# Decompositions of graphs of nonnegative characteristic with some forbidden subgraphs 

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#### Abstract

A $(d, h)$-decomposition of a graph $G$ is an order pair $(D, H)$ such that $H$ is a subgraph of $G$ where $H$ has the maximum degree at most $h$ and $D$ is an acyclic orientation of $G-$ $E(H)$ of maximum out-degree at most $d$. A graph $G$ is $(d, h)$-decomposable if $G$ has a $(d, h)$ decomposition. Let $G$ be a graph embeddable in a surface of nonnegative characteristic. In this paper, we prove the following results. (1) If $G$ has no chord 5 -cycles or no chord 6 -cycles or no chord 7 -cycles and no adjacent 4 -cycles, then $G$ is ( 3,1 )-decomposable, which generalizes the results of Chen, Zhu and Wang [Comput. Math. Appl, 56 (2008) 2073-2078] and the results of Zhang [Comment. Math. Univ. Carolin, 54(3) (2013) 339-344]. (2) If $G$ has no $i$-cycles nor $j$-cycles for any subset $\{i, j\} \subseteq\{3,4,6\}$ is (2,1)-decomposable, which generalizes the results of Dong and Xu [Discrete Math. Alg. and Appl., 1(2) (2009), 291-297].


## 1 Introduction

Graphs considered here are finite and simple. A graph is $d$-generate if every subgraph has a vertex of degree at most $d$. For two integers $d, h \in \mathbb{N}$, a $(d, h)$-decomposition of $G$ is a pair $\left(H_{1}, H_{2}\right)$ such that $H_{2}$ is a subgraph of $G$ of maximum degree at most $h$ and $H_{1}$ is $d$-degenerate. A graph $G$ is $(d, h)$-decomposable if $G$ has a $(d, h)$-decomposition. Decomposing a graph into subgraphs with simple structure is a fundamental problem in graph theory. The classical Theorem of Tutte $\frac{18}{18}$ and, independent by, Nash-Williams 15$]$ provides a necessary and sufficient condition for which a graph can be decomposed into forests. A proper coloring of $G$ is a decomposition of $G$ into independent sets. The problems of decomposing a graph $G$ into start forests, linear forests and some others are studied widely in the literature.

A proper $k$-coloring is a mapping $\varphi: V(G) \rightarrow\{1,2, \ldots, k\}$ such that $\varphi(u) \neq \varphi(v)$ where $u v \in E(G)$. The chromatic number, denoted by $\chi(G)$, of $G$ is the minimum $k$ such that $G$ is $k$-colorable. A $d$-defective $k$-coloring of $G$ is a mapping $\varphi: V(G) \rightarrow\{1,2, \ldots, k\}$ such that for each vertex $v \in V(G), v$ has at most $d$ neighbors of the same color as itself. A $k$-list assignment of $G$ is a function $L$ that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$ where $|L(v)|=k$. A $d$-defective $L$-coloring is a mapping $\varphi$ that assigns a color $\varphi(v) \in L(v)$ to each vertex $v \in V(G)$ such that $v$ has at most $d$ neighbors of the same color as itself. A graph $G$ is $d$-defective $k$-choosable

[^0]if there exists an $(L, d)$－coloring for every list assignment $L$ with $|L(v)|=k$ for all $v \in V(G)$ ．A graph is 0 －defective $k$－choosable if and only if it is $k$－choosabe．The choosable number，denoted by $\operatorname{ch}(G)$ ，of $G$ is the minimum $k$ such that $G$ is $k$－choosable．

The Alon－Tarsi number of $G$ ，denoted by $A T(G)$ ，was defined by Jensen and Toft lifil．It follows from the Alon－Tarsi Theorem［1］ 1 that $\operatorname{ch}(G) \leq A T(G)$ for any graph $G$ ．It is proved that the difference $A T(G)-c h(G)$ can be arbitrarily large．DP－coloring was introduced by Dvořák and Postle $\mathbb{\mathbb { P }} \mathbb{7}$ as a generation of list coloring．Clearly， $\operatorname{ch}(G) \leq \chi_{D P}(G)$ ，where $\chi_{D P}(G)$ is the DP－chromatic number of a graph $G$ ．A painting coloring was introduced by Schauz［16］and it is proved that $\operatorname{ch}(G) \leq \chi_{P}(G)$ for any graph $G$ ，where $\chi_{P}(G)$ is the paint number of $G$ ．

It is well－known that a graph $H_{1}$ has an acyclic orientation $D$ with $\Delta_{D}^{+} \leq d$ if and only if $H_{1}$ is $d$－degenerate，where $\Delta_{D}^{+}$is the maximum degree $D$ ．If $G$ is $d$－degenerate，then each of choosable number $\operatorname{ch}(G)$ ，Alon－Tarsi number $A T(G)$ ，paint number $\chi_{P}(G)$ and DP－chromatic number $\chi_{D P}$ is at most $d+1$ ．This implies that if $G$ is $(d, h)$－decomposable，then there is a subgraph $H$ of $G$ where $\Delta_{H} \leq h$ such that $G-E(H)$ is $h$－defective－$(d+1)$－choosable，$(d+1)$－DP－colorable and $A T(G-E(H)) \leq d+1$ ．

Defective coloring of graphs was considered by Cowen，Cowen and Woodall［罂］who proved that every planar graph is 2－defective 3 －colorable，which was improved by Eaton and Hull $\frac{\mathrm{EH} \bar{Z}]}{\underline{Z}}$ ， independently，Škrekovski $\frac{S_{1}}{17}$ ，who proved that every planar graph is 2－defective 3 －choosable． Cushing and Kierstead 识 and Zhu［票 10$]$ strengthen the result and proved that every planar graph $G$ has a matching $M$ such that $A T(G-M) \leq 4$ ．Lih，Song，Wang and Zhang li2］proved that every planar graph $G$ without 4 －cycles and $l$－cycles for some $l \in\{5,6,7\}$ is 1－defective 3 －choosable．Dong and Xu $\frac{X u}{[6]}$ showed that such result is also true for some $l \in\{8,9\}$ ．Lu and Zhu $\left[\frac{\pi 20}{[14]}\right.$ proved that every planar graph without 4－and $l$－cycles $G$ ，where $l=5,6,7$ ，has a matching $M$ such that $G-M$ is $A T(G-M) \leq 3$ ． Gonçalves $\frac{[G 009}{[9]}$ proved that every planar graph is（3，4）－decomposable．Zhu $\frac{\text { Zhu00 }}{Z 0]}$ proved that every planar graph is（2，8）－decomposable．Recently，Li，Lu，Wang and Zhu［13］improve this result and prove that for $l \in\{5,6,7,8,9\}$ ，every planar graph without 4 －and $l$－cycles is（ 2,1 ）－decomposable． Cho，Choi，Kim，Park，Shan and Zhu Zhu21 prove that every planar graph is $(4,1)-,(3,2)-,(2,6)$－ decomposable and that there are planar graphs which are not（2，3）－decomposable and there are also planar graphs which are not $(1, h)$－decomposable．

We are interested in decompositions of graphs of nonnegative characteristic in this paper．The characteristic of a surface $\Sigma$ is defined to be $|V(G)|-|E(G)|+|F(G)|$ for any graph $G$ which is 2－cell embedded in $\Sigma$ ．All the surfaces of nonnegative characteristic are the Euclidean plane，the projective plane，the torus and the Klein bottle．A graph of nonnegative characteristic means that it can be embedded on a surface of nonnegative characteristic．Throughout this paper，a graph of nonnegative characteristic is called a $N C$－graph．In this paper，we prove the following results．
th0 Theorem 1．1 A NC－graph $G$ is $(3,1)$－decomposable if one of the following hold：
（1）$G$ has no chord 5 －cycles．
（2）$G$ has no chord 6 －cycles．
（3）G has no chord 7－nor adjacent 4－cycles．
For simplicity，we define a family $\mathcal{G}$ of NC－graphs such that $G \in \mathcal{G}$ if and only if $G$ has neither
chord 5 -cycles nor chord 6 -cycles nor chord 7 - and adjacent 4 -cycles. From Theorem thi.1, next corollary follows immediately.

Corollary 1.2 Every graph $G \in \mathcal{G}$ has a matching $M$ such that each of choice number, paint number, $D P$-number and Alon-Tarsi number of $G-M$ is at most 4.

A graph $G$ is toroidal if $G$ can be drawn on the torus so that the edges meet only at the vertices of the graph.

Corollary 1.3 (1) (Chen, Zhu and Wang, 渍期 Every graph of nonnegative characteristic without either chord 5 -cycles or chord 6 -cycles is 1-defective 4-choosable.
(2) (Zhang, Zhang Every toroidal graph $G$ without chord 7-cycles and adjacent 4-cycles is 1defective 4-choosable.
th1 Theorem 1.4 A NC-graph $G$ is $(2,1)$-decomposable if one of the following hold:
(1) $G$ has neither 3- nor 4-cycles.
(2) $G$ has neither 3-nor 6-cycles.
(3) $G$ has neither 4- nor 6-cycles.

Similarly, we define a family $\mathcal{H}$ of NC-graphs such that $G \in \mathcal{H}$ if and only if $G$ has no $i$-cycles nor $j$-cycles for any $\{i, j\} \subseteq\{3,4,6\}$. We obtain the following corollary from Theorem th.

Corollary 1.5 Every graph $G \in \mathcal{H}$ has a matching $M$ such that each of choice number, paint number, $D P$-number and Alon-Tarsi number of $G-M$ is at most 3 .
 $j$-cycles for any subset $\{i, j\} \subseteq\{3,4,6\}$ is 1-defective 3-colorable.

