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Oscillation Results of Higher Order Nonlinear Neutral Delay Differential Equations with Oscillating Coefficients

Mustafa Kemal Yildiz
Kocatepe University, ANS Campus
Department of Mathematics, Faculty of Sciences and Arts
3200 Afyonkarahisar, Turkey
myildiz@aku.edu.tr

Abstract

In this paper, we shall consider higher order nonlinear neutral delay differential equation of the type

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)\left[x(\sigma(t))\right]^{\alpha}=0, \quad t\geq t_0,\ n\in\mathbb{N}, \tag{*}$$

where $p\in C\left([t_0,\infty),\mathbb{R}\right)$ is oscillatory and $\lim\limits_{t\to\infty}p(t)=0$, $q\in C\left([t_0,\infty),\mathbb{R}^+\right)$, $\tau,\sigma\in C\left([t_0,\infty),\mathbb{R}\right)$, $\tau(t),\sigma(t)<t$, $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$ and $\alpha\in(0,\infty)$ is a ratio of odd positive integers. If $\alpha\in(0,1)$, equation (*) is called a sublinear equation, when $\alpha\in(1,\infty)$, it is called a superlinear equation. We obtain sufficient conditions for the oscillation of all solutions of this equation.

AMS Subject Classifications: 39A10.
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1 Introduction

We consider the following higher order nonlinear neutral delay differential equation:

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)\left[x(\sigma(t))\right]^{\alpha}=0, \quad t\geq t_0,\ n\in\mathbb{N}, \tag{1.1}$$

where $p\in C\left([t_0,\infty),\mathbb{R}\right)$ is oscillatory and $\lim\limits_{t\to\infty}p(t)=0$, $q\in C\left([t_0,\infty),\mathbb{R}^+\right)$, $\tau,\sigma\in C\left([t_0,\infty),\mathbb{R}\right)$, $\tau(t),\sigma(t)<t$, $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$ and $\alpha\in(0,\infty)$ is a ratio of

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odd positive integers. If $\alpha \in (0,1)$, equation (1.1) is called a sublinear equation, when $\alpha \in (1,\infty)$, it is called a superlinear equation.

Recently, there have been a lot of studies concerning the oscillatory behavior of differential equations, see [1–10] and the references cited therein. In [3, 5, 7, 9] several authors have investigated the following first order nonlinear delay differential equation,

$$x'(t) + q(t)\left[x(\sigma(t))\right]^{\alpha} = 0, \quad t \geq t_0, \tag{1.2}$$

where $q \in C\left([t_0,\infty),\mathbb{R}^+\right)$, $\sigma \in C\left([t_0,\infty),\mathbb{R}\right)$, $\sigma(t) < t$, $\lim\limits_{t\to\infty}\sigma(t) = \infty$ and $\alpha \in (0,\infty)$ is a ratio of odd positive integers.

When $\alpha \in (0,1)$, it is shown that every solution of the sublinear equation (1.2) oscillates if and only if

$$\int_{t_0}^{\infty} q(s)ds = \infty. \tag{1.3}$$

When $\alpha = 1$, (1.2) reduces to the linear delay differential equation

$$x'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0. \tag{1.4}$$

Recently, the oscillatory behavior of (1.4) has been discussed extensively in the literature. A classical result is (see [3–5]) that every solution of (1.4) oscillates if

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} q(s)ds > \frac{1}{\mathrm{e}}.$$

In [9], when $\alpha \in (1,\infty)$, Tang obtained the oscillatory behavior of equation (1.2). The following is shown: Let σ be continuously differentiable and $\sigma' \geq 0$. Further, suppose that there exists a continuously differentiable function φ such that

$$\varphi'(t) > 0 \quad \text{and} \quad \lim_{t\to\infty}\varphi(t) = \infty,$$

$$\limsup_{t\to\infty}\frac{\alpha\varphi'(\sigma(t))\sigma'(t)}{\varphi'(t)} < 1,$$

