Decompositions of graphs of nonnegative characteristic with some forbidden subgraphs

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Abstract

A (d, h)-decomposition of a graph G is an order pair (D, H) such that H is a subgraph of G where H has the maximum degree at most h and D is an acyclic orientation of G - E(H) of maximum out-degree at most d. A graph G is (d, h)-decomposable if G has a (d, h)decomposition. Let G be a graph embeddable in a surface of nonnegative characteristic. In this paper, we prove the following results. (1) If G has no chord 5-cycles or no chord 6-cycles or no chord 7-cycles and no adjacent 4-cycles, then G is (3, 1)-decomposable, which generalizes the results of Chen, Zhu and Wang [Comput. Math. Appl, 56 (2008) 2073–2078] and the results of Zhang [Comment. Math. Univ. Carolin, 54(3) (2013) 339–344]. (2) If G has no i-cycles nor j-cycles for any subset $\{i, j\} \subseteq \{3, 4, 6\}$ is (2, 1)-decomposable, which generalizes the results of Dong and Xu [Discrete Math. Alg. and Appl., 1(2) (2009), 291–297].

1 Introduction

Graphs considered here are finite and simple. A graph is *d*-generate if every subgraph has a vertex of degree at most *d*. For two integers $d, h \in \mathbb{N}$, a (d, h)-decomposition of *G* is a pair (H_1, H_2) such that H_2 is a subgraph of *G* of maximum degree at most *h* and H_1 is *d*-degenerate. A graph *G* is (d, h)-decomposable if *G* has a (d, h)-decomposition. Decomposing a graph into subgraphs with simple structure is a fundamental problem in graph theory. The classical Theorem of Tutte [18] and, independent by, Nash-Williams [15] provides a necessary and sufficient condition for which a graph can be decomposed into forests. A proper coloring of *G* is a decomposition of *G* into independent sets. The problems of decomposing a graph *G* into start forests, linear forests and some others are studied widely in the literature.

A proper k-coloring is a mapping $\varphi : V(G) \to \{1, 2, ..., k\}$ such that $\varphi(u) \neq \varphi(v)$ where $uv \in E(G)$. The chromatic number, denoted by $\chi(G)$, of G is the minimum k such that G is k-colorable. A d-defective k-coloring of G is a mapping $\varphi : V(G) \to \{1, 2, ..., k\}$ such that for each vertex $v \in V(G)$, v has at most d neighbors of the same color as itself. A k-list assignment of G is a function L that assigns a list L(v) of colors to each vertex $v \in V(G)$ where |L(v)| = k. A d-defective L-coloring is a mapping φ that assigns a color $\varphi(v) \in L(v)$ to each vertex $v \in V(G)$ such that v has at most d neighbors of the same color as itself. A graph G is d-defective k-choosable

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if there exists an (L, d)-coloring for every list assignment L with |L(v)| = k for all $v \in V(G)$. A graph is 0-defective k-choosable if and only if it is k-choosable. The choosable number, denoted by ch(G), of G is the minimum k such that G is k-choosable.

The Alon-Tarsi number of G, denoted by AT(G), was defined by Jensen and Toft [II]. It follows from the Alon-Tarsi Theorem [I] that $ch(G) \leq AT(G)$ for any graph G. It is proved that the difference AT(G) - ch(G) can be arbitrarily large. DP-coloring was introduced by Dvořák and Postle [I] as a generation of list coloring. Clearly, $ch(G) \leq \chi_{DP}(G)$, where $\chi_{DP}(G)$ is the DP-chromatic number of a graph G. A painting coloring was introduced by Schauz [IG] and it is proved that $ch(G) \leq \chi_P(G)$ for any graph G, where $\chi_P(G)$ is the paint number of G.

It is well-known that a graph H_1 has an acyclic orientation D with $\Delta_D^+ \leq d$ if and only if H_1 is d-degenerate, where Δ_D^+ is the maximum degree D. If G is d-degenerate, then each of choosable number ch(G), Alon-Tarsi number AT(G), paint number $\chi_P(G)$ and DP-chromatic number χ_{DP} is at most d + 1. This implies that if G is (d, h)-decomposable, then there is a subgraph H of G where $\Delta_H \leq h$ such that G - E(H) is h-defective-(d + 1)-choosable, (d + 1)-DP-colorable and $AT(G - E(H)) \leq d + 1$.

Defective coloring of graphs was considered by Cowen, Cowen and Woodall $\begin{bmatrix} Cow}{3} \end{bmatrix}$ who proved that every planar graph is 2-defective 3-colorable, which was improved by Eaton and Hull $\begin{bmatrix} EH\\8 \end{bmatrix}$, independently, Škrekovski $\begin{bmatrix} 17\\7 \end{bmatrix}$, who proved that every planar graph is 2-defective 3-choosable. Cushing and Kierstead $\begin{bmatrix} Cush\\4 \end{bmatrix}$ proved that every planar graph is 1-defective 4-choosable. Grytczuk and Zhu $\begin{bmatrix} Gry\\10 \end{bmatrix}$ strengthen the result and proved that every planar graph G has a matching M such that $AT(G-M) \leq 4$. Lih, Song, Wang and Zhang $\begin{bmatrix} 12\\7 \end{bmatrix}$ proved that every planar graph G without 4-cycles and l-cycles for some $l \in \{5, 6, 7\}$ is 1-defective 3-choosable. Dong and Xu $\begin{bmatrix} Wu\\10 \end{bmatrix}$ showed that such result is also true for some $l \in \{8, 9\}$. Lu and Zhu $\begin{bmatrix} Z20\\14 \end{bmatrix}$ proved that every planar graph without 4- and l-cycles G, where l = 5, 6, 7, has a matching M such that G-M is $AT(G-M) \leq 3$. Gonçalves $\begin{bmatrix} 90\\9 \end{bmatrix}$ proved that every planar graph is (3, 4)-decomposable. Zhu $\begin{bmatrix} 200\\120 \end{bmatrix}$ proved that every planar graph is (2, 8)-decomposable. Recently, Li, Lu, Wang and Zhu $\begin{bmatrix} 113\\12 \end{bmatrix}$ improve this result and prove that for $l \in \{5, 6, 7, 8, 9\}$, every planar graph without 4- and l-cycles is (2, 1)-decomposable. Cho, Choi, Kim, Park, Shan and Zhu $\begin{bmatrix} 50\\9 \\ 10 \end{bmatrix}$ prove that every planar graphs which are not (2, 3)-decomposable and there are also planar graphs which are not (1, h)-decomposable.

We are interested in decompositions of graphs of nonnegative characteristic in this paper. The characteristic of a surface Σ is defined to be |V(G)| - |E(G)| + |F(G)| for any graph G which is 2-cell embedded in Σ . All the surfaces of nonnegative characteristic are the Euclidean plane, the projective plane, the torus and the Klein bottle. A graph of nonnegative characteristic means that it can be embedded on a surface of nonnegative characteristic. Throughout this paper, a graph of nonnegative characteristic is called a *NC-graph*. In this paper, we prove the following results.

Theorem 1.1 A NC-graph G is (3,1)-decomposable if one of the following hold:

- (1) G has no chord 5-cycles.
- (2) G has no chord 6-cycles.
- (3) G has no chord 7- nor adjacent 4-cycles.

For simplicity, we define a family \mathcal{G} of NC-graphs such that $G \in \mathcal{G}$ if and only if G has neither

chord 5-cycles nor chord 6-cycles nor chord 7- and adjacent 4-cycles. From Theorem 1.7, next corollary follows immediately.

Corollary 1.2 Every graph $G \in \mathcal{G}$ has a matching M such that each of choice number, paint number, DP-number and Alon-Tarsi number of G - M is at most 4.

A graph G is *toroidal* if G can be drawn on the torus so that the edges meet only at the vertices of the graph.

Corollary 1.3 (1) (Chen, Zhu and Wang, $\begin{bmatrix} Chen \\ [2] \end{bmatrix}$ Every graph of nonnegative characteristic without either chord 5-cycles or chord 6-cycles is 1-defective 4-choosable.

(2) (Zhang, [19]) Every toroidal graph G without chord 7-cycles and adjacent 4-cycles is 1defective 4-choosable.

th1 Theorem 1.4 A NC-graph G is (2,1)-decomposable if one of the following hold:

- (1) G has neither 3- nor 4-cycles.
- (2) G has neither 3- nor 6-cycles.
- (3) G has neither 4- nor 6-cycles.