In the end of this section, we introduce some terminology and notation. Let $G$ be a graph and denote by $V(G), E(G), F(G)$ (or $V, E, F$ for short) the sets of vertices, edges and faces of $G$, respectively. Let $G$ be a (di)graph. For a vertex $v$, denote by $d(v)\left(d^{+}(v)\right.$ or $d^{-}(v)$ in digraph) the degree (out-degree or in-degree in digraph) of $v$. Denote by $N_{G}(v)$ (or $N(v)$ for short) the set of neighbors of a vertex $v$ in $G$. A $k$-vertex $\left(k^{+}\right.$-vertex or $k^{-}$-vertex) is a vertex of degree $k$ (at least $k$ or at most $k$ ). Similarly, a $k$-face ( $k^{+}$-face or $k^{-}$-face) is a face of degree $k$ (at least $k$ or at most $k$ ). For $f \in F(G)$, denote by $d(f)$ the degree of face $f$ in $G$ which is the number of edges incident with $f$ and $b(f)$ the boundary walk of $f$ and write $f=\left[u_{1} u_{2} \ldots u_{l}\right]$ when $u_{1}, u_{2}, \ldots, u_{l}$ are the boundary vertices of $f$ in clockwise order. A $l$-face $\left[u_{1} u_{2} \ldots u_{l}\right]$ is called an $\left(a_{1}, a_{2}, \ldots, a_{l}\right)$-face if $d\left(u_{i}\right)=a_{i}$ for $i=1,2, \ldots, l$. Two faces are adjacent if they share at least one common edge. For $v \in V(G)$ and $i \geq 3$, denote by $n_{i}(v)\left(n_{i^{+}}(v)\right.$ or $\left(n_{i^{-}}(v)\right)$ the number of all $i^{-}\left(i^{+}\right.$- or $\left(i^{-}\right)$faces incident to $v$. A cycle is a $k$-cycle if it contains $k$ vertices. For a cycle $C$, an edge $x y \in E(G) \backslash E(C)$ is called a chord of $C$ if $x, y \in V(C)$. Let $C$ be a $k$-cycle. Then $C$ is called chord $k$-cycle.

## 2 Reducible configurations

Suppose otherwise that Theorems and th. 1 are both false. Assume that

$$
\begin{equation*}
G \in \mathcal{G} \text { is a counterexample to Theorem with }|V(G)| \text { minimized. } \tag{1}
\end{equation*}
$$

In this case, $G$ has no chord 5 -cycles nor chord 6 -cycles nor chord 7 -cycles and adjacent 4 -cycles. Clearly, $G$ has no (3,1)-decomposition but any subgraph of $G$ does. Similarly, assume that

$$
\begin{equation*}
H \in \mathcal{H} \text { is a counterexample to Theorem wh1 }|V(H)| \text { minimized. } \tag{2}
\end{equation*}
$$

In this case, $H$ has neither $i$-cycle nor $j$-cycle, where $\{i, j\} \subset\{3,4,6\}$. Clearly, $H$ has no $(2,1)$ decomposition but any subgraph of $H$ does. In this section, we establish several lemmas. The following lemma is straightforward.
lem0 Lemma 2.1 Assume that $G$ is a $N C$-graph and $d(v) \geq 3$ for all $v \in V(G)$. If $G$ has no 6 -cycles, then two 4-faces are not adjacent.

Recall that a graph $H$ is $d$-degenerate if and only if $H$ has an acyclic orientation $D$ with $\Delta_{D}^{+} \leq d$. Thus, to prove that a graph $G$ is $(d, h)$-decomposable, it is sufficient to show that $G$ can be decomposed into $H_{1}$ and $H_{2}$ such that $H_{1}$ has an acyclic orientation with $\Delta_{D}^{+} \leq d$ and $H_{2}$ has
 (这)
lem1 Lemma 2.2 (1) $d(v) \geq 4$ for all $v \in V(G)$;
(2) $G$ does not contain two adjacent 4-vertices.

Proof. (1) Suppose otherwise that $v$ is a 3 -vertex and $N(v)=\left\{v_{1}, v_{2}, v_{3}\right\}$. By the minimality of $G$, there is a $(3,1)$-decomposition $\left(D^{*}, M^{*}\right)$ of $G-\{v\}$. Let $M=M^{*}$ and $D=D^{*} \cup\left\{\overrightarrow{v v_{1}}, \overrightarrow{v v_{2}}, \overrightarrow{v v_{3}}\right\}$. Then $(D, M)$ is a (3,1)-decomposition of $G$, a contradiction.
(2) Suppose otherwise that $u$ is a 4 -vertex adjacent to a 4 -vertex $v$. Let $N(u)=\left\{u_{1}, u_{2}, u_{3}, v\right\}$ and $N(v)=\left\{v_{1}, v_{2}, v_{3}, u\right\}$. By the minimality of $G$, there is a $(3,1)$-decomposition $\left(D^{*}, M^{*}\right)$ of $G-\{u, v\}$. Let $M=M^{*} \cup\{u v\}$ and $D=D^{*} \cup\left\{\overrightarrow{v v_{1}}, \overrightarrow{v v_{2}}, \overrightarrow{v v_{3}}, \overrightarrow{u u_{1}}, \overrightarrow{u u_{2}}, \overrightarrow{u u_{3}}\right\}$. Then $(D, M)$ is a $(3,1)$-decomposition of $G$, a contradiction.
lem2 Lemma 2.3 (1) A 5-vertex $v$ is incident with at most one (4,5,5)-face.
(2) A 5-vertex $v$ is not incident with three consecutively adjacent 3-faces, one of which is $(4,5,5)$-face and other two of which are (4,5,6)-faces.

Proof. Let $v_{1}, v_{2}, \ldots, v_{5}$ be the neighbors of $v$ in clockwise order, and $f_{1}, f_{2}, \ldots, f_{5}$ be the incident faces of $v$ with $v v_{i}, v v_{i+1} \in b\left(f_{i}\right)$ for $i=1,2, \ldots, 5$ where indices are taken modulo 5 .
(1) Suppose otherwise that $v$ is incident with two $(4,5,5)$-faces. There are two cases.

Case 1. $f_{1}$ and $f_{2}$ are $(4,5,5)$-faces.
We first assume that $d\left(v_{1}\right)=d\left(v_{3}\right)=5$ and $d\left(v_{2}\right)=4$. Let $N\left(v_{1}\right)=\left\{v_{11}, v_{12}, v_{13}, v, v_{2}\right\}, N\left(v_{2}\right)=$ $\left\{v_{21}, v, v_{1}, v_{3}\right\}$ and $N\left(v_{3}\right)=\left\{v_{31}, v_{32}, v_{33}, v, v_{2}\right\}$.

By the minimally of $G$, there is a (3,1)-decomposition $\left(D^{*}, M^{*}\right)$ of $G-\left\{v, v_{1}, v_{2}, v_{3}\right\}$. Let $M=$ $M^{*} \cup\left\{v v_{1}, v_{2} v_{3}\right\}$ and $D=D^{*} \cup\left\{\overrightarrow{v_{1} v_{11}}, \overrightarrow{v_{1} v_{12}}, \overrightarrow{v_{1} v_{13}}, \overrightarrow{v_{2} v_{21}}, \overrightarrow{v_{3} v_{31}}, \overrightarrow{v_{3} v_{32}}, \overrightarrow{v_{3} v_{33}}, \overrightarrow{v v_{3}}, \overrightarrow{v v_{4}}, \overrightarrow{v v_{5}}, \overrightarrow{v_{2} v}, \overrightarrow{v_{2} v_{1}}\right\}$. Then $(D, M)$ is a ( 3,1 )-decomposition of $G$, a contradiction.

We further assume that $d\left(v_{1}\right)=d\left(v_{3}\right)=4$ and $d\left(v_{2}\right)=5$. Let $N\left(v_{1}\right)=\left\{v_{11}, v_{12}, v, v_{2}\right\}, N\left(v_{2}\right)=$ $\left\{v_{21}, v_{22}, v, v_{1}, v_{3}\right\}$ and $N\left(v_{3}\right)=\left\{v_{31}, v_{32}, v, v_{2}\right\}$.

By the minimality of $G$, there is a (3,1)-decomposition ( $D^{*}, M^{*}$ ) of $G-\left\{v, v_{1}, v_{2}, v_{3}\right\}$. Let $M=M^{*} \cup\left\{v_{1} v_{2}, v v_{3}\right\}$ and $D=D^{*} \cup\left\{\overrightarrow{v_{1} v_{11}}, \overrightarrow{v_{1} v_{12}}, \overrightarrow{v_{2} v_{21}}, \overrightarrow{v_{2} v_{22}}, \overrightarrow{v_{3} v_{31}}, \overrightarrow{v_{3} v_{32}}, \overrightarrow{v v_{2}}, \overrightarrow{v v_{4}}, \overrightarrow{v v_{5}}, \overrightarrow{v_{1} v}, \overrightarrow{v_{3} v_{2}}\right\}$. Then $(D, M)$ is a (3,1)-decomposition of $G$, a contradiction.
Case 2. $f_{1}$ and $f_{3}$ are ( $4,5,5$ )-faces.
We assume, without loss of generality, that $d\left(v_{1}\right)=d\left(v_{3}\right)=4, d\left(v_{2}\right)=d\left(v_{4}\right)=5$ and $N\left(v_{1}\right)=$ $\left\{v_{11}, v_{12}, v, v_{2}\right\}, N\left(v_{2}\right)=\left\{v_{21}, v_{22}, v_{23}, v, v_{1}\right\}, N\left(v_{3}\right)=\left\{v_{31}, v_{32}, v, v_{4}\right\}, N\left(v_{4}\right)=\left\{v_{41}, v_{42}, v_{43}, v, v_{3}\right\}$.

