SYMMETRIC PROPERTY AND EDGE-DISJOINT HAMILTONIAN CYCLES OF THE SPINED CUBE

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Abstract. The spined cube SQ_n is a variant of the hypercube Q_n, introduced by Zhou et al. in [Information Processing Letters 111 (2011) 561-567] as an interconnection network for parallel computing. A graph Γ is an m-Cayley graph if its automorphism group $\mathrm{Aut}(\Gamma)$ has a semiregular subgroup acting on the vertex set with m orbits, and is a Caley graph if it is a 1-Cayley graph. It is well-known that Q_n is a Cayley graph of an elementary abelian 2-group $\mathbb{Z}_2^n$ of order 2^n. In this paper, we prove that SQ_n is a 4-Cayley graph of $\mathbb{Z}_2^{n-2}$ when $n \geq 6$, and is a $\lfloor n/2 \rfloor$-Cayley graph when $n \leq 5$. This symmetric property shows that an n-dimensional spined cube with $n \geq 6$ can be decomposed to eight vertex-disjoint $(n-3)$-dimensional hypercubes, and as an application, it is proved that there exist two edge-disjoint Hamiltonian cycles in SQ_n when $n \geq 4$. Moreover, we determine the vertex-transitivity of SQ_n, and prove that SQ_n is not vertex-transitive unless $n \leq 3$.

Keywords: Interconnection network, The spined cube, 4-Cayley graphs, Hamiltonian cycles.

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1. Introduction

An interconnection network, say network shortly, is the backbone of a parallel computing system, and connects the processors of the system. The computational cost of a parallel computing system is heavily dominated by the communication cost of the underlying network, which decides the overall performance of the system. This fact clearly emphasizes the significance of network topology and its efficient structural designs [6, 30]. The topological structure of the underlying network can be modeled as a graph where vertices correspond to processors, memory modules or switches, and edges correspond to communication links. It has been universally accepted and used by computer scientists and engineers [1, 30].

1.1. **Symmetric property of networks.** In the design of a network, it is desirable that the designed network can provide us with high regularity and symmetry, since it is advantageous to construction and simulation of some algorithms (see [30]). The class of vertex-transitive graphs possesses high regularity and symmetry, and thus is an important and ideal class of topological structures of interconnection networks [4, 30, 32]. A number of networks, including hypercubes [7], varietal hypercubes [29], balanced hypercubes [32], and some of their generalizations are all vertex-transitive [30].

A graph Γ is *vertex-transitive* if it looks the same when we take a view from every vertex [26]. The vertex-transitivity of graphs is usually measured by using group actions. Let $V(\Gamma)$ and $E(\Gamma)$ be the vertex set and edge set of Γ, respectively. An *automorphism* of Γ is a permutation π on $V(\Gamma)$ satisfying the adjacency-preserving condition

$$(u, v) \in E(\Gamma) \text{ if and only if } (u^\pi, v^\pi) \in E(\Gamma).$$

The set of all automorphisms of Γ forms a group under the operation of composition, denoted by $\mathrm{Aut}(\Gamma)$, and it is referred to as the *full automorphism group* of Γ. A subgroup G of $\mathrm{Aut}(\Gamma)$ is *transitive* on $V(\Gamma)$ if for any pair (u, v) of vertices in Γ there is some $\pi \in G$ such that $v = u^\pi$. (For a vertex u, the set $\{u^\alpha \mid \alpha \in G\}$ is an *orbit* of G acting on $V(\Gamma)$. The transitivity of G on $V(\Gamma)$ means that G has exactly one orbit on $V(\Gamma)$.) A graph Γ is *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on $V(\Gamma)$.

The class of Cayley graphs presents a very useful graph-theoretic model for designing, analyzing, and improving symmetric networks [4, 14, 32]. In particular, it plays an important role in constructing vertex-transitive graphs. For a graph Γ, a subgroup G of $\mathrm{Aut}(\Gamma)$ is *semiregular* on $V(\Gamma)$ if evey element in G, except the identity, cannot fix a vertex of Γ, and *regular* if G is both transitive and semiregular on $V(\Gamma)$. The graph Γ is a *Cayley graph* of a group G if there exists a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G (see [10, 30]).

The concept of Cayley graphs can be naturally generalized to m-Cayley graphs, where regular actions are replaced with semiregular actions. A graph Γ is said to be an *m-Cayley graph* of a group G if $\mathrm{Aut}(\Gamma)$ admits a

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semiregular subgroup isomorphic to G having m orbits on $V(\Gamma)$. Of course, 1-Cayley graphs are simply Cayley graphs. For additional results regarding m-Cayley graphs we refer the reader to [5, 19]. The class of m-Cayley graphs provides a useful tool to study non-vertex-transitive graphs, see [12] for example. It also has been used in the research of some networks, see [10, 21] for example.

The hypercube Q_n is one of the most popular, versatile and efficient topological structures of networks [30]. Because of its many excellent features, it becomes the first choice for parallel processing and computing systems, one of which is its small diameter [7, 30]. Communication efficiency is a critical metric in a parallel computing system, while the diameter of a network is an important metric for communication efficiency [33]. A superior nature of the hypercube Q_n is that its diameter is equal to its dimension n, which is logarithm-level with respect to the order of Q_n. To further improve the performance of the hypercube network in terms of diameter, numerous variant networks were put forward successively. The n-dimensional spined cube SQ_n is one variant of the hypercube, which was proposed by Zhou et al. [33]. The diameter of SQ_n is only $\lceil \frac{n}{3} \rceil + 3$, which is less than many known variants such as crossed cubes, twisted cubes, Möbius cubes, etc. The spined cube has attracted the attention of many researchers, and its various properties such as embedability [6], reliability [11], the shortest-path routing [28] have been investigated.

The symmetric property of the hypercube Q_n have been widely investigated. It is a Cayley graph of an elementary abelian 2-group $\mathbb{Z}_2^n$, and consequently it is vertex-transitive. The symmetric properties of many variants of the hypercube have been studied, and however, there are also some variants whose symmetric properties are not clear. One may see a summary in Table 1. The symmetric property of the spined cube is first considered in this paper. It is shown that an n-dimensional spined cube STQ_n is a 4-Cayley graph of an elementary abelian 2-group $\mathbb{Z}_2^{n-2}$ when $n \geq 6$, and is a $\lfloor n/2 \rfloor$-Cayley graph when $n \leq 5$. Moreover, we determine the vertex-transitivity of SQ_n, and prove that SQ_n is vertex-transitive only when $n \leq 3$.

Networks	Vertex-transitive	m-Cayley graph	Orb	Reference
Hypercube Q_n	Yes	$m = 1$	1	[13]
Folded hypercube FQ_n	Yes	$m = 1$	1	[13]
Balanced hypercube BH_n	Yes	$m = 1$	1	[31]
Varietal hypercube VQ_n	Yes	$m = 1$	1	[29]
Twisted cube TQ_n	No	?	?	[2]
Locally twisted cube LTQ_n ($n \geq 4$)	No	$m = 2$	2	[10]
Crossed cube CQ_n ($n \geq 5$)	No	?	?	[20]
Folded crossed cube FCQ_n ($n \geq 5$)	No	?	?	[26]
Twisted hypercube H_n	?	?	?	[34]
Data center network $D_{k,n}$ ($k \geq 2, n \geq 2$)	No	?	?	[23]
Spined cube SQ_n ($n \geq 6$)	No	$m = 4$	?	This paper

TABLE 1. Summary of symmetric properties of some networks.

For a network Γ, determining the number $Orb(\Gamma)$ of orbits of $\mathrm{Aut}(\Gamma)$ acting on vertices is an interesting problem in the study of symmetry. This has been attracted the attention of some researchers. For example, Pai et al. [26] put forward an open problem to determine the number of orbits for the crossed cube and the folded crossed cube. To a certain extent, determining the number $Orb(\Gamma)$ is related to the m-Cayley property of networks. There is a famous conjecture that almost all vertex-transitive graphs are Cayley graphs, see [25] for the detail. Moreover, the m-Cayley property of a network is helpful for us to analyze its internal structure and other properties in some time. For example, the Cayley property and bi-Cayley property have been used to analyze the reliability of networks in [32] and [10], respectively. One may also see [21] for other work. What's more, the 4-Cayley graphic structure of SQ_n with $n \geq 6$ obtained in this paper will be applied to construct the edge-disjoint Hamiltonian cycles in SQ_n in Section 4. These, combined with Table 1, prompt us to consider the following problem.

