

# A note on the complexity of $k$ -METRIC DIMENSION

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## Abstract

Two vertices  $u, v \in V$  of an undirected connected graph  $G = (V, E)$  are *resolved* by a vertex  $w$  if the distance between  $u$  and  $w$  and the distance between  $v$  and  $w$  are different. A set  $R \subseteq V$  of vertices is a  $k$ -*resolving set* for  $G$  if for each pair of vertices  $u, v \in V$  there are at least  $k$  distinct vertices  $w_1, \dots, w_k \in R$  such that each of them resolves  $u$  and  $v$ . The  $k$ -*Metric Dimension* of  $G$  is the size of a smallest  $k$ -resolving set for  $G$ . The decision problem  $k$ -METRIC DIMENSION is the question whether  $G$  has a  $k$ -resolving set of size at most  $r$ , for a given graph  $G$  and a given number  $r$ . In this paper, we prove the NP-completeness of  $k$ -METRIC DIMENSION for bipartite graphs and each  $k \geq 2$ .

## 1 Introduction

The metric dimension of graphs has been introduced in the 1970s independently by Slater [Sla75] and by Harary and Melter [HM76]. We consider simple undirected and connected graphs  $G = (V, E)$ , where  $V$  is the set of vertices and  $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$  is the set of edges. Such a graph has *metric dimension* at most  $r$  if there is a vertex set  $R \subseteq V$  such that  $|R| \leq r$  and  $\forall u, v \in V, u \neq v$ , there is a vertex  $w \in R$  such that  $d(w, u) \neq d(w, v)$ , where  $d(u, v)$  is the distance (the length of a shortest path in an unweighted graph) between  $u$  and  $v$ . The *metric dimension* of  $G$  is the smallest integer  $r$  such that  $G$  has metric dimension at most  $r$ .

If  $d(w, u) \neq d(w, v)$ , for three vertices  $u, v, w$ , we say that  $u$  and  $v$  are *resolved* or *distinguished* by vertex  $w$ . If every pair of vertices is resolved by at least one vertex of a vertex set  $R$ , then  $R$  is a *resolving set* or *metric generator* for  $G$ . In certain applications, the vertices of a resolving set are also called *landmark nodes* or *anchor nodes*. This is a common naming, particularly in the theory of sensor networks.

The metric dimension finds applications in various areas, including network discovery and verification [BEE<sup>+</sup>05], geographical routing protocols [LA06], combinatorial optimization [ST04], sensor networks [HW12], robot navigation [KRR96] and chemistry [CEJO00, Hay77].

There are several algorithms for computing a minimum resolving set in polynomial time for special classes of graphs, for example trees [CEJO00, KRR96], wheels [HMP<sup>+</sup>05], grid graphs [MT84],  $k$ -regular bipartite graphs [BBS<sup>+</sup>11], amalgamation of cycles [IBSS10] and outerplanar graphs [DPSL12]. The approximability of the metric dimension has been studied for bounded degree, dense and general graphs in [HSV12]. Upper and lower bounds on the metric dimension are considered in [CGH08, CPZ00] for further classes of graphs.

In this paper, we consider the  $k$ -*Metric Dimension* for some positive integer  $k$ . A set  $R \subseteq V$  of vertices is a  $k$ -*resolving set* for  $G$  if for each pair of vertices  $u, v \in V$  there are at least  $k$  vertices  $w_1, \dots, w_k \in R$  such that each of them resolves  $u$  and  $v$ . The  $k$ -*Metric Dimension* of  $G$  is the size of a smallest  $k$ -resolving set for  $G$ . The  $k$ -METRIC DIMENSION problem was introduced by Estrada-Moreno et al. in [EMRY13]. The 1-metric dimension is simply called metric dimension. The 2-metric dimension is also called *fault-tolerant metric dimension* and was introduced in [HMSW08].

Estrada-Moreno et al. analysed the  $(k, t)$ -METRIC DIMENSION [EMYRV16]. The  $(k, t)$ -METRIC DIMENSION is the  $k$ -METRIC DIMENSION, with the addition, that the distance between two vertices  $u, v$  of  $G$  is defined as the minimum of  $d(u, v)$  and  $t$ . Therefore, if  $t$  is set to the diameter of  $G$ , the  $(k, t)$ -METRIC DIMENSION is the same as the  $k$ -METRIC DIMENSION. Estrada-Moreno et al. showed the NP-completeness of  $(k, t)$ -METRIC DIMENSION for odd values of  $k$ .

The decision problem  $k$ -METRIC DIMENSION is defined as follows.

$k$ -METRIC DIMENSION	
<i>Instance:</i>	An undirected connected graph $G = (V, E)$ and a number $r$ .
<i>Question:</i>	Is there a $k$ -resolving set $R \subseteq V$ for $G$ of size at most $r$ ?