By the minimality of $G$, there is a ( 3,1 )-decomposition ( $D^{*}, M^{*}$ ) of $G-\left\{v, v_{1}, v_{2}, v_{3}, v_{4}\right\}$. Let $M=M^{*} \cup\left\{v_{1} v_{2}, v_{3} v_{4}\right\}$ and $D=D^{*} \cup\left\{\overrightarrow{v_{1} v_{11}}, \overrightarrow{v_{1} v_{12}}, \overrightarrow{v_{2} v_{21}}, \overrightarrow{v_{2} v_{22}}, \overrightarrow{v_{2} v_{23}}, \overrightarrow{v_{3} v_{31}}, \overrightarrow{v_{3} v_{32}}, \overrightarrow{v_{4} v_{41}}, \overrightarrow{v_{4} v_{42}}\right.$, $\left.\overrightarrow{v_{4} v_{43}}, \overrightarrow{v v_{2}}, \overrightarrow{v v_{4}}, \overrightarrow{v v_{5}}, \overrightarrow{v_{1} v}, \overrightarrow{v_{3}} \vec{v}\right\}$. Then $(D, M)$ is a $(3,1)$-decomposition of $G$, a contradiction.
(2) By (1) and by symmetry, suppose otherwise that $f_{1}$ is a $(4,5,5)$-face and $f_{2}, f_{3}$ are two (4,5,6)-faces. In this case, $d\left(v_{1}\right)=5, d\left(v_{2}\right)=d\left(v_{4}\right)=4$ and $d\left(v_{3}\right)=6$. Let $N\left(v_{1}\right)=$ $\left\{v_{11}, v_{12}, v_{13}, v, v_{2}\right\}, N\left(v_{2}\right)=\left\{v_{21}, v, v_{1}, v_{3}\right\}, N\left(v_{3}\right)=\left\{v_{31}, v_{32}, v_{33}, v, v_{2}, v_{4}\right\}$ and $N\left(v_{4}\right)=\left\{v_{41}, v_{42}\right.$, $\left.v, v_{3}\right\}$. By the minimality of $G$, there is a $(3,1)$-decomposition $\left(D^{*}, M^{*}\right)$ of $G-\left\{v, v_{1}, v_{2}, v_{3}, v_{4}\right\}$. Let $M=M^{*} \cup\left\{v_{1} v_{2}, v_{3} v_{4}\right\}$ and $D=D^{*} \cup\left\{\overrightarrow{v_{1} v_{11}}, \overrightarrow{v_{1} v_{12}}, \overrightarrow{v_{1} v_{13}}, \overrightarrow{v_{2} v_{21}}, \overrightarrow{v_{3} v_{31}}, \overrightarrow{v_{3} v_{32}}, \overrightarrow{v_{3} v_{33}}, \overrightarrow{v_{4} v_{41}}, \overrightarrow{v_{4} v_{42}}\right.$, $\left.\overrightarrow{v v_{1}}, \overrightarrow{v v_{3}}, \overrightarrow{v v_{5}}, \overrightarrow{v_{2} v}, \overrightarrow{v_{2} v_{3}}, \overrightarrow{v_{4}} \vec{v}\right\}$. Then $(D, M)$ is a (3,1)-decomposition of $G$, a contradiction.
lem3 Lemma 2.4 If $G$ is a NC-graph without either chord 5 -cycles or chord 7 - and adjacent 4-cycles, then every $4^{+}$-vertex $v$ is incident with at most two consecutively adjacent 3-faces. Moreover, $v$ is incident with at most $\left\lfloor\frac{2 d(v)}{3}\right\rfloor$ 3-faces.

Proof. Suppose otherwise that $v$ is a $4^{+}$-vertex incident with three consecutively adjacent 3 faces $\left[v_{1} v v_{2}\right],\left[v_{2} v v_{3}\right]$ and $\left[v_{3} v v_{4}\right]$. In this case, $G$ has a 5 -cycle $\left[v_{1} v_{2} v_{3} v_{4} v\right]$ with a chord $v v_{3}$, a contradiction. Observe that two adjacent 4-faces $f_{1}=\left[v_{1} v_{2} v_{3} v\right], f_{2}=\left[v_{2} v_{3} v_{4} v\right]$ have one common edge $v_{2} v_{3}$. Thus, $G$ has adjacent 4 -cycles, a contradiction. Therefore, $v$ is incident with at most $\left\lfloor\frac{2 d(v)}{3}\right\rfloor 3$-faces.

1 em4 Lemma 2.5 Let $G$ be a NC-graph without chord 6 -cycles. Then every $5^{+}$-vertex $v$ is incident to at most three consecutively adjacent 3-faces. Thus, $v$ is incident to at most $(d(v)-2)$ 3-faces.

Proof. Suppose otherwise that $v$ is a $5^{+}$-vertex incident to four consecutively adjacent 3 -faces $\left[v_{1} v v_{2}\right],\left[v_{2} v v_{3}\right],\left[v_{3} v v_{4}\right],\left[v_{4} v v_{5}\right]$. Then $\left[v_{1} v_{2} v_{3} v_{4} v_{5} v\right]$ is a 6 -cycle with a chord $v v_{3}$, a contradiction. Thus, $v$ is incident to at most $(d(v)-2) 3$-faces.

lem5 Lemma 2.6 (1) $d(v) \geq 3$ for all $v \in V(G)$;
(2) $G$ does not contain two adjacent 3 -vertices.

Proof. (1) Suppose otherwise that $v$ is a 2-vertex and $N(v)=\left\{v_{1}, v_{2}\right\}$. By the minimality of $G$, there is a $(2,1)$-decomposition $\left(D^{*}, M^{*}\right)$ of $G-\{v\}$. Let $M=M^{*}$ and $D=D^{*} \cup\left\{\overrightarrow{v v_{1}}, \overrightarrow{v v_{2}}\right\}$. Then $(D, M)$ is a ( 2,1 )-decomposition of $G$, a contradiction.
(2) Suppose otherwise that $u$ is a 3-vertex adjacent to a 3-vertex $v$. Let $N(u)=\left\{u_{1}, u_{2}, v\right\}$ and $N(v)=\left\{v_{1}, v_{2}, u\right\}$. By the minimally of $G$, there is a (2,1)-decomposition ( $D^{*}, M^{*}$ ) of $G-\{u, v\}$. Let $M=M^{*} \cup\{u v\}$ and $D=D^{*} \cup\left\{\overrightarrow{v v_{1}}, \overrightarrow{v v_{2}}, \overrightarrow{u u_{1}}, \overrightarrow{u u_{2}}\right\}$. Then $(D, M)$ is a (2,1)-decomposition of $G$, a contradiction.
$1 \mathrm{em7}$ Lemma 2.7 If A NC-graph G has has no 3-cycle nor 6-cycle, then it has no any underlying subgraph of $G$ in Fig.1.

Proof. Suppose otherwise that $G$ contains one of the figures in Fig.1. Let $X$ be all the labeled vertices of each figure. By the minimality of $G, G^{*}=G-X$ has a (2,1)-decomposition ( $D^{*}, M^{*}$ ).

