \Big[\dot{W}_{\varepsilon,\eta}(u, B_u^1) + \dot{W}_{\varepsilon,\eta}(u, B_u^2) \Big] du \,. \end{cases}$$

Then

$$I^{\varepsilon,\eta}(s_1,s_2) = \sum_{i=1}^d \mathbb{E}^W \Big\{ Z_{1,i}^{\varepsilon,\eta} Z_{2,i}^{\varepsilon,\eta} \exp\left(Y^{\varepsilon,\eta}\right) \Big\} .$$

As random variables of W (namely for fixed B^1, B^2), $Z_{1,i}^{\varepsilon,\eta}, Z_{2,i}^{\varepsilon,\eta}, Y^{\varepsilon,\eta}$ are jointly Gaussians, we shall use the following lemma to compute the above expectations.

Lemma 3.7. *Assume that X_1, X_2, Y are jointly mean zero Gaussians. Then*

$$\mathbb{E}\left[X_1 X_2 \exp(Y) \right] = \left(\mathbb{E}(X_1 Y) + \mathbb{E}(X_2 Y) + \mathbb{E}(X_1 X_2) \right) \exp\left[\frac{1}{2} \mathbb{E}(Y^2) \right] . \tag{3.17}$$

Proof. For any constants $s, t \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E} \exp(Y + sX_1 + tX_2) = \exp\left\{ \frac{1}{2} \mathbb{E}(Y + sX_1 + tX_2)^2 \right\} \\
&\quad = \exp\left[\left\{ \mathbb{E}(Y^2) + s^2 \mathbb{E}(X_1^2) + t^2 \mathbb{E}(X_2^2) + 2s\mathbb{E}(X_1 Y) + 2t\mathbb{E}(X_2 Y) + 2st\mathbb{E}(X_1 X_2) \right\} \Big/ 2 \right] .
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\left[X_1 X_2 \exp(Y) \right] &= \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \exp\left\{ \frac{1}{2} \mathbb{E}(Y + sX_1 + tX_2)^2 \right\} \\
&= \left(\mathbb{E}(X_1 Y) + \mathbb{E}(X_2 Y) + \mathbb{E}(X_1 X_2) \right) \exp\left[\frac{1}{2} \mathbb{E}(Y^2) \right] .
\end{aligned}$$

This is (3.17). $\square$

Applying the above Lemma 3.7 to evaluate $I^{\varepsilon,\eta}(s_1,s_2)$ yields

$$\begin{aligned}\mathbb{E}^W\left[Z_t^{1,\varepsilon,\eta}\right]^2 =& \int_t^T\int_t^T \mathbb{E}\Bigg[\xi(B^1)\xi(B^2)\left(\sum_{i=1}^d\left(A_{1,i}^{\varepsilon,\eta}+A_{2,i}^{\varepsilon,\eta}+A_{3,i}^{\varepsilon,\eta}\right)\right)\\ &\exp\left(\frac{A_4^{\varepsilon,\eta}}{2}\right) \mid \mathcal{F}_t^{B^1,B^2}\Bigg]\Bigg|_{B^1=B^2=B} ds_1ds_2\\ =&\sum_{j=1}^3\sum_{i=1}^d I_{j,i,t}^{\varepsilon,\eta}\,,\end{aligned}$$

where

$$\begin{cases} A_{1,i}^{\varepsilon,\eta} :=\mathbb{E}^W(Z_{1,i}^{\varepsilon,\eta}Z_{2,i}^{\varepsilon,\eta}), \quad A_{2,i}^{\varepsilon,\eta} := \mathbb{E}^W(Z_{1,i}^{\varepsilon,\eta}Y^{\varepsilon,\eta}),\\ A_{3,i}^{\varepsilon,\eta} :=\mathbb{E}^W(Z_{2,i}^{\varepsilon,\eta}Y^{\varepsilon,\eta}), \quad A_4^{\varepsilon,\eta} := \mathbb{E}^W((Y^{\varepsilon,\eta})^2)\end{cases} \tag{3.18}$$

and

$$I_{j,i,t}^{\varepsilon,\eta} := \int_t^T\int_t^T \mathbb{E}\left[\xi(B^1)\xi(B^2)A_{j,i}^{\varepsilon,\eta}\exp\left(\frac{A_4^{\varepsilon,\eta}}{2}\right) \mid \mathcal{F}_t^{B^1,B^2}\right]\Bigg|_{B^1=B^2=B} ds_1ds_2\,. \tag{3.19}$$

Let us consider $I_{1,i,t}^{\varepsilon,\eta}$ in details. The other terms can be treated in similar way. First, let us compute

$$\begin{aligned} A_{1,i}^{\varepsilon,\eta} =&\mathbb{E}^W\left[\nabla_{x_i}\dot W_{\varepsilon,\eta}(s_1,B_{s_1}^1)\nabla_{x_i}\dot W_{\varepsilon,\eta}(s_2,B_{s_2}^2)\right]\\ =&\alpha_{H_0}\int_0^{s_1}\int_0^{s_2}\varphi_\eta(s_1-r_1)\varphi_\eta(s_2-r_2)|r_2-r_1|^{2H_0-2}dr_1dr_2\\ &\int_{\mathbb{R}^{2d}}\nabla_{x_i}p_\varepsilon(B_{s_1}^1-w)\nabla_{x_i}p_\varepsilon(B_{s_2}^2-z)\rho(w)\rho(z)\prod_{i=1}^d R_{H_i}(w_i,z_i)dwdz\\ =&J_1^\eta(s_1,s_2)J_2^\varepsilon(s_1,s_2)\,,\end{aligned} \tag{3.20}$$

where $J_1^\eta(s_1,s_2)$ and $J_2^\varepsilon(s_1,s_2)$ are defined as follows.

$$\begin{cases} J_1^\eta(s_1,s_2) := \displaystyle\int_0^{s_1}\int_0^{s_2}\varphi_\eta(s_1-r_1)\varphi_\eta(s_2-r_2)|r_2-r_1|^{2H_0-2}dr_1dr_2\,,\\ J_2^\varepsilon(s_1,s_2) := \displaystyle\int_{\mathbb{R}^{2d}}\nabla_{w_i}p_\varepsilon(B_{s_1}^1-w)\nabla_{z_i}p_\varepsilon(B_{s_2}^2-z)\rho(w)\rho(z)\prod_{i=1}^d R_{H_i}(w_i,z_i)dwdz\\ \qquad\qquad = \displaystyle\int_{\mathbb{R}^{2d}}\nabla_{w_i}p_\varepsilon(B_{s_1}^1-w)\nabla_{z_i}p_\varepsilon(B_{s_2}^2-z)q(w,z)dwdz\,,\end{cases}$$

where we recall that $q(x,y)$ is the spatial covariance of noise given by (2.1). Notice that $J_2^\varepsilon(s_1,s_2)$ is independent of ε. It is elementary to see that

$$J_1^\eta(s_1,s_2) := \int_0^{s_1}\int_0^{s_2}\varphi_\eta(s_1-r_1)\varphi_\eta(s_2-r_2)|r_2-r_1|^{2H_0-2}dr_1dr_2 \to |s_2-s_1|^{2H_0-2} \quad \text{as } \varepsilon,\eta\to 0\,. \tag{3.21}$$

Moreover, for any $p<1/(2-2H_0)$ and $1/p+1/q=1$, by Hölder's inequality we have

$$\begin{aligned}|J_1^\eta(s_1,s_2)|^p \le& \int_0^{s_1}\int_0^{s_2}|r_2-r_1|^{(2H_0-2)p}dr_1dr_2\\ &\times\left(\int_0^{s_1}\int_0^{s_2}\varphi_\eta(s_1-r_1)\varphi_\eta(s_2-r_2)dr_1dr_2\right)^{p/q}\,.\end{aligned}$$

The above second factor is less than or equal to 1. Making substitutions $s_1 - r_1 \to r_1'\eta$ and $s_2 - r_2 \to r_2'\eta$ we have

$$\sup_{\eta\in(0,1]} |J_1^\eta(s_1,s_2)|^p \le \sup_{\eta\in(0,1]} \int_0^1\int_0^1 |s_2 - s_1 + \eta(r_1' - r_2')|^{(2H_0-2)p} dr_1' dr_2' < \infty\,. \tag{3.22}$$

Now we consider J_2^ε. Integration by parts yields

$$\begin{aligned}
J_2^\varepsilon(s_1,s_2) &= \int_{\mathbb{R}^{2d}} p_\varepsilon(B^1_{s_1} - w)p_\varepsilon(B^2_{s_2} - z)\nabla_{w_i}\nabla_{z_i} q(w,z)dwdz\\
&= \int_{\mathbb{R}^{2d}} p_\varepsilon(B^1_{s_1} - w)p_\varepsilon(B^2_{s_2} - z)\prod_{j\neq i}^d R_{H_j}(w_j,z_j)\Big[\nabla_{w_i}\rho(w)\nabla_{z_i}\rho(z)R_{H_i}(w_i,z_i)\\
&\qquad + \nabla_{w_i}\rho(w)\rho(z)\Big(H_i|z_i|^{2H_i-1}\mathrm{sign}(z_i) - H_i|w_i - z_i|^{2H_i-1}\mathrm{sign}(w_i - z_i)\Big)\\
&\qquad + \rho(w)\nabla_{z_i}\rho(z)\Big(H_i|w_i|^{2H_i-1}\mathrm{sign}(z_i) - H_i|w_i - z_i|^{2H_i-1}\mathrm{sign}(w_i - z_i)\Big)\\
&\qquad + \rho(z)\rho(w)\alpha_{H_i}|w_i - z_i|^{2H_i-2}\Big]dw_i dz_i\\
&= J_{21}^\varepsilon(s_1,s_2) + J_{22}^\varepsilon(s_1,s_2)\,,
\end{aligned}$$

where

$$\begin{aligned}
J_{21}^\varepsilon(s_1,s_2) :&= \mathbb{E}^{X,X'}\Big[\prod_{j\neq i}^d R_{H_j}(B^{1,j}_{s_1} + \varepsilon X_j, B^{2,j}_{s_2} + \varepsilon X_j')\\
&\quad\times\Big(\nabla_{x_i}\rho(B^1_{s_1} + \varepsilon X)\nabla_{x_i}\rho(B^2_{s_2} + \varepsilon X')R_{H_i}(B^{1,i}_{s_1} + \varepsilon X_i, B^{2,i}_{s_2} + \varepsilon X_i')\\
&\qquad + \nabla_{x_i}\rho(B^1_{s_1} + \varepsilon X)\rho(B^2_{s_2} + \varepsilon X')\Big(H_i|B^{2,i}_{s_2} + \varepsilon X_i'|^{2H_i-1}\mathrm{sign}(B^{2,i}_{s_2} + \varepsilon X_i')\\
&\qquad\quad - H_i|B^{1,i}_{s_1} + \varepsilon X_i - B^{2,i}_{s_2} - \varepsilon X_i'|^{2H_i-1}\mathrm{sign}(B^{1,i}_{s_1} + \varepsilon X_i - B^{2,i}_{s_2} - \varepsilon X_i')\Big)\\
&\qquad + \rho(B^1_{s_1} + \varepsilon X)\nabla_{x_i}\rho'(B^2_{s_2} + \varepsilon X')\Big(H_i|B^{1,i}_{s_1} + \varepsilon X_i|^{2H_i-1}\mathrm{sign}(B^{2,i}_{s_2} + \varepsilon X_i')\\
&\qquad\quad - H_i|B^{1,i}_{s_1} + \varepsilon X_i - B^{2,i}_{s_2} - \varepsilon X_i'|^{2H_i-1}\mathrm{sign}(B^{1,i}_{s_1} + \varepsilon X_i - B^{2,i}_{s_2} - \varepsilon X_i')\Big)\Big)\Big]
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
J_{22}^\varepsilon(s_1,s_2) := \alpha_{H_i}\mathbb{E}^{X,X'}\Big[&\prod_{j\neq i}^d R_{H_j}(B^{1,j}_{s_1} + \varepsilon X_j, B^{2,j}_{s_2} + \varepsilon X_j')\\
&\times\rho(B^1_{s_1} + \varepsilon X)\rho(B^2_{s_2} + \varepsilon X')|B^{1,i}_{s_1} + \varepsilon X_i - B^{2,i}_{s_2} - \varepsilon X_i'|^{2H_i-2}\Big]
\end{aligned}$$

with $X = (X_1,\cdots,X_d), X' = (X_1',\cdots,X_d')$ being independent standard Gaussian random variables, which are also independent of B^1, B^2. From the definition, we can consider $J_{21}^\varepsilon(s_1,s_2)$ as a random variable of B^1 and B^2. From the above expression it is easy to see that

$$\sup_{\varepsilon\in(0,1]} |J_{21}^\varepsilon(s_1,s_2)| \le C\left(1 + |B^1_{s_1}|^m + |B^2_{s_2}|^m\right)\,, \tag{3.24}$$

for some positive constants C and m.

As concerns for $J_{22}^{\varepsilon}(s_1,s_2)$, we can find two constants p,q satisfying $p<1/(2-2H_0)$ and $1/p+1/q=1$ such that by Hölder's inequality,

$$J_{22}^{\varepsilon}(s_1,s_2)\le\alpha_{H_i}\left\{\mathbb{E}^{X,X'}\left[\prod_{j\neq i}^{d}\left(R_{H_j}(B_{s_1}^{1,j}+\varepsilon X_j,B_{s_2}^{2,j}+\varepsilon X'_j)\right)^q\rho^q(B_{s_1}^1+\varepsilon X)\rho^q(B_{s_2}^2+\varepsilon X')\right]\right\}^{1/q}$$
$$\times\left\{\mathbb{E}^{X,X'}\left[|B_{s_1}^{1,i}+\varepsilon X_i-B_{s_2}^{2,i}-\varepsilon X'_i|^{(2H_i-2)p}\Big)\right]\right\}^{1/p}.$$

By the Lemma A.1 of [1] the above second expectation is bounded by $\left|B_{s_1}^{1,i}-B_{s_2}^{2,i}\right|^{(2H_i-2)p}$. Thus, by the assumption on ρ and by the definition of R_H we have

$$\sup_{\varepsilon\in(0,1]}|J_{22}^{\varepsilon}(s_1,s_2)|\le C\left(1+|B_{s_1}^1|^m+|B_{s_2}^2|^m\right)\cdot\left|B_{s_1}^{1,i}-B_{s_2}^{2,i}\right|^{(2H_i-2)p}. \tag{3.25}$$