The complexity of deciding  $k$ -METRIC DIMENSION has only been investigated for very few graph classes, such as trees and other simple graph classes. For general graph classes,  $k$ -METRIC DIMENSION is assumed to be NP-complete if  $k$  is given as part of the input. The decision problem 1-METRIC DIMENSION is known to be NP-complete, see [GJ79]. A proof can be found in [KRR96]. In this paper, we show the NP-completeness of  $k$ -METRIC DIMENSION for bipartite graphs and each  $k \geq 2$  by an alternative approach to [YER17], whose proof unfortunately is incorrect and does not offer any simple correction options.

## 2 The NP-completeness of $k$ -METRIC DIMENSION

In this section,  $k$ -METRIC DIMENSION is shown to be NP-complete for bipartite graphs and each  $k \geq 2$  by a reduction from 3-DIMENSIONAL  $k$ -MATCHING, which is defined as follows.

3-DIMENSIONAL $k$ -MATCHING (3D $k$ M)	
<i>Instance:</i>	A set $S \subseteq A \times B \times C$ , where $A$ , $B$ and $C$ are disjoint sets of the same size $n$ .
<i>Question:</i>	Does $S$ contain a $k$ -matching, i.e. a subset $M$ of size $k \cdot n$ such that each element of $A$ , $B$ and $C$ is contained in exactly $k$ triples of $M$ ?

For  $k = 1$ , the 3D1M problem is the well-known NP-complete 3-DIMENSIONAL MATCHING (3DM) problem, see [GJ79]. The next theorem shows that 3D $k$ M is also NP-complete for each  $k \geq 2$ .

**Theorem 1.** *3D $k$ M is NP-complete for each  $k \geq 2$ .*

*Proof.* The 3D $k$ M problem is obviously in NP, because it can be checked in polynomial time whether a selection of triples from  $S$  is a  $k$ -matching.

The NP-hardness is shown by a reduction from 3DM. Let

$$\begin{aligned} A &= \{a_1, \dots, a_n\}, & B &= \{b_1, \dots, b_n\}, \\ C &= \{c_1, \dots, c_n\}, & \text{and } S &= \{s_1, \dots, s_m\} \end{aligned}$$

be an instance for 3DM. Without loss of generality,  $n$  is assumed to be a multiple of  $(k - 1)$ , that is  $n = r(k - 1)$  for a positive integer  $r$ . If this is not the case, then expand  $A$ ,  $B$  and  $C$  by at most  $k - 2$  elements each and  $S$  by at most  $k - 2$  triples, which cover every additional element exactly once and none of the originally given elements.

Now consider the following instance for 3D $k$ M defined by

$$\begin{aligned} A' &= A \cup \{a_{n+1}, \dots, a_{3n}\}, & B' &= B \cup \{b_{n+1}, \dots, b_{3n}\}, \\ C' &= C \cup \{c_{n+1}, \dots, c_{3n}\}, & \text{and } S' &= S \cup R \cup T \end{aligned}$$

where  $R = \{(a_i, b_i, c_i) \mid n + 1 \leq i \leq 3n\}$  and  $T \subseteq A' \times B' \times C'$ . Set  $T$  is a set with  $3n(k - 1)$  triples, which will be defined later.

The set  $A'$ ,  $B'$  and  $C'$  is the set  $A$ ,  $B$  and  $C$  respectively, each expanded by additional  $2n$  elements. Set  $S'$  is the set  $S$  expanded by the  $2n$  triples of  $R$  and the  $3n(k - 1)$  triples of  $T$ .

Let  $U = A \cup B \cup C$  and  $U' = A' \cup B' \cup C'$ . The  $2n$  triples of  $R$  cover each element of  $U' \setminus U$  exactly once and no element of  $U$ . Set  $T$  will be defined such that its  $3n(k - 1)$  triples cover each element of  $U'$  exactly  $k - 1$  times. Each triple of  $T$  will have exactly one element from  $U$  and two elements from  $U' \setminus U$ .

If  $M$  is a matching for  $U$  then  $M \cup R \cup T$  is obviously a  $k$ -matching for  $U'$  for any  $k \geq 2$ . Any  $k$ -matching  $M'$  for  $U'$  contains all triples from  $R$  and  $T$ , because otherwise it is not possible to cover the elements of  $U' \setminus U$  at least  $k$  times. The triples of  $T$  cover the elements of  $U'$  exactly  $k - 1$  times. That is, if  $M'$  is a  $k$ -matching for  $U'$  then  $M = M' \setminus (R \cup T)$  is a matching for  $U$ .