Similarly, we define a family \mathcal{H} of NC-graphs such that $G \in \mathcal{H}$ if and only if G has no *i*-cycles nor *j*-cycles for any $\{i, j\} \subseteq \{3, 4, 6\}$. We obtain the following corollary from Theorem [1.4].

Corollary 1.5 Every graph $G \in \mathcal{H}$ has a matching M such that each of choice number, paint number, DP-number and Alon-Tarsi number of G - M is at most 3.

cor1 Corollary 1.6 (Dong and Xu [6]) Every toroidal graph G which contains neither i-cycles nor *j*-cycles for any subset $\{i, j\} \subseteq \{3, 4, 6\}$ is 1-defective 3-colorable.

In the end of this section, we introduce some terminology and notation. Let G be a graph and denote by V(G), E(G), F(G) (or V, E, F for short) the sets of vertices, edges and faces of G, respectively. Let G be a (di)graph. For a vertex v, denote by d(v) ($d^+(v)$ or $d^-(v)$ in digraph) the *degree* (*out-degree* or *in-degree* in digraph) of v. Denote by $N_G(v)$ (or N(v) for short) the set of *neighbors of a vertex* v in G. A k-vertex (k^+ -vertex or k^- -vertex) is a vertex of degree k (at least k or at most k). Similarly, a k-face (k^+ -face or k^- -face) is a face of degree k (at least k or at most k). For $f \in F(G)$, denote by d(f) the degree of face f in G which is the number of edges incident with f and b(f) the boundary walk of f and write $f = [u_1u_2 \dots u_l]$ when u_1, u_2, \dots, u_l are the boundary vertices of f in clockwise order. A l-face $[u_1u_2 \dots u_l]$ is called an (a_1, a_2, \dots, a_l) -face if $d(u_i) = a_i$ for $i = 1, 2, \dots, l$. Two faces are adjacent if they share at least one common edge. For $v \in V(G)$ and $i \ge 3$, denote by $n_i(v)$ ($n_{i+}(v)$ or ($n_{i-}(v)$) the number of all i- (i^+ - or (i^-) faces incident to v. A cycle is a k-cycle if it contains k vertices. For a cycle C, an edge $xy \in E(G) \setminus E(C)$ is called a *chord* of C if $x, y \in V(C)$. Let C be a k-cycle. Then C is called *chord* k-cycle.

2 Reducible configurations

Suppose otherwise that Theorems $\frac{th0}{1.1}$ and $\frac{th1}{1.4}$ are both false. Assume that

 $G \in \mathcal{G}$ is a counterexample to Theorem $\stackrel{\texttt{th0}}{\blacksquare.1}$ with |V(G)| minimized. (1)

In this case, G has no chord 5-cycles nor chord 6-cycles nor chord 7-cycles and adjacent 4-cycles. Clearly, G has no (3, 1)-decomposition but any subgraph of G does. Similarly, assume that

 $H \in \mathcal{H}$ is a counterexample to Theorem $[1.4]{\text{th}}$ with |V(H)| minimized. (2)

In this case, H has neither *i*-cycle nor *j*-cycle, where $\{i, j\} \subset \{3, 4, 6\}$. Clearly, H has no (2, 1)-decomposition but any subgraph of H does. In this section, we establish several lemmas. The following lemma is straightforward.

Lemma 2.1 Assume that G is a NC-graph and $d(v) \ge 3$ for all $v \in V(G)$. If G has no 6-cycles, then two 4-faces are not adjacent.

Recall that a graph H is d-degenerate if and only if H has an acyclic orientation D with $\Delta_D^+ \leq d$. Thus, to prove that a graph G is (d, h)-decomposable, it is sufficient to show that G can be decomposed into H_1 and H_2 such that H_1 has an acyclic orientation with $\Delta_D^+ \leq d$ and H_2 has the maximum degree at most k. From Lemma 2.2 to 2.5, we assume that G satisfies Assumption $(\stackrel{\textbf{len}}{\textbf{l}})$.

lem1 Lemma 2.2 (1) $d(v) \ge 4$ for all $v \in V(G)$; (2) G does not contain two adjacent 4-vertices.

Proof. (1) Suppose otherwise that v is a 3-vertex and $N(v) = \{v_1, v_2, v_3\}$. By the minimality of G, there is a (3, 1)-decomposition (D^*, M^*) of $G - \{v\}$. Let $M = M^*$ and $D = D^* \cup \{\overrightarrow{vv_1}, \overrightarrow{vv_2}, \overrightarrow{vv_3}\}$. Then (D, M) is a (3, 1)-decomposition of G, a contradiction.

(2) Suppose otherwise that u is a 4-vertex adjacent to a 4-vertex v. Let $N(u) = \{u_1, u_2, u_3, v\}$ and $N(v) = \{v_1, v_2, v_3, u\}$. By the minimality of G, there is a (3, 1)-decomposition (D^*, M^*) of $G - \{u, v\}$. Let $M = M^* \cup \{uv\}$ and $D = D^* \cup \{\overrightarrow{vv_1}, \overrightarrow{vv_2}, \overrightarrow{vv_3}, \overrightarrow{uu_1}, \overrightarrow{uu_2}, \overrightarrow{uu_3}\}$. Then (D, M) is a (3, 1)-decomposition of G, a contradiction.

lem2 Lemma 2.3 (1) A 5-vertex v is incident with at most one (4, 5, 5)-face.

(2) A 5-vertex v is not incident with three consecutively adjacent 3-faces, one of which is (4,5,5)-face and other two of which are (4,5,6)-faces.

Proof. Let v_1, v_2, \ldots, v_5 be the neighbors of v in clockwise order, and f_1, f_2, \ldots, f_5 be the incident faces of v with $vv_i, vv_{i+1} \in b(f_i)$ for $i = 1, 2, \ldots, 5$ where indices are taken modulo 5.

(1) Suppose otherwise that v is incident with two (4, 5, 5)-faces. There are two cases. **Case 1.** f_1 and f_2 are (4, 5, 5)-faces.

We first assume that $d(v_1) = d(v_3) = 5$ and $d(v_2) = 4$. Let $N(v_1) = \{v_{11}, v_{12}, v_{13}, v, v_2\}, N(v_2) = \{v_{21}, v, v_1, v_3\}$ and $N(v_3) = \{v_{31}, v_{32}, v_{33}, v, v_2\}.$

By the minimally of G, there is a (3, 1)-decomposition (D^*, M^*) of $G - \{v, v_1, v_2, v_3\}$. Let $M = M^* \cup \{vv_1, v_2v_3\}$ and $D = D^* \cup \{\overrightarrow{v_1v_{11}}, \overrightarrow{v_1v_{12}}, \overrightarrow{v_1v_{13}}, \overrightarrow{v_2v_{21}}, \overrightarrow{v_3v_{31}}, \overrightarrow{v_3v_{32}}, \overrightarrow{vv_3}, \overrightarrow{vv_4}, \overrightarrow{vv_5}, \overrightarrow{v_2v}, \overrightarrow{v_2v_1}\}$. Then (D, M) is a (3, 1)-decomposition of G, a contradiction.

We further assume that $d(v_1) = d(v_3) = 4$ and $d(v_2) = 5$. Let $N(v_1) = \{v_{11}, v_{12}, v, v_2\}, N(v_2) = \{v_{21}, v_{22}, v, v_1, v_3\}$ and $N(v_3) = \{v_{31}, v_{32}, v, v_2\}.$

By the minimality of G, there is a (3,1)-decomposition (D^*, M^*) of $G - \{v, v_1, v_2, v_3\}$. Let $M = M^* \cup \{v_1v_2, vv_3\}$ and $D = D^* \cup \{\overrightarrow{v_1v_{11}}, \overrightarrow{v_1v_{12}}, \overrightarrow{v_2v_{21}}, \overrightarrow{v_2v_{22}}, \overrightarrow{v_3v_{31}}, \overrightarrow{v_3v_{32}}, \overrightarrow{vv_2}, \overrightarrow{vv_4}, \overrightarrow{vv_5}, \overrightarrow{v_1v}, \overrightarrow{v_3v_2}\}$. Then (D, M) is a (3, 1)-decomposition of G, a contradiction.

Case 2. f_1 and f_3 are (4, 5, 5)-faces.