and

$$\liminf_{t\to\infty}\frac{q(t)\mathrm{e}^{-\varphi(t)}}{\varphi'(t)} > 0.$$

Then every solution of the superlinear equation (1.2) oscillates. Furthermore, Tang considered the special form of (1.2),

$$x'(t) + q(t)\left[x(t-\sigma)\right]^{\alpha} = 0, \quad t \geq t_0 \tag{1.5}$$

for which the following results was obtained: If there exists $\lambda \in \left(\sigma^{-1}\ln\alpha,\infty\right)$ such that

$$\liminf_{t\to\infty} q(t)\mathrm{e}^{-\lambda t} > 0, \tag{1.6}$$

then every solution of (1.5) oscillates. In [10], Zein and Abu-Kaff have investigated the higher order nonlinear delay differential equation,

$$[x(t)+p(t)x(\tau(t))]^{(n)}+f(t,x(t),x(\sigma(t)))=s(t), \tag{1.7}$$

where $p \in C([t_0,\infty),\mathbb{R})$, $\lim\limits_{t\to\infty} p(t)=0$, $\sigma,\tau \in C([t_0,\infty),\mathbb{R})$, $\tau(t),\sigma(t)<t$, $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$, $f:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, $yf(t,x,y)>0$ for $xy>0$, there exists an oscillatory function $r\in C^n(\mathbb{R}_+,\mathbb{R})$, such that $r^{(n)}=s$, $\lim\limits_{t\to\infty} r(t)=0$.

In [1] Agarwal and Grace, in [4] Grace and Lalli studied oscillatory behavior of certain higher order differential equations.

Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of (1.1).

We need the following result for our subsequent discussion.

Lemma 1.1 (See [9]). *Assume that for large* t

$$q(s)\neq 0 \text{ for all } s\in[t,t^*],$$

where t^* satisfies $\sigma(t^*)=t$. Then

$$x'(t)+q(t)[x(\sigma(t))]^\alpha=0, \quad t\geq t_0$$

has an eventually positive solution if and only if the corresponding inequality

$$x'(t)+q(t)[x(\sigma(t))]^\alpha\leq 0, \quad t\geq t_0$$

has an eventually positive solution.

Lemma 1.2 (See [6]). *Let z be a positive and n-times differentiable function on $[t_0,\infty)$. If $z^{(n)}$ is of constant sign for $t\geq t_0$ and not identically zero on any interval $[t_*,\infty)$ for some $t_*\geq t_0$, then there exists a $t_z\geq t_0$ and an integer m, $0\leq m\leq n$ with $(n+m)$ odd for $z^{(n)}(t)\leq 0$, or $(n+m)$ even for $z^{(n)}(t)\geq 0$, and such that for every $t_z\geq t_0$,*

$$m\leq n-1 \text{ implies } (-1)^{m+k}z^{(k)}(t)>0, \quad k=m,m+1,\ldots,n-1,$$

and

$$m>0 \text{ implies } z^{(k)}(t)>0, \quad k=0,1,\ldots,m-1.$$

Lemma 1.3 (See [8]). *Let z be as in Lemma 1.2. If in addition $\lim\limits_{t\to\infty} z(t)\neq 0$ and $z^{(n-1)}(t)z^{(n)}(t)\leq 0$ for every $t\geq t_z$, then for every $\lambda\in(0,1)$, the following holds:*

$$z(t)\geq\frac{\lambda}{(n-1)!}t^{n-1}z^{(n-1)}(t), \quad \text{for all large } t.$$

2 Sufficient Conditions for Oscillation of (1.1)

Theorem 2.1. *Let n be even and $\lim_{t\to\infty} p(t) = 0$. If the differential equation*

$$w'(t) + c(t)\,[w(\sigma(t))]^{\alpha} = 0, \tag{2.1}$$

where

$$c(t) = q(t)\left(\frac{1}{2}\frac{\lambda}{(n-1)!}(\sigma(t))^{n-1}\right)^{\alpha}, \quad \lambda \in (0,1). \tag{2.2}$$

is oscillatory, then every bounded solution x of equation (1.1) *is oscillatory.*