Moreover, from (3.20), (3.21) and (3.23) we have

$$\begin{aligned}\lim_{\eta,\varepsilon\to0}A_{1,i}^{\varepsilon,\eta}=&\lim_{\eta,\varepsilon\to0}\mathbb{E}^W\left[\nabla_x\dot W_{\varepsilon,\eta}(s_1,B_{s_1}^1)\nabla_x\dot W_{\varepsilon,\eta}(s_2,B_{s_2}^2)\right]\\
=&\alpha_{H_0}|s_2-s_1|^{2H_0-2}\prod_{j\neq i}^{d}R_{H_j}(B_{s_1}^{1,j},B_{s_2}^{2,j})\Big[\nabla_{x_i}\rho(B_{s_1}^1)\nabla_{x_i}\rho(B_{s_2}^2)R_{H_i}(B_{s_1}^{1,i},B_{s_2}^{2,i})\\
&+\nabla_{x_i}\rho(B_{s_1}^1)\rho(B_{s_2}^2)\Big(2H_i|B_{s_2}^{2,i}|^{2H_i-1}\mathrm{sign}(B_{s_2}^{2,i})-2H_i|B_{s_1}^{1,i}-B_{s_2}^{2,i}|^{2H_i-1}\mathrm{sign}(B_{s_1}^{1,i}-B_{s_2}^{2,i})\Big)\\
&+\rho(B_{s_1}^1)\nabla_{x_i}\rho(B_{s_2}^2)\Big(2H_i|B_{s_1}^{1,i}|^{2H_i-1}\mathrm{sign}(B_{s_2}^{2,i})+2H_i|B_{s_1}^{1,i}-B_{s_2}^{2,i}|^{2H_i-1}\mathrm{sign}(B_{s_1}^{1,i}-B_{s_2}^{2,i})\Big)\\
&+\alpha_{H_i}\rho(B_{s_1}^1)\rho(B_{s_2}^2)|B_{s_1}^{1,i}-B_{s_2}^{2,i}|^{2H_i-2}\Big].\end{aligned}$$

Using the spatial covariance $q(x,y)$, we can write

$$\lim_{\eta,\varepsilon\to0}A_{1,i}^{\varepsilon,\eta}=\alpha_{H_0}|s_2-s_1|^{2H_0-2}\frac{\partial^2}{\partial x_i\partial y_i}q(x,y)\Big|_{x=B_{s_1}^1,y=B_{s_2}^2}. \tag{3.26}$$

Analogously to (3.20), (3.22), (3.24) and (3.25), we can show the boundedness of other $A_{ji}^{\varepsilon,\eta}$'s uniformly w.r.t. ε,η. So we can apply the dominated convergence theorem below. In particular, we have

$$\lim_{\eta,\varepsilon\to0}A_{2,i}^{\varepsilon,\eta}=\alpha_{H_0}\int_t^T|s_1-u|^{2H_0-2}\left[\frac{\partial}{\partial x_i}q(x,y)\Big|_{x=B_{s_1}^1,y=B_u^2}+\frac{\partial}{\partial x_i}q(x,y)\Big|_{x=B_{s_1}^1,y=B_u^1}\right]du.$$

$$\lim_{\eta,\varepsilon\to0}A_{3,i}^{\varepsilon,\eta}=\alpha_{H_0}\int_t^T|s_2-u|^{2H_0-2}\left[\frac{\partial}{\partial x_i}q(x,y)\Big|_{x=B_{s_1}^2,y=B_u^2}+\frac{\partial}{\partial x_i}q(x,y)\Big|_{x=B_{s_1}^2,y=B_u^1}\right]du.$$

As for $A_4^{\varepsilon,\eta}$, we have by definition of $Y^{\varepsilon,\eta}$

$$\begin{aligned}A_4^{\varepsilon,\eta}&=\mathbb{E}^W\left[\int_t^T\int_t^T\Big(\dot W_{\varepsilon,\eta}(u,B_u^1)+\dot W_{\varepsilon,\eta}(u,B_u^2)\Big)\Big(\dot W_{\varepsilon,\eta}(v,B_v^1)+\dot W_{\varepsilon,\eta}(v,B_v^2)\Big)dudv\right]\\
&=\sum_{i=1}^{3}A_{4,i}^{\varepsilon,\eta},\end{aligned}$$

where

$$
\begin{aligned}
A_{41}^{\varepsilon,\eta} :=& \mathbb{E}^W\left[\int_t^T\int_t^T \dot W_{\varepsilon,\eta}(u,B_u^1)\dot W_{\varepsilon,\eta}(v,B_v^1)dudv\right],\\
A_{42}^{\varepsilon,\eta} :=& 2\mathbb{E}^W\left[\int_t^T\int_t^T \dot W_{\varepsilon,\eta}(u,B_u^2)\dot W_{\varepsilon,\eta}(v,B_v^1)dudv\right],\\
A_{43}^{\varepsilon,\eta} :=& \mathbb{E}^W\left[\int_t^T\int_t^T \dot W_{\varepsilon,\eta}(u,B_u^2)\dot W_{\varepsilon,\eta}(v,B_v^2)dudv\right].
\end{aligned}
$$

Similar to the proof of Proposition 2.1 we can show that $A_{4,i}^{\varepsilon,\eta}$, $i=1,2,3$ can be bounded by a bound analogous to (2.10). Thus, we have

$$
\lim_{\eta,\varepsilon\to 0} A_4^{\varepsilon,\eta} = \int_t^T\int_t^T \alpha_{H_0}|u-v|^{2H_0-2}\left[q(B_u^1,B_v^1)+2q(B_u^1,B_v^2)+q(B_u^2,B_v^2)\right]dudv\,. \tag{3.27}
$$

Combining the above with (3.22)-(3.27) enables us to apply the dominated convergence theorem to obtain

$$
\begin{aligned}
\lim_{\varepsilon,\eta\to 0} I_{1,i,t}^{\varepsilon,\eta} =& \alpha_{H_0}\int_t^T\int_t^T \mathbb{E}\big[\xi(B^1)\xi(B^2)|s_2-s_1|^{2H_0-2}(\partial_{i,i}q)(B_{s_1}^1,B_{s_2}^2)\\
&\Upsilon(t,T,B^1,B^2)\big|\mathcal{F}_t^{B^1,B^2}\big]\big|_{B^1=B^2=B}ds_1ds_2\,,
\end{aligned} \tag{3.28}
$$

where $\partial_{i,i}q(x,y)=\frac{\partial^2}{\partial x_i\partial y_i}q(x,y)$ and

$$
\Upsilon := \exp\left\{\int_t^T\int_t^T \alpha_{H_0}|u-v|^{2H_0-2}\big[q(B_u^1,B_v^1)+2q(B_u^1,B_v^2)+q(B_u^2,B_v^2)\big]dudv\right\}. \tag{3.29}
$$

In a similar way we can show the existence of the limits of $I_{2,i,t}^{\varepsilon,\eta}$, $I_{3,i,t}^{\varepsilon,\eta}$, and we can further identity these limits.

Thus, we can easily deduce that $\mathbb{E}\displaystyle\int_0^T \left|Z_t^{\varepsilon,\eta}\right|^2dt$ exists. In order to take the limit, it would be sufficient to show that, along a subsequence, $Z^{\varepsilon,\eta}$ converges to some $Z\in\mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$. But this is guaranteed by the fact that $\mathbb{E}\displaystyle\int_0^T \left|Z_t^{\varepsilon,\eta}\right|^2dt$ is bounded w.r.t. $\varepsilon,\eta>0$. Indeed, as before we can also show that $Z^{\varepsilon,\eta}$ is a Cauchy sequence in $\mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$, whose limit is denoted by $Z=\{Z_t,\,t\in[0,T]\}$. We can also write Z as (3.13) and (3.14) (whose justification is given through our above approximation). □

After we have found the limit Y (Theorem 3.5) and the limit Z (Theorem 3.6), we want to show that they are the solution to (1.2). To this end we shall take limit in equation (3.1). Since we have shown the convergence of $Y_t^{\varepsilon,\eta}$ and $Z_t^{\varepsilon,\eta}$ as in Theorems 3.5 and Theorems 3.6, we only need to discuss the limit of $\int_t^T Y_s^{\varepsilon,\eta}\dot W_{\varepsilon,\eta}(s,B_s)ds$. Before discussing this limit we give the definition of a (Stratonovich) stochastic integral with respect to $\int_0^t F_sW(ds,B_s)$.

Definition 3.8. *Let be given a random field $F=\{F_t,t\ge 0\}$ such that $\int_0^T|F_s|ds<\infty$ almost surely, for all $T>0$. Then the Stratonovich integral $\int_t^T F_sW(ds,B_s)$ is defined as the following limit in probability if it exists (compared this with Proposition 2.1 when $F_s\equiv 1$):*

$$
\int_t^T F_s\dot W_{\varepsilon,\eta}(s,B_s)ds.
$$

Theorem 3.9. *Suppose* $\sum_{i=1}^d (2H_i - \beta_i) < 2$ *and* $\xi \in L^q(\Omega)$ *for* $q > \dfrac{2}{2\underline{H}-1}$, *where* $\underline{H} = \min\{H_0, \ldots, H_d\}$. *Then for any* $t \in [0,T]$, *we have*

$$\int_t^T Y_s^{\varepsilon,\eta} \dot{W}_{\varepsilon,\eta}(s, B_s) ds \to \int_t^T Y_s W(ds, B_s)$$

in L^2 *sense, as* $\varepsilon, \eta \downarrow 0$.

Proof. By (3.1), Lemma 3.1 and Theorem 3.6, we know

$$\int_t^T Y_s^{\varepsilon,\eta} \dot{W}_{\varepsilon,\eta}(s, B_s) ds = Y_t^{\varepsilon,\eta} - \xi + \int_t^T Z_s^{\varepsilon,\eta} dB_s$$

converges in L^2 sense to the random field $A_t := Y_t - \xi + \int_t^T Z_s dB_s$ as ε, η tend to zero. Hence, if

$$B_t^{\varepsilon,\eta} := \int_t^T \big(Y_s^{\varepsilon,\eta} - Y_s\big) \dot{W}_{\varepsilon,\eta}(s, B_s) ds \to 0 \tag{3.30}$$

in $L^2(\Omega)$, then we have $\int_t^T Y_s \dot{W}_{\varepsilon,\eta}(s, B_s) ds = \int_t^T Y_s^{\varepsilon,\eta} \dot{W}_{\varepsilon,\eta}(s, B_s) ds - B_t^{\varepsilon,\eta}$ will converge to A in $L^2(\Omega)$. Previously, we have proved A is well-defined, and then Y_s will be Stratonovich integrable. Thus, by Definition 3.8, we directly have

$$\int_t^T Y_s W(ds, B_s) = \lim_{\varepsilon,\eta \downarrow 0} \int_t^T Y_s \dot{W}_{\varepsilon,\eta}(s, B_s) ds = A,$$

i.e., the equation (1.2) is satisfied.

In the remaining part of the proof, we shall show (3.30). First we note that, recalling the definition of $\dot{W}_{\varepsilon,\eta}$ in (2.3) we have

$$\int_t^T \dot{W}_{\varepsilon,\eta}(s, B_s) ds = \int_t^T \int_{\mathbb{R}^d} \int_t^T \varphi_\eta(s-r) p_\varepsilon(B_s - y)\, W(dr, y) dy ds\,. \tag{3.31}$$

Recall $F \cdot W(\phi) = \delta(F\phi) + \langle D^W F, \phi \rangle_{\mathcal{H}}$. Then, we obtaion

$$\begin{aligned}\big(Y_s^{\varepsilon,\eta} - Y_s\big) \dot{W}_{\varepsilon,\eta}(s, B_s) &= \big(Y_s^{\varepsilon,\eta} - Y_s\big) \int_{\mathbb{R}^d} \int_t^s \varphi_\eta(s-r) p_\varepsilon(B_s - y) W(dr, y) dy \\ &= \int_t^s \int_{\mathbb{R}^d} \big(Y_s^{\varepsilon,\eta} - Y_s\big) \varphi_\eta(s-r) p_\varepsilon(B_s - y)\; W(\delta r, y) dy \\ &\quad + \big\langle D^W(Y_s^{\varepsilon,\eta} - Y_s), \varphi_\eta(s - \cdot) p_\varepsilon(B_s - \cdot) \big\rangle_{\mathcal{H}}.\end{aligned}$$

Hence, by stochastic Fubini's Theorem, $B_t^{\varepsilon,\eta}$ can be written as

$$\begin{aligned}B_t^{\varepsilon,\eta} &= \int_{\mathbb{R}^d} \int_t^T \int_t^T \big(Y_s^{\varepsilon,\eta} - Y_s\big) \varphi_\eta(s-r) p_\varepsilon(B_s - y) ds\, W(\delta r, y) dy \\ &\quad + \int_t^T \big\langle D^W(Y_s^{\varepsilon,\eta} - Y_s), \varphi_\eta(s - \cdot) p_\varepsilon(B_s - \cdot) \big\rangle_{\mathcal{H}} ds \\ &:= B_t^{\varepsilon,\eta,1} + B_t^{\varepsilon,\eta,2}.\end{aligned} \tag{3.32}$$

For the term $B_t^{\varepsilon,\eta,1}$, we define

$$\phi_{r,y}^{\varepsilon,\eta} = \int_t^T \big(Y_s^{\varepsilon,\eta} - Y_s\big)\varphi_\eta(s-r)p_\varepsilon(B_s - y)ds,$$

and with the help of L^2 estimate for Skorokhod type stochastic integral, it yields:

$$\mathbb{E}\big[\big(B_t^{\varepsilon,\eta,1}\big)^2\big] \le \mathbb{E}\big[\big\|\phi^{\varepsilon,\eta}\big\|_{\mathcal{H}}^2\big] + \mathbb{E}\big[\big\|D^W\phi^{\varepsilon,\eta}\big\|_{\mathcal{H}\otimes\mathcal{H}}^2\big]. \tag{3.33}$$

The above first term can be estimated as follows:

$$\begin{aligned}\mathbb{E}\big[\big\|\phi^{\varepsilon,\eta}\big\|_{\mathcal{H}}^2\big] = \mathbb{E}\Big[\int_{[t,T]^2}\big(Y_s^{\varepsilon,\eta} - Y_s\big)\big(Y_r^{\varepsilon,\eta} - Y_r\big)\\ \times\big\langle\varphi_\eta(s-\cdot)p_\varepsilon(B_s-\cdot),\varphi_\eta(r-\cdot)p_\varepsilon(B_r-\cdot)\big\rangle_{\mathcal{H}}dsdr\Big].\end{aligned} \tag{3.34}$$

Recalling the definition in (2.2), and combining with the proof in Proposition 2.1 (refer to (2.8) and (2.10)) we deduce that

$$\begin{aligned}&\big\langle\varphi_\eta(s-\cdot)p_\varepsilon(B_s-\cdot),\varphi_\eta(r-\cdot)p_\varepsilon(B_r-\cdot)\big\rangle_{\mathcal{H}}\\ &=\alpha_{H_0}\int_{[t,T]^2}\int_{\mathbb{R}^{2d}}\varphi_\eta(s-u)\varphi_\eta(r-v)p_\varepsilon(B_s-y)p_\varepsilon(B_r-z)\\ &\qquad\times|u-v|^{2H_0-2}\rho(y)\rho(z)\prod_{i=1}^d R_{H_i}(y_i,z_i)dudvdydz\\ &\le C|r-s|^{2H_0-2}\rho(B_s)\rho(B_r)\prod_{i=1}^d R_{H_i}(B_s^i,B_r^i).\end{aligned} \tag{3.35}$$

Substituting this into (3.34) and with the help of (2.9) we have

$$\begin{aligned}\mathbb{E}\big[\big\|\phi^{\varepsilon,\eta}\big\|_{\mathcal{H}}^2\big] \le& C\,\mathbb{E}\Big[\int_{[t,T]^2}\big(Y_s^{\varepsilon,\eta}-Y_s\big)\big(Y_r^{\varepsilon,\eta}-Y_r\big)|r-s|^{2H_0-2}\rho(B_s)\rho(B_r)\prod_{i=1}^d R_{H_i}(B_s^i,B_r^i)\ dsdr\Big]\\ \le& C\,\mathbb{E}\Big[\sup_{s\in[0,T]}\big(Y_s^{\varepsilon,\eta}-Y_s\big)^4\Big]^{1/2}\\ &\times\mathbb{E}\Big[\Big(\int_{[t,T]^2}|r-s|^{2H_0-2}\prod_{i=1}^d\Big[(1+|B_s^i|)^{2H_i-\beta_i}(1+|B_r^i|)^{2H_i-\beta_i}dsdr\Big)^2\Big]^{1/2}.\end{aligned} \tag{3.36}$$

Thanks to Theorem 3.5, Proposition 2.2 and the dominated convergence theorem, we see that $\mathbb{E}\big[\big\|\phi^{\varepsilon,\eta}\big\|_{\mathcal{H}}^2\big]$ converges to zero as $\varepsilon,\ \eta$ tend to zero.