We assume, without loss of generality, that $d(v_1) = d(v_3) = 4$, $d(v_2) = d(v_4) = 5$ and $N(v_1) = \{v_{11}, v_{12}, v, v_2\}$, $N(v_2) = \{v_{21}, v_{22}, v_{23}, v, v_1\}$, $N(v_3) = \{v_{31}, v_{32}, v, v_4\}$, $N(v_4) = \{v_{41}, v_{42}, v_{43}, v, v_3\}$.

By the minimality of G, there is a (3,1)-decomposition (D^*, M^*) of $G - \{v, v_1, v_2, v_3, v_4\}$. Let $M = M^* \cup \{v_1v_2, v_3v_4\}$ and $D = D^* \cup \{\overrightarrow{v_1v_{11}}, \overrightarrow{v_1v_{12}}, \overrightarrow{v_2v_{21}}, \overrightarrow{v_2v_{22}}, \overrightarrow{v_2v_{23}}, \overrightarrow{v_3v_{31}}, \overrightarrow{v_3v_{32}}, \overrightarrow{v_4v_{41}}, \overrightarrow{v_4v_{42}}, \overrightarrow{v_4v_{43}}, \overrightarrow{vv_2}, \overrightarrow{vv$

(2) By (1) and by symmetry, suppose otherwise that f_1 is a (4,5,5)-face and f_2, f_3 are two (4,5,6)-faces. In this case, $d(v_1) = 5, d(v_2) = d(v_4) = 4$ and $d(v_3) = 6$. Let $N(v_1) = \{v_{11}, v_{12}, v_{13}, v, v_2\}, N(v_2) = \{v_{21}, v, v_1, v_3\}, N(v_3) = \{v_{31}, v_{32}, v_{33}, v, v_2, v_4\}$ and $N(v_4) = \{v_{41}, v_{42}, v, v_3\}$. By the minimality of G, there is a (3, 1)-decomposition (D^*, M^*) of $G - \{v, v_1, v_2, v_3, v_4\}$. Let $M = M^* \cup \{v_1v_2, v_3v_4\}$ and $D = D^* \cup \{\overline{v_1v_11}, \overline{v_1v_12}, \overline{v_1v_{13}}, \overline{v_2v_{21}}, \overline{v_3v_{31}}, \overline{v_3v_{32}}, \overline{v_3v_{33}}, \overline{v_4v_{41}}, \overline{v_4v_{42}}, \overline{vv_1}, \overline{vv_3}, \overline{vv_5}, \overline{v_2v}, \overline{v_2v_3}, \overline{v_4v}\}$. Then (D, M) is a (3, 1)-decomposition of G, a contradiction.

Lemma 2.4 If G is a NC-graph without either chord 5-cycles or chord 7- and adjacent 4-cycles, then every 4^+ -vertex v is incident with at most two consecutively adjacent 3-faces. Moreover, v is incident with at most $\lfloor \frac{2d(v)}{3} \rfloor$ 3-faces.

Proof. Suppose otherwise that v is a 4⁺-vertex incident with three consecutively adjacent 3-faces $[v_1vv_2], [v_2vv_3]$ and $[v_3vv_4]$. In this case, G has a 5-cycle $[v_1v_2v_3v_4v]$ with a chord vv_3 , a contradiction. Observe that two adjacent 4-faces $f_1 = [v_1v_2v_3v], f_2 = [v_2v_3v_4v]$ have one common edge v_2v_3 . Thus, G has adjacent 4-cycles, a contradiction. Therefore, v is incident with at most $\lfloor \frac{2d(v)}{3} \rfloor$ 3-faces.

Lemma 2.5 Let G be a NC-graph without chord 6-cycles. Then every 5^+ -vertex v is incident to at most three consecutively adjacent 3-faces. Thus, v is incident to at most (d(v) - 2) 3-faces.

Proof. Suppose otherwise that v is a 5⁺-vertex incident to four consecutively adjacent 3-faces $[v_1vv_2], [v_2vv_3], [v_3vv_4], [v_4vv_5]$. Then $[v_1v_2v_3v_4v_5v]$ is a 6-cycle with a chord vv_3 , a contradiction. Thus, v is incident to at most (d(v) - 2) 3-faces.

From Lemma $\frac{1 \text{ em5}}{2.6 \text{ to } 2.8}$, we assume that G satisfies Assumption (2).

lem5 Lemma 2.6 (1) $d(v) \ge 3$ for all $v \in V(G)$; (2) G does not contain two adjacent 3-vertices.

Proof. (1) Suppose otherwise that v is a 2-vertex and $N(v) = \{v_1, v_2\}$. By the minimality of G, there is a (2,1)-decomposition (D^*, M^*) of $G - \{v\}$. Let $M = M^*$ and $D = D^* \cup \{\overrightarrow{vv_1}, \overrightarrow{vv_2}\}$. Then (D, M) is a (2,1)-decomposition of G, a contradiction.

(2) Suppose otherwise that u is a 3-vertex adjacent to a 3-vertex v. Let $N(u) = \{u_1, u_2, v\}$ and $N(v) = \{v_1, v_2, u\}$. By the minimally of G, there is a (2, 1)-decomposition (D^*, M^*) of $G - \{u, v\}$. Let $M = M^* \cup \{uv\}$ and $D = D^* \cup \{\overrightarrow{vv_1}, \overrightarrow{vv_2}, \overrightarrow{uu_1}, \overrightarrow{uu_2}\}$. Then (D, M) is a (2, 1)-decomposition of G, a contradiction.

lem7 Lemma 2.7 If A NC-graph G has has no 3-cycle nor 6-cycle, then it has no any underlying subgraph of G in Fig.1.

Proof. Suppose otherwise that G contains one of the figures in Fig.1. Let X be all the labeled vertices of each figure. By the minimality of G, $G^* = G - X$ has a (2, 1)-decomposition (D^*, M^*) .

In Fig.1 (1), $X = \{v_1, \dots, v_{11}\}$. Let $M' = \{v_1v_5, v_2v_3, v_6v_7, v_8v_9, v_{10}v_{11}\}$ and $D' = \{\overline{v_1v_2}, \overline{v_1v_7}, \overline{v_2v_9}, \overline{v_3v_4}, \overline{v_3v_8}, \overline{v_4v_5}, \overline{v_5v_6}, \overline{v_8v_{10}}, \overline{v_{11}v_4}\}$. In Fig.1 (2), $X = \{v_1, \dots, v_{11}\}$. Let $M' = \{v_1v_5, v_2v_3, v_6v_7, v_8v_9, v_4v_{11}\}$ and $D' = \{\overline{v_1v_2}, \overline{v_1v_7}, \overline{v_2v_9}, \overline{v_3v_4}, \overline{v_3v_8}, \overline{v_4v_5}, \overline{v_5v_6}, \overline{v_8v_{10}}, \overline{v_{10}v_{11}}\}$. In Fig.1 (3), $X = \{v_1, \dots, v_{11}\}$. Let $M' = \{v_1v_5, v_6v_7, v_2v_{11}, v_4v_8, v_9v_{10}\}$ and $D' = \{\overline{v_1v_2}, \overline{v_1v_7}, \overline{v_2v_3}, \overline{v_3v_4}, \overline{v_3v_9}, \overline{v_4v_5}, \overline{v_5v_6}, \overline{v_9v_8}, \overline{v_1v_7}, \overline{v_2v_3}, \overline{v_3v_4}, \overline{v_3v_9}, \overline{v_4v_5}, \overline{v_5v_6}, \overline{v_9v_8}, \overline{v_1v_7}, \overline{v_2v_3}, \overline{v_3v_4}, \overline{v_3v_9}, \overline{v_4v_5}, \overline{v_5v_6}, \overline{v_9v_8}, \overline{v_1v_1}\}$. In Fig.1 (4), $X = \{v_1, \dots, v_{11}\}$. Let $M' = \{v_1v_5, v_6v_7, v_2v_{11}, v_4v_8, v_9v_{10}\}$ and $D' = \{\overline{v_1v_2}, \overline{v_1v_7}, \overline{v_2v_3}, \overline{v_3v_4}, \overline{v_3v_9}, \overline{v_4v_5}, \overline{v_5v_6}, \overline{v_9v_8}, \overline{v_{10}v_{11}}\}$. In Fig.1 (5), $X = \{v_1, v_2, \dots, v_7, v_9\}$. Let $M' = \{v_1v_2, v_4v_5, v_6v_7\}$ and $D' = \{\overline{v_1v_5}, \overline{v_1v_7}, \overline{v_2v_6}, \overline{v_2v_5}, \overline{v_3v_4}, \overline{v_7v_9}\}$. In Fig.1 (6), $X = \{v_1, \dots, v_9\}$. Let $M' = \{v_1v_5, v_3v_4, v_6v_7, v_8v_9\}$ and $D' = \{\overline{v_1v_2}, \overline{v_1v_7}, \overline{v_2v_6}, \overline{v_3v_2}, \overline{v_3v_4}, \overline{v_3v_9}, \overline{v_4v_8}, \overline{v_5v_4}, \overline{v_5v_4}, \overline{v_5v_4}, \overline{v_5v_6}, \overline{v_9v_9}\}$. In Fig.1 (7), $X = \{v_1, \dots, v_{11}\}$. Let $M' = \{v_2v_3, v_4v_5, v_6v_7, v_8v_9, v_{10}v_{11}\}$ and $D' = \{\overline{v_1v_5}, \overline{v_1v_7}, \overline{v_2v_6}, \overline{v_2v_6}, \overline{v_3v_4}, \overline{v_5v_9}, \overline{v_3v_4}, \overline{v_3v_9}, \overline{v_4v_8}, \overline{v_9v_10}\}$.