Proof. Let x be a bounded nonoscillatory solution of (1.1). Without loss in the generality we may assume that

$$x(t), x(\tau(t)), x(\sigma(t)) > 0$$

for all $t \geq t_1$ where $t_1 \geq t_0$. Set

$$z(t) = x(t) + p(t)x(\tau(t)), \tag{2.3}$$

and

$$z^{(n)}(t) = -q(t)\,[x(\sigma(t))]^{\alpha} \leq 0, \tag{2.4}$$

for all $t \geq t_0$. It follows that $z^{(i)}$ $(i = 0, 1, \ldots, n-1)$ is strictly monotonic and of constant sign eventually. Since x is bounded, and using the fact that $\lim_{t\to\infty} p(t) = 0$, it follows from (2.3) that z is also bounded. Because n is even, we have by Lemma 1.2 that $m = 1$ (otherwise, z is not bounded) there exists a $t_2 \geq t_1$ such that for $t \geq t_2$

$$(-1)^{k+1} z^{(k)}(t) > 0\,, \quad (k = 1, \ldots, n-1)\,. \tag{2.5}$$

In particular, since $z'(t) > 0$ for all $t \geq t_2$ and so z is increasing. Since x is bounded, $\lim_{t\to\infty} p(t)x(\tau(t)) = 0$. Then there exists a $t_3 \geq t_2$ by (2.3),

$$x(t) = z(t) - p(t)x(\tau(t)) \geq \frac{1}{2}z(t) > 0$$

for all $t \geq t_3$. Also note that z does not tend to zero since it is increasing. We may find a $t_4 \geq t_3$ such that

$$x(\sigma(t)) \geq \frac{1}{2}z(\sigma(t)) > 0 \quad \text{and} \quad [x(\sigma(t))]^{\alpha} \geq \left[\frac{1}{2}x(\sigma(t))\right]^{\alpha}$$

hold for all $t \geq t_4$. From (2.4) and (2.6), we obtain the result of

$$z^{(n)}(t) + q(t)\left[\frac{1}{2}z(\sigma(t))\right]^{\alpha} \leq 0 \tag{2.6}$$

for all $t \geq t_4$. By Lemma 1.3, inequality (2.6) can be written as

$$z^{(n)}(t) + q(t) \left[\frac{1}{2}\frac{\lambda}{(n-1)!}(\sigma(t))^{n-1}\right]^{\alpha} \left[z^{(n-1)}(\sigma(t))\right]^{\alpha} \leq 0 \tag{2.7}$$

for all $t \geq t_4$. If we chose $z^{(n-1)} = w$, then

$$w'(t) + q(t) \left(\frac{1}{2}\frac{\lambda}{(n-1)!}(\sigma(t))^{n-1}\right)^{\alpha} [w(\sigma(t))]^{\alpha} \leq 0, \quad \text{for } t \geq t_4. \tag{2.8}$$

Therefore by Lemma 1.1, (2.8) has an eventually positive solution. This is a contradiction. The proof is complete. □

Theorem 2.2. *Let n be odd and $\lim_{t\to\infty} p(t) = 0$. If the differential equation*

$$w'(t) + c(t)\,[w(\sigma(t))]^{\alpha} = 0, \tag{2.9}$$

where

$$c(t) = q(t) \left(\frac{1}{2}\frac{\lambda}{(n-1)!}(\sigma(t))^{n-1}\right)^{\alpha}, \quad \lambda \in (0,1) \tag{2.10}$$

is oscillatory, then every bounded solution x of equation (1.1) *is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let x be a bounded nonoscillatory solution of (1.1), with

$$x(t), x(\tau(t)), x(\sigma(t)) > 0$$

for all $t \geq t_1$ where $t_1 \geq t_0$. Further, we assume that x does not tend to zero as $t \to \infty$. Set $z(t) = x(t) + p(t)x(\tau(t))$, and by (2.4), $z^{(i)}$ $(i = 0, 1, \ldots, n-1)$ is strictly monotonic and of constant sign eventually. Since p is an oscillating function, $\lim_{t\to\infty} p(t) = 0$, and x is bounded, there exists a $t_2 \geq t_1$ such that $z(t) > 0$ for all $t_2 \geq t_1$. Since x is bounded, by using $\lim_{t\to\infty} p(t) = 0$, it follows from (2.3) that z is also bounded. Because n is odd, by Lemma 1.2, since $m = 0$, there exists a $t_3 \geq t_2$ such that