The set  $T$  of triples can be easily defined with the help of a set

$$T_{p,q} \subseteq (A \times B) \cup (A \times C) \cup (B \times C)$$

of tuples defined by

$$T_{p,q} = \cup \left\{ \begin{array}{l} \{(a_i, b_j) \mid i \in \{p, \dots, p+q-1\}, j \in \{p+q, \dots, p+2q-1\}\} \\ \{(b_i, c_j) \mid i \in \{p, \dots, p+q-1\}, j \in \{p+q, \dots, p+2q-1\}\} \\ \{(c_i, a_j) \mid i \in \{p, \dots, p+q-1\}, j \in \{p+q, \dots, p+2q-1\}\}. \end{array} \right.$$

These  $3q^2$  tuples cover each element of

$$\{a_p, \dots, a_{p+2q-1}, b_p, \dots, b_{p+2q-1}, c_p, \dots, c_{p+2q-1}\}$$

exactly  $q$  times. There are

- $q^2$  tuples between the elements of  $\{a_p, \dots, a_{p+q-1}\}$  and  $\{b_{p+q}, \dots, b_{p+2q-1}\}$ ,
- $q^2$  tuples between the elements of  $\{b_p, \dots, b_{p+q-1}\}$  and  $\{c_{p+q}, \dots, c_{p+2q-1}\}$ , and
- $q^2$  tuples between the elements of  $\{c_p, \dots, c_{p+q-1}\}$  and  $\{a_{p+q}, \dots, a_{p+2q-1}\}$ .

Now let  $T'$  be the set of tuples defined by

$$T' = \bigcup_{i=0}^{r-1} T_{n+1+i2(k-1), k-1}, \quad \text{with } r = \frac{n}{k-1}.$$

$T'$  contains  $r3(k-1)^2 = \frac{n}{k-1} \cdot 3(k-1)^2 = 3n(k-1)$  tuples. It is the union of  $r = \frac{n}{k-1}$  sets  $T_{p,q}$  where index  $p$  is running from  $n+1$  to  $3n+1-2(k-1)$  in steps of width  $2(k-1)$  and  $q = k-1$ . These tuples of  $T'$  cover each element of  $U' \setminus U$  exactly  $(k-1)$  times.

In the last step, the  $3n(k-1)$  tuples of  $T'$  are expanded to  $3n(k-1)$  triples for  $T$ , by including each element from  $U$  to exactly  $k-1$  tuples from  $T'$ , such that each generated triple is from the set  $A' \times B' \times C'$ . Each tuple from  $T'$  is extended by exactly one element from  $U$ . The result is the set  $T$  of triples with the required properties. This transformation can obviously be done in polynomial time, see also Example 1.  $\square$

**Example 1.** Let  $A = \{a_1, \dots, a_4\}$ ,  $B = \{b_1, \dots, b_4\}$ ,  $C = \{c_1, \dots, c_4\}$  and

$$S = \{(a_1, b_1, c_1), (a_1, b_2, c_3), (a_2, b_3, c_3), (a_2, b_4, c_1), (a_3, b_1, c_2), (a_4, b_3, c_4)\}$$

be an instance for 3DM. The triple  $(a_1, b_2, c_3), (a_2, b_4, c_1), (a_3, b_1, c_2), (a_4, b_3, c_4)$  form a 3-dimensional matching and thus a solution for 3DM.

It follows the construction of an instance for 3DkM for  $k = 4$  as defined in the proof of Theorem 1. Integer  $n$  has to be a multiple of  $k-1 = 3$ . To ensure this,  $A$  is extended by  $a_5$  and  $a_6$ ,  $B$  is extended by  $b_5$  and  $b_6$ ,  $C$  is extended by  $c_5$  and  $c_6$  and  $S$  is extended by  $(a_5, b_5, c_5)$  and  $(a_6, b_6, c_6)$ . Now  $n = 6$  and  $r = \frac{n}{k-1} = 2$ .

Then  $A' = \{a_1, \dots, a_{18}\}$ ,  $B' = \{b_1, \dots, b_{18}\}$ ,  $C' = \{c_1, \dots, c_{18}\}$  and  $R = \{(a_i, b_i, c_i) \mid i = 7, \dots, 18\}$ . Set  $T'$  is defined as  $T' = T_{7,3} \cup T_{13,3}$ . Finally, set  $S'$  is defined as

$$S' = S \cup R \cup T,$$

where, for example,

$$T_{7,3} = \left\{ \begin{array}{l} (a_7, b_{10}), (a_7, b_{11}), (a_7, b_{12}), (a_8, b_{10}), (a_8, b_{11}), (a_8, b_{12}), (a_9, b_{10}), (a_9, b_{11}), (a_9, b_{12}), \\ (b_7, c_{10}), (b_7, c_{11}), (b_7, c_{12}), (b_8, c_{10}), (b_8, c_{11}), (b_8, c_{12}), (b_9, c_{10}), (b_9, c_{11}), (b_9, c_{12}), \\ (c_7, a_{10}), (c_7, a_{11}), (c_7, a_{12}), (c_8, a_{10}), (c_8, a_{11}), (c_8, a_{12}), (c_9, a_{10}), (c_9, a_{11}), (c_9, a_{12}) \end{array} \right\},$$