Let $M = M^* \cup M'$ and D be the orientation of G - M obtained by adding arcs in D' and all the edges between X and $V \setminus X$ oriented from X to $V \setminus X$. Then $\Delta(M) \leq 1$ and $\Delta_D^+ \leq 2$. Moreover, D is an acyclic orientation of G - M. Thus (D, M) is a (2, 1)-decomposition of G, a contradiction.



Fig. 1: Reducible configurations

lem6 Lemma 2.8 A NC-graph $G \in \mathcal{H}$ has no a (3, 4, 3, 4)-face,

Proof. Suppose otherwise that G has a (3, 4, 3, 4)-face $[v_1v_2v_3v_4]$. Let $N(v_1) = \{v_{11}, v_2, v_4\}, N(v_2) = \{v_{21}, v_{22}, v_1, v_3\}, N(v_3) = \{v_{31}, v_2, v_4\}$ and $N(v_4) = \{v_{41}, v_{42}, v_1, v_3\}$. By the minimality of G, there is a (2, 1)-decomposition (D^*, M^*) of $G - \{v_1, v_2, v_3, v_4\}$. Let $M = M^* \cup \{v_1v_2, v_3v_4\}$ and $D = D^* \cup \{\overrightarrow{v_1v_{11}}, \overrightarrow{v_2v_{21}}, \overrightarrow{v_2v_{22}}, \overrightarrow{v_3v_{31}}, \overrightarrow{v_4v_{41}}, \overrightarrow{v_4v_{42}}, \overrightarrow{v_1v_4}, \overrightarrow{v_3v_2}\}$. Then (D, M) is a (2, 1)-decomposition of G, a contradiction.

3 Proofs of Theorem 1.1 and 1.4

We are now ready to complete the proof of Theorem [1,1] and [1,4]. We define initial charge $\mu(x) = d(x) - 4$ for each $x \in V \cup F$. By Euler's Formula $|V(G)| + |F(G)| - |E(G)| \ge 0$,

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) \le 0.$$

Let $\mu'(x)$ be the charge of $x \in V(G) \cup F(G)$ after the discharge procedure. In order to prove the Theorems [1,1] and [1,4], we shall design some discharging rules so that after discharging. Since the total sum of weights is kept unchanged, the new weight function μ' satisfies

- (I) $\mu'(x) \ge 0$ for all $x \in V(G) \cup F(G)$;
- (II) There exists some $x^* \in V(G) \cup F(G)$ such that $\mu'(x^*) > 0$.

Thus

$$0 < \sum_{x \in V(G) \cup F(G)} \mu'(x) = \sum_{x \in V(G) \cup F(G)} \mu(x) = 0.$$

This contradiction completes our proofs.

3.1 Proofs of Theorem $\overset{\text{th0}}{1.1}(1)$ and (3).

In this section, we prove Theorem $[1,1]^{th0}(1)$ and (3). Now we define the discharge rules as follows.

- (R1) Every 5-vertex sends $\frac{1}{3}$ to each incident $(5^+, 5^+, 5^+)$ -face, $\frac{1}{2}$ to each incident (4, 5, 5)-face and $\frac{5}{12}$ to each incident $(4, 5, 6^+)$ -face.
- (R2) Every 6⁺-vertex sends $\frac{7}{12}$ to each incident 3-face.
- (R3) Every 5⁺-face sends $\frac{11}{60}$ to each incident vertex.

It suffices to show that the new weight function μ' satisfies Properties (I) and (II). We first check $\mu'(v) \ge 0$ for all $v \in V(G)$. By Lemma $\frac{\texttt{lem1}}{\texttt{2.2}}(1), d(v) \ge 4$.

1. d(v) = 4. Since no 4-vertex is involved in the discharge procedure, $\mu'(v) = \mu(v) = 4 - 4 = 0$.

2. d(v) = 5. Then $\mu(v) = 1$. By Lemma $\frac{|lem3}{2.4}$, $n_3(v) \leq 3$. If $n_3(v) \leq 2$, then v is incident with at most one (4, 5, 5)-face by Lemma $\frac{2.3}{2.3}(1)$ and is not incident with any $(4, 4, 5^-)$ -face by Lemma $\frac{2.2}{2.2}(2)$. By (R1), $\mu'(v) \geq 1 - \frac{1}{2} - \frac{5}{12} = \frac{1}{12} > 0$. Let $n_3(v) = 3$. Then v is incident with two 4⁺-faces. If v is incident with one 4-face, then G has a chord 5-face and so does a chord

7-face and adjacent 4-cycles, contrary to our assumption. Thus, $n_4(v) = 0$. This implies that $n_{5^+}(v) = 2$. By Lemmas 2.3(1) and 2.2(2), v is incident with at most one (4, 5, 5)-face and is not incident with any (4, 4, 5⁻)-face. Thus $\mu'(v) \ge 1 - \frac{1}{2} - 2 \times \frac{5}{12} + 2 \times \frac{11}{60} = \frac{1}{30} > 0$ by (R1) and (R3).

- 3. d(v) = 6. Then $\mu(v) = 2$. By Lemma $\frac{\text{Lem3}}{2.4}$, $n_3(v) \le 4$. If $n_3(v) \le 3$, then $\mu'(v) \ge 2 3 \times \frac{7}{12} = \frac{1}{4} > 0$ by (R2). Thus, assume that $n_3(v) = 4$. In this case, v is incident with two 4⁺-faces. If v is indeed incident one 4-face, then G has a chord 5-cycle and so does a chord 7-cycle and adjacent 4-cycles, contrary to our assumption. Thus, $n_4(v) = 0$. This implies that $n_{5^+}(v) = 2$. Thus $\mu'(v) \ge 2 4 \times \frac{7}{12} + 2 \times \frac{11}{60} = \frac{1}{30} > 0$ by (R2) and (R3).
- 4. $d(v) \ge 7$. By Lemma $\frac{[lem3]}{2.4}$, $n_3(v) \le \lfloor \frac{2d(v)}{3} \rfloor$. Thus $\mu'(v) \ge d(v) 4 \frac{7}{12} \times \lfloor \frac{2d(v)}{3} \rfloor \ge d(v) 4 \frac{7}{12} \times \frac{2d(v)}{3} = \frac{11}{18}d(v) 4 \ge \frac{5}{18} > 0$ by (R2).

Then we check $\mu'(f) \ge 0$ for all $f \in F(G)$.

1. d(f) = 3. Then $\mu(f) = -1$. By Lemma $\frac{\text{lem1}}{2.2(2)}$, v is not incident with any $(4, 4, 5^-)$ -face. If f is a (4, 5, 5)-face, then $\mu'(f) \ge -1 + 2 \times \frac{1}{2} = 0$ by (R1). If f is a $(4, 5^+, 6^+)$ -face, then $\mu'(f) \ge -1 + \frac{5}{12} + \frac{7}{12} = 0$ by (R1) and (R2). If f is a $(5^+, 5^+, 5^+)$ -face, then $\mu'(f) \ge -1 + 3 \times \frac{1}{3} = 0$ by (R1) and (R2).

2. d(f) = 4. Since no 4-face is involved in the discharge procedure, $\mu(f) = \mu'(f) = 4 - 4 = 0$.