Problem 1.1. *For some variants of hypercubes, including the crossed cube, the folded crossed cube and the twisted cube,*

(1) *determining the numbers of orbits of their automorphism groups acting on vertices;*
(2) *determining the minimum number m such that the variant network is an m-Cayley graph.*

We note that the answers of Problem 1.1 for the spined cube SQ_n are both 4 when $n \geq 6$. We also determine the full automorphism group of the spined cube based on its 4-Cayley property. Since the proof involves some combinatorial group theory, it will be presented in other place.

1.2. **Edge-disjoint Hamiltonian cycles in networks.** The ring structure is important for distributed computing, and one may see [22] for its benefits. The Hamiltonian cycles can provide an advantage for algorithms using ring structures [27]. A cycle in a graph Γ is *Hamiltonian* if it contains every vertex of Γ. Two Hamiltonian cycles in a graph are *edge-disjoint* if they have no common edge. For the convenience in writing, an edge-disjoint Hamiltonian cycle is abbreviated to an EDHC. EDHCs can be applied on the problem of all-to-all communication algorithm. If a network contains d EDHCs, then the time complexity of the algorithm can be improved by a factor of d. One may see [17, 27] for the detail. Moreover, EDHCs are also useful in fault-tolerant routing. A network can tolerate a large number of edge failures if it has more EDHCs. When faults occur to edges of a Hamiltonian cycle, then vertices can communicate with any other vertices along another Hamiltonian cycle [17].

How to search for EDHCs in a network is an active and popular filed in the literature. A number of networks having at least two EDHCs have been investigated, including the transposition network and some hypercube-like networks [16], the Eisenstein-Jacobi network [17], and the balanced hypercube [24]. For various results and constructions of EDHCs in networks, we refer the reader to [3, 15, 18] and all the references therein. In the end of this paper, we aim to apply the symmetric property to the search of EDHCs in the spined cube, and finally, through using the 4-Cayley graphic structure of the spine cube and EDHCs in hypercubes, we prove that there exist two EDHCs in SQ_n when $n \geq 4$.

The layout of this paper is as follows: In Section 2, some definitions and notations are introduced. We then discuss the symmetric property of the spined cube in Section 3, and construct two EDHCs in the spined cube in Section 4. Section 5 is the conclusion.

2. Preliminaries

All graphs in this paper are finite, simple and undirected.

2.1. **Fundamental graph and group terminologies.** We follow [8, 10] for some terminology and definitions related to graphs and groups. Some notations are listed in Table 2.

Notations	Meaning
Γ	A graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$
$N_\Gamma(u)$	The neighborhood of the vertex u in Γ
$d_\Gamma(u,v)$	The distance between the vertices u and v in Γ
I_m	$\{1,2,\ldots,m\}$
$\mathbb{Z}_n$	A cyclic group of order n
$\mathbb{Z}_2^n$	An elementary abelian 2-group of order 2^n
$M \times N$	The product of groups M and N
$M \rtimes N$	A semi-product of groups M and N
1_H	The identity of a group H
$\langle a_1,\ldots,a_n\rangle$	The group generated by $\{a_1,\ldots,a_n\}$
$G \cong H$	The groups G and H are isomorphic

Table 2. Some notations.

Let Γ be a graph, and let F be a subset of $V(\Gamma)$. The subgraph of Γ induced by F is the graph whose vertex set is F and edge set is $\{(u,v) \in E(\Gamma) \mid u,v \in F\}$, denoted by $\Gamma[F]$. The notation $\Gamma - F$ represents the subgraph of Γ after deleting all vertices and edges in $\Gamma[F]$ from Γ.

Let $P_1 = (x_1, x_2, \ldots, x_m)$ and $P_2 = (y_1, y_2, \ldots, y_n)$ be two paths in a graph Γ such that all vertices in $V(P_1) \cup V(P_2)$ are all distinct except $x_m = y_1$. One can use $P_1 + P_2$ to denote the *path-concatenation* of P_1 and P_2 as the path $(x_1, x_2, \ldots, x_m, y_2, \ldots, y_t)$, and use $P_1 - (x_1, x_2)$ to denote the path $(x_2, x_3, \ldots, x_m)$.

For two graphs Γ and Σ, an *isomorphism* from Γ to Σ is a bijection $\phi : V(\Gamma) \to V(\Sigma)$ such that $(u,v) \in E(\Gamma)$ if and only if $(u^\phi, v^\phi) \in E(\Sigma)$. The graphs Γ and Σ are *isomorphic*, write $\Gamma \cong \Sigma$, if there is an isomorphism from Γ to Σ.

2.2. **The m-Cayley graph.** Let G be a finite group, and let S be a subset of G such that $1_G \notin S$ and $S = S^{-1} = \{s^{-1} \mid s \in S\}$. The *Cayley graph* of G with respect to S, write $\mathrm{Cay}(G, S)$, is the graph with vertex set G and edge set

$$\{(g, sg) \mid g \in G, s \in S\}.$$

For the definition of m-Cayley graphs, we follow [19].

Definition 2.1. *Let H be a group, and let T_{ij} be a subset of H such that $T_{ij}^{-1} = T_{ji}$ and $1_H \notin T_{ii}$, where $i, j \in I_m$. The m-*Cayley graph *of H relative to the subsets T_{ij}s is the graph having vertex set $\{h_i : h \in H, i \in I_m\}$, and the vertex h_i is adjacent to g_j if and only if there exists an element $t \in T_{ij}$ such that $g_j = (th)_j$.*

Clearly, a 1-Cayley graph is just a Cayley graph, and a 2-Cayley graph is also called a *bi-Cayley graphs.* A bi-Cayley graph of a group H relative to the subsets T_{11}, T_{22}, T_{12} is often denoted by $\mathrm{BiCay}(H, T_{11}, T_{22}, T_{12})$. For an m-Cayley graph Γ of a group H relative to the subsets T_{ij}s, where $T_{ij} \subset H$ and $i, j \in I_m$, it follows from Definition 2.1 that the induced subgraph by H_i in Γ is isomorphic to the Cayley graph $\mathrm{Cay}(H, T_{ii})$.

2.3. **The hypercube and the spined cube.** Let n be a positive integer. An *n-dimensional hypercube* Q_n is a graph with 2^n vertices. Each vertex is labeled with an n-bit binary string $x_1x_2 \cdots x_{n-1}x_n$, where $x_i = 0$ or 1 for each $1 \leq i \leq n$, and two vertices are adjacent if they have exactly one bit distinct. The following proposition about Q_n is well-known and can be also checked easily.

Proposition 2.2. *Let $n \geq 3$ be an integer, and let $\{a_1, \ldots, a_n\}$ be a generating subset of $\mathbb{Z}_2^n$. Then $Q_n \cong \mathrm{Cay}(\mathbb{Z}_2^n, \{a_1, a_2, \ldots, a_n\})$.*

The spined cube were defined by Zhou et al. [33] in the following way.

Definition 2.3. *Let n be a positive integer. An n-dimensional spined cube, denoted by SQ_n, is defined recursively as follows:*

(1) *SQ_1 is a complete graph on two vertices 0 and 1.*
(2) *For $n \geq 2$, SQ_n consists of two copies of SQ_{n-1}, denoted by $0SQ_{n-1}$ and $1SQ_{n-1}$. Each vertex $x = 0x_2 \cdots x_n$ in $0SQ_{n-1}$ connects exactly one vertex x' in $1SQ_{n-1}$, where*
 (2.1) *$x' = 1x_2$ for $n = 2$;*
 (2.2) *$x' = 1((x_2 + x_n) \pmod 2)x_3 \cdots x_n$ for $n = 3$ or 4;*
 (2.3) *$x' = 1((x_2 + x_{n-1}) \pmod 2)((x_3 + x_n) \pmod 2)x_4 \cdots x_n$ for $n \geq 5$.*

By Definition 2.3, SQ_2 is a 4-cycle, that is, a cycle of length 4. For notation convenience, "(mod 2)" will not appear in similar expressions in the rest of the paper.

An equivalent definition of the spined cube can be obtained easily from Definition 2.3, as follows.