In Fig. $1(1), X=\left\{v_{1}, \ldots, v_{11}\right\}$. Let $M^{\prime}=\left\{v_{1} v_{5}, v_{2} v_{3}, v_{6} v_{7}, v_{8} v_{9}, v_{10} v_{11}\right\}$ and $D^{\prime}=\left\{\overrightarrow{v_{1} v_{2}}, \overrightarrow{v_{1} v_{7}}\right.$, $\left.\overrightarrow{v_{2} v_{9}}, \overrightarrow{v_{3} v_{4}}, \overrightarrow{v_{3} v_{8}}, \overrightarrow{v_{4} v_{5}}, \overrightarrow{v_{5} v_{6}}, \overrightarrow{v_{8} v_{10}}, \overrightarrow{v_{11} v_{4}}\right\}$. In Fig. $1(2), X=\left\{v_{1}, \ldots, v_{11}\right\}$. Let $M^{\prime}=\left\{v_{1} v_{5}, v_{2} v_{3}, v_{6} v_{7}\right.$, $\left.v_{8} v_{9}, v_{4} v_{11}\right\}$ and $D^{\prime}=\left\{\overrightarrow{v_{1} v_{2}}, \overrightarrow{v_{1} v_{7}}, \overrightarrow{v_{2} v_{9}}, \overrightarrow{v_{3} v_{4}}, \overrightarrow{v_{3} v_{8}}, \overrightarrow{v_{4} v_{5}}, \overrightarrow{v_{5} v_{6}}, \overrightarrow{v_{8} v_{10}}, \overrightarrow{v_{10} v_{11}}\right\}$. In Fig. (3), $X=$ $\left\{v_{1}, \ldots, v_{11}\right\}$. Let $M^{\prime}=\left\{v_{1} v_{5}, v_{6} v_{7}, v_{2} v_{11}, v_{4} v_{8}, v_{9} v_{10}\right\}$ and $D^{\prime}=\left\{\overrightarrow{v_{1} v_{2}}, \overrightarrow{v_{1} v_{7}}, \overrightarrow{v_{2} v_{3}}, \overrightarrow{v_{3} v_{4}}, \overrightarrow{v_{3} v_{9}}, \overrightarrow{v_{4} v_{5}}\right.$, $\left.\overrightarrow{v_{5} v_{6}}, \overrightarrow{v_{9} v_{8}}, \overrightarrow{v_{11} v_{10}}\right\}$. In Fig. 1 (4), $X=\left\{v_{1}, \ldots, v_{11}\right\}$. Let $M^{\prime}=\left\{v_{1} v_{5}, v_{6} v_{7}, v_{2} v_{11}, v_{4} v_{8}, v_{9} v_{10}\right\}$ and $D^{\prime}=\left\{\overrightarrow{v_{1} v_{2}}, \overrightarrow{v_{1} v_{7}}, \overrightarrow{v_{2} v_{3}}, \overrightarrow{v_{3} v_{4}}, \overrightarrow{v_{3} v_{9}}, \overrightarrow{v_{4} v_{5}}, \overrightarrow{v_{5} v_{6}}, \overrightarrow{v_{9} v_{8}}, \overrightarrow{v_{10} v_{11}}\right\}$. In Fig. 1 (5), $X=\left\{v_{1}, v_{2}, \ldots, v_{7}, v_{9}\right\}$. Let $M^{\prime}=\left\{v_{1} v_{2}, v_{4} v_{5}, v_{6} v_{7}\right\}$ and $D^{\prime}=\left\{\overrightarrow{v_{1} v_{5}}, \overrightarrow{v_{1} v_{7}}, \overrightarrow{v_{2} v_{6}}, \overrightarrow{v_{2} v_{9}}, \overrightarrow{v_{3} v_{2}}, \overrightarrow{v_{3} v_{4}}, \overrightarrow{v_{7} v_{3}}, \overrightarrow{v_{7} v_{9}}\right\}$. In Fig. 1 (6), $X=\left\{v_{1}, \ldots, v_{9}\right\}$. Let $M^{\prime}=\left\{v_{1} v_{5}, v_{3} v_{4}, v_{6} v_{7}, v_{8} v_{9}\right\}$ and $D^{\prime}=\left\{\overrightarrow{v_{1} v_{2}}, \overrightarrow{v_{1} v_{7}}, \overrightarrow{v_{2} v_{6}}, \overrightarrow{v_{3} v_{2}}, \overrightarrow{v_{3} v_{9}}, \overrightarrow{v_{4} v_{8}}, \overrightarrow{v_{5} v_{4}}\right.$, $\left.\overrightarrow{v_{5} v_{9}}, \overrightarrow{v_{6} v_{5}}\right\}$. In Fig. $1(7), X=\left\{v_{1}, \ldots, v_{11}\right\}$. Let $M^{\prime}=\left\{v_{2} v_{3}, v_{4} v_{5}, v_{6} v_{7}, v_{8} v_{9}, v_{10} v_{11}\right\}$ and $D^{\prime}=$ $\left\{\overrightarrow{v_{1} v_{5}}, \overrightarrow{v_{1} v_{7}}, \overrightarrow{v_{2} v_{1}}, \overrightarrow{v_{2} v_{6}}, \overrightarrow{v_{3} v_{4}}, \overrightarrow{v_{3} v_{9}}, \overrightarrow{v_{4} v_{8}}, \overrightarrow{v_{9} v_{10}}, \overrightarrow{v_{11} v_{2}}\right\}$.

Let $M=M^{*} \cup M^{\prime}$ and $D$ be the orientation of $G-M$ obtained by adding arcs in $D^{\prime}$ and all the edges between $X$ and $V \backslash X$ oriented from $X$ to $V \backslash X$. Then $\Delta(M) \leq 1$ and $\Delta_{D}^{+} \leq 2$. Moreover, $D$ is an acyclic orientation of $G-M$. Thus $(D, M)$ is a $(2,1)$-decomposition of $G$, a contradiction.


Fig. 1: Reducible configurations
lem6 Lemma 2.8 A NC-graph $G \in \mathcal{H}$ has no a (3, 4, 3, 4)-face,
Proof. Suppose otherwise that $G$ has a $(3,4,3,4)$-face $\left[v_{1} v_{2} v_{3} v_{4}\right]$. Let $N\left(v_{1}\right)=\left\{v_{11}, v_{2}, v_{4}\right\}, N\left(v_{2}\right)$ $=\left\{v_{21}, v_{22}, v_{1}, v_{3}\right\}, N\left(v_{3}\right)=\left\{v_{31}, v_{2}, v_{4}\right\}$ and $N\left(v_{4}\right)=\left\{v_{41}, v_{42}, v_{1}, v_{3}\right\}$. By the minimality of $G$, there is a $(2,1)$-decomposition $\left(D^{*}, M^{*}\right)$ of $G-\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$. Let $M=M^{*} \cup\left\{v_{1} v_{2}, v_{3} v_{4}\right\}$ and $D=D^{*} \cup\left\{\overrightarrow{v_{1} v_{11}}, \overrightarrow{v_{2} v_{21}}, \overrightarrow{v_{2} v_{22}}, \overrightarrow{v_{3} v_{31}}, \overrightarrow{v_{4} v_{41}}, \overrightarrow{v_{4} v_{42}}, \overrightarrow{v_{1} v_{4}}, \overrightarrow{v_{3} v_{2}}\right\}$. Then $(D, M)$ is a (2,1)-decomposition of $G$, a contradiction.

## 3 Proofs of Theorem ${ }^{\text {tho }} 1.1$ and ${ }^{\frac{4}{\text { th }} 1.4}$

We are now ready to complete the proof of Theorem and th. We define initial charge $\mu(x)=$ $d(x)-4$ for each $x \in V \cup F$. By Euler's Formula $|V(G)|+|F(G)|-|E(G)| \geq 0$,

$$
\sum_{v \in V(G)}(d(v)-4)+\sum_{f \in F(G)}(d(f)-4) \leq 0 .
$$

Let $\mu^{\prime}(x)$ be the charge of $x \in V(G) \cup F(G)$ after the discharge procedure. In order to prove the Theorems th.1 and th. 4 , we shall design some discharging rules so that after discharging. Since the total sum of weights is kept unchanged, the new weight function $\mu^{\prime}$ satisfies
(I) $\mu^{\prime}(x) \geq 0$ for all $x \in V(G) \cup F(G)$;
(II) There exists some $x^{*} \in V(G) \cup F(G)$ such that $\mu^{\prime}\left(x^{*}\right)>0$.

Thus

$$
0<\sum_{x \in V(G) \cup F(G)} \mu^{\prime}(x)=\sum_{x \in V(G) \cup F(G)} \mu(x)=0 .
$$

This contradiction completes our proofs.

### 3.1 Proofs of Theorem 1.1 (1) and (3).

In this section, we prove Theorem (1) and (3). Now we define the discharge rules as follows.
(R1) Every 5 -vertex sends $\frac{1}{3}$ to each incident $\left(5^{+}, 5^{+}, 5^{+}\right)$-face, $\frac{1}{2}$ to each incident $(4,5,5)$-face and $\frac{5}{12}$ to each incident $\left(4,5,6^{+}\right)$-face.
(R2) Every $6^{+}$-vertex sends $\frac{7}{12}$ to each incident 3-face.
(R3) Every $5^{+}$-face sends $\frac{11}{60}$ to each incident vertex.

It suffices to show that the new weight function $\mu^{\prime}$ satisfies Properties (I) and (II).
We first check $\mu^{\prime}(v) \geq 0$ for all $v \in V(G)$. By Lemma $\frac{\text { em } 1}{2.2}(1), d(v) \geq 4$.

1. $d(v)=4$. Since no 4-vertex is involved in the discharge procedure, $\mu^{\prime}(v)=\mu(v)=4-4=0$.
2. $d(v)=5$. Then $\mu(v)=1$. By Lemma $n_{2}$ at most one $(4,5,5)$-face by Lemma $\frac{\text { hem } 2}{2}(1)$ and is not incident with any $\left(4,4,5^{-}\right)$-face by Lemma $2(2)$. By (R1), $\mu^{\prime}(v) \geq 1-\frac{1}{2}-\frac{5}{12}=\frac{1}{12}>0$. Let $n_{3}(v)=3$. Then $v$ is incident with two $4^{+}$-faces. If $v$ is incident with one 4 -face, then $G$ has a chord 5 -face and so does a chord