3.
$$d(f) \ge 5$$
. Then $\mu'(f) \ge d(f) - 4 - \frac{11}{60}d(f) = \frac{49}{60}d(f) - 4 \ge \frac{1}{12} > 0$ by (R3).

So far, we have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu'(x) = 0$ for all $x \in V(G) \cup F(G)$. We observe the above proof and have each of the following holds.

(a) For each vertex $v \in V(G)$, d(v) = 4;

(b) For each face $f \in F(G)$, $3 \le d(f) \le 4$.

By (a), G has no 5⁺-vertices. Thus G is 4-regular, which is contrary to Lemma $\frac{\text{Lem1}}{2.2}(2)$. This completes the proofs of Theorem I.1 (1) and (3).

3.2 Proof of Theorem $\frac{1}{1.1}(2)$

In this section, we prove Theorem $\frac{\text{th0}}{1.1}(2)$. Now we define the discharge rules as follows.

- (R1) Every 5-vertex sends $\frac{1}{3}$ to each incident $(5^+, 5^+, 5^+)$ -face, $\frac{1}{2}$ to each incident (4, 5, 5)-face, $\frac{5}{12}$ to each incident (4, 5, 6)-face and $\frac{61}{150}$ to each incident $(4, 5, 7^+)$ -face.
- (R2) Every 6-vertex sends $\frac{7}{12}$ to each incident 3-face.
- (R3) Every 7⁺-vertex sends $\frac{89}{150}$ to each incident 3-face.
- (R4) Every 5-face sends $\frac{11}{60}$ to each incident vertex.
- (R5) Every 6^+ -face sends $\frac{49}{150}$ to each incident vertex.

It suffices to show that the new weight function μ' satisfies Properties (I) and (II). Note that each 3-face is not adjacent to 5-face since G has no chord 6-cycles.

We first check $\mu'(v) \ge 0$ for all $v \in V(G)$. By Lemma $\frac{\mu e m 1}{2.2}(1), d(v) \ge 4$.

- 1. d(v) = 4. Since no 4-vertex is involved in the discharge procedure, $\mu'(v) = \mu(v) = 4 4 = 0$.
- 2. d(v) = 5. By Lemma $\frac{1 \text{ em4}}{2.5}$, $n_3(v) \le 3$. If $n_3(v) \le 2$, then v is incident with at most one (4, 5, 5)-face by Lemma $\frac{2.3}{2.3}(1)$ and is not incident with any $(4, 4, 5^-)$ -face by Lemma $\frac{1 \text{ em1}}{2.2}(2)$. By (R1), $\mu'(v) \ge 1 \frac{1}{2} \frac{5}{12} = \frac{1}{12} > 0$ by (R1). Thus, assume that $n_3(v) = 3$.

Suppose that v_1, v_2, \ldots, v_5 are the neighbors of v in clockwise order, and f_1, f_2, \ldots, f_5 are the incident faces of v with $vv_i, vv_{i+1} \in b(f_i)$ for $i = 1, 2, \ldots, 5$ where indices are taken modulo 5. By symmetry, there are two cases: either f_1, f_2 and f_3 or f_1, f_2 and f_4 are 3-faces.

In the former case, since G has no chord 6-cycles, each of f_4 and f_5 is not a 5-face. Thus, $n_5(v) = 0$. We claim that at most one of f_4 and f_5 is a 4-face. Suppose otherwise. Let $f_4 = [vv_4xv_5]$ and $f_5 = [v_1vv_5y]$. Since G has no chord 6-cycle, $x, y \in \{v, v_1, v_2, v_3, v_4, v_5\}$. Since G is a simple graph, $x \notin \{v, v_2, v_3\}$. Since $n_3(v) = 3$, $x \notin \{v_1, v_4, v_5\}$. Similarly, $y \notin \{v, v_5, v_2, v_4, v_1\}$. Thus, $x = v_2$ and $y = v_3$. In this case, G has a chord 6-cycle $vv_4v_3v_1v_2v_5v$, a contradiction. Thus $n_4(v) \leq 1$. This implies that $1 \leq n_{6^+}(v) \leq 2$. By Lemma $\boxed{2.3(1)}$, v is incident with at most one (4, 5, 5)-face. If v is not incident with (4, 5, 5)-face, then $\mu'(v) \geq 1 - 3 \times \frac{5}{12} + \frac{49}{150} = \frac{23}{300} > 0$ by (R1) and (R5). Thus, assume that v is incident with one (4, 5, 5)-face. By Lemma $\boxed{2.3(2)}$, v is incident with at most one (4, 5, 6)-faces. By (R1) and (R5), $\mu'(v) \geq 1 - \frac{1}{2} - \frac{5}{12} - \frac{61}{150} + \frac{49}{150} = \frac{1}{300} > 0$.

In the latter case, since G has no chord 6-cycles, none of f_3 and f_5 is a 5-face. Thus $n_5(v) = 0$. If $f_3 = [vv_3xv_4]$ is a 4-face, then $x \notin \{v, v_2, v_3, v_4, v_5\}$ since G is a simple graph and by Lemma 2.2(1). If $x \neq v_1$, then $vv_2v_3xv_4v_5v$ is a 6-cycle with a chord vv_3 , a contradiction. If $x = v_1$, then $v_1v_4v_5vv_3v_2v_1$ is a 6-cycle with a chord vv_4 , a contradiction. By symmetry, f_5 is not a 4-face. Thus, $n_4(v) = 0$. This implies that $n_{6^+}(v) = 2$. Thus $\mu'(v) \geq 1 - 3 \times \frac{1}{2} + 2 \times \frac{49}{150} = \frac{23}{150} > 0$ by (R1) and (R5).

3. d(v) = 6. Then $\mu(v) = 2$. By Lemma $\frac{1 \text{ em4}}{2.5}$, $n_3(v) \le 4$. If $n_3(v) \le 3$, then $\mu'(v) \ge 2 - 3 \times \frac{7}{12} = \frac{1}{4} > 0$ by (R2). Thus, assume that $n_3(v) = 4$.

We now prove $n_4(v) = n_5(v) = 0$. Assume that v_1, v_2, \ldots, v_6 are the neighbors of v in clockwise order, and f_1, f_2, \ldots, f_6 are the incident faces of v with $vv_i, vv_{i+1} \in b(f_i)$ for $i = 1, 2, \ldots, 6$ where indices are taken modulo 6. Since G has no chord 6-cycles, $n_5(v) = 0$ by Lemma \mathbb{Z} . By Lemma \mathbb{Z} . By Lemma \mathbb{Z} . By Lemma \mathbb{Z} . For f_1, f_2, f_4, f_5 are four 3-faces.

In the former case, assume that $f_4 = [vv_4xv_5]$ is a 4-face. Since G is a simple graph, $x \notin \{v, v_3, v_4, v_5, v_6\}$ by Lemma 2.2(1). If $x = v_2$, then $v_2v_4v_3vv_6v_5v_2$ is a 6-cycle with a chord vv_4 , a contradiction. If $x = v_1$, then $v_1v_2v_3v_4vv_5v_1$ is a 6-cycle with a chord vv_3 , a contradiction. If $x \neq v_1$ and $x \neq v_2$, then $xv_5vv_2v_3v_4x$ is a 6-cycle with a chord vv_4 , a contradiction. Thus f_4 is not a 4-face. By symmetry, f_6 is not a 4-face. Thus $n_4(v) = 0$.

In the latter case, assume that $f_3 = [vv_3xv_4]$ is a 4-face. Since G is a simple graph, $x \notin \{v, v_2, v_3, v_4, v_5\}$ by Lemma $\frac{\texttt{lem1}}{\texttt{2.2}(1)}$. If $x = v_1$, then $v_1v_2vv_6v_5v_4v_1$ is a 6-cycle with a chord vv_4 , a contradiction. By symmetry, $x \neq v_6$. If $x \neq v_1$ and $x \neq v_6$, then $v_1v_2v_3xv_4v_5v_1$ is a 6-cycle with a chord vv_5 , a contradiction. Thus f_3 is not a 4-face. By symmetry, f_6 is not a 4-face. Thus, $n_4(v) = 0$.

So far, we have proved that $n_4(v) = n_5(v) = 0$. This implies that $n_{6^+}(v) = 2$. Thus $\mu'(v) \ge 2 - 4 \times \frac{7}{12} + 2 \times \frac{49}{150} = \frac{8}{25} > 0$ by (R2) and (R5).