Secondly, we have to deal with $\mathbb{E}\big[\big\|D^W\phi^{\varepsilon,\eta}\big\|_{\mathcal{H}\otimes\mathcal{H}}^2\big]$, the second term in (3.33). By Malliavin calculus and (3.31) we have

$$\begin{aligned}D^W Y_t^{\varepsilon,\eta} &= \mathbb{E}\big[\xi\, D^W\exp\big(\int_t^T \dot W_{\varepsilon,\eta}(s,B_s)ds\big)\big|\mathcal{F}_t\big]\\ &= \mathbb{E}\big[\xi\ \exp\big(\int_t^T\dot W_{\varepsilon,\eta}(s,B_s)ds\big)\int_t^T\varphi_\eta(s-\cdot)p_\varepsilon(B_s-\cdot)ds\big|\mathcal{F}_t\big]\,.\end{aligned} \tag{3.37}$$

We denote $\mathcal{F}_{t,s}^{B^1,B^2} = \sigma(B_u^1, B_v^2, 0 \le u \le t, 0 \le v \le s; W(t,x), t \ge 0, x \in \mathbb{R}^d)$ the σ-algebra generated by B^1, B^2 and W. Recalling the definition (2.3) we know that, for random variable W (namely for fixed B), $\int_t^T \dot{W}_{\varepsilon,\eta}(s,B_s)ds$ is Gaussians. Then Proposition 2.1 and (3.35) tell us

$$
\begin{aligned}
&\mathbb{E}^W\langle D^W Y_t^{\varepsilon,\eta}, D^W Y_t^{\varepsilon',\eta'}\rangle_{\mathcal{H}} \\
&\quad = \mathbb{E}^W\mathbb{E}\Big[\xi(B^1)\xi(B^2)\,\exp\big(\int_t^T \dot{W}_{\varepsilon,\eta}(s,B_s^1)ds + \int_t^T \dot{W}_{\varepsilon',\eta'}(s,B_s^2)ds\big) \\
&\qquad \times \int_{[t,T]^2}\langle\varphi_\eta(s-\cdot)p_\varepsilon(B_s^1-\cdot), \varphi_{\eta'}(r-\cdot)p_{\varepsilon'}(B_r^2-\cdot)\rangle_{\mathcal{H}}dsdr\Big|\mathcal{F}_t^{B^1,B^2}\Big]\Big|_{B^1=B^2=B} \\
&\quad \le \alpha_{H_0}\mathbb{E}\Bigg[\xi(B^1)\xi(B^2)\,\mathbb{E}^W\Big[\exp\big(\int_t^T \dot{W}_{\varepsilon,\eta}(s,B_s^1)ds + \int_t^T \dot{W}_{\varepsilon',\eta'}(s,B_s^2)ds\big)\Big] \qquad (3.38)\\
&\qquad \times \int_{[t,T]^2}|s-r|^{2H_0-2}\rho(B_s^1)\rho(B_r^2)\prod_{i=1}^d R_{H_i}(B_s^{1,i}, B_r^{2,i})dsdr\Big|\mathcal{F}_t^{B^1,B^2}\Bigg]\Bigg|_{B^1=B^2=B} \\
&\quad = \alpha_{H_0}\mathbb{E}\Big[\xi(B^1)\xi(B^2)\,\exp\Big(\sum_{j,k=1}^2\int_t^T\int_t^T |s-r|^{2H_0-2}\rho(B_s^j)\rho(B_r^k)\prod_{i=1}^d R_{H_i}(B_s^{j,i}, B_r^{k,i})dsdr\Big) \\
&\qquad \times \int_{[t,T]^2}|s-r|^{2H_0-2}\rho(B_s^1)\rho(B_r^2)\prod_{i=1}^d R_{H_i}(B_s^{1,i}, B_r^{2,i})dsdr\Big|\mathcal{F}_t^{B^1,B^2}\Big]\Bigg|_{B^1=B^2=B}.
\end{aligned}
$$

We have to prove the integrability of (3.38). Put a, b be two positive constants such that $1/a + 1/b = 1$ and $2a < q$. With the help of Proposition 2.2 and Hölder's inequality,

$$
\begin{aligned}
&\mathbb{E}\Big[\xi(B^1)\xi(B^2)\,\exp\big(\sum_{j,k=1}^2\int_t^T\int_t^T |s-r|^{2H_0-2}\prod_{i=1}^d R_{H_i}(B_s^{j,i}, B_r^{k,i})\rho(B_s^{j,i})\rho(B_r^{k,i})dsdr\big) \\
&\qquad \times \int_{[t,T]^2}|s-r|^{2H_0-2}\rho(B_s^1)\rho(B_r^2)\prod_{i=1}^d R_{H_i}(B_s^{1,i}, B_r^{2,i})dsdr\Big] \\
&\le \|\xi\|_q^2\Bigg(\mathbb{E}\Big[\exp\big(2b\sum_{j,k=1}^2\int_t^T\int_t^T |s-r|^{2H_0-2}\prod_{i=1}^d R_{H_i}(B_s^{j,i}, B_r^{k,i})\rho(B_s^{j,i})\rho(B_r^{k,i})dsdr\big)\Big]\Bigg)^{1/2b} \qquad (3.39)\\
&\qquad \times \Bigg(\mathbb{E}\Big[\Big|\int_{[t,T]^2}|s-r|^{2H_0-2}\rho(B_s^1)\rho(B_r^2)\prod_{i=1}^d R_{H_i}(B_s^{1,i}, B_r^{2,i})dsdr\Big|^{2b}\Big]\Bigg)^{1/2b} \\
&<\infty.
\end{aligned}
$$

That is, we get $\mathbb{E}\,\langle D^W Y_t^{\varepsilon,\eta}, D^W Y_t^{\varepsilon',\eta'}\rangle_{\mathcal{H}}$ is integrable. Hence, in a similar idea as that shown in (3.36), we obtain $Y_t^{\varepsilon,\eta}$ also converges to Y_t in $\mathbb{D}_W^{1,2}$ as $\varepsilon,\ \eta \downarrow 0$. Then putting $\varepsilon = \varepsilon',\ \eta = \eta'$,

$$
\sup_{\varepsilon,\eta\in(0,1]}\sup_{t\in[0,T]}\mathbb{E}\|D^W Y_t^{\varepsilon,\eta}\|_{\mathcal{H}}^2 < \infty.
$$

Hence, combining (3.35), (3.37) and (3.38) we have

$$
\mathbb{E}\Big[\|D^W\phi^{\varepsilon,\eta}\|_{\mathcal{H}\otimes\mathcal{H}}^2\Big]
$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\int_{[t,T]^2} \big\langle D^W(Y_s^{\varepsilon,\eta}-Y_s), D^W(Y_r^{\varepsilon,\eta}-Y_r)\big\rangle_{\mathcal{H}} \\
&\qquad \times \big\langle \varphi_\eta(s-\cdot)p_\varepsilon(B_s-\cdot), \varphi_\eta(r-\cdot)p_\varepsilon(B_r-\cdot)\big\rangle_{\mathcal{H}} dsdr\Big] \\
&= \alpha_{H_0}\mathbb{E}\Big[\int_{[t,T]^2} \langle D^W(Y_s^{\varepsilon,\eta}-Y_s), D^W(Y_r^{\varepsilon,\eta}-Y_r)\rangle_{\mathcal{H}} \qquad (3.40)\\
&\qquad \times |s-r|^{2H_0-2}\rho(B_s)\rho(B_r)\prod_{i=1}^d R_{H_i}(B_s^i,B_r^i)dsdr\Big] \\
&\le C\,\mathbb{E}\Big[\int_{[t,T]^2} \langle D^W(Y_s^{\varepsilon,\eta}-Y_s), D^W(Y_r^{\varepsilon,\eta}-Y_r)\rangle_{\mathcal{H}} \\
&\qquad \times |s-r|^{2H_0-2}\prod_{i=1}^d \big[(1+|B_s^i|)^{2H_i-\beta_i}(1+|B_r^i|)^{2H_i-\beta_i}\big]dsdr\Big].
\end{aligned}
$$

In a similar method as in the proof of Theorem 3.6: there are two positive constants p', q', $1/p' + 1/q' = 1$ such that $1 < p' < \dfrac{1}{2-2H_0}$ and $2q' < q$ for which we can deduce

$$
\begin{aligned}
&\mathbb{E}\Big[\|D^W\phi^{\varepsilon,\eta}\|^2_{\mathcal{H}\otimes\mathcal{H}}\Big] \\
&\le \Big(\int_{[t,T]^2}\mathbb{E}\Big[|\langle D^W(Y_s^{\varepsilon,\eta}-Y_s), D^W(Y_r^{\varepsilon,\eta}-Y_r)\rangle_{\mathcal{H}}|^{q'}\Big]dsdr\Big)^{1/q'} \qquad (3.41)\\
&\qquad \times \Big(\int_{[t,T]^2}\mathbb{E}\Big[|s-r|^{(2H_0-2)p'}\prod_{i=1}^d \big[(1+|B_s^i|)^{2H_i-\beta_i}(1+|B_r^i|)^{2H_i-\beta_i}\big]^{p'}\Big]dsdr\Big)^{1/p'},
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_{[t,T]^2}\mathbb{E}\Big[|s-r|^{(2H_0-2)p'}\prod_{i=1}^d \big[(1+|B_s^i|)^{2H_i-\beta_i}(1+|B_r^i|)^{2H_i-\beta_i}\big]^{p'}\Big]dsdr \qquad (3.42)\\
&\le C\int_{[t,T]^2}|s-r|^{(2H_0-2)p'}dsdr < \infty.
\end{aligned}
$$

Now we only need to study the integrability of the first term on the right side of (3.41). Pick two constants $a, b > 1$, $1/a + 1/b = 1$ such that a is sufficiently small to satisfy $2aq' < q$. With the help of proof in Proposition 2.2 and (3.39), we have that

$$
\begin{aligned}
&\Big(\int_{[t,T]^2}\mathbb{E}\Big[\big|\big\langle D^W Y_s^{\varepsilon,\eta}, D^W Y_r^{\varepsilon,\eta}\big\rangle_{\mathcal{H}}\big|^{q'}\Big]dsdr\Big)^{1/q'} \\
&\le \|\xi\|_q^2\Big(\int_{[t,T]^2}\mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(bq'\sum_{j,k=1}^2\int_s^T\int_r^T |u-v|^{2H_0-2}\rho(B_u^j)\rho(B_v^k)\prod_{i=1}^d R_{H_i}(B_u^{j,i},B_v^{k,i})dudv\Big) \\
&\qquad \times \Big(\int_s^T\int_r^T |u-v|^{2H_0-2}\rho(B_u^1)\rho(B_v^2)\prod_{i=1}^d R_{H_i}(B_u^{1,i},B_v^{2,i})dudv\Big)^{bq'}\big|\mathcal{F}_{s,r}^{B^1,B^2}\Big]\Big|_{B^1=B^2=B}\Big]^{1/b}dsdr\Big)^{1/q'} \\
&< \infty. \qquad (3.43)
\end{aligned}
$$

It yields that $\mathbb{E}\Big[\|D^W\phi^{\varepsilon,\eta}\|^2_{\mathcal{H}\otimes\mathcal{H}}\Big]$ is integrable. Since we have deduced that $Y^{\varepsilon,\eta} \to Y$ in $\mathbb{D}_W^{1,2}$, $\varepsilon, \eta \to 0$, therefore $\mathbb{E}\Big[\|D^W\phi^{\varepsilon,\eta}\|^2_{\mathcal{H}\otimes\mathcal{H}}\Big]$ converges to zero as ε, η tend to zero. Thus, we get $B_t^{\varepsilon,\eta,1}$

defined in (3.32) converges to zero in L^2 as $\varepsilon,\ \eta$ tend to zero.

Now we are going to bound $B_t^{\varepsilon,\eta,2}$. We have

$$
\begin{aligned}
D^W Y_s &= D^W \mathbb{E}\big[\xi \exp(\int_s^T W(dr, B_r))\big|\mathcal{F}_s\big] \\
&= D^W \mathbb{E}\big[\xi \exp(\int_{\mathbb{R}^d}\int_s^T \delta(B_r - y) W(dr, y)dy)\big|\mathcal{F}_s\big] \\
&= \mathbb{E}\big[\xi \exp\big(\int_s^T W(dr, B_r)\big)\delta(B_\cdot - \cdot)|\mathcal{F}_s\big].
\end{aligned}
\tag{3.44}
$$

Thus, by (3.37) and (3.44) we have

$$
\begin{aligned}
B_t^{\varepsilon,\eta,2} &= \int_t^T \big\langle D^W(Y_s^{\varepsilon,\eta} - Y_s), \varphi_\eta(s-\cdot)p_\varepsilon(B_s - \cdot)\big\rangle_{\mathcal{H}}\, ds \\
&= \int_t^T \mathbb{E}\big[\xi \exp\big(\int_s^T \dot{W}_{\varepsilon,\eta}(r, B_r)dr\big) \\
&\qquad\qquad \times \int_s^T \big\langle \varphi_\eta(r-\cdot)p_\varepsilon(B_r - \cdot), \varphi_\eta(s-\cdot)p_\varepsilon(B_s - \cdot)\big\rangle_{\mathcal{H}} dr|\mathcal{F}_s\big]ds \\
&\quad - \int_t^T \mathbb{E}\big[\xi \exp\big(\int_s^T W(dr, B_r)\big)\big\langle \delta(B_\cdot - \cdot), \varphi_\eta(s-\cdot)p_\varepsilon(B_s-\cdot)\big\rangle_{\mathcal{H}}|\mathcal{F}_s\big]ds \\
&:= B_t^{\varepsilon,\eta,3} - B_t^{\varepsilon,\eta,4}.
\end{aligned}
\tag{3.45}
$$

Note that,

$$
\begin{aligned}
&\big\langle \delta(B_\cdot - \cdot), \varphi_\eta(s-\cdot)p_\varepsilon(B_s - \cdot)\big\rangle_{\mathcal{H}} \\
= &\int_{[s,T]^2}\int_{\mathbb{R}^{2d}} |u-v|^{2H_0-2}\delta(B_u - y)\varphi_\eta(s-v)p_\varepsilon(B_s - z)\rho(y)\rho(z)\prod_{i=1}^d R_{H_i}(y^i, z^i)drdv \\
= &\int_{[s,T]^2}\int_{\mathbb{R}^{d}} |u-v|^{2H_0-2}\varphi_\eta(s-v)p_\varepsilon(B_s - z)\rho(B_u)\rho(z)\prod_{i=1}^d R_{H_i}(B_u^i, y^i)dudvdz.
\end{aligned}
$$