7 －face and adjacent 4－cycles，contrary to our assumption．Thus，$n_{4}(v)=0$ ．This implies that $n_{5^{+}}(v)=2$ ．By Lemmas $\boxed{2} .3(1)$ and $\boxed{2} .2(2), v$ is incident with at most one $(4,5,5)$－face and is not incident with any $\left(4,4,5^{-}\right)$－face．Thus $\mu^{\prime}(v) \geq 1-\frac{1}{2}-2 \times \frac{5}{12}+2 \times \frac{11}{60}=\frac{1}{30}>0$ by（R1）and（R3）．
3．$d(v)=6$ ．Then $\mu(v)=2$ ．By Lemma 尼 $n_{3}(v) \leq 4$ ．If $n_{3}(v) \leq 3$ ，then $\mu^{\prime}(v) \geq 2-3 \times \frac{7}{12}=$ $\frac{1}{4}>0$ by（R2）．Thus，assume that $n_{3}(v)=4$ ．In this case，$v$ is incident with two $4^{+}$－faces． If $v$ is indeed incident one 4 －face，then $G$ has a chord 5 －cycle and so does a chord 7 －cycle and adjacent 4 －cycles，contrary to our assumption．Thus，$n_{4}(v)=0$ ．This implies that $n_{5^{+}}(v)=2$ ．Thus $\mu^{\prime}(v) \geq 2-4 \times \frac{7}{12}+2 \times \frac{11}{60}=\frac{1}{30}>0$ by（R2）and（R3）．
4．$d(v) \geq 7$ ．By Lemma $\left\lfloor\frac{\lfloor\text { 这3 }}{2.4} n_{3}(v) \leq\left\lfloor\frac{2 d(v)}{3}\right\rfloor\right.$ ．Thus $\mu^{\prime}(v) \geq d(v)-4-\frac{7}{12} \times\left\lfloor\frac{2 d(v)}{3}\right\rfloor \geq$ $d(v)-4-\frac{7}{12} \times \frac{2 d(v)}{3}=\frac{11}{18} d(v)-4 \geq \frac{5}{18}>0$ by（R2）．

Then we check $\mu^{\prime}(f) \geq 0$ for all $f \in F(G)$ ．
1．$d(f)=3$ ．Then $\mu(f)=-1$ ．By Lemma 这 2 ，$v$ is not incident with any $\left(4,4,5^{-}\right)$－ face．If $f$ is a $(4,5,5)$－face，then $\mu^{\prime}(f) \geq-1+2 \times \frac{1}{2}=0$ by（R1）．If $f$ is a $\left(4,5^{+}, 6^{+}\right)$－ face，then $\mu^{\prime}(f) \geq-1+\frac{5}{12}+\frac{7}{12}=0$ by（R1）and（R2）．If $f$ is a $\left(5^{+}, 5^{+}, 5^{+}\right)$－face，then $\mu^{\prime}(f) \geq-1+3 \times \frac{1}{3}=0$ by（R1）and（R2）．

2．$d(f)=4$ ．Since no 4 －face is involved in the discharge procedure，$\mu(f)=\mu^{\prime}(f)=4-4=0$ ．
3．$d(f) \geq 5$ ．Then $\mu^{\prime}(f) \geq d(f)-4-\frac{11}{60} d(f)=\frac{49}{60} d(f)-4 \geq \frac{1}{12}>0$ by（R3）．
So far，we have proved Property（I）．Assume that Property（II）does not hold．This implies that $\mu^{\prime}(x)=0$ for all $x \in V(G) \cup F(G)$ ．We observe the above proof and have each of the following holds．
（a）For each vertex $v \in V(G), d(v)=4$ ；
（b）For each face $f \in F(G), 3 \leq d(f) \leq 4$ ．
By（a），$G$ has no $5^{+}$－vertices．Thus $G$ is 4－regular，which is contrary to Lemma $\frac{\| \text { em1 }}{2.2}(2)$ ．This completes the proofs of Theorem（1）and（3）．

### 3.2 Proof of Theorem $\frac{\operatorname{th} 0}{1.1(2)}$

In this section，we prove Theorem 迤．
（R1）Every 5 －vertex sends $\frac{1}{3}$ to each incident $\left(5^{+}, 5^{+}, 5^{+}\right)$－face，$\frac{1}{2}$ to each incident $(4,5,5)$－face， $\frac{5}{12}$ to each incident $(4,5,6)$－face and $\frac{61}{150}$ to each incident $\left(4,5,7^{+}\right)$－face．
（R2）Every 6－vertex sends $\frac{7}{12}$ to each incident 3 －face．
（R3）Every $7^{+}$－vertex sends $\frac{89}{150}$ to each incident 3 －face．
（R4）Every 5 －face sends $\frac{11}{60}$ to each incident vertex．
（R5）Every $6^{+}$－face sends $\frac{49}{150}$ to each incident vertex．

It suffices to show that the new weight function $\mu^{\prime}$ satisfies Properties (I) and (II). Note that each 3 -face is not adjacent to 5 -face since $G$ has no chord 6 -cycles.

We first check $\mu^{\prime}(v) \geq 0$ for all $v \in V(G)$. By Lemma

1. $d(v)=4$. Since no 4-vertex is involved in the discharge procedure, $\mu^{\prime}(v)=\mu(v)=4-4=0$.
2. $d(v)=5$. By Lemma $\frac{\frac{\text { nem } 4}{2.5},}{\text { em }} n_{3}(v) \leq 3$. If $n_{3}(v) \leq 2$, then $v$ is incident with at most one $(4,5,5)$-face by Lemma By (R1), $\mu^{\prime}(v) \geq 1-\frac{1}{2}-\frac{5}{12}=\frac{1}{12}>0$ by (R1). Thus, assume that $n_{3}(v)=3$.

Suppose that $v_{1}, v_{2}, \ldots, v_{5}$ are the neighbors of $v$ in clockwise order, and $f_{1}, f_{2}, \ldots, f_{5}$ are the incident faces of $v$ with $v v_{i}, v v_{i+1} \in b\left(f_{i}\right)$ for $i=1,2, \ldots, 5$ where indices are taken modulo 5. By symmetry, there are two cases: either $f_{1}, f_{2}$ and $f_{3}$ or $f_{1}, f_{2}$ and $f_{4}$ are 3 -faces.

In the former case, since $G$ has no chord 6 -cycles, each of $f_{4}$ and $f_{5}$ is not a 5 -face. Thus, $n_{5}(v)=0$. We claim that at most one of $f_{4}$ and $f_{5}$ is a 4 -face. Suppose otherwise. Let $f_{4}=\left[v v_{4} x v_{5}\right]$ and $f_{5}=\left[v_{1} v v_{5} y\right]$. Since $G$ has no chord 6 -cycle, $x, y \in\left\{v, v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right\}$. Since $G$ is a simple graph, $x \notin\left\{v, v_{2}, v_{3}\right\}$. Since $n_{3}(v)=3, x \notin\left\{v_{1}, v_{4}, v_{5}\right\}$. Similarly, $y \notin\left\{v, v_{5}, v_{2}, v_{4}, v_{1}\right\}$. Thus, $x=v_{2}$ and $y=v_{3}$. In this case, $G$ has a chord 6 -cycle $v v_{4} v_{3} v_{1} v_{2} v_{5} v$, a contradiction. Thus $n_{4}(v) \leq 1$. This implies that $1 \leq n_{6^{+}}(v) \leq 2$. By Lemma face, then $\mu^{\prime}(v) \geq 1-3 \times \frac{5}{12}+\frac{49}{150}=\frac{23}{300}>0$ by (R1) and (R5). Thus, assume that $v$ is incident with one ( $4,5,5$ )-face. By Lemma (R1) and (R5), $\mu^{\prime}(v) \geq 1-\frac{1}{2}-\frac{5}{12}-\frac{61}{150}+\frac{49}{150}=\frac{1}{300}>0$.

In the latter case, since $G$ has no chord 6 -cycles, none of $f_{3}$ and $f_{5}$ is a 5 -face. Thus $n_{5}(v)=0$. If $f_{3}=\left[v v_{3} x v_{4}\right]$ is a 4 -face, then $x \notin\left\{v, v_{2}, v_{3}, v_{4}, v_{5}\right\}$ since $G$ is a simple graph and by Lemma $\frac{\text { nem }}{2.2}(1)$. If $x \neq v_{1}$, then $v v_{2} v_{3} x v_{4} v_{5} v$ is a 6 -cycle with a chord $v v_{3}$, a contradiction. If $x=v_{1}$, then $v_{1} v_{4} v_{5} v v_{3} v_{2} v_{1}$ is a 6 -cycle with a chord $v v_{4}$, a contradiction. By symmetry, $f_{5}$ is not a 4 -face. Thus, $n_{4}(v)=0$. This implies that $n_{6^{+}}(v)=2$. Thus $\mu^{\prime}(v) \geq 1-3 \times \frac{1}{2}+2 \times \frac{49}{150}=\frac{23}{150}>0$ by (R1) and (R5).
 $\frac{1}{4}>0$ by (R2). Thus, assume that $n_{3}(v)=4$.

We now prove $n_{4}(v)=n_{5}(v)=0$. Assume that $v_{1}, v_{2}, \ldots, v_{6}$ are the neighbors of $v$ in clockwise order, and $f_{1}, f_{2}, \ldots, f_{6}$ are the incident faces of $v$ with $v v_{i}, v v_{i+1} \in b\left(f_{i}\right)$ for $i=1,2, \ldots, 6$ where indices are taken modulo 6 . Since $G$ has no chord 6 -cycles, $n_{5}(v)=0$ by Lemma By Lemma and symmetry, we consider two cases: either $f_{1}, f_{2}, f_{3}, f_{5}$ or $f_{1}, f_{2}, f_{4}, f_{5}$ are four 3 -faces.

In the former case, assume that $f_{4}=\left[v v_{4} x v_{5}\right]$ is a 4 -face. Since $G$ is a simple graph, $x \notin\left\{v, v_{3}, v_{4}, v_{5}, v_{6}\right\}$ by Lemma $\frac{12.21}{2.21} 1$. If $x=v_{2}$, then $v_{2} v_{4} v_{3} v v_{6} v_{5} v_{2}$ is a 6 -cycle with a chord $v v_{4}$, a contradiction. If $x=v_{1}$, then $v_{1} v_{2} v_{3} v_{4} v v_{5} v_{1}$ is a 6 -cycle with a chord $v v_{3}$, a contradiction. If $x \neq v_{1}$ and $x \neq v_{2}$, then $x v_{5} v v_{2} v_{3} v_{4} x$ is a 6 -cycle with a chord $v v_{4}$, a contradiction. Thus $f_{4}$ is not a 4 -face. By symmetry, $f_{6}$ is not a 4 -face. Thus $n_{4}(v)=0$.