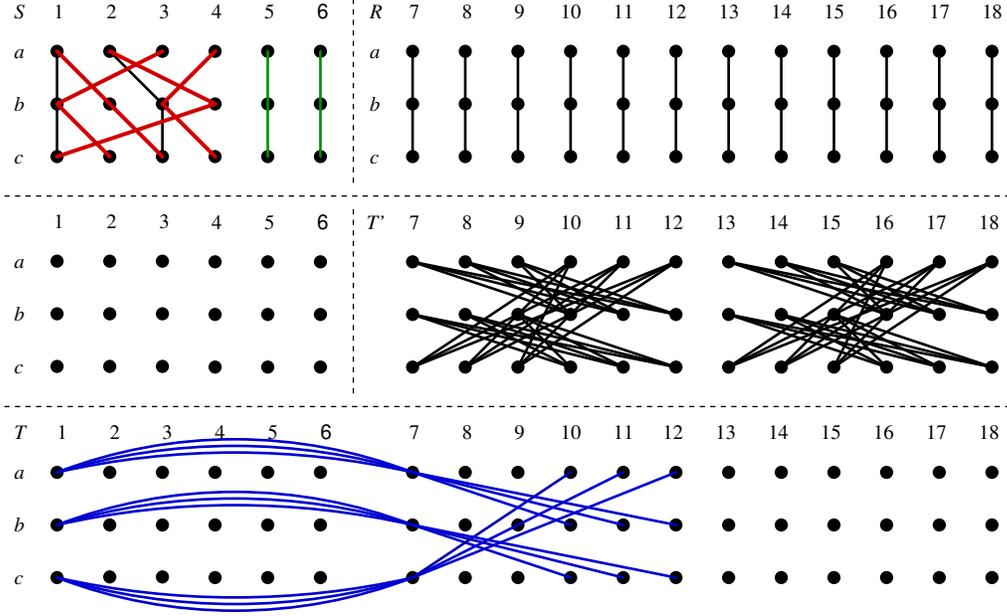


Figure 1: This graphic illustrates the transformation from 3DM to 3D $k$ M for  $k = 4$  as explained in Example 1. The drawing on the top left visualizes an instance with 6 triples in  $S$  that cover the elements  $\{a_1, \dots, a_4, b_1, \dots, b_4, c_1, \dots, c_4\}$ . The triples are indicated by 6 red and 2 black lines, each covering 3 elements. Set  $S$  contains a matching indicated by the red lines. Each set  $A$ ,  $B$  and  $C$  is extended by two element  $a_5, a_6$ ,  $b_5, b_6$  and  $c_5, c_6$  respectively, and set  $S$  is extended by two triples  $(a_5, b_5, c_5), (a_6, b_6, c_6)$ , such that the number of elements in the new sets  $A$ ,  $B$  and  $C$  is a multiple of  $(k - 1) = 3$ . These two triples are indicated by green lines. The drawing in the middle right visualizes the  $2 \cdot 6 = 12$  triples of  $R$  indicated by black lines. The drawing at the bottom visualizes the 54 tuples of  $T' = T_{7,3} \cup T_{13,3}$ , also indicated by black lines, each covering 2 elements. The set  $T$  is formed from set  $T'$  by adding each element of  $A$ ,  $B$  and  $C$  to  $k - 1 = 3$  tuples of  $T'$ . For the sake of clarity, only the triples from  $T$  for the elements  $a_1$ ,  $b_1$  and  $c_1$  are shown in the figure. These triples are indicated by blue lines.

$$T_{13,3} = \left\{ \begin{array}{l} (a_{13}, b_{16}), (a_{13}, b_{17}), (a_{13}, b_{18}), (a_{14}, b_{16}), (a_{14}, b_{17}), (a_{14}, b_{18}), (a_{15}, b_{16}), (a_{15}, b_{17}), (a_{15}, b_{18}), \\ (b_{13}, c_{16}), (b_{13}, c_{17}), (b_{13}, c_{18}), (b_{14}, c_{16}), (b_{14}, c_{17}), (b_{14}, c_{18}), (b_{15}, c_{16}), (b_{15}, c_{17}), (b_{15}, c_{18}), \\ (c_{13}, a_{16}), (c_{13}, a_{17}), (c_{13}, a_{18}), (c_{14}, a_{16}), (c_{14}, a_{17}), (c_{14}, a_{18}), (c_{15}, a_{16}), (c_{15}, a_{17}), (c_{15}, a_{18}) \end{array} \right\},$$