Thus, by Fubini's Theorem and previous estimates, we have

$$
|B_t^{\varepsilon,\eta,3}| \le \int_t^T \mathbb{E}\big[\xi\ \exp\big(\int_s^T \dot{W}_{\varepsilon,\eta}(r, B_r)dr\big)\int_s^T |s-r|^{2H_0-2}\rho(B_s)\rho(B_r)\prod_{i=1}^d R_{H_i}(B_s^i, B_r^i)dr|\mathcal{F}_s\big]ds
\tag{3.46}
$$

and

$$
\begin{aligned}
|B_t^{\varepsilon,\eta,4}| &= \int_t^T \mathbb{E}\big[\xi \exp\big(\int_s^T W(dr, B_r)\big)\int_{[s,T]^2}\int_{\mathbb{R}^d} |u-v|^{2H_0-2}\varphi_\eta(s-v) \\
&\qquad\qquad \times p_\varepsilon(B_s - y)\rho(B_u)\rho(y)\prod_{i=1}^d R_{H_i}(B_u^i, y_i)dudvdy|\mathcal{F}_s\big]ds \\
&\le \int_t^T \mathbb{E}\big[\xi\ \exp\big(\int_s^T W(dr, B_r)\big)\int_s^T |s-v|^{2H_0-2}\rho(B_v)\rho(B_s)\prod_{i=1}^d R_{H_i}(B_v^i, B_s^i)dv|\mathcal{F}_s\big]ds.
\end{aligned}
\tag{3.47}
$$

Proposition 2.2 and dominated convergence theorem guarantee the integrability of these two expressions. Now, with the help of dominated convergence theorem we get $B_t^{\varepsilon,\eta,3}$ and $B_t^{\varepsilon,\eta,4}$ converge in L^2 to

$$\int_t^T \mathbb{E}\big[\xi \exp\big(\int_s^T W(dr, B_r)\big) \int_s^T |s-r|^{2H_0-2}\rho(B_s)\rho(B_r)\prod_{i=1}^d R_{H_i}(B_s^i, B_r^i)dr|\mathcal{F}_s\big]ds$$

as $\varepsilon,\ \eta$ tend to zero which also mean that $B_t^{\varepsilon,\eta,2}$ converges in L^2 to zero as $\varepsilon,\ \eta$ tend to zero. $\square$

4 Hölder continuity of Y and Z

Let the Assumption (2) in Theorem 1.1 be satisfied. Now we can prove the Hölder continuity of Y and Z.

Proof. First we prove the Hölder continuity of Y. Recall $q > \dfrac{2}{2\underline{H}-1}$, where $\underline{H} = \min\{H_0, \dots, H_d\}$. Thus for all $a \in (1, q)$, we have

$$\begin{aligned}
\mathbb{E}\big|Y_t - Y_s\big|^a &= \mathbb{E}\left|\mathbb{E}\big[\xi\exp(V_t)\big|\mathcal{F}_t\big] - \mathbb{E}\big[\xi\exp(V_s)\big|\mathcal{F}_s\big]\right|^a \\
&\le 2\bigg(\mathbb{E}\Big|\mathbb{E}\big[\xi\exp(V_t)\big|\mathcal{F}_t\big] - \mathbb{E}^B\big[\xi\exp(V_s)\big|\mathcal{F}_t\big]\Big|^a \\
&\quad + \mathbb{E}\Big|\mathbb{E}\big[\xi\exp(V_s)\big|\mathcal{F}_t\big] - \mathbb{E}\big[\xi\exp(V_s)\big|\mathcal{F}_s\big]\Big|^a\bigg) \\
&=: 2\Big(I_1 + I_2\Big).
\end{aligned} \tag{4.1}$$

For I_1, one can use Jensen's inequality and the exponential integrability of Proposition 2.2 to get, for two positive constants p', q' satisfying $1/p' + 1/q' = 1$ and $aq' < q$,

$$\begin{aligned}
I_1 &= \mathbb{E}\Big|\mathbb{E}\big[\xi\exp(V_t)\big|\mathcal{F}_t\big] - \mathbb{E}\big[\xi\exp(V_s)\big|\mathcal{F}_t\big]\Big|^a \\
&\le \mathbb{E}\Big|\mathbb{E}\left[\xi\big(|V_t - V_s|\exp\left(\max\{V_t, V_s\}\right)\big)\big|\mathcal{F}_t\right]\Big|^a \\
&\le \mathbb{E}\bigg[\Big(\mathbb{E}\left[\xi^{q'}\exp(q'\max\{V_t, V_s\})\big|\mathcal{F}_t\right]\Big)^{a/q'}\Big(\mathbb{E}\big[\big|V_t - V_s\big|^{p'}\big|\mathcal{F}_t\big]\Big)^{\frac{a}{p'}}\bigg] \\
&\le C\Big(\mathbb{E}\big[\big|V_t - V_s\big|^{ap'}\big]\Big)^{\frac{1}{p'}}.
\end{aligned} \tag{4.2}$$

By (2.5), (2.6) and the equivalence between the L^2-norm and the L^p-norm for a Gaussian random variable, it yields that

$$\begin{aligned}
I_1 &\le \Big(\mathbb{E}\big[\big|V_t - V_s\big|^{ap'}\big]\Big)^{\frac{1}{p'}} = \bigg(\mathbb{E}\bigg[\Big|\int_s^t W(dr, B_r)\Big|^{ap'}\bigg]\bigg)^{\frac{1}{p'}} \le C\bigg(\mathbb{E}\Big(\mathbb{E}^W\Big|\int_s^t W(dr, B_r)\Big|^2\Big)^{ap'/2}\bigg)^{\frac{1}{p'}} \\
&\le C\bigg(\mathbb{E}\Big(\int_s^t\int_s^t \alpha_{H_0}|u-v|^{2H_0-2}\rho(B_u)\rho(B_v)\prod_{i=1}^d R_{H_i}(B_u^i, B_v^i)dudv\Big)^{ap'/2}\bigg)^{\frac{1}{p'}} \\
&\le C\bigg(\Big(\int_s^t\int_s^t |u-v|^{(2H_0-2)m}dudv\Big)^{\frac{ap'}{2m}}\mathbb{E}\Big(\int_s^t\int_s^t \big(\rho(B_u)\rho(B_v)\prod_{i=1}^d R_{H_i}(B_u^i, B_v^i)\big)^n dudv\Big)^{\frac{ap'}{2n}}\bigg)^{\frac{1}{p'}}
\end{aligned} \tag{4.3}$$

$$\leq C\left|t-s\right|^{aH_0-\varepsilon},$$

where ε is an arbitrary positive constant, $n, m > 1$ such that $\frac{1}{n} + \frac{1}{m} = 1$.

For I_2, denote by $\psi_t = \exp\big(\int_0^t W(ds, B_s)\big)$. Proposition 2.2 tells us that, $\xi\psi_T$ is $L^q(\Omega)$-integrable for $q > \dfrac{2}{2\underline{H}-1}$. Moreover, Clark-Ocone formula implies that,

$$\xi\psi_T = \mathbb{E}^B[\xi\psi_T] + \int_0^T f_r dB_r,$$

where

$$f_r = \mathbb{E}[D_r^B(\xi\psi_T)|\mathcal{F}_r^B] = \mathbb{E}[\psi_T D_r^B(\xi)|\mathcal{F}_r] + \mathbb{E}[\xi D_r^B(\psi_T)|\mathcal{F}_r]. \tag{4.4}$$

Thus, from the Burkholder-Davis-Gundy inequality and the fact that $a > 2$ we deduce that

$$\begin{aligned} I_2 &= \mathbb{E}\left|\psi_s^{-1}\Big(\mathbb{E}\big[\xi\psi_T\big|\mathcal{F}_t\big] - \mathbb{E}\big[\xi\psi_T\big|\mathcal{F}_s\big]\Big)\right|^a \\ &= \mathbb{E}\left|\psi_s^{-1}\int_s^t f_r dB_r\right|^a \leq \left(\mathbb{E}\left[\psi_s^{-2a}\right]\right)^{1/2}\left(\mathbb{E}\left[\int_s^t \left|f_r\right|^2 dr\right]^a\right)^{1/2} \\ &\leq C\left(\mathbb{E}\left[\int_s^t \left|f_r\right|^2 dr\right]^a\right)^{1/2}. \end{aligned} \tag{4.5}$$

Taking (4.4) into above formula yields that

$$\begin{aligned} &\mathbb{E}\left[\int_s^t \left|f_r\right|^2 dr\right]^a \\ &\leq C\left(\mathbb{E}\int_s^t \left|\mathbb{E}\left[\psi_T D_r^B(\xi)\big|\mathcal{F}_r\right]\right|^2 dr\right)^a + C\left(\mathbb{E}\int_s^t \left|\mathbb{E}\left[\xi D_r^B(\psi_T)\big|\mathcal{F}_r\right]\right|^2 dr\right)^{a/2} \\ &\leq C\left(\int_s^t \mathbb{E}\left(D_r^B\xi\right)^{2q'} dr\right)^{a/q'}\left(\int_s^t \mathbb{E}\left(\psi_T\right)^{2p'} dr\right)^{a/p'} \\ &\qquad + C\left(\int_s^t \mathbb{E}[\xi^{2q'}]dr\right)^{a/q'}\left(\int_s^t \mathbb{E}\left(D_r^B\psi_T\right)^{2p'} dr\right)^{a/p'} \\ &\leq C|t-s|^{a/q'}\left(\left(\int_s^t \mathbb{E}\left(\mathbb{E}^W\left(\psi_T\right)^2\right)^{p'} dr\right)^{a/p'} + \left(\int_s^t \mathbb{E}\left(\mathbb{E}^W\left(D_r^B\psi_T\right)^2\right)^{p'} dr\right)^{a/p'}\right), \end{aligned}$$

where C is a constant only depends on $p', q', \|D_r^B\xi\|_{L_q}^2$ and $\|\xi\|_{L_q}^2$. We recall ψ_s and $D_r^B\psi_s$ are centralized Gaussian processes given B. Moreover, $\mathbb{E}^W\left(D_r^B\psi_T\right)^2 = \mathbb{E}^W\left(D_r^B\exp\big(\int_0^t W(ds, B_s)\big)\right)^2$ can be treated in a similar way as we did in the proof of Theorem 3.6. By Proposition 2.1 and Theorem 3.6, we can directly obtain the boundedness of $\mathbb{E}\left(\mathbb{E}^W\left(\psi_T\right)^2\right)^{p'}$ and $\mathbb{E}\left(\mathbb{E}^W\left(D_r^B\psi_T\right)^2\right)^{p'}$. Thus, we deduce

$$I_2 \leq \left(\mathbb{E}\left[\int_s^t \left|f_r\right|^2 dr\right]^a\right)^{1/2} \leq C\left|t-s\right|^{a/2}.$$

Because we assume that $H > \dfrac{1}{2}$, the Hölder continuous coefficient can only be $\dfrac{1}{2}$.

Next we have to consider the Hölder continuity of Z. Recall (3.14) for the expression of Z:

$$
\begin{aligned}
Z_t = D_t^B Y_t &= \mathbb{E}\left[e^{\int_t^T W(d\tau, B_\tau)d\tau} D_t^B \xi + \xi \, \exp\Big(\int_t^T W(du, B_u)du \Big) \int_t^T \nabla_x W(ds, B_s) \,\Big|\, \mathcal{F}_t \right] \\
&= \mathbb{E}\left[e^{V_t} D_t^B \xi + \xi \, \exp\big(V_t\big) \nabla_x V_t \,\Big|\, \mathcal{F}_t \right] =: Z_t^1 + Z_t^2 ,
\end{aligned}
$$

where we recall the definition of (2.5) for V_t and where we denote $\nabla_x V_t = \int_t^T \nabla_x W(ds, B_s)$. Z^1 is easy to deal with. In fact, similar to the way to treating (4.1), (4.3), (4.5), and by the assumption that $D^B \xi \in L^q(\Omega)$ and $\mathbb{E}|D_t \xi - D_s \xi|^q \le C|t-s|^{\kappa q/2}$ for some $\kappa > 0$, we see

$$
\mathbb{E}\big|Z_t^1 - Z_s^1\big|^2 \le C|t-s|^{\kappa \wedge 1}.
$$

We shall focus on Z^2.

$$
\begin{aligned}
\mathbb{E}\big|Z_t^2 - Z_s^2\big|^2 &= \mathbb{E}\left(\mathbb{E}\left[\xi \exp(V_t) \nabla_x V_t \,\middle|\, \mathcal{F}_t \right] - \mathbb{E}\left[\xi \exp(V_s) \nabla_x V_s \,\middle|\, \mathcal{F}_s \right] \right)^2 \\
&\le 2\mathbb{E}\left(\mathbb{E}\left[\xi \exp(V_t) \nabla_x V_t \,\middle|\, \mathcal{F}_t \right] - \mathbb{E}\left[\xi \exp(V_s) \nabla_x V_s \,\middle|\, \mathcal{F}_t \right] \right)^2 \\
&\qquad + 2\mathbb{E}\left(\mathbb{E}\left[\xi \exp(V_s) \nabla_x V_s \,\middle|\, \mathcal{F}_t \right] - \mathbb{E}\left[\xi \exp(V_s) \nabla_x V_s \,\middle|\, \mathcal{F}_s \right] \right)^2 \\
&:= 2(I_1 + I_2).
\end{aligned}
\tag{4.6}
$$

For I_1, with the help of Jensen's inequality we have

$$
\begin{aligned}
I_1 &\le \mathbb{E}\left[|\xi \exp(V_t) \nabla_x V_t - \xi \exp(V_s) \nabla_x V_s|^2 \right] \\
&\le 2\mathbb{E}\left[|\xi \left(\exp(V_t) - \exp(V_s)\right) \nabla_x V_t|^2 + |\xi \exp(V_s)(\nabla_x V_t - \nabla_x V_s)|^2 \right] \\
&\le 2\mathbb{E}\left[|\xi \exp(\max\{V_t, V_s\}) \nabla_x V_t (V_t - V_s)|^2 \right] + 2\mathbb{E}\left[|\xi \exp(V_s)(\nabla_x V_t - \nabla_x V_s)|^2 \right] \\
&:= 2(I_{1,1} + I_{1,2}).
\end{aligned}
$$

We can find two constant a, b such that $1/a + 1/b = 1$, $1 < a < \frac{1}{2-2\underline{H}}$ and $2b < q$. Then we have

$$
\begin{aligned}
I_{1,1} &\le \left(\mathbb{E}\left[(\nabla_x V_t)^{2a} \right] \right)^{1/a} \left(\mathbb{E}\, \xi^{2b} \left(V_t - V_s\right)^{2b} \right)^{1/b} \\
&\le C \left(\mathbb{E}\, \xi^{2b} \left(V_t - V_s\right)^{2b} \right)^{1/b} \\
&\qquad \left(\sum_{i=1}^d \mathbb{E}^B \left(\int_t^T \int_t^T |u-v|^{2H_0 - 2} |B_u^i - B_v^i|^{2H_i - 2} \prod_{j \ne i}^d R_{H_j}(B_u^j - B_v^j) \rho(B_u^j) \rho(B_v^j) du dv \right)^a \right)^{1/a} \\
&\le C|t-s|^{2H_0 - \varepsilon}.
\end{aligned}
\tag{4.7}
$$