Definition 2.4. *An n-dimensional spined cube SQ_n is an undirected graph with 2^n vertices with addresses $x_1 \cdots x_n$, where $x_i = 0$ or 1 for each $1 \leq i \leq n$. Two vertices $x = x_1 \cdots x_n$ and y are adjacent if and only if one of the following conditions is satisfied:*

(1) *$y = (1 + x_1)$ with $n = 1$;*
(2) *$y = (1 + x_1)x_2$ or $x_1(1 + x_2)$ with $n = 2$;*
(3) *$y \in \{(1 + x_1)(x_2 + x_3)x_3, x_1(1 + x_2)x_3, x_1x_2(1 + x_3)\}$ with $n = 3$;*
(4) *$y \in \{(1 + x_1)(x_2 + x_4)x_3x_4, x_1(1 + x_2)(x_3 + x_4)x_4, x_1x_2(1 + x_3)x_4, x_1x_2x_3(1 + x_4)\}$ with $n = 4$;*
(5) *$y \in \{(1 + x_1)(x_2 + x_4)(x_3 + x_5)x_4x_5, x_1(1 + x_2)(x_3 + x_5)x_4x_5, x_1x_2(1 + x_3)(x_4 + x_5)x_5, x_1x_2x_3(1 + x_4)x_5, x_1x_2x_3x_4(1 + x_5)\}$ with $n = 5$;*
(6) *$y \in \{x_1 \cdots x_{n-1}(1 + x_n), x_1 \cdots x_{n-2}(1 + x_{n-1})x_n, x_1 \cdots x_{n-3}(1 + x_{n-2})(x_{n-1} + x_n)x_n, x_1 \cdots x_{n-4}(1 + x_{n-3})(x_{n-2} + x_n)x_{n-1}x_n, x_1 \cdots x_{k-1}(1 + x_k)(x_{k+1} + x_{n-1})(x_{k+2} + x_n)x_{k+3} \cdots x_n, (1 + x_1)(x_2 + x_{n-1})(x_3 + x_n)x_4 \cdots x_n \mid 2 \leq k \leq n - 4\}$ with $n \geq 6$.*

3. Symmetric property of the spined cube

This section is divided into two parts, in which the m-Cayley property and the vertex-transitivity of the spined cube SQ_n are considered, respectively. Since SQ_2 is a 4-cycle and this case is trivial, we always assume $n \geq 3$ in this section.

3.1. **m-Cayley property of SQ_n.** For the case of $n \le 5$, we first introduce a Cayley graph and two bi-Cayley graphs.

Definition 3.1. *Let $H = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle \cong \mathbb{Z}_2^3$, $G = \langle a\rangle \cong \mathbb{Z}_8$, and $K = \langle b_1, b_2, b_3, b_4 \mid b_1^2 = b_2^2 = b_3^2 = b_4^2 = [b_1, b_2] = [b_1, b_3] = [b_2, b_3] = [b_2, b_4] = [b_3, b_4] = 1_K, b_4b_1 = b_1b_3b_4\rangle \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_2$ a non-abelian group of order 16. Define a Cayley graph Γ_3 and two bi-Cayley graphs Γ_4, Γ_5 as follows:*

$$\begin{aligned}
&\Gamma_3 = \mathrm{Cay}(G, \{a, a^{-1}, a^4\});\\
&\Gamma_4 = \mathrm{BiCay}(H, \{a_1, a_2, a_3\}, \{a_1a_2, a_2a_3, a_3\}, \{1_H\});\\
&\Gamma_5 = \mathrm{BiCay}(K, \{b_1, b_2, b_3, b_4\}, \{b_1b_2, b_2b_3, b_2b_4, b_4\}, \{1_K\}).
\end{aligned}$$

Clearly, Γ_3 is vertex-transitive, as it is a Cayley graph. It can be checked easily that $SQ_3 \cong \Gamma_3$. Note that a 3-dimensional spined cube SQ_3 is also called a 3-dimensional locally twisted cube (see [10, Figure 2]). For Γ_4 and Γ_5, it can be easily checked by using the software Magma [9] that $SQ_4 \cong \Gamma_4$ and $SQ_5 \cong \Gamma_5$. Moreover, neither of them is vertex-transitive.

Lemma 3.2. *The following hold.*

1. *SQ_3 is vertex-transitive and is a Cayley graph of a cyclic group $\mathbb{Z}_8$.*
2. *SQ_4 is a bi-Cayley graph of an elementary abelian group $\mathbb{Z}_2^3$, and is not vertex-transitive.*
3. *SQ_5 is a bi-Cayley graph of a non-abelian group $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_2$, and is not vertex-transitive.*

Now, we turn to the case $n \ge 6$, and we describe a family of 4-Cayley graphs of the elementary abelian 2-groups.

Definition 3.3. *Let $n \ge 6$ be an integer, and let $H = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{n-2}\rangle \cong \mathbb{Z}_2^{n-2}$. Set*

$$\begin{aligned}
&T_{11} = \{a_1, a_2, \ldots, a_{n-2}\};\ T_{22} = \{a_1a_3, a_2a_4, \ldots, a_{n-4}a_{n-2}, a_{n-3}a_{n-2}\};\\
&T_{33} = \{a_1a_2a_3, a_2a_3a_4, \ldots, a_{n-4}a_{n-3}a_{n-2}, a_{n-3}a_{n-2}\};\\
&T_{44} = \{a_1a_2, a_2a_3, \ldots, a_{n-4}a_{n-3}, a_{n-3}, a_{n-2}\};\\
&T_{12} = T_{14} = T_{34} = \{1_H\};\ T_{23} = \{1_H, a_{n-2}\};\ T_{13} = T_{24} = \emptyset.
\end{aligned}$$

Define a 4-Cayley graph of H relative to T_{ij}s, and denote it by Γ_n.

By Definition 2.1, the 4-Cayley graph Γ_n has order 2^n and valency n. The vertex set $V(\Gamma_n)$ is $\bigcup_{i=1}^4 H_i$, where $H_i = \{h_i \mid h \in H\}$ for each $i \in I_4$. The neighborhood of a vertex h_i in Γ_n for each $h \in H$ and $i \in I_4$ are

$$N_{\Gamma_n}(h_1) = \{h_2,\ h_4,\ (a_kh)_1 \mid 1 \le k \le n-2\}; \tag{3.1}$$
$$N_{\Gamma_n}(h_2) = \{h_1,\ h_3,\ (a_{n-2}h)_3,\ (a_{n-3}a_{n-2}h)_2,\ (a_ka_{k+2}h)_2 \mid 1 \le k \le n-4\}; \tag{3.2}$$
$$N_{\Gamma_n}(h_3) = \{h_4,\ h_2,\ (a_{n-2}h)_2,\ (a_{n-3}a_{n-2}h)_3,\ (a_ka_{k+1}a_{k+2}h)_3 \mid 1 \le k \le n-4\}; \tag{3.3}$$
$$N_{\Gamma_n}(h_4) = \{h_1,\ h_3,\ (a_{n-2}h)_4,\ (a_{n-3}h)_4,\ (a_ka_{k+1}h)_4 \mid 1 \le k \le n-4\}. \tag{3.4}$$

We note that the induced subgraphs by H_1 and H_4 in Γ_n are Cayley graphs of $H \cong \mathbb{Z}_2^{n-2}$ with respect to T_{11} and T_{44}, respectively (see Definition 2.1). Since $H = \langle T_{11}\rangle = \langle T_{44}\rangle$, the Proposition 2.2 implies that the two induced subgraphs are isomorphic to Q_{n-2}. For $i = 2$ or 3, the induced subgraph by H_i in Γ_n consists of two components, and each of them is isomorphic to Q_{n-3}. Therefore, there are eight vertex-disjoint $(n-3)$-dimensional hypercubes in Γ_n. This observation will be also put in the end of Section 3.1 and used in Section 4.

In the following, we will prove that a spined cube SQ_n with $n \ge 6$ is isomorphic to Γ_n. The following well-known fact relative to elementary abelian groups will be frequently used in the later proof.

Fact *For any $1 \le i, j \le n-2$, $a_i^0 = a_i^2 = 1_H$ and $a_ia_j = a_ja_i$. Every element in H can be uniquely written as $a_1^{x_1}a_2^{x_2}\cdots a_{n-2}^{x_{n-2}}$, where $x_i \in \{0, 1\}$ for $1 \le i \le n-2$.*

Lemma 3.4. *For $n \ge 6$, we have $SQ_n \cong \Gamma_n$.*

Proof. Define a map from $V(SQ_n)$ to $V(\Gamma_n)$ as following:

$$\begin{aligned}
\phi:\quad &x_1\cdots x_{n-2}00 \mapsto (a_1^{x_1}\cdots a_{n-2}^{x_{n-2}})_1,\\
&x_1\cdots x_{n-2}01 \mapsto (a_1^{x_1}\cdots a_{n-2}^{x_{n-2}})_2,\\
&x_1\cdots x_{n-2}11 \mapsto (a_1^{x_1}\cdots a_{n-2}^{x_{n-2}})_3,\\
&x_1\cdots x_{n-2}10 \mapsto (a_1^{x_1}\cdots a_{n-2}^{x_{n-2}})_4.
\end{aligned}$$

where $x_i \in \{0,1\}$ for $1 \leq i \leq n-2$. By the Fact above, it can be checked easily that ϕ is a bijection. To show that ϕ is an isomorphism from SQ_n to Γ_n, we need to show that $(x,y) \in E(SQ_n)$ if and only if $(x^\phi, y^\phi) \in E(\Gamma_n)$. Since SQ_n and Γ_n have some valency, they have the same number of edges, and since ϕ is a bijection, to finish the proof, it suffices to show that $[N_{SQ_n}(x)]^\phi = N_{\Gamma_n}(x^\phi)$ for any $x = x_1 \cdots x_n \in V(SQ_n)$. We consider the following four cases depending on $x_{n-1}x_n = 00$, 01, 11 or 10.