$$T = \left\{ \begin{array}{l} (a_1, b_7, c_{10}), (a_1, b_7, c_{11}), (a_1, b_7, c_{12}), (a_2, b_8, c_{10}), (a_2, b_8, c_{11}), (a_2, b_8, c_{12}), \\ (a_3, b_9, c_{10}), (a_3, b_9, c_{11}), (a_3, b_9, c_{12}), (a_4, b_{13}, c_{16}), (a_4, b_{13}, c_{17}), (a_4, b_{13}, c_{18}), \\ (a_5, b_{14}, c_{16}), (a_5, b_{14}, c_{17}), (a_5, b_{14}, c_{18}), (a_6, b_{15}, c_{16}), (a_6, b_{15}, c_{17}), (a_6, b_{15}, c_{18}), \\ (a_7, b_{10}, c_1), (a_7, b_{11}, c_1), (a_7, b_{12}, c_1), (a_8, b_{10}, c_2), (a_8, b_{11}, c_2), (a_8, b_{12}, c_2), \\ (a_9, b_{10}, c_3), (a_9, b_{11}, c_3), (a_9, b_{12}, c_3), (a_{10}, b_1, c_7), (a_{10}, b_2, c_8), (a_{10}, b_3, c_9), \\ (a_{11}, b_1, c_7), (a_{11}, b_2, c_8), (a_{11}, b_3, c_9), (a_{12}, b_1, c_7), (a_{12}, b_2, c_8), (a_{12}, b_3, c_9), \\ (a_{13}, b_{16}, c_4), (a_{13}, b_{17}, c_4), (a_{13}, b_{18}, c_4), (a_{14}, b_{16}, c_5), (a_{14}, b_{17}, c_5), (a_{14}, b_{18}, c_5), \\ (a_{15}, b_{16}, c_6), (a_{15}, b_{17}, c_6), (a_{15}, b_{18}, c_6), (a_{16}, b_4, c_{13}), (a_{16}, b_5, c_{14}), (a_{16}, b_6, c_{15}), \\ (a_{17}, b_4, c_{13}), (a_{17}, b_5, c_{14}), (a_{17}, b_6, c_{15}), (a_{18}, b_4, c_{13}), (a_{18}, b_5, c_{14}), (a_{18}, b_6, c_{15}) \end{array} \right\},$$

see also Figure 1.

**Theorem 2.**  $k$ -MD is NP-complete for bipartite graphs  $G$  and each  $k \geq 2$ .

*Proof.* The  $k$ -MD problem is obviously in NP, because it can be checked in polynomial time whether a set of vertices is a  $k$ -resolving set.

The NP-hardness is proven by a reduction from 3D $(k-1)$ M. Let  $A = \{a_1, \dots, a_n\}$ ,  $B = \{b_1, \dots, b_n\}$ ,  $C = \{c_1, \dots, c_n\}$ ,  $S = \{s_1, \dots, s_m\}$  be an instance  $I$  for 3D $(k-1)$ M where  $k \geq 2$

and  $n > k$ . The aim is to define a graph  $G = (V, E)$  and a number  $x$  such that  $G$  has a  $k$ -resolving set of size  $x$  if and only if instance  $I$  has a  $(k-1)$ -matching.

Graph  $G$  is defined as follows, see also Figure 2. It has a vertex  $a_i, b_i$  and  $c_i$  for  $i = 1, \dots, n$  and a vertex  $s_i$  for  $i = 1, \dots, m$ . Graph  $G$  additionally contains vertices denoted by  $a_0, b_0, c_0, v_0, v_A, v_B, v_C$  and  $d_1, \dots, d_{m'}$  where  $m' = \lceil \log(m) \rceil$ .

1. Each vertex  $a_i, 0 \leq i \leq n$ , is connected with
  - (a) vertex  $v_A$ ,
  - (b) vertex  $v_0$ , and
  - (c) vertex  $s_j, 1 \leq j \leq m$  if and only if triple  $s_j$  contains element  $a_i$ .
2. Each vertex  $b_i, 0 \leq i \leq n$ , is connected with
  - (a) vertex  $v_B$ ,
  - (b) vertex  $v_0$ , and
  - (c) vertex  $s_j, 1 \leq j \leq m$  if and only if triple  $s_j$  contains element  $b_i$ .
3. Each vertex  $c_i, 0 \leq i \leq n$ , is connected with
  - (a) vertex  $v_C$ ,
  - (b) vertex  $v_0$ , and
  - (c) vertex  $s_j, 1 \leq j \leq m$  if and only if triple  $s_j$  contains element  $c_i$ .
4. Each vertex  $d_i, 1 \leq i \leq m'$ , is connected with
  - (a) vertex  $v_0$  and
  - (b) vertex  $s_j, 1 \leq j \leq m$ , if and only if the  $i$ -th bit of the binary representation of  $j$  is 1.

Graph  $G$  contains additionally so-called *leg vertices*. These leg vertices form paths (*legs*) with  $\lceil k/2 \rceil$  or  $\lfloor k/2 \rfloor$  vertices. Two such legs, one with  $\lceil k/2 \rceil$  vertices and one with  $\lfloor k/2 \rfloor$  vertices, are attached to each vertex of  $L_{\text{root}} = \{v_A, v_B, v_C, v_0, d_1, \dots, d_{m'}\}$ , see Figure 2. Set  $L_{\text{root}}$  is the set of *root vertices* of the legs. Let  $L_v$  be the set of vertices of the two legs at vertex  $v$  and

$$L = L_{v_A} \cup L_{v_B} \cup L_{v_C} \cup L_{v_0} \cup L_{d_1} \cup \dots \cup L_{d_{m'}}$$

be the set of all leg vertices of  $G$ . Set  $L_{\text{root}}$  has  $4 + m'$  vertices, each set  $L_v, v \in L_{\text{root}}$ , has  $k$  vertices and  $L$  has  $(4 + m')k$  vertices.