As for $I_{1,2}$, we deduce similarly that

$$
\begin{aligned}
I_{1,2} &\le \mathbb{E}\left[|\xi|^{2b} \exp(2bV_t) \right]^{1/b} \left(\mathbb{E}\left[\nabla_x V_t - \nabla_x V_s \right]^{2a} \right)^{1/a} \\
&\le C_a \mathbb{E}\left[|\xi|^{2b} \exp(2bV_t) \right]^{1/b} \left(\mathbb{E}\left[\mathbb{E}^W \left\{ \int_s^t \int_s^t \left(\nabla_x W(du, B_u)\right)^T \left(\nabla_x W(dv, B_v)\right) \right\} \right]^a \right)^{1/a} \\
&\le C \left[\mathbb{E}\left(\left| \sum_{i=1}^d \int_s^t \int_s^t |u-v|^{2H_0 - 2} |B_u^i - B_v^i|^{2H_i - 2} \prod_{j \ne i}^d R_{H_j}(B_u^j, B_v^j) \rho(B_u^j) \rho(B_u^j) du dv \right|^a \right) \right]^{1/a}
\end{aligned}
\tag{4.8}
$$

$$\leq C \sum_{i=1}^{d} \left(\int_s^t \int_s^t |u-v|^{(2H_0-2)a+(H_i-1)a} dudv \right)^{1/a}$$
$$\leq C|t-s|^{2H_0+\underline{H}-1-\varepsilon}.$$

As I_2, the Clark-Ocone formula yields

$$\xi \exp(V_s)\nabla_x V_s = \mathbb{E}^B \left[\xi \exp(V_s)\nabla_x V_s\right] + \int_s^T \mathbb{E}\left[D_r^B\left(\xi \exp(V_s)\nabla_x V_s\right)\Big|\mathcal{F}_r\right] dB_r. \tag{4.9}$$

Thus we have

$$\begin{aligned} I_2 &= \mathbb{E}\left|\int_s^t \mathbb{E}\left[D_r^B\left(\xi \exp(V_s)\nabla_x V_s\right)\Big|\mathcal{F}_r\right] dB_r\right|^2 = \mathbb{E}\int_s^t \left(\mathbb{E}\left[D_r^B\left(\xi \exp(V_s)\nabla_x V_s\right)\Big|\mathcal{F}_r\right]\right)^2 dr \\ &= \mathbb{E}\int_s^t \left(\mathbb{E}\left[\xi \nabla_x V_s D_r^B\left(\exp(V_s)\right)\Big|\mathcal{F}_r\right]\right)^2 dr + \mathbb{E}\int_s^t \left(\mathbb{E}\left[\xi \exp(V_s) D_r^B\left(\nabla_x V_s\right)\Big|\mathcal{F}_r\right]\right)^2 dr \\ &\qquad + \mathbb{E}\int_s^t \left(\mathbb{E}\left[\exp(V_s)\nabla_x V_s\, D_r^B \xi\Big|\mathcal{F}_r\right]\right)^2 dr \\ &=: I_{2,1} + \int_s^t I_{2,2} dr + I_{2,3}. \end{aligned} \tag{4.10}$$

The integrability inside the integral of $I_{2,3}$ is obvious due to (4.7). For $I_{2,1}$ we have

$$\begin{aligned} \mathbb{E}\int_s^t \left(\mathbb{E}\left[D_r^B\left(\exp(V_s)\right)\xi\, \nabla_x V_s\Big|\mathcal{F}_r\right]\right)^2 dr &= \mathbb{E}\int_s^t \left(\mathbb{E}\left[\xi \exp(V_s)\ \left(\nabla_x V_s\right)^2\Big|\mathcal{F}_r\right]\right)^2 dr \\ &\leq \int_s^t \left(\mathbb{E}[\xi^b \exp(bV_s)]\right)^{2/b} \left(\mathbb{E}(\nabla_x V_s)^{2a}\right)^{2/a} ds \leq C|t-s|\,. \end{aligned}$$

Finally, we deal with $I_{2,2}$. We shall use the technique as in (3.15). Notice that,

$$D_r^B\left(\nabla_x V_s\right) = D_r^B\left(\int_s^T \nabla_x W(du, B_u) 1_{[0,u]}(r)\right) = \int_{s\vee r}^T \nabla_x^2 W(du, B_u)\,.$$

We have analogously to (3.15)

$$\begin{aligned} I_{2,2} = \mathbb{E}\Bigg(\mathbb{E}\Bigg[&\xi(B^1)\,\xi(B^2)\exp\left(\sum_{j,k=1}^{2}\frac{\alpha_{H_0}}{2}\int_s^T\int_s^T |u-v|^{2H_0-2} R_{H_i}(B_u^j, B_v^k)\rho(B_u^j)\rho(B_v^k) dudv\right) \\ &\times \left(\int_{s\vee r}^T \int_{s\vee r}^T \mathrm{Tr}\left[\left(\nabla_x^2 W(du, B_u^1)\right)^T \nabla_x^2 W(dv, B_v^2)\right] dudv\right)\Big|\mathcal{F}_r^{B^1,B^2}\Bigg]\Bigg|_{B^1=B^2=B}\Bigg). \end{aligned} \tag{4.11}$$

Using the Hölder inequality, we have

$$\begin{aligned} I_{2,2} = \Bigg[\mathbb{E}\Bigg(\mathbb{E}\Bigg[\Bigg|&\xi(B^1)\,\xi(B^2)\exp\Bigg(\sum_{j,k=1}^{2}\frac{\alpha_{H_0}}{2}\int_{s\vee r}^T\int_{s\vee r}^T |u-v|^{2H_0-2} \\ &R_{H_i}(B_u^j, B_v^k)\rho(B_u^j)\rho(B_v^k) dudv\Bigg)\Bigg|^b \Bigg|\mathcal{F}_r^{B^1,B^2}\Bigg]\Bigg|_{B^1=B^2=B}\Bigg)\Bigg]^{1/b} \end{aligned} \tag{4.12}$$

$$\times \left[\mathbb{E}\left(\mathbb{E}\left[\left|\int_{s\vee r}^T\int_{s\vee r}^T \operatorname{Tr}\left[\left(\nabla_x^2 W(du,B_u^1)\right)^T \nabla_x^2 W(dv,B_v^2)\right]dudv\right|^a\Big|\mathcal{F}_r^{B^1,B^2}\right]\Big|_{B^1=B^2=B}\right)\right]^{1/a}$$
$$\le CI_{2,2,1}^{1/a},$$

where

$$I_{2,2,1}=\mathbb{E}\left(\mathbb{E}\left[\left|\int_{s\vee r}^T\int_{s\vee r}^T \operatorname{Tr}\left[\left(\nabla_x^2 W(du,B_u^1)\right)^T \nabla_x^2 W(dv,B_v^2)\right]dudv\right|^a\Big|\mathcal{F}_r^{B^1,B^2}\right]\Big|_{B^1=B^2=B}\right).$$

We shall consider the term that contains $J=\int_{s\vee r}^T\int_{s\vee r}^T \frac{\partial^2}{\partial x_i^2}W(du,B_u^1)\frac{\partial^2}{\partial x_i^2}W(dv,B_v^2)$ (denote the corresponding term by J_i) since the other terms can be treated in similar way. When $r\ge s$, we have for any $a>1$,

$$\begin{aligned}
J_i =&\mathbb{E}\left(\mathbb{E}\left[\left|\int_r^T\int_r^T \frac{\partial^2}{\partial x_i^2}W(du,B_u^1)\frac{\partial^2}{\partial x_i^2}W(dv,B_v^2)\right|^a \mathcal{F}_r^{B^1,B^2}\right]\Big|_{B^1=B^2=B}\right)\\
\le& C_a\mathbb{E}\left(\mathbb{E}\left[\left(\mathbb{E}^W\left|\int_r^T\int_r^T \frac{\partial^2}{\partial x_i^2}W(du,B_u^1)\frac{\partial^2}{\partial x_i^2}W(dv,B_v^2)\right|^2\right)^{a/2}\Big|\mathcal{F}_r^{B^1,B^2}\right]\Big|_{B^1=B^2=B}\right)\\
\le& C_a\mathbb{E}\Bigg(\mathbb{E}\Bigg[\Bigg(\int_r^T\int_r^T |u-v|^{2H_0-2}|B_u^{1,i}-B_v^{2,i}|^{2H_i-4}\\
&\qquad \rho(B_u^1)\rho(B_v^2)\prod_{j\neq i}|R_{H_j}(B_u^{1,j},B_v^{2,j})|dudv\Bigg)^{a/2}\Big|\mathcal{F}_r^{B^1,B^2}\Bigg]\Big|_{B^1=B^2=B}\Bigg)+C_a\,,
\end{aligned}$$

where in the above first inequality, we used the hypercontractivity for $\mathbb{E}^W$ and in the above last inequality, there are terms such as the derivatives with respect to $\partial_{x_i}^2\rho$ and $\partial_{x_i}\rho\partial_{x_i}R_{H_i}$ which are easy to be bounded. By using Hölder's inequality again, the above expectation is bounded by a multiple of $1/a'$ power of (for any $a'>1$)

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{E}\left[\left(\int_r^T\int_r^T |u-v|^{2H_0-2}|B_u^{1,i}-B_v^{2,i}|^{2H_i-4}dudv\right)^{aa'/2}\Big|\mathcal{F}_r^{B^1,B^2}\right]\Big|_{B^1=B^2=B}\right)\\
=&\mathbb{E}\Bigg(\mathbb{E}\Bigg[\Bigg|\int_r^T\int_r^T |u-v|^{2H_0-2}|(B_u^{1,i}-B_r^{1,i})-(B_v^{2,i}-B_r^{2,i})\\
&\qquad +B_r^{1,i}-B_r^{2,i}|^{2H_i-4}dudv\Bigg|^{aa'/2}\Big|\mathcal{F}_r^{B^1,B^2}\Bigg]\Big|_{B^1=B^2=B}\Bigg)\\
=&\mathbb{E}\Bigg(\mathbb{E}^{X,Y}\Bigg[\int_r^T\int_r^T |u-v|^{2H_0-2}|\sqrt{u-r}X-\sqrt{v-r}Y\\
&\qquad +B_r^{1,i}-B_r^{2,i}|^{2H_0-4}dudv\Bigg]^{aa'/2}\Big|_{B^1=B^2=B}\Bigg),
\end{aligned}$$

where X and Y are two independent standard Gaussians. The above expectation in X and Y are bounded by (denoting $Z=B_r^{1,i}-B_r^{2,i}$ and choosing $aa'/2<1$)

$$\mathbb{E}^{X,Y}\left[\int_r^T\int_r^T |u-v|^{2H_0-2}|\sqrt{u-r}X-\sqrt{v-r}Y+Z|^{2H_0-4}dudv\right]^{aa'/2}$$

$$
\begin{aligned}
&\le \left[\int_r^T\int_r^T |u-v|^{2H_0-2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}|\sqrt{u-r}x-\sqrt{v-r}y+Z|^{2H_i-4}dxdydudv\right]^{aa'/2}\\
&\le \Bigg[\int_r^T\int_r^T |u-v|^{2H_0-2}\frac{1}{\sqrt{(u-r)(v-r)}}\\
&\qquad\left[\int_{\mathbb{R}^2}\frac{|xy|}{4}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}|\sqrt{u-r}x-\sqrt{v-r}y+Z|^{2H_i-2}dxdy\right]dudv\Bigg]^{aa'/2}\\
&\le \Bigg[\int_r^T\int_r^T |u-v|^{2H_0-2}\frac{1}{\sqrt{(u-r)(v-r)}}\\
&\qquad\left[\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi}}e^{-x^2/4}\frac{1}{\sqrt{2\pi}}e^{-y^2/4}|\sqrt{u-r}x-\sqrt{v-r}y+Z|^{2H_i-2}dxdy\right]dudv\Bigg]^{aa'/2}\\
&= C\left[\int_r^T\int_r^T |u-v|^{2H_0-2}\frac{1}{\sqrt{(u-r)(v-r)}}\mathbb{E}^{X,Y}|\sqrt{u-r}X-\sqrt{v-r}Y+Z|^{2H_i-2}dudv\right]^{aa'/2}\\
&\le C\left[\int_r^T\int_r^T |u-v|^{2H_0-2}|u-r|^{-1/2}|v-r|^{-1/2}|u-r+v-r|^{2H_i-2}dudv\right]^{aa'/2}\\
&\le C\left[\int_r^T\int_r^T |u-v|^{2H_0-2}|u-r|^{-1/2}|v-r|^{-1/2}|u-r|^{H_i-1}|v-r|^{H_i-1}dudv\right]^{aa'/2}\\
&<\infty\,,
\end{aligned}
$$

where the third last inequality follows from Lemma A.1 of [1] and the last inequality holds true since $H_i > 1/2$. This proves that $I_{2,2}$ is bounded and hence $I_2 \le C|t-s|$. Hence, Combing (4.7), (4.8) and (4.10), we have

$$
\mathbb{E}\big|Z_t^2 - Z_s^2\big|^2 \le C|t-s|^{2H_0-1+\underline{H}-\varepsilon},
$$

and finally we deduce

$$
\mathbb{E}\big|Z_t - Z_s\big|^2 \le C|t-s|^{(2H_0-1+\underline{H}-\varepsilon)\wedge\kappa}, \text{ for all } \varepsilon>0.
$$

□

5 Uniqueness of solution

We have proved parts (1) and (2) of Theorem 1.1. In this section, we are going to prove part (3), the uniqueness of BSDEs (1.2). We need the following proposition.

Proposition 5.1. *Suppose that the conditions in Theorem 1.1 are satisfied. Let* $(Y,Z)\in\mathcal{S}_{\mathbb{F}}^2(0,T;\mathbb{R})\times\mathcal{M}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$ *be the solution of BSDEs* (1.2) *so that* Y, D^BY *are* $\mathbb{D}^{1,2}$. *Then the solution has the explicit expression* (1.7) *and hence the BSDEs* (1.2) *has a unique solution.*

Before we prove Proposition 5.1, we first need the following lemma.