4. $d(v) \ge 7$, then by Lemma $\frac{1 \text{ em4}}{2.5} v$ is incident with at most (d(v) - 2) 3-faces. Thus $\mu'(v) \ge d(v) - 4 - \frac{89}{150}(d(v) - 2) = \frac{61}{150}d(v) - \frac{422}{150} \ge \frac{1}{30} > 0$ by (R3).

Then we check $\mu'(f) \ge 0$ for all $f \in F(G)$.

- 1. d(f) = 3. By Lemma $\frac{\text{Lem1}}{2.2(2)}$, v is not incident with any $(4, 4, 4^+)$ -face. If f is a $(5^+, 5^+, 5^+)$ -face, then $\mu'(f) \ge -1 + 3 \times \frac{1}{3} = 0$ by (R1)–(R3). If f is a (4, 5, 5)-face, then $\mu'(f) \ge -1 + 2 \times \frac{1}{2} = 0$ by (R1). If f is a $(4, 5^+, 6)$ -face, then $\mu'(f) \ge -1 + \frac{5}{12} + \frac{7}{12} = 0$ by (R1)–(R3). If f is a $(4, 5^+, 7^+)$ -face, then $\mu'(f) \ge -1 + \frac{61}{150} + \frac{89}{150} = 0$ by (R1)–(R3).
- 2. d(f) = 4. Since 4-faces are not involved in discharge procedure, $\mu(f) = \mu'(f) = 0$.
- 3. d(f) = 5. Then $\mu(f) = 1$. By (R4), $\mu'(f) \ge 1 5 \times \frac{11}{60} = \frac{5}{60} > 0$.

4.
$$d(f) \ge 6$$
. By (R5), $\mu'(f) \ge d(f) - 4 - \frac{49}{150}d(f) = \frac{101}{150}d(f) - 4 \ge \frac{6}{150} > 0$.

We have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu'(x) = 0$ for all $x \in V(G) \cup F(G)$. We check above proof and obtain the following assertions.

- (a) For each vertex $v \in V(G)$, d(v) = 4;
- (b) For each face $f \in F(G)$, $3 \le d(f) \le 4$.

By (a), G has no 5⁺-vertices. Thus G is 4-regular, which is contrary to Lemma $\frac{1}{2.2}$ (2). This completes the proof of Theorem $\frac{1}{1.1}$ (2).

3.3 Proof of Theorem $\frac{\pm 1}{1.4}(1)$

In this section, we prove Theorem 1.4(1). Now we define the discharge rules as follows.

(R1) Every 5⁺-face sends $\frac{1}{3}$ to each incident 3-vertex.

It suffices to show that the new weight function μ' satisfies Properties (I) and (II). We first check $\mu'(v) \ge 0$ for all $v \in V(G)$. By Lemma $\frac{1 \text{ em5}}{2.6}(1), d(v) \ge 3$.

- 1. d(v) = 3. Then $\mu(v) = 3 4 = -1$. Since G has no 3- and 4-cycles, v is incident with three 5⁺-faces. Thus $\mu'(v) \ge -1 + 3 \times \frac{1}{3} = 0$ by (R1).
- 2. d(v) = 4. Then $\mu'(v) = \mu(v) = 4 4 = 0$.
- 3. d(v) = 5. Then $\mu'(v) = \mu(v) = d(v) 4 \ge 1 > 0$.

We further check $\mu'(f) \ge 0$ for all $f \in F(G)$. Note that $d(f) \ge 5$.

By Lemma $\frac{[\text{lem5}]}{2.6}(2)$, f is incident with at most $\lfloor \frac{d(f)}{2} \rfloor$ 3-vertices. Thus $\mu'(f) \ge d(f) - 4 - \frac{1}{3} \times \lfloor \frac{d(f)}{2} \rfloor \ge \frac{5}{6}d(f) - 4 \ge \frac{1}{6} > 0$.

We have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu'(x) = 0$ for all $x \in V(G) \cup F(G)$. Considering above proof, we obtain that G has no 5⁺-face and hence every face of G is a 4⁻-face, contrary to our assumption that G has no 3-cycle nor 4-cycle. This completes the proof of Theorem $\frac{l \pm 1}{l \cdot 4}(1)$.

3.4 Proof of Theorem $\frac{\text{th1}}{1.4}(2)$

In this section, we prove Theorem 1.4(2). Since G has no 6-cycle, each 3-vertex is incident with at most one 4-face by Lemma 2.1. A 3-vertex v is *bad* if v is incident with one 4-face and *good* otherwise.

Now we define the discharge rules as follows.

(R1) Every 5⁺-face sends $\frac{1}{3}$ to each incident good 3-vertex and $\frac{1}{2}$ to each incident bad 3-vertex.

It suffices to show that the new weight function μ' satisfies Properties (I) and (II). Note that each 4-face is not adjacent to 4-face by Lemma 2.1 and 2.5 (1).

We first check $\mu'(v) \ge 0$ for all $v \in V(G)$. By Lemma $\overset{\text{lem5}}{2.6(1)}, d(v) \ge 3$.

- 1. d(v) = 3. If v is good, then v is incident with three 5⁺-faces. Thus $\mu'(v) \ge -1 + 3 \times \frac{1}{3} = 0$ by (R1). If v is bad, then v is incident with two 5⁺-faces. Thus $\mu'(v) \ge -1 + 2 \times \frac{1}{2} = 0$ by (R1).
- 2. d(v) = 4. Since any 4-vertex does not involved in discharge procedure, $\mu'(v) = \mu(v) = 4 4 = 0$.
- 3. $d(v) \ge 5$. Then $\mu(v) = d(v) 4$. Since any 5-vertex does not involved in discharge procedure, $\mu'(v) = \mu(v) \ge 1 > 0$.

We further check $\mu'(f) \ge 0$ for all $f \in F(G)$. Note that $d(f) \ge 4$ and $d(f) \ne 6$.

- 1. d(f) = 4. Since any 4-face does not involved in discharge procedure, $\mu'(f) = \mu(f) = 4 4 = 0$.
- 2. d(f) = 5. Then $\mu(f) = 5 4 = 1$. By Lemma $\frac{1 \text{em5}}{2.6(2)}$, f is incident with at most two 3-vertices. If v is incident with at most one 3-vertex, then $\mu'(f) \ge 1 \frac{1}{2} = \frac{1}{2} > 0$ by (R1). Let v be incident with two 3-vertices v_1 and v_2 . If one of v_1 and v_2 is bad and the other is good, then $\mu'(v) \ge 1 \frac{1}{2} \frac{1}{3} = \frac{1}{6} > 0$ by (R1). If both v_1 and v_2 are bad 3-vertices, then $\mu'(v) \ge 1 2 \times \frac{1}{2} = 0$ by (R1).
- 3. $d(f) \ge 7$. By Lemma $\frac{|\underline{lem5}|}{2.6(2)}$, f is incident with at most $\lfloor \frac{d(f)}{2} \rfloor$ 3-vertices. Thus $\mu'(v) \ge d(f) 4 \frac{1}{2} \times \lfloor \frac{d(f)}{2} \rfloor \ge \frac{3}{4}d(f) 4 \ge \frac{5}{4} > 0$.

We have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu'(x) = 0$ for all $x \in V(G) \cup F(G)$. Considering above proof, we establish the following claims. Claim 1. Each of the following holds.

(1) For each vertex $v \in V(G)$, $3 \le d(v) \le 4$.

(2) For each face $f \in F(G)$, f is either a 5-face incident with two bad 3-vertices or a 4-face.

Claim 2. Let f be a 5-face. Then each of the following holds.

- (1) f is not incident with three consecutively adjacent 4-vertices.
- (2) If f is adjacent to a 4-face g, then g is a (3, 4, 4, 4)-face.

Proof of Claim 2. (1) By Claim 1(2), f is incident two bad 3-vertices. By Lemma $\frac{12\text{em5}}{2.6(2)}$, f is a (4, 3, 4, 3, 4)-face and hence f is not incident with three consecutively adjacent 4-vertices.

(2) it follows by Lemma 2.8.

Claim 3. G has a 5-face.

Proof of Claim 3. Suppose otherwise that G has no 5-face. By Claim 1(2), G has only 4-faces. If G has more than one 4-face, then G contains two adjacent 4-faces, contrary to Lemma 2.1.