Case 1: $x_{n-1}x_n = 00$, that is, $x = x_1 \cdots x_{n-2}00$.

By Definition 2.4, the neighborhood of x in SQ_n is

$$N_{SQ_n}(x) = \{x_1 \cdots x_{n-2}01, x_1 \cdots x_{n-2}10, x_1 \cdots x_{k-1}(1+x_k)x_{k+1} \cdots x_{n-2}00, (1+x_1)x_2 \cdots x_{n-2}00 \mid 2 \leq k \leq n-2\}.$$

Since $x^\phi = (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_1$, Eq. (3.1) implies that the neighborhood of x^ϕ in Γ_n is

$$\begin{aligned} N_{\Gamma_n}(x^\phi) &= \{(a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_2,\ (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_4,\ (a_k \cdot a_1^{x_1} \cdots a_{n-1}^{x_{n-1}})_1 \mid 1 \leq k \leq n-2\} \\ &= \{(a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_2,\ (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_4,\ (a_1^{x_1} \cdots a_{k-1}^{x_{k-1}} a_k^{1+x_k} a_{k+1}^{x_{k+1}} \cdots a_{n-2}^{x_{n-2}})_1, \\ &\quad (a_1^{x_1+1} a_2^{x_2} \cdots a_{n-2}^{x_{n-2}})_1 \mid 2 \leq k \leq n-2\}. \end{aligned}$$

An easy checking yields that $[N_{SQ_n}(x)]^\phi = N_{\Gamma_n}(x^\phi)$, as required.

Case 2: $x_{n-1}x_n = 01$, that is, $x = x_1 \cdots x_{n-2}01$.

In this case, the neighborhood of x in SQ_n is

$$\begin{aligned} N_{SQ_n}(x) = \{ \quad & x_1 \cdots x_{n-2}00,\ x_1 \cdots x_{n-2}11,\ x_1 \cdots x_{n-3}(1+x_{n-2})11, \\ & x_1 \cdots x_{n-4}(1+x_{n-3})(1+x_{n-2})01,\ (1+x_1)x_2(1+x_3)x_4 \cdots x_{n-2}01, \\ & x_1 \cdots x_{k-1}(1+x_k)x_{k+1}(1+x_{k+2})x_{k+3} \cdots x_{n-2}01 \mid 2 \leq k \leq n-4\}, \end{aligned}$$

and from Eq. (3.2), the neighborhood of $x^\phi = (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_2$ in Γ_n is

$$\begin{aligned} N_{\Gamma_n}(x^\phi) = \{ \quad & (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_1,\ (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_3,\ (a_1^{x_1} \cdots a_{n-3}^{x_{n-3}} a_{n-2}^{1+x_{n-2}})_3, \\ & (a_1^{x_1} \cdots a_{n-4}^{x_{n-4}} a_{n-3}^{1+x_{n-3}} a_{n-2}^{1+x_{n-2}})_2,\ (a_1^{1+x_1} a_2^{x_2} a_3^{1+x_3} a_4^{x_4} \cdots a_n^{x_{n-2}})_2, \\ & (a_1^{x_1} \cdots a_{k-1}^{x_{k-1}} a_k^{1+x_k} a_{k+1}^{x_{k+1}} a_{k+2}^{1+x_{k+2}} a_{k+3}^{x_{k+3}} \cdots a_{n-2}^{x_{n-2}})_2 \mid 2 \leq k \leq n-4\}. \end{aligned}$$

Again, by an easy checking, we have $[N_{SQ_n}(x)]^\phi = N_{\Gamma_n}(x^\phi)$, as required.

Case 3: $x_{n-1}x_n = 11$, that is, $x = x_1 \cdots x_{n-2}11$.

In this case, the neighborhood of x in SQ_n is

$$\begin{aligned} N_{SQ_n}(x) = \{ \quad & x_1 \cdots x_{n-2}10,\ x_1 \cdots x_{n-2}01,\ x_1 \cdots x_{n-3}(1+x_{n-2})01, \\ & x_1 \cdots x_{n-4}(1+x_{n-3})(1+x_{n-2})11,\ (1+x_1)(1+x_2)(1+x_3)x_4 \cdots x_{n-2}11, \\ & x_1 \cdots x_{k-1}(1+x_k)(1+x_{k+1})(1+x_{k+2})x_{k+3} \cdots x_{n-2}01 \mid 2 \leq k \leq n-4\}, \end{aligned}$$

and from Eq. (3.3), the neighborhood of $x^\phi = (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_3$ in Γ_n is

$$\begin{aligned} N_{\Gamma_n}(x^\phi) = \{ \quad & (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_4,\ (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_2,\ (a_1^{x_1} \cdots a_{n-3}^{x_{n-3}} a_{n-2}^{1+x_{n-2}})_2, \\ & (a_1^{x_1} \cdots a_{n-4}^{x_{n-4}} a_{n-3}^{1+x_{n-3}} a_{n-2}^{1+x_{n-2}})_3,\ (a_1^{1+x_1} a_2^{1+x_2} a_3^{1+x_3} a_4^{x_4} \cdots a_n^{x_{n-2}})_3, \\ & (a_1^{x_1} \cdots a_{k-1}^{x_{k-1}} a_k^{1+x_k} a_{k+1}^{1+x_{k+1}} a_{k+2}^{1+x_{k+2}} a_{k+3}^{x_{k+3}} \cdots a_{n-2}^{x_{n-2}})_3 \mid 2 \leq k \leq n-4\}. \end{aligned}$$

Hence $[N_{SQ_n}(x)]^\phi = N_{\Gamma_n}(x^\phi)$, as required.

Case 4: $x_{n-1}x_n = 10$, that is, $x = x_1 \cdots x_{n-2}10$.

In this case, the neighborhood of x in SQ_n is

$$\begin{aligned} N_{SQ_n}(x) = \{ \quad & x_1 \cdots x_{n-2}11,\ x_1 \cdots x_{n-2}00,\ x_1 \cdots x_{n-3}(1+x_{n-2})10, x_1 \cdots x_{n-4}(1+x_{n-3})x_{n-2}10, \\ & (1+x_1)(1+x_2)x_3 \cdots x_{n-2}10, x_1 \cdots x_{k-1}(1+x_k)(1+x_{k+1})x_{k+2} \cdots x_{n-2}10 \mid 2 \leq k \leq n-4\}, \end{aligned}$$

and from Eq. (3.2), the neighborhood of $x^\phi = (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_4$ in Γ_n is

$$\begin{aligned} N_{\Gamma_n}(x^\phi) = \{ \quad & (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_1,\ (a_1^{x_1} \cdots a_{n-2}^{x_{n-2}})_3,\ (a_1^{x_1} \cdots a_{n-3}^{x_{n-3}} a_{n-2}^{1+x_{n-2}})_4, (a_1^{x_1} \cdots a_{n-4}^{x_{n-4}} a_{n-3}^{1+x_{n-3}} a_{n-2}^{x_{n-2}})_4, \\ & (a_1^{1+x_1} a_2^{1+x_2} a_3^{x_3} \cdots a_n^{x_{n-2}})_4, (a_1^{x_1} \cdots a_{k-1}^{x_{k-1}} a_k^{1+x_k} a_{k+1}^{1+x_{k+1}} a_{k+2}^{x_{k+2}} \cdots a_{n-2}^{x_{n-2}})_4 \mid 2 \leq k \leq n-4\}. \end{aligned}$$

Hence $[N_{SQ_n}(x)]^\phi = N_{\Gamma_n}(x^\phi)$, as required. □

In SQ_n with $n \geq 6$, a vertex $x = x_1 \ldots x_{n-2}x_{n-1}x_n$ is said to be of $x_{n-2}x_{n-1}x_n$-*type*. In view of the proof of Lemma 3.4, we have the following corollary.