In the latter case, assume that $f_{3}=\left[v v_{3} x v_{4}\right]$ is a 4 -face. Since $G$ is a simple graph, $x \notin$ $\left\{v, v_{2}, v_{3}, v_{4}, v_{5}\right\}$ by Lemma $\left.\frac{\text { en }}{2.2} 1\right)$. If $x=v_{1}$, then $v_{1} v_{2} v v_{6} v_{5} v_{4} v_{1}$ is a 6 -cycle with a chord $v v_{4}$, a contradiction. By symmetry, $x \neq v_{6}$. If $x \neq v_{1}$ and $x \neq v_{6}$, then $v_{1} v_{2} v_{3} x v_{4} v_{5} v_{1}$ is a 6 -cycle with a chord $v v_{5}$, a contradiction. Thus $f_{3}$ is not a 4 -face. By symmetry, $f_{6}$ is not a 4 -face. Thus, $n_{4}(v)=0$.

So far, we have proved that $n_{4}(v)=n_{5}(v)=0$. This implies that $n_{6^{+}}(v)=2$. Thus $\mu^{\prime}(v) \geq 2-4 \times \frac{7}{12}+2 \times \frac{49}{150}=\frac{8}{25}>0$ by (R2) and (R5).
4. $d(v) \geq 7$, then by Lemma $\frac{\text { nem } 4}{2.5,} v$ is incident with at most $\left(d(v)-2\right.$ ) 3-faces. Thus $\mu^{\prime}(v) \geq$ $d(v)-4-\frac{89}{150}(d(v)-2)=\frac{61}{150} d(v)-\frac{422}{150} \geq \frac{1}{30}>0$ by (R3).

Then we check $\mu^{\prime}(f) \geq 0$ for all $f \in F(G)$.

1. $d(f)=3$. By Lemma $\frac{\text { em } 12(2)}{}$, $v$ is not incident with any $\left(4,4,4^{+}\right)$-face. If $f$ is a $\left(5^{+}, 5^{+}, 5^{+}\right)$face, then $\mu^{\prime}(f) \geq-1+3 \times \frac{1}{3}=0$ by (R1)-(R3). If $f$ is a $(4,5,5)$-face, then $\mu^{\prime}(f) \geq$ $-1+2 \times \frac{1}{2}=0$ by (R1). If $f$ is a $\left(4,5^{+}, 6\right)$-face, then $\mu^{\prime}(f) \geq-1+\frac{5}{12}+\frac{7}{12}=0$ by (R1)-(R3). If $f$ is a $\left(4,5^{+}, 7^{+}\right)$-face, then $\mu^{\prime}(f) \geq-1+\frac{61}{150}+\frac{89}{150}=0$ by (R1)-(R3).
2. $d(f)=4$. Since 4 -faces are not involved in discharge procedure, $\mu(f)=\mu^{\prime}(f)=0$.
3. $d(f)=5$. Then $\mu(f)=1$. By $(\mathrm{R} 4), \mu^{\prime}(f) \geq 1-5 \times \frac{11}{60}=\frac{5}{60}>0$.
4. $d(f) \geq 6$. By (R5), $\mu^{\prime}(f) \geq d(f)-4-\frac{49}{150} d(f)=\frac{101}{150} d(f)-4 \geq \frac{6}{150}>0$.

We have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu^{\prime}(x)=0$ for all $x \in V(G) \cup F(G)$. We check above proof and obtain the following assertions.
(a) For each vertex $v \in V(G), d(v)=4$;
(b) For each face $f \in F(G), 3 \leq d(f) \leq 4$.

By (a), $G$ has no $5^{+}$-vertices. Thus $G$ is 4 -regular, which is contrary to Lemma $\frac{\sqrt{2} .21}{2.2}(2)$. This completes the proof of Theorem th.1 (2).

### 3.3 Proof of Theorem 1.4

In this section, we prove Theorem (1.4 (1). Now we define the discharge rules as follows.
(R1) Every $5^{+}$-face sends $\frac{1}{3}$ to each incident 3 -vertex.

It suffices to show that the new weight function $\mu^{\prime}$ satisfies Properties (I) and (II).
We first check $\mu^{\prime}(v) \geq 0$ for all $v \in V(G)$. By Lemma

1. $d(v)=3$. Then $\mu(v)=3-4=-1$. Since $G$ has no 3 - and 4 -cycles, $v$ is incident with three $5^{+}$-faces. Thus $\mu^{\prime}(v) \geq-1+3 \times \frac{1}{3}=0$ by (R1).
2. $d(v)=4$. Then $\mu^{\prime}(v)=\mu(v)=4-4=0$.
3. $d(v)=5$. Then $\mu^{\prime}(v)=\mu(v)=d(v)-4 \geq 1>0$.

We further check $\mu^{\prime}(f) \geq 0$ for all $f \in F(G)$. Note that $d(f) \geq 5$.
By Lemma $\left\lfloor\frac{d(f)}{2}\right\rfloor 3$-vertices. Thus $\mu^{\prime}(f) \geq d(f)-4-\frac{1}{3} \times$ $\left\lfloor\frac{d(f)}{2}\right\rfloor \geq \frac{5}{6} d(f)-4 \geq \frac{1}{6}>0$.

We have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu^{\prime}(x)=0$ for all $x \in V(G) \cup F(G)$. Considering above proof, we obtain that $G$ has no $5^{+}$-face and hence every face of $G$ is a $4^{-}$-face, contrary to our assumption that $G$ has no 3 -cycle nor 4 -cycle. This completes the proof of Theorem th.4 (1).

### 3.4 Proof of Theorem $\frac{\text { th1 }}{1.4(2)}$

In this section, we prove Theorem
 otherwise.

Now we define the discharge rules as follows.
(R1) Every $5^{+}$-face sends $\frac{1}{3}$ to each incident good 3 -vertex and $\frac{1}{2}$ to each incident bad 3 -vertex.

It suffices to show that the new weight function $\mu^{\prime}$ satisfies Properties (I) and (II). Note that


We first check $\mu^{\prime}(v) \geq 0$ for all $v \in V(G)$. By Lemma em 1 ), $d(v) \geq 3$.

1. $d(v)=3$. If $v$ is good, then $v$ is incident with three $5^{+}$-faces. Thus $\mu^{\prime}(v) \geq-1+3 \times \frac{1}{3}=0$ by (R1). If $v$ is bad, then $v$ is incident with two $5^{+}$-faces. Thus $\mu^{\prime}(v) \geq-1+2 \times \frac{1}{2}=0$ by (R1).
2. $d(v)=4$. Since any 4 -vertex does not involved in discharge procedure, $\mu^{\prime}(v)=\mu(v)=$ $4-4=0$.
3. $d(v) \geq 5$. Then $\mu(v)=d(v)-4$. Since any 5 -vertex does not involved in discharge procedure, $\mu^{\prime}(v)=\mu(v) \geq 1>0$.

We further check $\mu^{\prime}(f) \geq 0$ for all $f \in F(G)$. Note that $d(f) \geq 4$ and $d(f) \neq 6$.

1. $d(f)=4$. Since any 4-face does not involved in discharge procedure, $\mu^{\prime}(f)=\mu(f)=4-4=$ 0 .
 3 -vertices. If $v$ is incident with at most one 3 -vertex, then $\mu^{\prime}(f) \geq 1-\frac{1}{2}=\frac{1}{2}>0$ by (R1). Let $v$ be incident with two 3 -vertices $v_{1}$ and $v_{2}$. If one of $v_{1}$ and $v_{2}$ is bad and the other is good, then $\mu^{\prime}(v) \geq 1-\frac{1}{2}-\frac{1}{3}=\frac{1}{6}>0$ by (R1). If both $v_{1}$ and $v_{2}$ are bad 3 -vertices, then $\mu^{\prime}(v) \geq 1-2 \times \frac{1}{2}=0$ by (R1).
2. $d(f) \geq$ 7. By Lemma $\frac{\text { nem } 5}{2.6}(2), f$ is incident with at most $\left\lfloor\frac{d(f)}{2}\right\rfloor 3$-vertices. Thus $\mu^{\prime}(v) \geq$ $d(f)-4-\frac{1}{2} \times\left\lfloor\frac{d(f)}{2}\right\rfloor \geq \frac{3}{4} d(f)-4 \geq \frac{5}{4}>0$.

We have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu^{\prime}(x)=0$ for all $x \in V(G) \cup F(G)$. Considering above proof, we establish the following claims.
Claim 1. Each of the following holds.
(1) For each vertex $v \in V(G), 3 \leq d(v) \leq 4$.
(2) For each face $f \in F(G), f$ is either a 5 -face incident with two bad 3 -vertices or a 4 -face.

Claim 2. Let $f$ be a 5 -face. Then each of the following holds.
(1) $f$ is not incident with three consecutively adjacent 4 -vertices.
(2) If $f$ is adjacent to a 4 -face $g$, then $g$ is a $(3,4,4,4)$-face.