Lemma 5.2. *Recall the notation*

$$
\alpha_s^t = \exp\left\{\int_s^t W(dr,B_r)\right\}\,. \tag{5.1}
$$

Then α_s^t satisfies the following equation.

$$\alpha_0^t = \alpha_0^s + \int_0^t \alpha_0^r W(dr, B_r). \tag{5.2}$$

Proof. Define $K_t = \int_0^t W(dr, B_r)$. Consider a sequence of partitions $\pi_n = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ such that $|\pi_n| = \max_{0 \le i \le n-1}(t_{i+1} - t_i) \to 0$ when $n \to \infty$ (the t_i's depend on n and we omit this explicit dependence to simplify notation). Since $H \in (1/2, 1)$ and since W satisfies (1.1), by Markov's inequality, Proposition 2.1 and the estimate (2.11), it is easy to obtain

$$\begin{aligned}
\lim_{n\to\infty} P\left\{ \sum_{i=1}^n \left| \int_{t_i}^{t_{i+1}} W(dr, B_r) \right|^2 > \varepsilon \right\} &\le \lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}\left[| \int_{t_i}^{t_{i+1}} W(dr, B_r)|^2 \right]}{\varepsilon} \\
&\le \lim_{n\to\infty} \frac{C \sum_{i=1}^n |t_{i+1} - t_i|^{2H_0}}{\varepsilon} \\
&= 0,
\end{aligned} \tag{5.3}$$

for any $\varepsilon > 0$. On the other hand, we have

$$\alpha_0^t - 1 = \sum_{i=0}^n \left[e^{K_{t_{i+1}}} - e^{K_{t_i}} \right] = \sum_{i=0}^n \alpha_0^{t_i} \left(K_{t_{i+1}} - K_{t_i} \right) + R_t^n, \tag{5.4}$$

where

$$R_t^n = \sum_{i=0}^n \left(K_{t_{i+1}} - K_{t_i} \right) \int_0^1 \left[e^{K_{t_i} + (K_{t_{i+1}} - K_{t_i})u} - e^{K_{t_i}} \right] du.$$

Combining (5.3) with (3.12) yields

$$|R_t^n| \le C \sup_{0 \le r \le t} e^{K_r} \cdot \sum_{i=0}^n |K_{t_{i+1}} - K_{t_i}|^2 \xrightarrow{P} 0, \quad n \to \infty.$$

This proves that α_0^t satisfies (5.2). □

Lemma 5.3. *Let (Y, Z) satisfy (1.2) and let α_t be given as above. Suppose the conditions of Proposition 5.1 are satisfied. Then*

$$\alpha_0^T \xi - \alpha_0^t Y_t = \int_t^T \alpha_0^s Z_s dB_s \,. \tag{5.5}$$

Proof. Let (Y, Z) satisfy (1.2) and we use partition $\pi_n = \{t = t_0 < t_1 < \ldots < t_n = T\}$. Taking (5.2) into account we have

$$\begin{aligned}
\alpha_0^T \xi - \alpha_0^t Y_t &= \sum_{i=1}^n \left(\alpha_0^{t_{i+1}} Y_{t_{i+1}} - \alpha_0^{t_i} Y_{t_i} \right) = \sum_{i=1}^n \left(\alpha_0^{t_{i+1}} (Y_{t_{i+1}} - Y_{t_i}) + Y_{t_i} (\alpha_0^{t_{i+1}} - \alpha_0^{t_i}) \right) \\
&= \sum_{i=1}^n \left(\alpha_0^{t_{i+1}} \big(- \int_{t_i}^{t_{i+1}} Y_r W(dr, B_r) + \int_{t_i}^{t_{i+1}} Z_r dB_r \big) \right) \\
&\qquad + \sum_{i=1}^n Y_{t_i} \int_{t_i}^{t_{i+1}} \alpha_0^r W(dr, B_r)
\end{aligned} \tag{5.6}$$

$$= \sum_{i=1}^{n} \left(-\alpha_0^{t_i} Y_{t_i} \int_{t_i}^{t_{i+1}} W(dr, B_r) + \alpha_0^{t_i} Z_{t_i} \int_{t_i}^{t_{i+1}} dB_r) \right) \\ + \sum_{i=1}^{n} Y_{t_i} \alpha_0^{t_i} \int_{t_i}^{t_{i+1}} W(dr, B_r) + \tilde{R}_t^n \\ = \sum_{i=1}^{n} \alpha_0^{t_i} Z_{t_i} \int_{t_i}^{t_{i+1}} dB_r + \tilde{R}_t^n ,$$

where

$$\begin{aligned}
\tilde{R}_t^n =& \sum_{i=1}^{n} \left(\alpha_0^{t_{i+1}} \int_{t_i}^{t_{i+1}} [Y_r - Y_{t_i}] \, W(dr, B_r) \right) + \sum_{i=1}^{n} \left(\alpha_0^{t_{i+1}} - \alpha_0^{t_i} \right) Y_{t_i} \int_{t_i}^{t_{i+1}} W(dr, B_r) \\
&+ \sum_{i=1}^{n} \alpha_0^{t_i} \int_{t_i}^{t_{i+1}} [Z_r - Z_{t_i}] \, dB_r + \sum_{i=1}^{n} \left(\alpha_0^{t_{i+1}} - \alpha_0^{t_i} \right) \int_{t_i}^{t_{i+1}} Z_r dB_r \\
&+ \sum_{i=1}^{n} Y_{t_i} \int_{t_i}^{t_{i+1}} \left[\alpha_0^{r} - \alpha_0^{t_i} \right] W(dr, B_r) \\
=& \sum_{i=1}^{n} R_{1,i} + R_{2,i} + R_{3,i} + R_{4,i} + R_{5,i}.
\end{aligned} \tag{5.7}$$

For $R_{1,i}$, using (7.1) we get

$$\begin{aligned}
|R_{1,i}|^2 \lesssim & \, \mathbb{E} \left[\int_{t_i}^{t_{i+1}} [Y_r - Y_{t_i}] \, W(dr, B_r) \right]^2 \\
=& \, \mathbb{E} \left[\int_{[t_i, t_{i+1}]^2} (Y_r - Y_{t_i})(Y_s - Y_{t_i}) |s - r|^{2H_0 - 2} q(B_r, B_s) dr ds \right] \\
&+ \mathbb{E} \Bigg[\int_{[t_i, t_{i+1}]^2} \int_{[r, t_{i+1}]} \int_{[t_i, s]} \int_{\mathbb{R}^{2d}} D_{r,y}^W (Y_u - Y_{t_i}) \, D_{v,w}^W (Y_s - Y_{t_i}) \\
&\qquad \times |u - v|^{2H_0 - 2} |s - r|^{2H_0 - 2} q(B_u, w) q(B_s, y) dw dy dv du dr ds \Bigg] .
\end{aligned} \tag{5.8}$$

Recalling (2.9) that covariance $q(x, y)$ satisfies

$$|q(x, y)| \leq C \prod_{i=1}^{d} (1 + |x_i|)^{2H_i - \beta_i} (1 + |y_i|)^{2H_i - \beta_i}, \tag{5.9}$$

where $\beta_i > 2H_i + 1, i = 1, \ldots, d$, it yields

$$\begin{aligned}
|R_{1,i}|^2 \lesssim & \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathbb{E} \Big[|Y_r - Y_{t_i}|^2 \Big]^{1/2} \mathbb{E} \Big[|Y_s - Y_{t_i}|^2 \Big]^{1/2} |s - r|^{2H_0 - 2} \sup_{\omega, s, r} \Big| q(B_s, B_r) \Big| ds dr \\
&+ \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_{r}^{t_{i+1}} \int_{t_i}^{s} \mathbb{E} \Big[|D_{r,y}^W (Y_r - Y_{t_i})|^2 \Big]^{1/2} \mathbb{E} \Big[D_{v,w}^W (|Y_s - Y_{t_i})|^2 \Big]^{1/2} \\
&\qquad \times |u - v|^{2H_0 - 2} |s - r|^{2H_0 - 2} \sup_{\omega, u, s, w, y} \Big| q(B_u, w) q(B_s, y) \Big| dv du ds dr.
\end{aligned} \tag{5.10}$$

If Y satisfies condition (3) in Theorem 1.1, that is, the continuity coefficient of Y is $\frac{1}{2}$, then we can directly obtain

$$|R_{1,i}|^2 \lesssim C\Big(|t_{i+1}-t_i|^{1+2H_0} + |t_{i+1}-t_i|^{4H_0}\Big) \tag{5.11}$$

If Y satisfies condition (4) in Theorem 1.1, then from (1.2), we have

$$Y_r - Y_{t_i} = \int_r^{t_i} Y_s W(ds, B_s) - \int_r^{t_i} Z_s dB_s, \quad r \in [t_i, t_{t+1}]\,. \tag{5.12}$$

With the help of (7.1) again and the fact that $(Y,Z) \in \mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ as well as that Y also belongs to $\mathbb{D}^{1,2}$, it holds

$$\begin{aligned}
\mathbb{E}\,|Y_r - Y_{t_i}|^2 \le\, & 2\mathbb{E}\left[\int_{[t_i,r]^2} Y_s Y_u |s-u|^{2H_0-2} q(B_s,B_u) ds du\right] \\
& + 2\mathbb{E}\Bigg[\int_{[t_i,r]^2}\int_{[u,r]}\int_{[t_i,s]}\int_{\mathbb{R}^{2d}} D^W_{s',y} Y_u\, D^W_{v,w} Y_s \\
& \qquad \times |u-v|^{2H_0-2}|s-s'|^{2H_0-2} q(B_u,w) q(B_s,y) dw dy du dv ds' ds\Bigg] \\
& + \mathbb{E}\left|\int_r^{t_i} Z_s dB_s\right|^2 \\
\le\, & 2\int_{[t_i,r]^2} \mathbb{E}\left[|Y_s|^2\right]^{1/2} \mathbb{E}\left[|Y_u|^2\right]^{1/2} |s-u|^{2H_0-2} \sup_{\omega,s,u} \big|q(B_s,B_u)\big| ds du \\
& + 2\int_{[t_i,r]^2}\int_{[u,r]}\int_{[t_i,s]}\int_{\mathbb{R}^{2d}} \mathbb{E}\left[|D^W_{s',y}Y_u|^2\right]^{1/2} \mathbb{E}\left[|D^W_{v,w}Y_s|^2\right]^{1/2} |u-v|^{2H_0-2} \\
& \qquad \times |s-s'|^{2H_0-2} \sup_{\omega,s,u,w,y} \big|q(B_u,w)q(B_s,y)\big| dw dy dv du ds' ds\,. \\
& + \int_r^{t_i} \mathbb{E}\,|Z_s|^2\, ds \\
\le\, & C(|r-t_i|^{2H_0} + |r-t_i|^{4H_0} + |r-t_i|).
\end{aligned} \tag{5.13}$$

Taking this result back to (5.10) we get

$$\begin{aligned}
|R_{1,i}|^2 \lesssim\, & C\Bigg(\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} |r-t_i|^{1/2}|s-t_i|^{1/2}|s-r|^{2H_0-2} ds dr \\
& \qquad + \int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\int_r^{t_{i+1}}\int_s^{t_{i+1}} |u-v|^{2H_0-2}|s-r|^{2H_0-2} du dv ds dr\Bigg). \\
\le\, & C\Bigg(|t_{i+1}-t_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} |s-r|^{2H_0-2} ds dr \\
& \qquad + \int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\int_r^{t_{i+1}}\int_s^{t_{i+1}} |u-v|^{2H_0-2}|s-r|^{2H_0-2} du dv ds dr\Bigg). \\
\le\, & C\Big(|t_{i+1}-t_i|^{1+2H_0} + |t_{i+1}-t_i|^{4H_0}\Big).
\end{aligned} \tag{5.14}$$

Hence we have

$$\sum_{i=0}^{n-1} |R_{1,i}| \lesssim \sum_{i=0}^{n-1} (t_{i+1}-t_i)^{1/2+H_0} \le \max_{0\le i\le n-1} (t_{i+1}-t_i)^{1/2+H_0-1} \sum_{i=0}^{n-1} (t_{i+1}-t_i) \to 0\,,\ n\to\infty.$$

Using (7.1) again, we get

$$
\begin{aligned}
|R_{5,i}|^2 &\lesssim \mathbb{E}\left(\int_{t_i}^{t_{i+1}} (\alpha_0^s - \alpha_0^{t_i}) W(ds, B_s)\right)^2 \\
&= \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} (\alpha_0^s - \alpha_0^{t_i})(\alpha_0^r - \alpha_0^{t_i}) |s-r|^{2H_0-2} q(B_s, B_r) ds dr\right] \\
&\qquad + \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_r^{t_{i+1}} \int_{t_i}^s \int_{\mathbb{R}^{2d}} D_{r,y}^W (\alpha_0^u - \alpha_0^{t_i}) \cdot D_{v,w}^W (\alpha_0^s - \alpha_0^{t_i}) \\
&\qquad\qquad \times |u-v|^{2H_0-2} |s-r|^{2H_0-2} q(B_u, w) q(B_s, y) dw dy dv du dr ds\Big] \\
&:= R_{5,1,i} + R_{5,2,i}.
\end{aligned} \tag{5.15}
$$

For $R_{5,2,i}$, recalling (5.1) we have

$$
\begin{aligned}
R_{5,2,i} &= \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_r^{t_{i+1}} \int_{t_i}^s \int_{\mathbb{R}^{2d}} (\alpha_0^u - \alpha_0^{t_i})\, \delta(B_r - y) \cdot (\alpha_0^s - \alpha_0^{t_i})\, \delta(B_v - w) \\
&\qquad \times |u-v|^{2H_0-2} |s-r|^{2H_0-2} q(B_u, w) q(B_s, y) dw dy dv du dr ds\Big] \\
&= \Big[\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_r^{t_{i+1}} \int_{t_i}^s (\alpha_0^u - \alpha_0^{t_i})\, (\alpha_0^s - \alpha_0^{t_i}) \\
&\qquad \times |u-v|^{2H_0-2} |s-r|^{2H_0-2} q(B_u, B_v) q(B_s, B_r) dv du dr ds\Big].
\end{aligned} \tag{5.16}
$$

Taking this result back to (5.15), and with the help of (4.2), (4.3), we obtain

$$
\begin{aligned}
|R_{5,i}|^2 &\lesssim \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[(\alpha_0^r - \alpha_0^{t_i})\, (\alpha_0^s - \alpha_0^{t_i})\, |s-r|^{2H_0-2} q(B_r, B_s)\right] ds dr \\
&\quad + \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_r^{t_{i+1}} \int_{t_i}^s \mathbb{E}\Big[(\alpha_0^u - \alpha_0^{t_i}) \cdot (\alpha_0^s - \alpha_0^{t_i}) \\
&\qquad\qquad \times |u-v|^{2H_0-2} |s-r|^{2H_0-2} q(B_u, B_v)\, q(B_r, B_s)\Big] du dv ds dr \\
&\le C \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathbb{E}\big[|\alpha_0^r - \alpha_0^{t_i}|^4\big]^{1/4} \mathbb{E}\big[|\alpha_0^r - \alpha_0^{t_i}|^4\big]^{1/4} |r-s|^{2H_0-2} \mathbb{E}\big[|q(B_r, B_s)|^2\big]^{1/2} ds dr \\
&\quad + \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_r^{t_{i+1}} \int_{t_i}^s \mathbb{E}\big[|\alpha_0^u - \alpha_0^{t_i}|^4\big]^{1/4} \mathbb{E}\big[|\alpha_0^s - \alpha_0^{t_i}|^4\big]^{1/4} \\
&\qquad\qquad \times |r-s|^{2H_0-2} |u-v|^{2H_0-2} \mathbb{E}\big[|q(B_u, B_v) q(B_r, B_s)|^2\big]^{1/2} du dv ds dr \\
&\le C \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} (r-t_i)^{H_0} (s-t_i)^{H_0} (r-s)^{2H_0-2} ds dr \\
&\quad + C \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_r^{t_{i+1}} \int_{t_i}^s (u-t_i)^{H_0} (s-t_i)^{H_0} |r-s|^{2H_0-2} |u-v|^{2H_0-2} du dv ds dr \\
&\le C (t_{i+1} - t_i)^{2H_0} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} (r-s)^{2H_0-2} ds dr \\
&\quad + C (t_{i+1} - t_i)^{2H_0} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_r^{t_{i+1}} \int_{t_i}^s |r-s|^{2H_0-2} |u-v|^{2H_0-2} du dv ds dr.
\end{aligned} \tag{5.17}
$$

$$\leq C\big((t_{i+1}-t_i)^{4H_0}+(t_{i+1}-t_i)^{6H_0}\big).$$

Thus

$$\sum_{i=0}^{n-1}|R_{5,i}|\lesssim\sum_{i=0}^{n-1}(t_{i+1}-t_i)^{2H_0}\leq\max_{0\leq i\leq n-1}(t_{i+1}-t_i)^{2H_0-1}\sum_{i=0}^{n-1}(t_{i+1}-t_i)\to 0,\ \ n\to\infty.$$