Corollary 3.5. *Let $n \geq 6$. An n-dimensional spined cube SQ_n can be decomposed to eight hypercubes Q_{n-3}s of dimension $n-3$ and three perfect matchings. Furthermore,*

(1) *the subgraph induced by the vertices of type* 000 *and* 100 (*or* 010 *and* 110) *is isomorphic to Q_{n-2};*
(2) *the subgraphs induced by the vertices of type* 000, 100, 001, 101, 011, 111, 010, *and* 110 *are all isomorphic to Q_{n-3}.*

3.2. **Vertex-transitivity of SQ_n.** In this subsection, we derive in two lemmas that the spined cube SQ_n is not vertex-transitive when $n \geq 6$.

Lemma 3.6. *Let $n \geq 6$. There are exactly $\frac{n^2-5n+12}{2}$ 4-cycles going through the vertex $(1_H)_1$ in Γ_n, which are listed in Table 3. In particular, the following hold.*

(1) *There is only one 4-cycle going through the edge $((1_H)_1, (1_H)_2)$ in Γ_n.*
(2) *There are exactly three 4-cycles going through the edge $((1_H)_1, (1_H)_4)$ in Γ_n.*
(3) *The number of 4-cycles going through the edge $((1_H)_1, (a_i)_1)$ in Γ_n is $n-2$ for $n-3 \leq i \leq n-2$, and $n-3$ for $1 \leq i \leq n-4$.*

Row	4-cycles
1	$((1_H)_1, (1_H)_2, (1_H)_3, (1_H)_4, (1_H)_1)$
2	$((1_H)_1, (1_H)_4, (a_{n-3})_4, (a_{n-3})_1, (1_H)_1)$
3	$((1_H)_1, (1_H)_4, (a_{n-2})_4, (a_{n-2})_1, (1_H)_1)$
4	$((1_H)_1, (a_i)_1, (a_ia_j)_1, (a_j)_1, (1_H)_1)$, $1 \leq j < i \leq n-2$

TABLE 3. All 4-cycles going through $(1_H)_1$ in Γ_n

Proof. Let $C = ((1_H)_1, w, u, v, (1_H)_1)$ be a 4-cycle going through $(1_H)_1$. We have $w \in N_{\Gamma_n}((1_H)_1)$, and since

$$N_{\Gamma_n}((1_H)_1) = \{(1_H)_2, (1_H)_4, (a_i)_1 \mid 1 \leq i \leq n-2\} \tag{3.5}$$

by Eq. (3.1), we have $w = (1_H)_2$, $(1_H)_4$ or $(a_i)_1$. Clearly, $w \neq v$ and $u \neq (1_H)_1$.

(1). Assume $w = (1_H)_2$, that is, $C = ((1_H)_1, (1_H)_2, u, v, (1_H)_1)$. It follows that $v \in N_{\Gamma_n}((1_H)_1) \setminus \{(1_H)_2\}$ and $u \in (N_{\Gamma_n}((1_H)_2) \cap N_{\Gamma_n}(v)) \setminus \{(1_H)_1\}$. By Eq. (3.5), $v = (1_H)_4$ or $(a_i)_1$ with $1 \leq i \leq n-2$. Suppose $v = (a_i)_1$. By Eqs. (3.2) and (3.1), we have

$$N_{\Gamma_n}((1_H)_2) = \{(1_H)_1, (1_H)_3, (a_{n-2})_3, (a_{n-3}a_{n-2})_2, (a_ka_{k+2})_2 \mid 1 \leq k \leq n-4\}; \tag{3.6}$$

$$N_{\Gamma_n}((a_i)_1) = \{(a_i)_2, (a_i)_4, (a_ka_i)_1 \mid 1 \leq k \leq n-2\}, \tag{3.7}$$

implying that $(N_{\Gamma_n}((1_H)_2) \cap N_{\Gamma_n}(v)) \setminus \{(1_H)_1\} = \emptyset$. A contradiction occurs. Hence $v = (1_H)_4$. Since Eq. (3.4) implies that

$$N_{\Gamma_n}((1_H)_4) = \{(1_H)_1, (1_H)_3, (a_{n-2})_4, (a_{n-3})_4, (a_ka_{k+1})_4 \mid 1 \leq k \leq n-4\}, \tag{3.8}$$

we have $(N_{\Gamma_n}((1_H)_2) \cap N_{\Gamma_n}(v)) \setminus \{(1_H)_1\} = \{(1_H)_3\}$. We conclude that $u = (1_H)_3$. The (1) holds.

(2). Assume $w = (1_H)_4$. Now, $v \in N_{\Gamma_n}((1_H)_1) \setminus \{(1_H)_4\}$ and $u \in (N_{\Gamma_n}((1_H)_4) \cap N_{\Gamma_n}(v)) \setminus \{(1_H)_1\}$. By Eq. (3.5), either $v = (1_H)_2$ or $v = (a_i)_1$ with $1 \leq i \leq n-2$. For the former case, the (1) implies that $C = ((1_H)_1, (1_H)_4, (1_H)_3, (1_H)_2, (1_H)_1)$. For the latter case, we have $u \in (N_{\Gamma_n}((a_i)_1) \cap N_{\Gamma_n}((1_H)_4)) \setminus \{(1_H)_1\}$. Since $N_{\Gamma_n}((a_i)_1) \cap N_{\Gamma_n}((1_H)_4) = \{(1_H)_1\}$ for $1 \leq i \leq n-4$ and $\{(1_H)_1, (a_i)_4\}$ for $i = n-3$ or $n-2$ by Eqs. (3.7) and (3.8), we have $i = n-3$ or $n-2$ and $u = (a_i)_4$. In this case, there are three 4-cycles going through $((1_H)_1, (1_H)_4)$ in Γ_n, which are $((1_H)_1, (1_H)_4, (1_H)_3, (1_H)_2, (1_H)_1)$, $((1_H)_1, (1_H)_4, (a_{n-3})_4, (a_{n-3})_1, (1_H)_1)$ and $((1_H)_1, (1_H)_4, (a_{n-2})_4, (a_{n-2})_1, (1_H)_1)$. The (2) holds.

(3). Assume $w = (a_i)_1$ with $1 \leq i \leq n-2$, that is, $C = ((1_H)_1, (a_i)_1, u, v, (1_H)_1)$. Now, $v \in N_{\Gamma_n}((1_H)_1) \setminus \{(a_i)_1\}$ and $u \in (N_{\Gamma_n}((a_i)_1) \cap N_{\Gamma_n}(v)) \setminus \{(1_H)_1\}$. It follows from Eq. (3.5) that $v = (1_H)_2$, $(1_H)_4$ or $(a_j)_1$ with $1 \leq j \neq i \leq n-2$.

By (1), we have $v \neq (1_H)_2$. If $v = (1_H)_4$, then the (2) implies that $C = ((1_H)_1, (a_{n-3})_1, (a_{n-3})_4, (1_H)_4, (1_H)_1)$ or $((1_H)_1, (a_{n-2})_1, (a_{n-2})_4, (1_H)_4, (1_H)_1)$. Finally, let $v = (a_j)_1$ with $1 \leq j \neq i \leq n-2$. Now, $u \in (N_{\Gamma_n}((a_i)_1) \cap N_{\Gamma_n}((a_j)_1)) \setminus \{(1_H)_1\}$, and then by Eq. (3.7), we have that $u = (a_ia_j)_1$ and $C = ((1_H)_1, (a_i)_1, (a_ia_j)_1, (a_j)_1, (1_H)_1)$. Hence the number of 4-cycles going through $((1_H)_1, (a_i)_1)$ is $n-2$ when $i = n-3$ or $n-2$, and $n-3$ when $1 \leq i \leq n-4$. The (3) holds.