Proof of Claim 2. (1) By Claim 1(2), $f$ is incident two bad 3 -vertices. By Lemma Rem 2 , $f$ is a (4, 3, 4, 3, 4)-face and hence $f$ is not incident with three consecutively adjacent 4 -vertices.
(2) it follows by Lemma

Claim 3. $G$ has a 5 -face.
Proof of Claim 3. Suppose otherwise that $G$ has no 5 -face. By Claim 1(2), $G$ has only 4 -faces. If $G$ has more than one 4 -face, then $G$ contains two adjacent 4 -faces, contrary to Lemma emo

By Claims $1(2)$ and 3 , we assume that $G$ has a 5 -face $f=\left[v_{1} v_{2} v_{3} v_{4} v_{5}\right]$ incident with two bad 3 -vertices. By Claim 2(1) and by symmetry, assume that $v_{1}$ and $v_{3}$ are two 3 -vertices. Thus, $v_{1}$ and $v_{3}$ are incident with one 4 -face and two 5 -faces. Assume that $v_{1}$ is incident with $f, f_{1}, f_{2}$ and $v_{3}$ is incident with $f, f_{3}, f_{4}$ where $f_{1}, f_{3}$ are two 4 -faces and $f_{2}, f_{4}$ are two 5 -faces. By Lemma $f_{1}$ and $f_{3}$ are two ( $3,4,4,4$ )-faces. We observe $f_{1}$ and consider two following cases.
Case 1. $v_{1} v_{2} \in b\left(f_{1}\right)$.
Let $f_{1}=\left[v_{1} v_{2} v_{6} v_{7}\right]$. We first claim that $v_{6}, v_{7} \notin\left\{v_{1}, \ldots, v_{5}\right\}$. Note that $v_{6} \notin\left\{v_{1}, v_{2}, v_{3}\right\}$ and $v_{7} \notin\left\{v_{1}, v_{2}, v_{5}\right\}$. Since $G$ has no 3 -cycle, $v_{6} \notin\left\{v_{4}, v_{5}\right\}$ and $v_{7} \notin\left\{v_{3}, v_{4}\right\}$. In this case, we consider two cases for $f_{3}$ : either $v_{2} v_{3} \in b\left(f_{3}\right)$ or $v_{3} v_{4} \in b\left(f_{3}\right)$.

In the former case, let $f_{3}=\left[v_{2} v_{3} v_{8} v_{9}\right]$. We claim that $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$. Since $d\left(v_{1}\right)=3$, $v_{8} \neq v_{1}$ and $v_{9} \neq v_{1}$. Clearly, $v_{8} \notin\left\{v_{2}, v_{3}, v_{4}\right\}$ and $v_{9} \notin\left\{v_{2}, v_{3}, v_{6}\right\}$. Since $G$ has no 3-cycle, $v_{8} \notin\left\{v_{5}, v_{6}\right\}$ and $v_{9} \notin\left\{v_{4}, v_{5}, v_{7}\right\}$. If $v_{8}=v_{7}$, then $G$ contains Configuration (5) in Fig.1, contrary to Lemma Lem7 Thus, $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$.

Let $v_{2}$ be incident with $f, f_{1}, f_{5}, f_{3}$ in clockwise order. By Lemma fem0 $f_{5}$ is a 5 -face. By Claim 2(2), $d\left(v_{2}\right)=d\left(v_{6}\right)=d\left(v_{9}\right)=4$. Therefore, $f_{5}$ is incident with three consecutively adjacent 4 -vertices which is contrary to Claim 2(1).

In the latter case, let $f_{3}=\left[v_{4} v_{3} v_{9} v_{8}\right]$. We first claim that $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$. Recall that $v_{1}$ is a 3 -vertex, $v_{8} \neq v_{1}$ and $v_{9} \neq v_{1}$. Obviously, $v_{8} \notin\left\{v_{3}, v_{4}, v_{5}\right\}$ and $v_{9} \notin\left\{v_{2}, v_{3}, v_{4}\right\}$ since $G$ is simple. Since $G$ has no 3 -cycle, $v_{8} \neq v_{2}$ and $v_{9} \notin\left\{v_{5}, v_{6}\right\}$. If $v_{8}=v_{6}$, then $v_{4} v_{3} v_{2} v_{1} v_{7} v_{6} v_{4}$ is a 6 -cycle, a contradiction. Thus $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{6}\right\}$. If $v_{8}=v_{7}$, then $v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$ and $v_{7} v_{1} v_{5} v_{4} v_{3} v_{9} v_{7}$ is a 6 -cycle, a contradiction. If $v_{9}=v_{7}$, then $v_{8} \notin\left\{v_{1}, \ldots, v_{7}\right\}$ and $v_{7} v_{6} v_{2} v_{3} v_{4} v_{8} v_{7}$ is a 6 -cycle, a contradiction. Thus, $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$.

Let $v_{2}$ be incident with $f, f_{1}, f_{5}, f_{4}$ in clockwise order. By Lemma $f_{4}$ is a 5 -face. Let $f_{4}=\left[v_{2} v_{3} v_{9} v_{10} v_{11}\right]$. We first assume that $v_{10}, v_{11} \notin\left\{v_{1}, \ldots, v_{9}\right\}$. In this case, $d\left(v_{2}\right)=d\left(v_{9}\right)=4$ and $d\left(v_{3}\right)=4$. By Claim 1(2), only one vertex in $\left\{v_{10}, v_{11}\right\}$ is a 3 -vertex. Since $G$ does not contain Configuration (7) of Fig. 1 in Lemma 2.7. $d\left(v_{11}\right)=4$ and $d\left(v_{10}\right)=3$. By Lemma 2.1 and Claim 1
(2), $f_{5}$ is a 5 -face. Moreover, $f_{5}$ is incident with three consecutively adjacent 4 -vertices $v_{6}, v_{2}$ and $v_{11}$, contrary to Claim 2 (1). Thus, assume that $v_{10} \in\left\{v_{1}, \ldots, v_{9}\right\}$ or $v_{11} \in\left\{v_{1}, \ldots, v_{9}\right\}$.

Since $v_{1}$ is a 3 -vertex, $v_{10} \neq v_{1}$ and $v_{11} \neq v_{1}$. Obviously, $v_{10} \notin\left\{v_{2}, v_{3}, v_{8}, v_{9}\right\}$ and $v_{11} \notin$ $\left\{v_{2}, v_{3}, v_{6}\right\}$ since $G$ is simple. Since $G$ has no 3 -cycle, $v_{10} \neq v_{4}$ and $v_{11} \notin\left\{v_{4}, v_{5}, v_{7}, v_{9}\right\}$. If $v_{10}=v_{6}$, then $v_{1} v_{2} v_{3} v_{9} v_{6} v_{7} v_{1}$ is a 6 -cycle, a contradiction. If $v_{10}=v_{7}$, then $v_{7} v_{9} v_{3} v_{4} v_{5} v_{1} v_{7}$ is a 6 cycle, a contradiction. Thus $v_{10} \notin\left\{v_{1}, \ldots, v_{4}, v_{6}, \ldots, v_{9}\right\}$ and $v_{11} \notin\left\{v_{1}, \ldots, v_{7}, v_{9}\right\}$. Let $v_{10}=v_{5}$. If $v_{11}=v_{8}$, then $v_{5} v_{4} v_{8} v_{5}$ is a 3 -cycle, a contradiction. If $v_{11} \notin\left\{v_{1}, \ldots, v_{9}\right\}$, then $v_{1} v_{5} v_{11} v_{2} v_{6} v_{7} v_{1}$ is a 6 -cycle, a contradiction. If $v_{10} \notin\left\{v_{1}, \ldots, v_{9}\right\}$, then $v_{11}=v_{8}$ and $v_{8} v_{9} v_{10} v_{8}$ is a 3 -cycle, a contradiction.
Case 2. $v_{1} v_{5} \in b\left(f_{1}\right)$.
Let $f_{1}=\left[v_{1} v_{5} v_{6} v_{7}\right]$. We first claim that $v_{6}, v_{7} \notin\left\{v_{1}, \ldots, v_{5}\right\}$. Obviously, $v_{6} \notin\left\{v_{1}, v_{4}, v_{5}\right\}$ and $v_{7} \notin\left\{v_{1}, v_{2}, v_{5}\right\}$. Since $G$ has no 3 -cycle, $v_{6} \notin\left\{v_{2}, v_{3}\right\}$ and $v_{7} \notin\left\{v_{3}, v_{4}\right\}$. In this case, we consider two cases of $f_{3}$ : either $v_{2} v_{3} \in b\left(f_{3}\right)$ or $v_{3} v_{4} \in b\left(f_{3}\right)$.

In the former case, let $f_{3}=\left[v_{2} v_{3} v_{8} v_{9}\right]$. We claim that $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$. Since $v_{1}$ is a 3 -vertex, $v_{8} \neq v_{1}$ and $v_{9} \neq v_{1}$. Obviously, $v_{8} \notin\left\{v_{2}, v_{3}, v_{4}\right\}$ and $v_{9} \notin\left\{v_{2}, v_{3}\right\}$ since $G$ is simple. Since $G$ has no 3 -cycle, $v_{8} \neq v_{5}$ and $v_{9} \notin\left\{v_{4}, v_{5}, v_{7}\right\}$. If $v_{8}=v_{7}$, then $v_{7} v_{6} v_{5} v_{1} v_{2} v_{3} v_{7}$ is a 6 -cycle, a contradiction. Thus $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{5}, v_{7}\right\}$. If $v_{8}=v_{6}$, then $v_{9} \neq v_{6}$. In this case, $v_{6} v_{5} v_{4} v_{3} v_{2} v_{9} v_{6}$ is a 6 -cycle, a contradiction. If $v_{9}=v_{6}$, then $v_{8} \neq v_{6}$. In this case, $v_{6} v_{5} v_{1} v_{2} v_{3} v_{8} v_{6}$ is a 6 -cycle, a contradiction. So far, we have proved that $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$.