The graph  $G$  can obviously be constructed in polynomial time from instant  $I$ .

First of all, let us note some properties of  $G$ .

P1:  $G$  is bipartite.

P2: The distance between

- (a) two vertices of  $\{v_B, v_B, v_C\}$  is 4,
- (b) two vertices of  $\{d_1, \dots, d_{m'}\}$  is 2,
- (c) a vertex of  $\{v_B, v_B, v_C\}$  and a vertex of  $\{d_1, \dots, d_{m'}\}$  is 3,
- (d) vertex  $v_0$  and a vertex of  $\{v_B, v_B, v_C\}$  is 2, and
- (e) vertex  $v_0$  and a vertex of  $\{d_1, \dots, d_{m'}\}$  is 1.

P3: Every  $k$ -resolving set for  $G$  contains all vertices of  $L$ . This follows from the observation that for each vertex  $v \in L_{\text{root}}$  the two vertices of  $L_v$  adjacent with  $v$  are only resolved by the  $k$  vertices of  $L_v$ .

Now we will prove that  $S$  has a  $(k-1)$ -matching for instance  $I$  if and only if  $G$  has a resolving set of size

$$x = (4 + m')k + 3 + (k - 1)n.$$

" $\Rightarrow$ :" Let  $M \subseteq S$  be a  $(k-1)$ -matching for instance  $I$ . The aim is to show that

$$R = L \cup \{a_0, b_0, c_0\} \cup M$$

is a  $k$ -resolving set for  $G$  of size

$$x = (4 + m')k + 3 + (k - 1)n,$$

that is, each pair of two distinct vertices  $u_1, u_2$  of  $G$  is resolved by at least  $k$  vertices of  $U$ . Here the triple  $s_j$  of  $M$  are considered as vertices of  $G$ .

Consider the following case distinctions for two vertices  $u_1$  and  $u_2$ .

1.  $u_1, u_2 \in L_v, v \in L_{\text{root}}$ .
  - (a)  $d(u_1, v) = d(u_2, v)$ . Each of the  $k$  vertices of  $L_v$  resolves  $u_1$  and  $u_2$ .
  - (b)  $d(u_1, v) \neq d(u_2, v)$ . Each of the  $k$  vertices of  $L_{v'}, v' \in L_{\text{root}} \setminus \{v\}$ , resolves  $u_1$  and  $u_2$ .
2.  $u_1 \in L_{v_1}, u_2 \in L_{v_2}, v_1, v_2 \in L_{\text{root}}, v_1 \neq v_2$ , and  $d(u_1, v_1) \leq d(u_2, v_2)$ . Each of the  $k$  vertices of  $L_{v_1}$  resolves  $u_1$  and  $u_2$ .

Up to this point all pairs of vertices  $u_1, u_2$  are considered of which both are in  $L$ .

3.  $u_1 \in L_{v_A} \cup L_{v_B} \cup L_{v_C}$  and  $u_2 \notin L$ . Each of the  $k$  vertices of  $L_{v_0}$  resolves  $u_1$  and  $u_2$ .
4.  $u_1 \in L_{d_1} \cup \dots \cup L_{d_{m'}}$  and  $u_2 \notin L$ .
  - (a)  $u_2 \notin \{v_B, v_C\}$ . Each of the  $k$  vertices of  $L_{v_A}$  resolves  $u_1$  and  $u_2$ .
  - (b)  $u_2 \notin \{v_A, v_C\}$ . Each of the  $k$  vertices of  $L_{v_B}$  resolves  $u_1$  and  $u_2$ .
  - (c)  $u_2 \notin \{v_A, v_B\}$ . Each of the  $k$  vertices of  $L_{v_C}$  resolves  $u_1$  and  $u_2$ .
5.  $u_1 \in L_{v_0}$  and  $u_2 \notin L$ .
  - (a)  $u_2 \in \{v_A, a_0, \dots, a_n\}$ . Each of the  $k$  vertices of  $L_{v_A}$  resolves  $u_1$  and  $u_2$ .
  - (b)  $u_2 \in \{v_B, b_0, \dots, b_n\}$ . Each of the  $k$  vertices of  $L_{v_B}$  resolves  $u_1$  and  $u_2$ .
  - (c)  $u_2 \in \{v_C, c_0, \dots, c_n\}$ . Each of the  $k$  vertices of  $L_{v_C}$  resolves  $u_1$  and  $u_2$ .
  - (d)  $u_2 \in \{d_i\} \cup \{s_j \mid \text{the } i\text{-th bit in the binary representation of } j \text{ is } 1\}$ . Each of the  $k$  vertices of  $L_{d_i}$  resolves  $u_1$  and  $u_2$ .

Up to this point all pairs of vertices  $u_1, u_2$  are considered of which at least one of them is in  $L$ .