Summing up, there are $3+\frac{(n-3)(n-2)}{2}=\frac{n^2-5n+12}{2}$ 4-cycles going through $(1_H)_1$ in Γ_n in total, and all of them are listed in Table 3. □

Lemma 3.7. *Let $n \geq 6$. There are exactly $\frac{n^2-7n+20}{2}$ 4-cycles going through the vertex $(1_H)_2$ in Γ_n, which are listed in Table 4.*

Row	4-cycles
1	$((1_H)_1,(1_H)_2,(1_H)_3,(1_H)_4,(1_H)_1)$
2	$((1_H)_2,(1_H)_3,(a_{n-2})_2,(a_{n-2})_3,(1_H)_2)$
3	$((1_H)_2,(1_H)_3,(a_{n-3}a_{n-2})_3,(a_{n-3}a_{n-2})_2,(1_H)_2))$
4	$(1_H)_2,(a_{n-2})_3,(a_{n-3})_3,(a_{n-3}a_{n-2})_2,(1_H)_2))$
5	$((1_H)_2,(a_{n-3}a_{n-2})_2,(a_{n-3}a_{n-2}a_ia_{i+2})_2,(a_ia_{i+2})_2,(1_H)_2)),\ 1\leq i\leq n-4$
6	$((1_H)_2,(a_ia_{i+2})_2,(a_ja_{j+2}a_ia_{i+2})_2,(a_ja_{j+2})_2,(1_H)_2)),\ 1\leq j<i\leq n-4.$

TABLE 4. All 4-cycles going through $(1_H)_2$ in Γ_n

Proof. Let $C=((1_H)_2,w,u,v,(1_H)_2)$ be a 4-cycle going through $(1_H)_2$ in Γ_n. Now, $w,v\in N_{\Gamma_n}((1_H)_2)=\{(1_H)_1,(1_H)_3,(a_{n-2})_3,(a_{n-3}a_{n-2})_2,(a_ka_{k+2})_2 \mid 1\leq k\leq n-4\}$ (see Eq. (3.6)), $w\neq v$ and $u\neq(1_H)_2$. If $w=(1_H)_1$ or $v=(1_H)_1$, then Lemma 3.6 (1) implies that $C=((1_H)_2,(1_H)_1,(1_H)_4,(1_H)_3,(1_H)_2)$.

Assume $w,v\in\{(1_H)_3,(a_{n-2})_3,(a_{n-3}a_{n-2})_2,(a_ia_{i+2})_2\}$ for some $1\leq i\leq n-4$. Note that $u\in(N_{\Gamma_n}(w)\cap N_{\Gamma_n}(v))\setminus\{(1_H)_2\}$. For all possible w and v, we list their neighborhoods. By Eq. (3.2) we have

$$N_{\Gamma_n}((1_H)_3) = \{(1_H)_4,(1_H)_2,(a_{n-2})_2,(a_{n-3}a_{n-2})_3,(a_ka_{k+1}a_{k+2})_3 \mid 1\leq k\leq n-4\}; \tag{3.9}$$

$$N_{\Gamma_n}((a_{n-2})_3) = \{(a_{n-2})_4,(a_{n-2})_2,(1_H)_2,(a_{n-3})_3,(a_ka_{k+1}a_{k+2}a_{n-2})_3 \mid 1\leq k\leq n-4\}; \tag{3.10}$$

and by Eq. (3.3) we have

$$N_{\Gamma_n}((a_{n-3}a_{n-2})_2)=\{(a_{n-3}a_{n-2})_1,(a_{n-3}a_{n-2})_3,(a_{n-3})_3,(1_H)_2,(a_ka_{k+2}a_{n-3}a_{n-2})_2 \mid 1\leq k\leq n-4\};$$
$$N_{\Gamma_n}((a_ia_{i+2})_2)=\{(a_ia_{i+2})_1,(a_ia_{i+2})_3,(a_{n-2}a_ia_{i+2})_3,(a_{n-3}a_{n-2}a_ia_{i+2})_2,(a_ka_{k+2}a_ia_{i+2})_2 \mid 1\leq k\leq n-4\},$$

where $1\leq i\leq n-4$. By an easy check, we have

$$N_{\Gamma_n}((1_H)_3)\cap N_{\Gamma_n}((a_{n-2})_3)=\{(1_H)_2,(a_{n-2})_2\};$$
$$N_{\Gamma_n}((1_H)_3)\cap N_{\Gamma_n}((a_{n-3}a_{n-2})_2)=\{(1_H)_2,(a_{n-3}a_{n-2})_3\},$$
$$N_{\Gamma_n}((1_H)_3)\cap N_{\Gamma_n}((a_ia_{i+2})_2)=\{(1_H)_2\},$$
$$N_{\Gamma_n}((a_{n-2})_3)\cap N_{\Gamma_n}((a_{n-3}a_{n-2})_2)=\{(1_H)_2,(a_{n-3})_3\},$$
$$N_{\Gamma_n}((a_{n-2})_3)\cap N_{\Gamma_n}((a_ia_{i+2})_2)=\{(1_H)_2\},$$
$$N_{\Gamma_n}((a_{n-3}a_{n-2})_2)\cap N_{\Gamma_n}((a_ia_{i+2})_2)=\{(1_H)_2,(a_{n-3}a_{n-2}a_ia_{i+2})_2\},$$
$$N_{\Gamma_n}((a_ia_{i+2})_2)\cap N_{\Gamma_n}((a_ja_{j+2})_2)=\{(1_H)_2,(a_ja_{j+2}a_ia_{i+2})_2\} \text{ with } 1\leq i\neq j\leq n-4.$$

Hence when $w,v\neq(1_H)_1$, the 4-cycles going through $(1_H)_2$ are:

$$((1_H)_2,(1_H)_3,(a_{n-2})_2,(a_{n-2})_3,(1_H)_2));$$
$$((1_H)_2,(1_H)_3,(a_{n-3}a_{n-2})_3,(a_{n-3}a_{n-2})_2,(1_H)_2));$$
$$((1_H)_2,(a_{n-2})_3,(a_{n-3})_3,(a_{n-3}a_{n-2})_2,(1_H)_2));$$
$$((1_H)_2,(a_{n-3}a_{n-2})_2,(a_{n-3}a_{n-2}a_ia_{i+2})_2,(a_ia_{i+2})_2,(1_H)_2)),\ 1\leq i\leq n-4;$$
$$((1_H)_2,(a_ia_{i+2})_2,(a_ja_{j+2}a_ia_{i+2})_2,(a_ja_{j+2})_2,(1_H)_2)),\ 1\leq j<i\leq n-4.$$

Summing up, there are exactly $\frac{n^2-7n+20}{2}$ 4-cycles passing through $(1_H)_2$. □

Clearly, $\frac{n^2-5n+12}{2}\neq\frac{n^2-7n+20}{2}$ when $n\geq 6$. It follows from Lemmas 3.6 and 3.7 that there are different number of 4-cycles going through the vertices $(1_H)_1$ and $(1_H)_2$ in Γ_n, and so Γ_n is not vertex-transitive when $n\geq 6$. Combined with Lemmas 3.2 and 3.4, we have the following theorem.

Theorem 3.8. *The n-dimensional spined cube SQ_n is not vertex-transitive unless $n\leq 3$.*

4. Edge-disjoint Hamiltonian cycles in SQ_n

In this section, we aim to prove that there exist two EDHCs in the spined cube SQ_n with $n \geq 4$.

Theorem 4.1. *There exist two EDHCs in SQ_n when $4 \leq n \leq 6$.*

Proof. By Definition 2.3, it can be easily checked that the $C_n^{(1)}$ and $C_n^{(2)}$ listed below are two EDHCs in SQ_n for $4 \leq n \leq 6$. $\square$

$$
\begin{aligned}
C_4^{(1)} = \quad & (0000, 0010, 1010, 1011, 1101, 1111, 0011, 0001, 0111, 0110, 1110, 1100, 0100, 0101, 1001, 1000, 0000); \\
C_4^{(2)} = \quad & (0000, 0100, 0110, 0010, 0011, 0101, 0111, 1011, 1001, 1111, 1110, 1010, 1000, 1100, 1101, 0001, 0000); \\
C_5^{(1)} = \quad & (00000, 00001, 00011, 00010, 00110, 00100, 00101, 00111, 01011, 01001, 01000, 01010, \\
& 01110, 01100, 01101, 01111, 10011, 10001, 10111, 10101, 10100, 10110, 10010, 11010, \\
& 11011, 11101, 11100, 11110, 11111, 11001, 11000, 10000, 00000); \\
C_5^{(2)} = \quad & (00000, 00010, 01010, 01011, 01101, 00001, 00111, 00110, 01110, 01111, 00011, 11111, \\
& 11101, 01001, 00101, 10001, 10000, 10010, 10011, 10101, 11001, 11011, 10111, 10110, \\
& 11110, 11010, 11000, 01000, 01100, 11100, 10100, 00100, 00000); \\
C_6^{(1)} = \quad & (000000, 100000, 110000, 111000, 111010, 111011, 111101, 111100, 111110, 111111, 111001, \\
& 110101, 110100, 100100, 100110, 100111, 101011, 101101, 101111, 110011, 110001, 110111, \\
& 110110, 110010, 101010, 101110, 101100, 001100, 001101, 100101, 101001, 101000, 001000, \\
& 011000, 011100, 011101, 011111, 011110, 011010, 000010, 000110, 001110, 001010, 001011, \\
& 001001, 100001, 100011, 100010, 010010, 010011, 001111, 000011, 000001, 000111, 011011, \\
& 011001, 010101, 010111, 010110, 010100, 010000, 010001, 000101, 000100, 000000); \\
C_6^{(2)} = \quad & (000000, 010000, 110000, 110100, 111100, 111000, 111001, 111011, 110111, 110101, 110011, \\
& 111111, 111101, 101000, 100101, 100111, 100001, 101101, 101100, 101000, 101010, 101011, \\
& 101001, 101111, 100011, 011011, 011101, 010001, 010111, 001011, 001101, 011001, 011000, \\
& 011010, 010010, 010110, 100110, 100010, 100000, 100100, 000100, 010100, 011100, 001100, \\
& 001110, 001111, 001001, 001000, 001010, 000010, 110010, 111010, 111110, 110110, 101110, \\
& 011110, 000110, 000111, 000101, 000011, 011111, 010011, 010101, 000001, 000000).
\end{aligned}
$$