By Claims 1(2) and 3, we assume that G has a 5-face $f = [v_1v_2v_3v_4v_5]$ incident with two bad 3-vertices. By Claim 2(1) and by symmetry, assume that v_1 and v_3 are two 3-vertices. Thus, v_1 and v_3 are incident with one 4-face and two 5-faces. Assume that v_1 is incident with f, f_1, f_2 and v_3 is incident with f, f_3, f_4 where f_1, f_3 are two 4-faces and f_2, f_4 are two 5-faces. By Lemma $\frac{1 \text{ em} 6}{2.8}$, f_1 and f_3 are two (3, 4, 4, 4)-faces. We observe f_1 and consider two following cases. **Case 1.** $v_1v_2 \in b(f_1)$.

Let $f_1 = [v_1v_2v_6v_7]$. We first claim that $v_6, v_7 \notin \{v_1, \ldots, v_5\}$. Note that $v_6 \notin \{v_1, v_2, v_3\}$ and $v_7 \notin \{v_1, v_2, v_5\}$. Since G has no 3-cycle, $v_6 \notin \{v_4, v_5\}$ and $v_7 \notin \{v_3, v_4\}$. In this case, we consider two cases for f_3 : either $v_2v_3 \in b(f_3)$ or $v_3v_4 \in b(f_3)$.

In the former case, let $f_3 = [v_2v_3v_8v_9]$. We claim that $v_8, v_9 \notin \{v_1, \ldots, v_7\}$. Since $d(v_1) = 3$, $v_8 \neq v_1$ and $v_9 \neq v_1$. Clearly, $v_8 \notin \{v_2, v_3, v_4\}$ and $v_9 \notin \{v_2, v_3, v_6\}$. Since G has no 3-cycle, $v_8 \notin \{v_5, v_6\}$ and $v_9 \notin \{v_4, v_5, v_7\}$. If $v_8 = v_7$, then G contains Configuration (5) in Fig.1, contrary to Lemma $\frac{\text{Lem7}}{2.7}$. Thus, $v_8, v_9 \notin \{v_1, \ldots, v_7\}$.

Let v_2 be incident with f, f_1, f_5, f_3 in clockwise order. By Lemma $\frac{\text{LemO}}{2.1}, f_5$ is a 5-face. By Claim 2(2), $d(v_2) = d(v_6) = d(v_9) = 4$. Therefore, f_5 is incident with three consecutively adjacent 4-vertices which is contrary to Claim 2(1).

In the latter case, let $f_3 = [v_4v_3v_9v_8]$. We first claim that $v_8, v_9 \notin \{v_1, \ldots, v_7\}$. Recall that v_1 is a 3-vertex, $v_8 \neq v_1$ and $v_9 \neq v_1$. Obviously, $v_8 \notin \{v_3, v_4, v_5\}$ and $v_9 \notin \{v_2, v_3, v_4\}$ since G is simple. Since G has no 3-cycle, $v_8 \neq v_2$ and $v_9 \notin \{v_5, v_6\}$. If $v_8 = v_6$, then $v_4v_3v_2v_1v_7v_6v_4$ is a 6-cycle, a contradiction. Thus $v_8, v_9 \notin \{v_1, \ldots, v_6\}$. If $v_8 = v_7$, then $v_9 \notin \{v_1, \ldots, v_7\}$ and $v_7v_1v_5v_4v_3v_9v_7$ is a 6-cycle, a contradiction. If $v_9 = v_7$, then $v_8 \notin \{v_1, \ldots, v_7\}$ and $v_7v_6v_2v_3v_4v_8v_7$ is a 6-cycle, a contradiction. Thus, $v_8, v_9 \notin \{v_1, \ldots, v_7\}$.

Let v_2 be incident with f, f_1, f_5, f_4 in clockwise order. By Lemma $\frac{1 \text{ em0}}{2.1}, f_4$ is a 5-face. Let $f_4 = [v_2 v_3 v_9 v_{10} v_{11}]$. We first assume that $v_{10}, v_{11} \notin \{v_1, \ldots, v_9\}$. In this case, $d(v_2) = d(v_9) = 4$ and $d(v_3) = 4$. By Claim 1(2), only one vertex in $\{v_{10}, v_{11}\}$ is a 3-vertex. Since G does not contain Configuration (7) of Fig.1 in Lemma $\frac{1 \text{ em7}}{2.7}, d(v_{11}) = 4$ and $d(v_{10}) = 3$. By Lemma $\frac{1 \text{ em0}}{2.1}$ and Claim 1

(2), f_5 is a 5-face. Moreover, f_5 is incident with three consecutively adjacent 4-vertices v_6, v_2 and v_{11} , contrary to Claim 2 (1). Thus, assume that $v_{10} \in \{v_1, \ldots, v_9\}$ or $v_{11} \in \{v_1, \ldots, v_9\}$.

Since v_1 is a 3-vertex, $v_{10} \neq v_1$ and $v_{11} \neq v_1$. Obviously, $v_{10} \notin \{v_2, v_3, v_8, v_9\}$ and $v_{11} \notin \{v_2, v_3, v_6\}$ since G is simple. Since G has no 3-cycle, $v_{10} \neq v_4$ and $v_{11} \notin \{v_4, v_5, v_7, v_9\}$. If $v_{10} = v_6$, then $v_1v_2v_3v_9v_6v_7v_1$ is a 6-cycle, a contradiction. If $v_{10} = v_7$, then $v_7v_9v_3v_4v_5v_1v_7$ is a 6-cycle, a contradiction. Thus $v_{10} \notin \{v_1, \ldots, v_4, v_6, \ldots, v_9\}$ and $v_{11} \notin \{v_1, \ldots, v_7, v_9\}$. Let $v_{10} = v_5$. If $v_{11} = v_8$, then $v_5v_4v_8v_5$ is a 3-cycle, a contradiction. If $v_{11} \notin \{v_1, \ldots, v_9\}$, then $v_1v_5v_{11}v_2v_6v_7v_1$ is a 6-cycle, a contradiction. If $v_{11} \notin \{v_1, \ldots, v_9\}$, then $v_1v_5v_{11}v_2v_6v_7v_1$ is a 6-cycle, a contradiction. If $v_{10} \notin \{v_1, \ldots, v_9\}$, then $v_1v_5v_{10}v_8$ is a 3-cycle, a contradiction.

Case 2. $v_1v_5 \in b(f_1)$.

Let $f_1 = [v_1v_5v_6v_7]$. We first claim that $v_6, v_7 \notin \{v_1, \ldots, v_5\}$. Obviously, $v_6 \notin \{v_1, v_4, v_5\}$ and $v_7 \notin \{v_1, v_2, v_5\}$. Since G has no 3-cycle, $v_6 \notin \{v_2, v_3\}$ and $v_7 \notin \{v_3, v_4\}$. In this case, we consider two cases of f_3 : either $v_2v_3 \in b(f_3)$ or $v_3v_4 \in b(f_3)$.

In the former case, let $f_3 = [v_2v_3v_8v_9]$. We claim that $v_8, v_9 \notin \{v_1, \ldots, v_7\}$. Since v_1 is a 3-vertex, $v_8 \neq v_1$ and $v_9 \neq v_1$. Obviously, $v_8 \notin \{v_2, v_3, v_4\}$ and $v_9 \notin \{v_2, v_3\}$ since G is simple. Since G has no 3-cycle, $v_8 \neq v_5$ and $v_9 \notin \{v_4, v_5, v_7\}$. If $v_8 = v_7$, then $v_7v_6v_5v_1v_2v_3v_7$ is a 6-cycle, a contradiction. Thus $v_8, v_9 \notin \{v_1, \ldots, v_5, v_7\}$. If $v_8 = v_6$, then $v_9 \neq v_6$. In this case, $v_6v_5v_4v_3v_2v_9v_6$ is a 6-cycle, a contradiction. If $v_9 = v_6$, then $v_8 \neq v_6$. In this case, $v_6v_5v_1v_2v_3v_8v_6$ is a 6-cycle, a contradiction. So far, we have proved that $v_8, v_9 \notin \{v_1, \ldots, v_7\}$.