6.  $u_1 \in L_{\text{root}}$  and  $u_2 \notin L$ . Each of the  $k$  vertices of  $L_{u_1}$  resolves  $u_1$  and  $u_2$ .

Up to this point all pairs of vertices  $u_1, u_2$  are considered of which at least one of them is in  $L \cup L_{\text{root}}$ .

7.  $u_1 = s_{i_1} \in \{s_1, \dots, s_{m'}\}$  and  $u_2 \notin L \cup L_{\text{root}}$ .
  - (a)  $u_2 = s_{i_2} \in \{s_1, \dots, s_{m'}\}$ . Each of the  $k$  vertices of  $L_{d_j}$  resolves  $u_1$  and  $u_2$ , if the binary representation of  $i_1$  and  $i_2$  differs in position  $j$ .
  - (b)  $u_2 \in \{a_0, \dots, a_n\}$ ,  $u_2 \in \{b_0, \dots, b_n\}$ , or  $u_2 \in \{c_0, \dots, c_n\}$ . Each of the  $k$  vertices of  $L_{v_A}$ ,  $L_{v_B}$ , or  $L_{v_C}$ , respectively, resolves  $u_1$  and  $u_2$ .

Up to this point all pairs of vertices  $u_1, u_2$  are considered of which at least one of them is in  $L \cup L_{\text{root}} \cup \{s_1, \dots, s_{m'}\}$ .

8.  $u_1 \in \{a_1, \dots, a_n\}$  and  $u_2 \notin L \cup L_{\text{root}} \cup \{s_1, \dots, s_{m'}\}$ .
  - (a)  $u_2 \in \{b_0, \dots, b_n, c_0, \dots, c_n\}$ . Each of the  $k$  vertices of  $L_{v_A}$  resolves  $u_1$  and  $u_2$ .
  - (b)  $u_2 \in \{a_1, \dots, a_n\}$ . Each vertex  $s_i$  for which triple  $s_i$  contains  $u_1$  or  $u_2$  resolves  $u_1$  and  $u_2$ . There are  $2(k-1) \geq k$  such vertices for  $k \geq 2$ .
  - (c)  $u_2 = a_0$ . Each vertex  $s_i$  for which triple  $s_i$  contains  $u_1$  resolves  $u_1$  and  $u_2$ , and vertex  $a_0$  resolves  $u_1$  and  $u_2$ . Altogether these are exactly  $(k-1) + 1 = k$  vertices.
9.  $u_1 \in \{b_1, \dots, b_n\}$  and  $u_2 \notin L \cup L_{\text{root}} \cup \{s_1, \dots, s_{m'}\}$ . (as in case 8)
10.  $u_1 \in \{c_1, \dots, c_n\}$  and  $u_2 \notin L \cup L_{\text{root}} \cup \{s_1, \dots, s_{m'}\}$ . (as in case 8)
11.  $u_1, u_2 \in \{a_0, b_0, c_0\}$ . Each of the  $k$  vertices of  $L_{v_A}$ ,  $L_{v_B}$  or  $L_{v_C}$  resolves  $u_1$  and  $u_2$ .

Now all pairs of vertices  $u_1, u_2$  of  $G$  are considered and it is shown that all of them are resolved by at least  $k$  vertices from  $R$ . Note that only the vertex pairs  $u_1, u_2 \in \{a_0, \dots, a_n\}$ ,  $u_1, u_2 \in \{b_0, \dots, b_n\}$  and  $u_1, u_2 \in \{c_0, \dots, c_n\}$  are not already resolved by  $k$  vertices of  $L$ . Strictly speaking, not a single vertex from  $L \cup \{v_A, v_B, v_C, v_0, d_1, \dots, d_{m'}\}$  resolves such a pair of vertices.

" $\Leftarrow$ ." Let  $R \subseteq V$  be a  $k$ -resolving set for  $G$  with  $x = (4 + m')k + 3 + (k - 1)n$  vertices. By Property P3,  $R$  contains all the  $(4 + m)k$  vertices of  $L$ . This leaves  $3 + (k - 1)n$  vertices of  $R$  that are not in  $L$ . Let us now consider the vertex pairs  $a_0, a_i$ , and  $b_0, b_i$ , and  $c_0, c_i$  for  $i = 1, \dots, n$ . The vertices of  $L$  and the vertices of  $\{v_A, v_B, v_C, v_0, d_1, \dots, d_{m'}\}$  do not resolve these vertex pairs. The only way to resolve these  $3n$  vertex pairs at least  $k$  times with  $3 + (k - 1)n$  vertices for  $n > k \geq 2$ , is to use  $k-1$  vertices from  $\{s_1, \dots, s_m\}$  that form a  $k-1$  matching and the three vertices  $a_0, b_0, c_0$ . This is the point where it is necessary that  $n$  is greater than  $k$ .  $\square$

In the introduction of this paper, we mentioned that the  $k$ -METRIC DIMENSION and the  $(k, t)$ -METRIC DIMENSION in [EMYRV16] are the same if  $t$  is set to the diameter of  $G$ . Since the constructed graph in Theorem 2 has diameter  $2 \cdot \lceil k/2 \rceil + 3$ , Theorem 2 also proves the NP-completeness of  $(k, t)$ -METRIC DIMENSION for bipartite graphs, each  $k \geq 2$  and  $t \geq 2 \cdot \lceil k/2 \rceil + 3$ .