Next, we begin to consider the Hamiltonian cycles in SQ_n when $n \geq 7$. Since an n-dimensional spined cube SQ_n can be decomposed to eight vertex-disjoint $(n-3)$-dimensional hypercubes Q_{n-3}s when $n \geq 7$ by Corollary 3.5, our main strategy to construct EDHCs in SQ_n consists of the following three steps:

- Step 1: Find two EDHCs in some Q_{n-3}s;
- Step 2: Find EDHCs in the other Q_{n-3}s under graph isomorphisms;
- Step 3: Concatenate the cycles.

In the first step, we have the following lemma about hypercubes. For notation convenience, we use the elements of the group $\mathbb{Z}_2^n$ to denote the vertices of the hypercube Q_n (see Proposition 2.2) and the spined cube SQ_n (see Lemma 3.4).

Lemma 4.2. *Let $n \geq 4$, and let $\{a_1, \ldots, a_n\}$ be a generating subset of $H = \mathbb{Z}_2^n$. In an n-dimensional hypercube $Q_n = \mathrm{Cay}(H, \{a_1, \ldots, a_n\})$, there exist two EDHCs containing the edges $(1_H, a_n)$ and $(a_n, a_{n-1}a_n)$, respectively.*

Proof. We proceed by induction on n. It can be easily checked that the following two cycles C_1 and C_2 are EDHCs in Q_4 containing the edges $(1_H, a_4)$ and (a_4, a_3a_4), respectively, and so the lemma is true when $n = 4$.

$$
\begin{aligned}
C_1 = \quad & (1_H, a_3, a_2a_3, a_1a_2a_3, a_1a_3, a_1, a_1a_2a_2, a_2a_4, a_1a_2a_4, a_1a_2a_3a_4, a_2a_3a_4, a_3a_4, a_1a_3a_4, a_1a_4, a_4, 1_H); \\
C_2 = \quad & (1_H, a_1, a_1a_4, a_1a_2a_4, a_1a_2, a_1a_2a_3, a_1a_2a_3a_4, a_1a_3a_4, a_1a_3, a_3, a_3a_4, a_4, a_2a_4, a_2a_3a_4, a_2a_3, a_2, 1_H).
\end{aligned}
$$

Assume that the lemma is true for some $k \geq 4$, and let $n = k + 1$. Let $H_1 = \langle a_1, \ldots, a_k \rangle$, the subgroup of $\mathbb{Z}_2^{k+1}$ generated by $a_1, \ldots, a_k$, and denote $H_2 = \{g a_{k+1} \mid g \in H_1\}$, the coset of H_1 in $\mathbb{Z}_2^{k+1}$. Clearly, $H_1 \cong \mathbb{Z}_2^k$, and $Q_{k+1}[H_1] \cong Q_{k+1}[H_2] \cong Q_k$ (see Proposition 2.2). The map $g \mapsto g a_{k+1}$, $\forall g \in H_1$, induces an isomorphism from $Q_{k+1}[H_1]$ to $Q_{k+1}[H_1]$, say α. Moreover, for each vertex g in $Q_{k+1}[H_1]$, $g^\alpha = ga_{k+1}$ is the unique neighbor of g in $Q_{k+1}[H_2]$.

By the induction hypothesis, there exist two EDHCs C_1 and C_2, containing the edges $(1_H, a_k)$ and $(a_k, a_{k-1}a_k)$, respectively. Since α is an isomorphism from $Q_{k+1}[H_1]$ to $Q_{k+1}[H_1]$, C_1^α and C_2^α are two EDHCs in $Q_{k+1}[H_2]$. Moreover, the cycles C_1^α contains the edge $(1_H, a_k)^\alpha = (a_{k+1}, a_k a_{k+1})$. Let

$$\widehat{C_1} = C_1 - (1_H, a_k) + (1_H, a_{k+1}) + C_1^\alpha - (a_{k+1}, a_k a_{k+1}) + (a_k a_{k+1}, a_k).$$

Now, $\widehat{C_1}$ is an Hamiltonian cycle in Q_{k+1}, and the edge $(1_H, a_{k+1})$ belongs to $\widehat{C_1}$.

Since C_1 and C_2 are edge-disjoint and $(1_H, a_k) \in E(C_1)$, the edge $(1_H, a_k) \notin E(C_2)$, that is, $d_{C_2}(1, a_k) \geq 2$. Noting that C_2 is a Hamiltonian cycle in $Q_n[H_1]$ with length $2^k \geq 2^4$, we may assume that $C_2 = (1_H, u, \ldots, a_k, v, \ldots, 1_H)$, where u and v are neighbors of 1_H and a_k in C_2, respectively. Clearly, $\{u, v\} \cap \{a_{k+1}, a_k, 1_H\} = \emptyset$. Denote P_1 be the subpath from u to a_k in C_2, and P_2 the subpath from 1 to v. Let

$$\begin{aligned}\widehat{C_2} &= P_1 + (a_k, 1_H) + P_2 + (v, v^\alpha) + P_2^\alpha + (1_H^\alpha, a_k^\alpha) + P_1^\alpha + (u^\alpha, u)\\ &= P_1 + (a_k, 1_H) + P_2 + (v, va_{k+1}) + P_2^\alpha + (a_{k+1}, a_{k+1}a_k) + P_1^\alpha + (ua_{k+1}, u).\end{aligned}$$

The $\widehat{C_2}$ is an Hamiltonian cycle in Q_n, containing the edge $(a_{k+1}, a_{k+1}a_k)$. Since $v \notin \{1_H, a_k\}$, we have $\{(1_H, a_{k+1}), (a_k a_{k+1}, a_k)\} \cap \{(v, va_{k+1}), (a_{k+1}, a_{k+1}a_k)\} = \emptyset$, and since $E(C_1) \cap [E(P_1) \cap E(P_2)] \subseteq E(C_1) \cap E(C_2) = \emptyset$, we conclude that $\widehat{C_1}$ and $\widehat{C_2}$ are edge-disjoint. The proof is complete. $\square$

Next, we aim to find some graph isomorphisms (see Step 2), and we need some symbols. Let $H = \langle a_1 \rangle \times \cdots \times \langle a_{n-2} \rangle \cong \mathbb{Z}_2^{n-2}$ with $n \geq 7$, and let

$$\begin{aligned}H^{11} &= \langle a_1, a_2, \ldots, a_{n-3} \rangle;\\ H^{21} &= \langle a_1a_3, a_2a_4, \ldots, a_{n-4}a_{n-2}, a_{n-3}a_{n-2} \rangle;\\ H^{31} &= \langle a_1a_2a_3, a_2a_3a_4, \ldots, a_{n-4}a_{n-3}a_{n-2}, a_{n-3}a_{n-2} \rangle;\\ H^{41} &= \langle a_1a_2, a_2a_3, \ldots, a_{n-4}a_{n-3}, a_{n-3} \rangle;\\ H^{i2} &= \{ha_{n-2} \mid h \in H^{i1}\},\ 1 \leq i \leq 4;\\ H_i^{ij} &= \{h_i \mid h \in H^{ij}\},\ 1 \leq i \leq 4,\ 1 \leq j \leq 2.\end{aligned}$$

By some primary knowledge in group theory, one can observe that H^{i1} is a subgroup of H isomorphic to $\mathbb{Z}_2^{n-3}$, H^{i2} is a coset of H^{i1} in H, and $H = H^{i1} \cup H^{i2}$, where $1 \leq i \leq 4$. In view of Lemma 3.4, SQ_n is a 4-Cayley graph of H, and by Definition 3.1, we have $V(SQ_n) = \bigcup_{i=1}^4 H_i = \bigcup_{i=1}^4 (H_i^{i1} \cup H_i^{i2})$. Moreover, the induced subgraph $SQ_n[H_i^{ij}]$ is isomorphic to Q_{n-3} (see Corollary 3.5), where $1 \leq i \leq 4$ and $1 \leq j \leq 2$. Define the map from H_i^{i1} to H_i^{i2} with $1 \leq i \leq 4$ as follows:

$$\mathcal{R}(a_{n-2}):\quad h_i \mapsto (ha_{n-2})_i,\ \forall h_i \in H_i^{i1}.$$

It can be checked easily that $\mathcal{R}(a_{n-2})$ is an isomorphism from $SQ_n[H_i^{i1}]$ to $SQ_n[H_i^{i2}]$.