For $R_{3,i}$, from the orthogonality of the increments of standard Brownian motion and the fact that $\alpha_0^{t_i}=\exp\left\{\int_0^{t_i}W(dr,B_r)\right\}$ is $\mathcal{F}_{t_i}$-adapted, we have

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{i=1}^{n}\alpha_0^{t_i}\int_{t_i}^{t_{i+1}}[Z_r-Z_{t_i}]\,dB_r\right|^2\right]\leq&\sum_{i=1}^{n}\mathbb{E}\left|\alpha_0^{t_i}\right|^2\mathbb{E}\left|\int_{t_i}^{t_{i+1}}[Z_r-Z_{t_i}]\,dB_r\right|^2\\ \leq& C\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\mathbb{E}\left|Z_r-Z_{t_i}\right|^2dr.\end{aligned}\tag{5.18}$$

If Z satisfies condition (3) in Theorem 1.1, we have easily

$$\left|\sum_{i=0}^{n-1}R_{3,i}\right|^2\lesssim C\sum_{i=0}^{n-1}|t_{i+1}-t_i|^{\kappa+1}.\tag{5.19}$$

Denote $\tilde{Y}_t=D_r^BY_t,\ \tilde{Z}_t=D_r^BZ_t$ (we fix r), and from (1.2) we obtain

$$\begin{aligned}D_r^BY_t=\tilde{Y}_t=&D_r^B\xi+\int_t^T\tilde{Y}_sW(ds,B_s)\\ &+\int_r^TY_s\nabla_xW(ds,B_s)-\int_t^T\tilde{Z}_sdB_s,\ \ 0\leq t\leq r\leq T.\end{aligned}\tag{5.20}$$

Therefore, we first need to verify the square integrability of $\int_t^TY_s\nabla_xW(ds,B_s)$, and then we can treat (5.20) in a similar way to that for (1.2). We can write

$$\mathbb{E}\Big|\int_t^TY_s\nabla_xW(ds,B_s)\Big|^2=\mathbb{E}\Big|\int_t^TY_s\nabla_x\delta(B_s-x)W(ds,x)dx\Big|^2\tag{5.21}$$

From (7.1) and by integration by parts, for all $0\leq t\leq T$,

$$\begin{aligned}&\mathbb{E}\Big|\int_t^TY_s\nabla_xW(ds,B_s)\Big|^2\\ =&\mathbb{E}\left[\int_{[t,T]^2}Y_rY_s|s-r|^{2H_0-2}\nabla_{x,y}q(B_r,B_s)dydrds\right]\\ &+\mathbb{E}\Bigg[\int_{[a,b]^2}\int_{\mathbb{R}^d}\int_{[r,b]}\int_{[s,b]}\int_{\mathbb{R}^d}D^W_{r,y}Y_u\,D^W_{v,w}Y_s\\ &\qquad\times|u-v|^{2H_0-2}|s-r|^{2H_0-2}\nabla_xq(B_u,w)\nabla_xq(B_s,y)dwdudvdydrds\Bigg]\\ \leq&\int_{[t,T]^2}\mathbb{E}\Big[\big|Y_r\big|^2\Big]^{1/2}\mathbb{E}\Big[\big|Y_s\big|^2\Big]^{1/2}|s-r|^{2H_0-2}\sup_{\omega,r,s}\big|\nabla_{x,y}q(B_r,B_s)\big|dydrds\end{aligned}\tag{5.22}$$

$$
\begin{aligned}
&+\mathbb{E}\Bigg[\int_{[t,T]^2}\int_{\mathbb{R}^d}\int_{[r,T]}\int_{[s,T]}\int_{\mathbb{R}^d}\mathbb{E}\Big[\big|D^W_{r,y}Y_u\big|^2\Big]^{1/2}\mathbb{E}\Big[\big|D^W_{v,w}Y_s\big|^2\Big]^{1/2}\\
&\qquad\times|u-v|^{2H_0-2}|s-r|^{2H_0-2}\sup_{\omega,u,s,w,y}\big|\nabla_x q(B_u,w)\nabla_x q(B_s,y)\big|dwdudvdydrds\Bigg]\\
\le& C\big(|T-t|^{2H_0}+|T-t|^{4H_0}\big)\le CT^{4H_0}.
\end{aligned}
$$

Thus we have $(\tilde{Y}_t,\tilde{Z}_t)$ of BSDE (5.20) is well-defined, i.e., $\mathbb{E}\Big[\int_0^T|\tilde{Y}_t|^2+|\tilde{Z}_t|^2dt\Big]<\infty$. Using the classical conclusion that $Z_t=D^B_tY_t$, $\forall t\in[0,T]$ (see e.g. [13]), we can treat $Z_r-Z_{t_i}$ as

$$
\begin{aligned}
Z_r-Z_{t_i}&=(D^B_r\xi-D^B_{t_i}\xi)+\int_r^{t_i}D^B_rY_sW(ds,B_s)\\
&\quad+\int_r^{t_i}Y_s\nabla_xW(ds,B_s)-\int_r^{t_i}D^B_rZ_sdB_s,\ 0\le t_i\le r\le T\\
&=\tilde{Z}_1+\tilde{Z}_2+\tilde{Z}_3+\tilde{Z}_4\,.
\end{aligned}
\tag{5.23}
$$

For the above first term $\tilde{Z}_1$ we can use the assumption $\mathbb{E}|D^B_r\xi-D^B_{t_i}\xi|^2\le C|r-t_i|^\kappa$ for some $\kappa>0$. We can deal with the second term $\tilde{Z}_2$ in (5.23) in the similar way as in (5.8). In fact, with the help of (7.1) again, it has

$$
\begin{aligned}
&\mathbb{E}\Big|\tilde{Z}_2\Big|^2\\
&\le\int_r^{t_{i+1}}\int_r^{t_{i+1}}\mathbb{E}\Big[|D^B_rY_s|^2\Big]^{1/2}\mathbb{E}\Big[|D^B_rY_{s'}|^2\Big]^{1/2}|s-s'|^{2H_0-2}\sup_{\omega,s,s'}\Big|q(B_s,B'_s)\Big|dsds'\\
&\quad+\int_r^{t_{i+1}}\int_r^{t_{i+1}}\int_{s'}^{t_{i+1}}\int_r^{s}\Bigg(\mathbb{E}\Big[|D^W_{s',y}(D^B_rY_{s'})|^2\Big]^{1/2}\mathbb{E}\Big[D^W_{v,w}(|D^B_rY_s)|^2\Big]^{1/2}\\
&\qquad\times|u-v|^{2H_0-2}|s-s'|^{2H_0-2}\sup_{\omega,u,s,w,y}\Big|q(B_u,w)q(B_s,y)\Big|\Bigg)dudvdsds'\,.
\end{aligned}
\tag{5.24}
$$

Since $Y,D^BY\in\mathbb{D}^{1,2}$, we have the estimate

$$
\begin{aligned}
\mathbb{E}\Big|\tilde{Z}_2\Big|^2&\le C\int_r^{t_{i+1}}\int_r^{t_{i+1}}|s-s'|^{2H_0-2}dsds'\\
&\quad+C\int_r^{t_{i+1}}\int_r^{t_{i+1}}\int_{s'}^{t_{i+1}}\int_r^{s}|u-v|^{2H_0-2}|s-s'|^{2H_0-2}dudvdsds'\\
&\le C\left(|t_{i+1}-r|^{2H_0}+|t_{i+1}-r|^{4H_0}\right).
\end{aligned}
\tag{5.25}
$$

From (5.22) we have

$$
\mathbb{E}\Big|\tilde{Z}_3\Big|^2\le C\left(|t_{i+1}-r|^{2H_0}+|t_{i+1}-r|^{4H_0}\right).
\tag{5.26}
$$

Finally it is easy to obtain

$$
\mathbb{E}|\tilde{Z}_4|^2\le\sup_{s\in[r,t_i]}\mathbb{E}|D^B_rZ_s|^2\,|t_i-r|\le C\,|t_i-r|.
\tag{5.27}
$$

Taking those estimates back to (5.18) we have

$$\left(\sum_{i=1}^n \int_{t_i}^{t_{i+1}} \mathbb{E}\left|Z_r - Z_{t_i}\right|^2 dr\right)^{1/2} \le C\left(\sum_{i=1}^n \big(|t_{i+1}-t_i|^{2H_0+1} + |t_{i+1}-t_i|^{4H_0+1} + |t_{i+1}-t_i|^{\kappa+1} + |t_{i+1}-t_i|^2\big)\right)^{1/2}, \tag{5.28}$$

which implies

$$\begin{aligned}\left|\sum_{i=1}^n R_{3,i}\right|^2 &\lesssim C\sum_{i=1}^n \big(|t_{i+1}-t_i|^{2H_0+1} + |t_{i+1}-t_i|^{4H_0+1} \\ &\quad + |t_{i+1}-t_i|^{\kappa+1} + |t_{i+1}-t_i|^2\big) \\ &\le \max_{0\le i\le n-1}(t_{i+1}-t_i)^{\kappa\wedge 1}\sum_{i=1}^n |t_{i+1}-t_i| \to 0,\ \ n\to\infty.\end{aligned} \tag{5.29}$$

For $R_{2,i}$ and $R_{4,i}$, it is easy to deduce that

$$\begin{aligned}|R_{2,i}|^2 &\lesssim C\left(\mathbb{E}\Big|Y_{t_i}\Big|^2 \mathbb{E}\left[\Big|\alpha_0^{t_{i+1}} - \alpha_0^{t_i}\Big|^4\right]^{1/2} \mathbb{E}\left[\Big|\int_{t_i}^{t_{i+1}} W(dr,B_r)\Big|^4\right]^{1/2}\right) \\ &\le C\,|t_{i+1}-t_i|^{4H_0},\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}|R_{4,i}|^2 &\lesssim C\mathbb{E}\left[\Big|\alpha_0^{t_{i+1}} - \alpha_0^{t_i}\Big|^4\right]^{1/2} \mathbb{E}\left[\Big|\int_{t_i}^{t_{i+1}} Z_r dB_r\Big|^2\right] \\ &\le C\,|t_{i+1}-t_i|^{2H_0+1}.\end{aligned} \tag{5.31}$$

Thus we have

$$\sum_{i=0}^{n-1}|R_{2,i}| \lesssim \sum_{i=0}^{n-1}(t_{i+1}-t_i)^{2H_0} \le \max_{0\le i\le n-1}(t_{i+1}-t_i)^{2H_0-1}\sum_{i=0}^{n-1}(t_{i+1}-t_i) \to 0,\ \ n\to\infty.$$

and

$$\sum_{i=0}^{n-1}|R_{4,i}| \lesssim \sum_{i=0}^{n-1}(t_{i+1}-t_i)^{H_0+1/2} \le \max_{0\le i\le n-1}(t_{i+1}-t_i)^{H_0-1/2}\sum_{i=0}^{n-1}(t_{i+1}-t_i) \to 0,\ \ n\to\infty.$$

Hence, letting the mesh size $|\pi_n|$ goes to zero yields $\tilde{R}_t^n \to 0$, P-a.s., and the right side of (5.6) converges to $\int_t^T \alpha_0^r Z_r dB_r$. This concludes the proof of the lemma. □

Proof of Proposition 5.1. In equation (5.5) we take the conditional expectation with respect to $\mathcal{F}_t^B$ we see to obtain

$$\alpha_0^t Y_t = \mathbb{E}\left[\alpha_0^T \xi | \mathcal{F}_t\right].$$

Thus,

$$Y_t = \left(\alpha_0^t\right)^{-1}\mathbb{E}^B\left[\alpha_0^T\xi|\mathcal{F}_t^B\right] = \mathbb{E}^B\left[\alpha_t^T\xi\big|\mathcal{F}_t^B\right] = \mathbb{E}^B\left[\xi\exp\big(\int_t^T W(dr,B_r)\big)\,\big|\,\mathcal{F}_t^B\right]. \tag{5.32}$$

From the general relationship between Z and Y (e.g. [13]) we have

$$Z_t = D_t^B Y_t = D_t^B \mathbb{E}^B \left[\xi \exp\Big(\int_t^T W(dr, B_r) \Big) \,\Big|\, \mathcal{F}_t^B \right] . \tag{5.33}$$

This concludes the proof of the proposition. □

6 BSDEs and semilinear SPDEs

In this section we obtain the regularity of the solution to the BSDE, and then establish the relationship between the SPDE

$$-du(t,x) = \frac{1}{2}\Delta u(t,x)dt + u(t,x)W(dt,x), \;\; u(T,x) = \phi(x) . \tag{6.1}$$

and our BSDE

$$Y_s^{t,x} = \phi(B_{T-t}^{t,x}) + \int_s^T Y_r^{t,x} W(dr, B_{r-t}^{t,x}) - \int_s^t Z_r^{t,x} dB_r \, , \;\; s \in [t,T]. \tag{6.2}$$

Theorem 6.1. *Suppose $\phi \in C^2(\mathbb{R}^d)$. Let $\{u(t,x) : t \in [0,T], x \in \mathbb{R}^d\}$ be a random field such that $u(t,x)$ is $\mathcal{F}_t$-measurable for each (t,x), $u \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$ a.s. and let $u(t,x)$ satisfy (6.1). Then $u(t,x) = Y_t^{t,x}$, $\nabla u(t,x) = Z_t^{t,x}$, where $(Y_t^{t,x}, Z_t^{t,x})$ is the solution of (6.2).*

Proof. It suffices to show that $(u(s, X_s^{t,x}), \nabla u(s, X_s^{t,x}))_{s\in[t,T]}$ solves BSDE (6.2). Since $W(t,x)$ is not differentiable in t and x, one could not apply Itô's formula to $u(s, X_s^{t,x})$. Let us consider

$$-du^{\varepsilon,\eta}(t,x) = \frac{1}{2}\Delta u^{\varepsilon,\eta}(t,x)dt + u^{\varepsilon,\eta}(t,x)\dot{W}_{\varepsilon,\eta}(dt,x), \;\; u(T,x) = \phi(x) . \tag{6.3}$$

Recall (2.3) we have

$$\dot{W}_{\varepsilon,\eta}(s,x) = \int_0^s \int_{R^d} \varphi_\eta(s-r) p_\varepsilon(x-y) W(dr,y)dy \, .$$

We see that $u(t,x)$ is differentiable with respect to x. Now we can use Itô's formula to $u^{\varepsilon,\eta}(s, X_s^{t,x})$ to deduce

$$u^{\varepsilon,\eta}(t, X_s^{t,x}) = \phi(X_T^{t,x}) + \int_s^T u^{\varepsilon,\eta}(r, X_r^{t,x}) \dot{W}_{\varepsilon,\eta}(r, X_r^{t,x}) dr - \int_s^T \nabla u^{\varepsilon,\eta}(r, X_r^{t,x}) dB_r. \tag{6.4}$$