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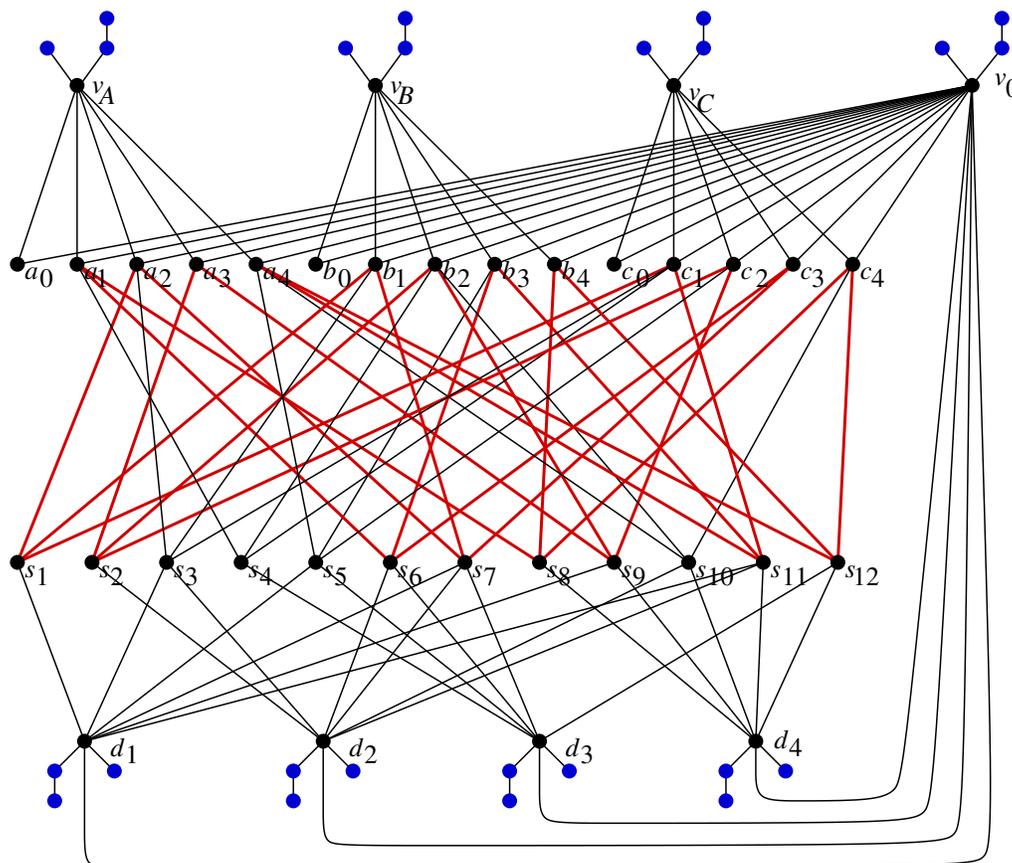


Figure 2: This graphic illustrates the transformation from 3D2M to 3-MD. The Instance  $I$  consisting of  $A = \{a_1, \dots, a_4\}$ ,  $B = \{b_1, \dots, b_4\}$ ,  $C = \{c_1, \dots, c_4\}$ ,  $S = \{s_1, \dots, s_{12}\}$  with  $s_1 = (a_2, b_1, c_1)$ ,  $s_2 = (a_3, b_2, c_2)$ ,  $s_3 = (a_2, b_1, c_1)$ ,  $s_4 = (a_1, b_2, c_1)$ ,  $s_5 = (a_4, b_3, c_2)$ ,  $s_6 = (a_1, b_3, c_3)$ ,  $s_7 = (a_2, b_1, c_3)$ ,  $s_8 = (a_1, b_4, c_4)$ ,  $s_9 = (a_3, b_2, c_2)$ ,  $s_{10} = (a_4, b_2, c_4)$ ,  $s_{11} = (a_4, b_3, c_1)$ ,  $s_{12} = (a_4, b_4, c_4)$  for 3D2M is transformed into the graph  $G$  and  $x = (4 + 4)3 + 3 + (3 - 1)n = 35$ . The set of triples  $M = \{s_1, s_2, s_6, s_7, s_8, s_9, s_{11}, s_{12}\}$ , indicated in the figure by the red lines, is a 2-matching for instance  $I$ , where  $L \cup \{a_0, b_0, c_0\} \cup M$  is a 3-resolving set for  $G$  of size  $x$ . Set  $L$  is the set of vertices of the legs attached at the vertices  $v_A, v_B, v_C, v_0, d_1, d_2, d_3, d_4$ . In the figure, the vertices of  $L$  are colored blue.

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