Now, we are ready to construct EDHCs in the spined cube SQ_n with $n \geq 7$.

Theorem 4.3. *There exist two EDHCs in SQ_n when $n \geq 7$.*

Proof. Since the induced subgraphs

$$\begin{aligned}SQ_n[H_1^{11}] &\cong \mathrm{Cay}(H^{11}, \{a_1, a_2, \ldots, a_{n-3}\}) \cong Q_{n-3};\\ SQ_n[H_2^{21}] &\cong \mathrm{Cay}(H^{21}, \{a_1a_3, a_2a_4, \ldots, a_{n-4}a_{n-2}, a_{n-3}a_{n-2}\}) \cong Q_{n-3};\\ SQ_n[H_3^{31}] &\cong \mathrm{Cay}(H^{31}, \{a_1a_2a_3, a_2a_3a_4, \ldots, a_{n-4}a_{n-3}a_{n-2}, a_{n-3}a_{n-2}\}) \cong Q_{n-3};\\ SQ_n[H_4^{41}] &\cong \mathrm{Cay}(H^{41}, \{a_1a_2, a_2a_3, \ldots, a_{n-4}a_{n-3}, a_{n-3}\}) \cong Q_{n-3},\end{aligned}$$

each of them admits two EDHCs by Lemma 4.2. Assume that C_{i1} and C_{i2} are two EDHCs in $SQ_n[H_i^{i1}]$, and by Lemma 4.2 we may further assume that

$$\begin{aligned}&((1_H)_1, (a_{n-3})_1) \in E(C_{11}),\ ((a_{n-3})_1, (a_{n-4}a_{n-3})_1) \in E(C_{12});\\ &((1_H)_2, (a_{n-3}a_{n-2})_2) \in E(C_{21}),\ ((a_{n-3}a_{n-2})_2, (a_{n-4}a_{n-3})_2) \in E(C_{22});\\ &((1_H)_3, (a_{n-3}a_{n-2})_3) \in E(C_{31}),\ ((a_{n-3}a_{n-2})_3, (a_{n-4})_3) \in E(C_{32});\\ &((1_H)_4, (a_{n-3})_4) \in E(C_{41}),\ ((a_{n-3})_4, (a_{n-4})_4) \in E(C_{42}).\end{aligned}$$

Let $C'_{i1} = C_{i1}^{\mathcal{R}(a_{n-2})}$ and $C'_{i2} = C_{i2}^{\mathcal{R}(a_{n-2})}$ for $1 \leq i \leq 4$. Since $\mathcal{R}(a_{n-2})$ is an isomorphism from $SQ_n[H_i^{i1}]$ to $SQ_n[H_i^{i2}]$, C'_{i1} and C'_{i2} are two EDHCs of $SQ_n[H_i^{i2}]$. Since $((1_H)_1, (a_{n-3})_1) \in E(C_{11})$ and $((a_{n-3})_1, (a_{n-4}a_{n-3})_1) \in E(C_{12})$, we have

$$\begin{aligned}
&((1_H)_1, (a_{n-3})_1)^{\mathcal{R}(a_{n-2})} = ((a_{n-2})_1, (a_{n-3}a_{n-2})_1) \in E(C'_{11}),\\
&((a_{n-3})_1, (a_{n-4}a_{n-3})_1)^{\mathcal{R}(a_{n-2})} = ((a_{n-3}a_{n-2})_1, (a_{n-4}a_{n-3}a_{n-2})_1) \in E(C_{12})'.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&((a_{n-2})_2, (a_{n-3})_2) \in E(C'_{21}),\ ((a_{n-3})_2, (a_{n-4}a_{n-3}a_{n-2})_2) \in E(C'_{22});\\
&((a_{n-2})_3, (a_{n-3})_3) \in E(C'_{31}),\ ((a_{n-3})_3, (a_{n-4}a_{n-2})_3) \in E(C'_{32});\\
&((a_{n-2})_4, (a_{n-3}a_{n-2})_4) \in E(C'_{41}),\ ((a_{n-3}a_{n-2})_4, (a_{n-4}a_{n-2})_4) \in E(C_{42'}).
\end{aligned}$$

Now, we have 16 cycles in SQ_n, and clearly, they are edge-disjoint. Finally, we concatenate the cycles. Let

$$\begin{aligned}
C_1 = \ & C_{11} - ((1_H)_1, (a_{n-3})_1) + ((a_{n-3})_1, (a_{n-3}a_{n-2})_1) + C'_{11} - ((a_{n-3}a_{n-2})_1, (a_{n-2})_1)\\
&+((a_{n-2})_1, (a_{n-2})_2) + C'_{21} - ((a_{n-2})_2, (a_{n-3})_2) + ((a_{n-3})_2, (a_{n-3}a_{n-2})_3) + C_{31}\\
&-((a_{n-3}a_{n-2})_3, (1_H)_3) + ((1_H)_3, (1_H)_2) + C_{21} - ((1_H)_2, (a_{n-3}a_{n-2})_2)\\
&+((a_{n-3}a_{n-2})_2, (a_{n-3})_3) + C'_{31} - ((a_{n-3})_3, (a_{n-2})_3) + ((a_{n-2})_3, (a_{n-2})_4)\\
&+C'_{41} - ((a_{n-2})_4, (a_{n-3}a_{n-2})_4) + ((a_{n-3}a_{n-2})_4, (a_{n-3})_4) + C_{41}\\
&-((a_{n-3})_4, (1_H)_4) + ((1_H)_4, (1_H)_1),
\end{aligned}$$

and let

$$\begin{aligned}
C_2 = \ & C_{12} - ((a_{n-3})_1, (a_{n-4}a_{n-3})_1) + ((a_{n-4}a_{n-3})_1, (a_{n-4}a_{n-3})_2) + C_{22}\\
&-((a_{n-3}a_{n-2})_2, (a_{n-4}a_{n-3})_2) + ((a_{n-3}a_{n-2})_2, (a_{n-3}a_{n-2})_1) + C'_{12}\\
&-((a_{n-3}a_{n-2})_1, (a_{n-4}a_{n-3}a_{n-2})_1) + ((a_{n-4}a_{n-3}a_{n-2})_1, (a_{n-4}a_{n-3}a_{n-2})_2) + C'_{22}\\
&-((a_{n-4}a_{n-3}a_{n-2})_2, (a_{n-3})_2) + ((a_{n-3})_2, (a_{n-3})_3) + C'_{32} - ((a_{n-3})_3, (a_{n-4}a_{n-2})_3)\\
&+((a_{n-4}a_{n-2})_3, (a_{n-4}a_{n-2})_4) + C'_{42} - ((a_{n-3}a_{n-2})_4, (a_{n-4}a_{n-2})_4)\\
&+((a_{n-3}a_{n-2})_4, (a_{n-3}a_{n-2})_3) + C_{32} - ((a_{n-3}a_{n-2})_3, (a_{n-4})_3)\\
&+((a_{n-4})_3, (a_{n-4})_4) + C_{42} - ((a_{n-4})_4, (a_{n-3})_4) + ((a_{n-3})_4, (a_{n-3})_1).
\end{aligned}$$

We conclude that C_1 and C_2 are EDHCs of SQ_n. The proof is complete. □

5. Conclusion

Graph symmetry is an important factor in the design of a network. The spined cube SQ_n was introduced by Zhou et al. [33] in 2011 as a variant of the hypercube Q_n, whose diameter is less than most known variants of hypercubes. The hypercube have been well-studied in the literature. A natural problem is how to use the numerous works about hypercubes to study the variants, that is, how to establish the connection between hypercube and its variants. This is also a reason why we consider the symmetric property of the spined cube in this paper. We first prove that SQ_n is a 4-Cayley graph of an elementary abelian 2-group $\mathbb{Z}_2^{n-2}$ when $n \geq 6$, and then have that it is not vertex-transitive unless $n \leq 3$. The symmetric property of SQ_n shows that it can be decomposed to eight vertex-disjoint $(n-3)$-dimensional hypercubes when $n \geq 6$. By using the existence of EDHCs in hypercubes, we show that there exists two EDHCs in SQ_n when $n \geq 4$.

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