Note that $X_r^{t,x} = x + B_r - B_t = B_{r-t}^x$, and by the uniqueness of BSDE we know $Y_s^{t,x,\varepsilon,\eta} = u^{\varepsilon,\eta}(s, X_s^{t,x})$, $Z_s^{t,x,\varepsilon,\eta} = \nabla u^{\varepsilon,\eta}(s, X_s^{t,x})$ satisfy

$$Y_s^{t,x,\varepsilon,\eta} = \phi(B_{T-t}^{t,x}) + \int_s^T Y_r^{t,x,\varepsilon,\eta} \dot{W}(r, B_{r-t}^{t,x}) dr - \int_s^t Z_r^{t,x,\varepsilon,\eta} dB_r \, , \;\; s \in [t,T]. \tag{6.5}$$

Theorem 1.1 yields

$$Y_s = \lim_{\varepsilon,\eta\to 0} u^{\varepsilon,\eta}(s, X_s^{t,x}) = \lim_{\varepsilon,\eta\to 0} Y_s^{t,x,\varepsilon,\eta}, \qquad Z_s = \lim_{\varepsilon,\eta\to 0} \nabla u^{\varepsilon,\eta}(s, X_s^{t,x}) = \lim_{\varepsilon,\eta\to 0} Z_s^{t,x,\varepsilon,\eta}$$

is a solution pair of BSDE (6.2). It remains to show SPDE (6.3) converges to (6.1). From the classical Feynman-Kac formula it follows

$$u^{\varepsilon,\eta}(t,x)=\mathbb{E}^B\left[\phi(X_T^{t,x})\exp\Big\{\int_t^T \dot{W}^{\varepsilon,\eta}(r,X_r^{t,x})dr\Big\}\right]=\mathbb{E}^B\left[\phi(B_{T-t}^x)\exp\Big\{\int_t^T \dot{W}^{\varepsilon,\eta}(r,(B_{T-t}^x))dr\Big\}\right]. \tag{6.6}$$

Define

$$u(t,x)=\mathbb{E}\left[\phi(B_{T-t}^x)\exp\Big\{\int_t^T W(dr,(B_{T-t}^x))\Big\}\right]. \tag{6.7}$$

Similar to the proof of Lemma 3.1, we can deduce

$$\lim_{\varepsilon,\eta\to 0}\mathbb{E}^W\big|u^{\varepsilon,\eta}(t,x)-u(t,x)\big|^p=0 \ \text{ for all } p\geq 2. \tag{6.8}$$

Since $u^{\varepsilon,\eta}$ satisfies (6.3) for any C^∞ function ψ with compact support, we have

$$\begin{aligned}\int_{\mathbb{R}^d}u^{\varepsilon,\eta}(t,x)\psi(x)dx=&\int_{\mathbb{R}^d}\phi(x)\psi(x)dx+\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}u^{\varepsilon,\eta}(s,x)\Delta\psi(x)dxds\\&+\int_t^T\int_{\mathbb{R}^d}u^{\varepsilon,\eta}(s,x)\psi(x)\dot{W}_{\varepsilon,\eta}(s,x)dsdx.\end{aligned} \tag{6.9}$$

Letting $\varepsilon,\eta\to 0$ will yield

$$\begin{aligned}\int_{\mathbb{R}^d}u(t,x)\psi(x)dx=&\int_{\mathbb{R}^d}\phi(x)\psi(x)dx+\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}u(s,x)\Delta\psi(x)dxds\\&+\int_t^T\int_{\mathbb{R}^d}u(s,x)\psi(x)W(ds,x)dx,\end{aligned} \tag{6.10}$$

since

$$\lim_{\varepsilon,\eta\to 0}\int_t^T\int_{\mathbb{R}^d}u^{\varepsilon,\eta}(s,x)\psi(x)\dot{W}_{\varepsilon,\eta}(s,x)dsdx=\int_t^T\int_{\mathbb{R}^d}u(s,x)\psi(x)W(ds,x)dx. \tag{6.11}$$

In fact, (6.11) can be deduced in a similar way to that of Theorem 3.9. This proves the conclusion. □

Theorem 6.2. *Suppose the same conditions as in Theorem 1.1 and let $(Y_s^{t,x},Z_s^{t,x})$ be the solution pair of BSDE* (6.2). *Then $u(t,x):=Y_t^{t,x}, t\in[0,T], x\in\mathbb{R}^d$ is in $C([0,T]\times\mathbb{R}^d,\mathbb{R})$ and is the solution of SPDE* (6.1).

Proof. Notice that $u\left(t+h,X_{t+h}^{t,x}\right)=Y_{t+h}^{t+h,X_{t+h}^{t,x}}=Y_{t+h}^{t,x}$. We still use the approximated BSDE (6.5). Define $u^{\varepsilon,\eta}(t,x):=Y_t^{\varepsilon,\eta,t,x}, t\in[0,T], x\in\mathbb{R}^d$. We want to show that $u^{\varepsilon,\eta}(t,x)$ satisfies (6.1). An application of Itô's formula yields that for $h>0$

$$\begin{aligned}u^{\varepsilon,\eta}\left(t+h,X_t^{t,x}\right)-u^{\varepsilon,\eta}\left(t+h,X_{t+h}^{t,x}\right)=&-\int_t^{t+h}\frac{1}{2}\Delta u^{\varepsilon,\eta}\left(t+h,X_s^{t,x}\right)ds\\&-\int_t^{t+h}\nabla u^{\varepsilon,\eta}\left(t+h,X_s^{t,x}\right)dB_s\,.\end{aligned} \tag{6.12}$$

Combining this with the backward SDE satisfied by $u^{\varepsilon,\eta}(t,x) := Y_t^{\varepsilon,\eta,t,x}, t \in [0,T], x \in \mathbb{R}^d$ we have

$$
\begin{aligned}
u^{\varepsilon,\eta}(t+h,x)-u^{\varepsilon,\eta}(t,x) &= u^{\varepsilon,\eta}\left(t+h, X_t^{t,x}\right) - u^{\varepsilon,\eta}\left(t+h, X_{t+h}^{t,x}\right) + u^{\varepsilon,\eta}\left(t+h, X_{t+h}^{t,x}\right) - u^{\varepsilon,\eta}(t,x) \\
&= -\int_t^{t+h} \frac{1}{2}\Delta u^{\varepsilon,\eta}\left(t+h, X_s^{t,x}\right) ds - \int_t^{t+h} \nabla u^{\varepsilon,\eta}\left(t+h, X_s^{t,x}\right) dB_s \\
&\quad - \int_t^{t+h} Y_s^{\varepsilon,\eta,t,x} \dot{W}_{\varepsilon,\eta}(s, X_s^{t,x}) ds + \int_t^{t+h} Z_s^{\varepsilon,\eta,t,x} dB_s.
\end{aligned}
\tag{6.13}
$$

Thus, let π_n be a partition $t = t_0 < t_1 < \cdots < t_n = T$. By (6.13), we have

$$
\begin{aligned}
\phi(x) - u^{\varepsilon,\eta}(t,x) = &-\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2}\Delta u^{\varepsilon,\eta}\left(t_i, X_s^{t_i,x}\right) ds - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} u^{\varepsilon,\eta}(s, X_s^{t_i,x}) \dot{W}_{\varepsilon,\eta}\left(s, X_s^{t_i,x}\right) ds \\
&+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[Z_s^{\varepsilon,\eta,t_i,x} - \nabla u^{\varepsilon,\eta}\left(t_i, X_s^{t_i,x}\right)\right] dB_s. .
\end{aligned}
\tag{6.14}
$$

On the other hand, it is elementary to show that random field $\{Z_s^{\varepsilon,\eta,t,x}, t \le s \le T\}$ has a continuous version (e.g., [7, Proposition 5.2]) such that

$$
Z_s^{\varepsilon,\eta,t,x} = D_s^B Y_s^{\varepsilon,\eta,t,x} = \nabla Y_s^{\varepsilon,\eta,t,x} (\nabla X_s^{t,x})^{-1}, \tag{6.15}
$$

and in particular, $Z_t^{\varepsilon,\eta,t,x} = \nabla Y_t^{\varepsilon,\eta,t,x}$. Thus, if we let mesh sizes of the partitions π_n go to zero, then it yields

$$
\phi(x) - u^{\varepsilon,\eta}(t,x) = -\int_0^t \frac{1}{2}\Delta u^{\varepsilon,\eta}(s,x)\, ds - \int_0^t u^{\varepsilon,\eta}(s,x) \dot{W}_{\varepsilon,\eta}(s,x)\, ds\,, \tag{6.16}
$$

or by Duhamel's principle we obtain

$$
\phi(x) - u^{\varepsilon,\eta}(t,x) = -\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u^{\varepsilon,\eta}(s,y) \dot{W}_{\varepsilon,\eta}(s,x)\, dyds\,. \tag{6.17}
$$

From Theorem 3.9, letting $\varepsilon, \eta \to 0$ we get

$$
\phi(x) - u(t,x) = -\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y) W(ds,y)\, dy\,. \tag{6.18}
$$

The above formula means that $u(t,x) := Y_t^{t,x}$ of BSDE (6.2) is a mild solution of SPDE (6.1). $\square$

7 Appendix

Proposition 7.1. *Let Y be a process such that its Malliavin derivative exists and assume that $D_{s,y}^W Y$ is integrable with respect to s. Then*

$$
\begin{aligned}
\mathbb{E}\left(\int_a^b Y_s W(ds, B_s)\right)^2 = &\ \mathbb{E}\left[\int_{[a,b]^2} Y_r Y_s |s-r|^{2H_0-2} q(B_r, B_s) drds\right] \\
&+ \mathbb{E}\Bigg[\int_{[a,b]^2} \int_{[r,b]} \int_{[a,s]} \int_{\mathbb{R}^{2d}} D_{r,y}^W Y_u\, D_{v,w}^W Y_s |u-v|^{2H_0-2} \\
&\qquad |s-r|^{2H_0-2} q(B_u, w) q(B_s, y) dwdydvdudrds\Bigg].
\end{aligned}
\tag{7.1}
$$

Proof. Recalling $W(\phi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \phi(t,x) W(dt,x)dx$. We have

$$\mathbb{E}\left(\int_a^b Y_s W(ds, B_s)\right)^2 = \mathbb{E}\left(\int_a^b \int_{\mathbb{R}^d} Y_s \delta(B_s - x) W(ds,x)dx\right)^2. \tag{7.2}$$

Denote by $F := \int_a^b Y_s W(ds, B_s)$ and we shall use $F \cdot W(\phi) = \delta(F\phi) + \langle D^W F, \phi \rangle_{\mathcal{H}}$. From the definition of spatial covariance (2.1), it follows

$$\begin{aligned}
\mathbb{E}\left(\int_a^b Y_s W(ds, B_s)\right)^2 &= \mathbb{E}\left(F \cdot \int_a^b \int_{\mathbb{R}^d} Y_s \delta(B_s - x) W(ds,x)dx\right) = \mathbb{E}\left(\left\langle D^W F, Y_\cdot \delta(B_\cdot - \cdot)\right\rangle_{\mathcal{H}}\right) \\
&= \mathbb{E}\left[\int_{[a,b]^2} \int_{\mathbb{R}^{2d}} D^W_{r,y} F \cdot Y_s \delta(B_s - z) |s-r|^{2H_0 - 2} q(y,z) dy dz dr ds\right] \\
&= \mathbb{E}\left[\int_{[a,b]^2} \int_{\mathbb{R}^d} D^W_{r,y} F \cdot Y_s |s-r|^{2H_0 - 2} q(y, B_s) dy dr ds\right].
\end{aligned} \tag{7.3}$$

Note that,

$$\begin{aligned}
D^W_{r,y} F &= D^W_{r,y}\left(\int_a^b \int_{\mathbb{R}^d} Y_s \delta(B_s - x) W(ds,x)dx\right) \\
&= \int_r^b \int_{\mathbb{R}^d} D^W_{r,y} Y_s \delta(B_s - x) W(ds,x)dx + Y_r \delta(B_r - y).
\end{aligned} \tag{7.4}$$

Substituting this computation into(7.1) we have

$$\mathbb{E}\left(\int_a^b Y_s W(ds, B_s)\right)^2 = I_1 + I_2\,. \tag{7.5}$$

For I_2, it is easy to deduce

$$\begin{aligned}
I_2 =& \mathbb{E}\left[\int_{[a,b]^2} \int_{\mathbb{R}^d} Y_r \delta(B_r - y) \cdot Y_s |s-r|^{2H_0 - 2} q(y, B_s) dy dr ds\right] \\
=& \mathbb{E}\left[\int_{[a,b]^2} Y_r Y_s |s-r|^{2H_0 - 2} q(B_r, B_s) dy dr ds\right].
\end{aligned} \tag{7.6}$$

I_1 has the following expression

$$\begin{aligned}
I_1 =& \mathbb{E}\left[\int_{[a,b]^2} \int_{\mathbb{R}^d} \int_r^b \int_{\mathbb{R}^d} D^W_{r,y} Y_s \delta(B_s - x) W(ds,x)dx \cdot Y_s |s-r|^{2H_0 - 2} q(y, B_s) dr ds dy\right] \\
=& \mathbb{E}\left[\int_{[a,b]^2} \int_{\mathbb{R}^d} I_3 |s-r|^{2H_0 - 2} q(y, B_s) dr ds dy\right]
\end{aligned} \tag{7.7}$$

where

$$I_3 = \int_r^b \int_{\mathbb{R}^d} D^W_{r,y} Y_s \delta(B_s - x) W(ds,x)dx \cdot Y_s\,.$$

Using $F \cdot W(\phi) = \delta(F\phi) + \langle D^W F, \phi \rangle_{\mathcal{H}}$ again we have

$$\begin{aligned}
I_3 =& \mathbb{E}^W \left[\int_r^b \int_{\mathbb{R}^d} D^W_{r,y} Y_s \delta(B_s - x) W(ds, x) dx \cdot Y_s \right] = \mathbb{E}^W \left\langle D^W_{r,y} Y_\cdot \delta(B_\cdot - \cdot), D^W Y_s \right\rangle_{\mathcal{H}} \\
=& \mathbb{E}^W \left[\int_{[r,b]} \int_{[a,s]} \int_{\mathbb{R}^{2d}} D^W_{r,y} Y_u \delta(B_u - x) D^W_{v,w} Y_s |u - v|^{2H_0 - 2} q(x, w) dx dw dv du \right] \\
=& \mathbb{E}^W \left[\int_{[r,b]} \int_{[a,s]} \int_{\mathbb{R}^d} D^W_{r,y} Y_u \, D^W_{v,w} Y_s |u - v|^{2H_0 - 2} q(B_u, w) dw dv du \right] .
\end{aligned} \tag{7.8}$$

Substituting this back to (7.7) we obtain

$$\begin{aligned}
I_2 =& \mathbb{E} \Big[\int_{[a,b]^2} \int_{[r,b]} \int_{[a,s]} \int_{\mathbb{R}^{2d}} D^W_{r,y} Y_u \, D^W_{v,w} Y_s |u - v|^{2H_0 - 2} \\
& \qquad |s - r|^{2H_0 - 2} q(B_u, w) q(y, B_s) dw dy du dv dr ds \Big] .
\end{aligned} \tag{7.9}$$

Inserting the expressions for I_1 and I_2 into (7.5) yields the proposition. □

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