Assume that $v_{3}$ is incident with $f, f_{3}$ and $f_{4}$ in clockwise order. Since $G$ has no 6 -cycle, by Lemma $f_{4}$ is a 5 -face. Let $f_{4}=\left[v_{4} v_{3} v_{8} v_{10} v_{11}\right]$. If $v_{10}, v_{11} \notin\left\{v_{1}, \ldots, v_{9}\right\}$, then $G$ contains Configuration (1) or (2) in Fig.1, contrary to Lemma Thus, assume that $v_{10} \in\left\{v_{1}, \ldots, v_{9}\right\}$ or $v_{11} \in\left\{v_{1}, \ldots, v_{9}\right\}$. Since $v_{1}$ is a 3 -vertex, $v_{10} \neq v_{1}$ and $v_{11} \neq v_{1}$. Obviously, $v_{10} \notin\left\{v_{3}, v_{8}, v_{9}\right\}$ and $v_{11} \notin\left\{v_{3}, v_{4}, v_{5}\right\}$ since $G$ is simple. Since $G$ has no 3 -cycle, $v_{10} \notin\left\{v_{2}, v_{4}\right\}$ and $v_{11} \notin\left\{v_{2}, v_{6}, v_{8}\right\}$. If $v_{10}=v_{5}$, then $v_{8} v_{5} v_{4} v_{3} v_{2} v_{9} v_{8}$ is a 6 -cycle, a contradiction. If $v_{10}=v_{6}$, then $v_{6} v_{7} v_{1} v_{2} v_{9} v_{8} v_{6}$ is a 6 -cycle, a contradiction. If $v_{10}=v_{7}$, then $v_{8} v_{7} v_{1} v_{5} v_{4} v_{3} v_{8}$ is a 6 -cycle, a contradiction. Thus $v_{10} \notin\left\{v_{1}, \ldots, v_{9}\right\}$. If $v_{11}=v_{7}$, then $v_{7} v_{1} v_{2} v_{9} v_{8} v_{10} v_{7}$ is a 6 -cycle, a contradiction. If $v_{11}=v_{9}$, then $v_{8} v_{9} v_{10} v_{8}$ is a 3 -cycle, a contradiction.

In the latter case, let $f_{3}=\left[v_{3} v_{4} v_{8} v_{9}\right]$. We claim that $v_{8}, v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$. Since $v_{1}$ is a 3 -vertex, $v_{9} \neq v_{1}$ and $v_{8} \neq v_{1}$. Since $G$ is simple, $v_{9} \notin\left\{v_{2}, v_{3}, v_{4}\right\}$ and $v_{8} \notin\left\{v_{3}, v_{4}, v_{5}\right\}$ by Lemma is a 6 -cycle, a contradiction. If $v_{9}=v_{7}$, then $v_{5} v_{6} v_{7} v_{3} v_{2} v_{1} v_{5}$ is a 6 -cycle, a contradiction. Thus $v_{9} \notin\left\{v_{1}, \ldots, v_{7}\right\}$. If $v_{8}=v_{7}$, then $v_{7} v_{6} v_{5} v_{4} v_{3} v_{9} v_{7}$ is a 6 -cycle, a contradiction.

Since $G$ has no 6 -cycle, by Lemma $f_{4}$ is a 5 -face. Let $f_{4}=\left[v_{2} v_{3} v_{9} v_{10} v_{11}\right]$. If $v_{10}, v_{11} \notin$ $\left\{v_{1}, \ldots, v_{9}\right\}$, then $G$ contains Configuration (3) or (4) of Fig.1, contrary to Lemma Them assume that either $v_{10} \in\left\{v_{1}, \ldots, v_{9}\right\}$ or $v_{11} \in\left\{v_{1}, \ldots, v_{9}\right\}$. Since $v_{1}$ is a 3 -vertex, $v_{10} \neq v_{1}$. Since $G$ is simple and by Lemma has no 3 -cycles, $v_{10} \notin\left\{v_{2}, v_{4}\right\}$ and $v_{11} \notin\left\{v_{4}, v_{5}, v_{7}, v_{9}\right\}$. If $v_{10}=v_{6}$, then $v_{6} v_{7} v_{1} v_{2} v_{3} v_{9} v_{6}$ is a 6 -cycle, a contradiction. If $v_{10}=v_{7}$, then $v_{9} v_{8} v_{4} v_{5} v_{6} v_{7} v_{9}$ is a 6 -cycle, a contradiction. Thus $v_{10} \notin\left\{v_{1}, \ldots, v_{4}, v_{6}, \ldots, v_{9}\right\}$ and $v_{11} \notin\left\{v_{1}, \ldots, v_{5}, v_{7}, v_{9}\right\}$. Assume that $v_{11}=v_{6}$. If $v_{10}=$ $v_{5}$, then $G$ contains Configuration (6) of Fig.1, contrary to Lemma 登.7? Thus, $v_{10} \neq v_{5}$. So, $v_{10} \notin\left\{v_{1}, \ldots, v_{9}\right\}$. In this case, $v_{10} v_{6} v_{5} v_{4} v_{8} v_{9} v_{10}$ is a 6 -cycle, a contradiction. Thus, assume that
$v_{11}=v_{8}$. If $v_{10} \notin\left\{v_{1}, \ldots, v_{9}\right\}$, then $v_{8} v_{9} v_{10} v_{8}$ is a 3-cycle, a contradiction. If $v_{10}=v_{5}$, then $v_{5} v_{4} v_{8} v_{5}$ is a 3 -cycle, a contradiction. Thus $v_{11} \notin\left\{v_{1}, \ldots, v_{9}\right\}$. If $v_{10}=v_{5}$, then $v_{5} v_{6} v_{7} v_{1} v_{2} v_{11} v_{5}$ is a 6 -cycle, a contradiction.

This implies that $G$ is not existence. We have proved Property (II). This completes the proof of Theorem thit (2).

### 3.5 Proof of Theorem ${ }^{\text {th1 }} 1$ (3)

In this section, we prove Theorem thit (3). A 3 -vertex $v$ is bad if $v$ is incident with one 3 -face and good otherwise.

Now we define the discharge rules as follows.
(R1) Every $5^{+}$-face sends $\frac{1}{3}$ to each incident good 3 -vertex, $\frac{1}{2}$ to each incident bad 3 -vertex and $\frac{1}{3}$ to each incident 3 -face.

It suffices to show that the new weight function $\mu^{\prime}$ satisfies Properties (I) and (II). We first check $\mu^{\prime}(v) \geq 0$ for all $v \in V(G)$. By Lemma 登而 $5(1), d(v) \geq 3$.

1. $d(v)=3$. If $v$ is bad, then $v$ is incident with two $7^{+}$-faces. By (R1), $\mu^{\prime}(v) \geq-1+2 \times \frac{1}{2}=0$. If $v$ is good, then $v$ is incident with three $5^{+}$-faces. By (R1), $\mu^{\prime}(v) \geq-1+3 \times \frac{1}{3}=0$.
2. $d(v)=4$. Since no 4-vertex is involved in discharge procedure, $\mu^{\prime}(v)=\mu(v)=4-4=0$.
3. $d(v) \geq 5$. By (R1), $\mu^{\prime}(v)=\mu(v)=d(v)-4 \geq 1>0$.

Further we check $\mu^{\prime}(f) \geq 0$ for all $f \in F(G)$.

1. $d(f)=3$. Then $\mu(f)=-1$. Since $G$ has no 4 -cycle, each face adjacent to $f$ is a $5^{+}$-face. By (R1), $\mu^{\prime}(v) \geq-1+3 \times \frac{1}{3}=0$.
2. $d(f)=5$. Then $\mu(f)=1$. Since $G$ has no 6 -cycle, $f$ is not adjacent to 3 -face. By
 is not adjacent to any 5 -cycle. Thus, $f$ is adjacent to at most two good 3 -vertices. By (R1), $\mu^{\prime}(f) \geq 1-2 \times \frac{1}{3}=\frac{1}{3}>0$.
3. $d(f) \geq 7$. Let $f$ be incident with $m$-vertices. Since $G$ has no 4 -cycles, no 3 -face is adjacent to any 3 -face. Then $v$ is incident with at most $d(f)-m 3$-faces. By Lemma $m \leq\left\lfloor\frac{d(f)}{2}\right\rfloor$. Thus $\mu^{\prime}(f) \geq d(f)-4-\frac{m}{2}-\frac{1}{3}(d(f)-m)=\frac{2}{3} d(f)-\frac{1}{6} m-4 \geq \frac{7}{12} d(f)-4 \geq \frac{1}{12}>0$ by (R1).

So far, we have proved Property (I). Assume that Property (II) does not hold. This implies that $\mu^{\prime}(x)=0$ for all $x \in V(G) \cup F(G)$. Observing above proof, we obtain the following statements.
(a) For each vertex $v \in V(G), 3 \leq d(v) \leq 4$;
(b) For each face $f \in F(G), d(f)=3$.

By (b), $G$ is one 3 -cycle $[u v w]$. Clearly, $G$ has a matching $M=\{u v\}$ such that $G-M$ is $(2,1)$ decomposable, a contradiction. This completes the proof of Theorem (1.43).

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[^0]:    *Supported in part by NSFC (12031018)