$$(-1)^k z^{(k)}(t) > 0, \quad k = 0, 1, \ldots, n-1 \tag{2.11}$$

for all $t \geq t_3$. In particular, since $z'(t) < 0$ for all $t \geq t_3$, z is decreasing. Since x is bounded, $\lim_{t\to\infty} p(t)x(\tau(t)) = 0$ by $\lim_{t\to\infty} p(t) = 0$. Then there exists a $t_4 \geq t_3$ such that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq \frac{1}{2}z(t) > 0$$

for $t \geq t_4$. Also note that z does not tend to zero as $t \to \infty$ since x does not tend to zero as $t \to \infty$. We may find a $t_5 \geq t_4$ such that for all $t \geq t_5$ we have

$$x(\sigma(t)) \geq \frac{1}{2}z(\sigma(t)) > 0 \quad \text{and} \quad [x(\sigma(t))]^{\alpha} \geq \left[\frac{1}{2}z(\sigma(t))\right]^{\alpha}. \tag{2.12}$$

From (2.4) and (2.12), we obtain

$$z^{(n)}(t) + q(t)\left[\frac{1}{2}z(\sigma(t))\right]^{\alpha} \leq 0$$

for all $t \geq t_5$. By Lemma 1.3, inequality (2.12) can be written as

$$z^{(n)}(t) + q(t)\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\left[\sigma(t)\right]^{n-1}\right)^{\alpha}\left[z^{(n-1)}(\sigma(t))\right]^{\alpha} \leq 0 \tag{2.13}$$

for all $t \geq t_5$. If we chose $z^{(n-1)} = w$, then

$$w'(t) + q(t)\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\left[\sigma(t)\right]^{n-1}\right)^{\alpha}\left[w(\sigma(t))\right]^{\alpha} \leq 0, \quad \text{for } t \geq t_5. \tag{2.14}$$

Therefore by Lemma 1.1, (2.14) has an eventually positive solution. This is a contradiction. The proof is complete. □

References

[1] R. P. Agarwal and S. R. Grace. Oscillation of certain functional-differential equations. *Comput. Math. Appl.*, 38(5-6):143–153, 1999.

[2] Ravi P. Agarwal, Martin Bohner, and Wan-Tong Li. *Nonoscillation and oscillation: theory for functional differential equations*, volume 267 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 2004.

[3] L. H. Erbe, Qingkai Kong, and B. G. Zhang. *Oscillation theory for functional-differential equations*, volume 190 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1995.

[4] S. R. Grace and B. S. Lalli. Oscillation theorems for nth-order delay differential equations. *J. Math. Anal. Appl.*, 91(2):352–366, 1983.

[5] I. Győri and G. Ladas. *Oscillation theory of delay differential equations*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. With applications, Oxford Science Publications.

[6] I. T. Kiguradze. The capability of certain solutions of ordinary differential equations to oscillate. *Dokl. Akad. Nauk SSSR*, 144:33–36, 1962.

[7] G. S. Ladde, V. Lakshmikantham, and B. G. Zhang. *Oscillation theory of differential equations with deviating arguments*, volume 110 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1987.

[8] Christos Philos. A new criterion for the oscillatory and asymptotic behavior of delay differential equations. *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 29(7-8):367–370, 1981.

[9] X. H. Tang. Oscillation for first order superlinear delay differential equations. *J. London Math. Soc. (2)*, 65(1):115–122, 2002.

[10] Ali Zein and Taha Abu-Kaff. Bounded oscillation of higher order neutral differential equations with oscillating coefficients. *Appl. Math. E-Notes*, 6:126–131 (electronic), 2006.