Assume that v_3 is incident with f, f_3 and f_4 in clockwise order. Since G has no 6-cycle, by Lemma $\underbrace{\mathbb{L}:m0}_{\mathbb{L}:\mathbb{R}} f_4$ is a 5-face. Let $f_4 = [v_4 v_3 v_8 v_{10} v_{11}]$. If $v_{10}, v_{11} \notin \{v_1, \ldots, v_9\}$, then G contains Configuration (1) or (2) in Fig.1, contrary to Lemma $\underbrace{\mathbb{L}:m7}_{\mathbb{L}:\mathbb{R}}$. Thus, assume that $v_{10} \in \{v_1, \ldots, v_9\}$ or $v_{11} \in \{v_1, \ldots, v_9\}$. Since v_1 is a 3-vertex, $v_{10} \neq v_1$ and $v_{11} \neq v_1$. Obviously, $v_{10} \notin \{v_3, v_8, v_9\}$ and $v_{11} \notin \{v_3, v_4, v_5\}$ since G is simple. Since G has no 3-cycle, $v_{10} \notin \{v_2, v_4\}$ and $v_{11} \notin \{v_2, v_6, v_8\}$. If $v_{10} = v_5$, then $v_8 v_5 v_4 v_3 v_2 v_9 v_8$ is a 6-cycle, a contradiction. If $v_{10} = v_6$, then $v_6 v_7 v_1 v_2 v_9 v_8 v_6$ is a 6-cycle, a contradiction. If $v_{10} = v_7$, then $v_8 v_7 v_1 v_5 v_4 v_3 v_8$ is a 6-cycle, a contradiction. Thus $v_{10} \notin \{v_1, \ldots, v_9\}$. If $v_{11} = v_7$, then $v_7 v_1 v_2 v_9 v_8 v_{10} v_7$ is a 6-cycle, a contradiction. If $v_{11} = v_9$, then $v_8 v_9 v_{10} v_8$ is a 3-cycle, a contradiction.

In the latter case, let $f_3 = [v_3v_4v_8v_9]$. We claim that $v_8, v_9 \notin \{v_1, \ldots, v_7\}$. Since v_1 is a 3-vertex, $v_9 \neq v_1$ and $v_8 \neq v_1$. Since G is simple, $v_9 \notin \{v_2, v_3, v_4\}$ and $v_8 \notin \{v_3, v_4, v_5\}$ by Lemma 2.6(1). Since G has no 3-cycle, $v_9 \neq v_5$ and $v_8 \notin \{v_2, v_6\}$. If $v_9 = v_6$, then $v_6v_7v_1v_5v_4v_3v_6$ is a 6-cycle, a contradiction. If $v_9 = v_7$, then $v_5v_6v_7v_3v_2v_1v_5$ is a 6-cycle, a contradiction. Thus $v_9 \notin \{v_1, \ldots, v_7\}$. If $v_8 = v_7$, then $v_7v_6v_5v_4v_3v_9v_7$ is a 6-cycle, a contradiction.

Since G has no 6-cycle, by Lemma $\overset{\text{lemU}}{2.1}$, f_4 is a 5-face. Let $f_4 = [v_2v_3v_9v_{10}v_{11}]$. If $v_{10}, v_{11} \notin \{v_1, \ldots, v_9\}$, then G contains Configuration (3) or (4) of Fig.1, contrary to Lemma $\overset{\text{lem7}}{2.7}$. Thus, assume that either $v_{10} \in \{v_1, \ldots, v_9\}$ or $v_{11} \in \{v_1, \ldots, v_9\}$. Since v_1 is a 3-vertex, $v_{10} \neq v_1$. Since G is simple and by Lemma $\overset{\text{lem5}}{2.6}(1)$, $v_{10} \notin \{v_3, v_8, v_9\}$ and $v_{11} \notin \{v_1, v_2, v_3\}$. Since G has no 3-cycles, $v_{10} \notin \{v_2, v_4\}$ and $v_{11} \notin \{v_4, v_5, v_7, v_9\}$. If $v_{10} = v_6$, then $v_6v_7v_1v_2v_3v_9v_6$ is a 6-cycle, a contradiction. If $v_{10} = v_7$, then $v_9v_8v_4v_5v_6v_7v_9$ is a 6-cycle, a contradiction. Thus $v_{10} \notin \{v_1, \ldots, v_9\}$ and $v_{11} \notin \{v_1, \ldots, v_5, v_7, v_9\}$. Assume that $v_{11} = v_6$. If $v_{10} = v_5$, then G contains Configuration (6) of Fig.1, contrary to Lemma $\overset{\text{lem7}}{2.7}$. Thus, $v_{10} \neq v_5$. So, $v_{10} \notin \{v_1, \ldots, v_9\}$. In this case, $v_{10}v_6v_5v_4v_8v_9v_{10}$ is a 6-cycle, a contradiction. Thus, assume that

 $v_{11} = v_8$. If $v_{10} \notin \{v_1, \ldots, v_9\}$, then $v_8v_9v_{10}v_8$ is a 3-cycle, a contradiction. If $v_{10} = v_5$, then $v_5v_4v_8v_5$ is a 3-cycle, a contradiction. Thus $v_{11} \notin \{v_1, \ldots, v_9\}$. If $v_{10} = v_5$, then $v_5v_6v_7v_1v_2v_{11}v_5$ is a 6-cycle, a contradiction. \Box

This implies that G is not existence. We have proved Property (II). This completes the proof of Theorem $\frac{\texttt{th1}}{\texttt{I.4}}(2)$.

3.5 Proof of Theorem $\frac{\pm h1}{1.4}(3)$

In this section, we prove Theorem $[\frac{th1}{1.4}(3)$. A 3-vertex v is *bad* if v is incident with one 3-face and *good* otherwise.

Now we define the discharge rules as follows.

(R1) Every 5⁺-face sends $\frac{1}{3}$ to each incident good 3-vertex, $\frac{1}{2}$ to each incident bad 3-vertex and $\frac{1}{3}$ to each incident 3-face.

It suffices to show that the new weight function μ' satisfies Properties (I) and (II). We first check $\mu'(v) \ge 0$ for all $v \in V(G)$. By Lemma $\frac{1 \text{ em5}}{2.6}(1), d(v) \ge 3$.

- 1. d(v) = 3. If v is bad, then v is incident with two 7⁺-faces. By (R1), $\mu'(v) \ge -1 + 2 \times \frac{1}{2} = 0$. If v is good, then v is incident with three 5⁺-faces. By (R1), $\mu'(v) \ge -1 + 3 \times \frac{1}{3} = 0$.
- 2. d(v) = 4. Since no 4-vertex is involved in discharge procedure, $\mu'(v) = \mu(v) = 4 4 = 0$.

3.
$$d(v) \ge 5$$
. By (R1), $\mu'(v) = \mu(v) = d(v) - 4 \ge 1 > 0$.

Further we check $\mu'(f) \ge 0$ for all $f \in F(G)$.

- 1. d(f) = 3. Then $\mu(f) = -1$. Since G has no 4-cycle, each face adjacent to f is a 5⁺-face. By (R1), $\mu'(v) \ge -1 + 3 \times \frac{1}{3} = 0$.
- 2. d(f) = 5. Then $\mu(f) = 1$. Since G has no 6-cycle, f is not adjacent to 3-face. By Lemma $\overline{2.6(2)}$, f is adjacent to at most two 3-vertices. Since G has no 6-cycles, no 3-cycle is not adjacent to any 5-cycle. Thus, f is adjacent to at most two good 3-vertices. By (R1), $\mu'(f) \ge 1 2 \times \frac{1}{3} = \frac{1}{3} > 0$.
- 3. $d(f) \geq 7$. Let f be incident with m 3-vertices. Since G has no 4-cycles, no 3-face is adjacent to any 3-face. Then v is incident with at most d(f) m 3-faces. By Lemma $\frac{1 \text{ Lem5}}{2.6}(2)$, $m \leq \lfloor \frac{d(f)}{2} \rfloor$. Thus $\mu'(f) \geq d(f) 4 \frac{m}{2} \frac{1}{3}(d(f) m) = \frac{2}{3}d(f) \frac{1}{6}m 4 \geq \frac{7}{12}d(f) 4 \geq \frac{1}{12} > 0$ by (R1).

So far, we have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu'(x) = 0$ for all $x \in V(G) \cup F(G)$. Observing above proof, we obtain the following statements.

- (a) For each vertex $v \in V(G)$, $3 \le d(v) \le 4$;
- (b) For each face $f \in F(G)$, d(f) = 3.

By (b), G is one 3-cycle [uvw]. Clearly, G has a matching $M = \{uv\}$ such that G - M is (2, 1) decomposable, a contradiction. This completes the proof of Theorem 1.4(3).

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