Group Analysis of Nonlinear Fin Equations

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Group classification of a class of nonlinear fin equations is carried out exhaustively. Additional equivalence transformations and conditional equivalence groups are also found. These allow to simplify results of classification and further applications of them. The derived Lie symmetries are used to construct exact solutions of truly nonlinear equations for the class under consideration. Nonclassical symmetries of the fin equations are discussed. Adduced results amend and essentially generalize recent works on the subject [M. Pakdemirli and A.Z. Sahin, Appl. Math. Lett., 2006, V.19, 378–384; A.H. Bokhari, A.H. Kara and F.D. Zaman, Appl. Math. Lett., 2006, V.19, 1356–1340].

1 Introduction

Investigation of heat conductivity and diffusion processes leads to interesting mathematical models which can be often formulated in terms of partial differential equations. In general case coefficients of model equations explicitly include both dependent and independent model variables that makes difficulties in studying such models.

In this letter the class of nonlinear fin equations of the general form

$$u_t = (D(u)u_x)_x + h(x)u, (1)$$

where $D_u \neq 0$, is investigated within the symmetry framework. Here u is treated as the dimensionless temperature, t and x the dimensionless time and space variables, D the thermal conductivity, $h = -N^2 f(x)$, N the fin parameter and f the heat transfer coefficient.

The condition $D_u = 0$ corresponds to the linear case of (1) which was completely investigated with the Lie symmetry point of view long time ago [6, 10]. Moreover, the sets of the linear and nonlinear equations of form (1) can be separately investigated under restriction with point symmetries. That is why the linear case is excluded from consideration in the present letter.

The problem of group classification for the degenerate case h=0 (i.e. the class of nonlinear one-dimensional diffusion equations) was first solved by Ovsiannikov [9, 10]. The equations of form (1) with h being a constant are in the class of diffusion–reaction equations classified by Dorodnitsyn [3, 5]. Group classification of the subclass where the thermal conductivity is a power function of the temperature was carried out in [15]. We keep the above cases in presentation of results for reasons of classification usability. Note also that Lie symmetries of the class of quasilinear parabolic equations in two independent variables, which has a wide equivalence group and covers all the mentioned classes, were classified in [1, 7].

Recently Lie symmetries and reductions of equations from class (1) were considered in a number of papers [2, 11, 12]. (See ibid for references on physical meaning and applications of equations (1).) In contrast to these papers, study in our letter is concentrated on rigorous and exhaustive group classification of the whole class (1) and construction of exact solutions for truly nonlinear 'variable-coefficient' equations from this class. Additional equivalence transformations and conditional equivalence groups are also found. These allow to simplify results of classification and further applications of them. To find exact solutions, we apply both classical Lie reduction and nonclassical symmetry approaches.

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2 Group classification and related problems

Group classification of class (1) is performed in the framework of the classical approach [10]. All necessary objects (the equivalence group, the kernel and all inequivalent extensions of maximal Lie invariance algebras) are found. Moreover, we extend the classical approach with additional equivalence transformations and conditional equivalence group for simplification of the classification results.

The equivalence group G^{\sim} of class (1) is formed by the transformations

$$\tilde{t} = \delta_1 t + \delta_2$$
, $\tilde{x} = \delta_3 x + \delta_4$, $\tilde{u} = \delta_5 u$, $\tilde{D} = \delta_1^{-1} \delta_3^2 D$, $\tilde{h} = \delta_1^{-1} h$,

where δ_i , i = 1, ..., 5, are arbitrary constants, $\delta_1 \delta_3 \delta_5 \neq 0$. The connected component of the unity in G^{\sim} is formed by continuous transformations having $\delta_1 > 0$, $\delta_3 > 0$ and $\delta_5 > 0$. The complement discrete component of G^{\sim} is generated by three involutive transformations of alternating sign in the sets $\{t, D, h\}$, $\{x\}$ and $\{u\}$.

The *kernel* of the maximal Lie invariance algebras of equations from class (1) coincides with the one-dimensional algebra $\langle \partial_t \rangle$.

All possible G^{\sim} -inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by ones adduced in Table 1, where

$$h^{1}(x) = \varepsilon \exp\left[\int \frac{q}{x^{2} + p} dx\right]; \quad p \in \{-1, 0, 1\}, \ \varepsilon = \pm 1, \ \alpha \in \{0, 1\} \mod G^{\sim}; \quad n, \ q \neq 0.$$

Table 1. Results of group classification

N	D(u)	h(x)	Basis of A^{\max}
1	\forall	\forall	∂_t
2	\forall	1	$\partial_t, \ \partial_x$
3	\forall	x^{-2}	$\partial_t, \ 2t\partial_t + x\partial_x$
4	u^n	εx^q	∂_t , $-qnt\partial_t + nx\partial_x + (q+2)u\partial_u$
5	u^n	εe^x	$\partial_t, -nt\partial_t + n\partial_x + u\partial_u$
6	$u^{-4/3}$	$h^1(x)$	∂_t , $-4qt\partial_t + 4(x^2+p)\partial_x - 3(4x+q)u\partial_u$
7	\forall	0	$\partial_t, \ \partial_x, \ 2t\partial_t + x\partial_x$
8	$(u+1)^{-1}$	ε	$\partial_t, \ \partial_x, \ e^{\varepsilon t}\partial_t + \varepsilon e^{\varepsilon t}(u+1)\partial_u$
9	e^u	0	$\partial_t, \ \partial_x, \ 2t\partial_t + x\partial_x, \ x\partial_x + 2\partial_u$
10	$u^n, n \neq -\frac{4}{3}$	ε	∂_t , ∂_x , $e^{-\varepsilon nt}(\partial_t + \varepsilon u \partial_u)$, $nx\partial x + 2u\partial_u$
11	$(u+\alpha)^n, \ n \neq -\frac{4}{3}$	0	∂_t , ∂_x , $2t\partial_t + x\partial_x$, $nx\partial_x + 2(u+\alpha)\partial_u$
12	$u^{-4/3}$	ε	∂_t , ∂_x , $e^{\frac{4}{3}\varepsilon t}(\partial_t + \varepsilon u\partial_u)$, $2x\partial_x - 3u\partial_u$, $x^2\partial_x - 3xu\partial_u$
13	$(u\!+\!\alpha)^{-4/3}$	0	∂_t , ∂_x , $2t\partial_t + x\partial_x$, $2x\partial_x - 3(u+\alpha)\partial_u$, $x^2\partial_x - 3x(u+\alpha)\partial_u$

The Case 6 was missed in [11] and in subsequent papers on the subject. The parameterfunction h^1 equals to the following functions depending on values of p:

$$p=-1$$
: $h^1=arepsilon\left|rac{x-1}{x+1}
ight|^{q/2}, \quad p=0$: $h^1=arepsilon e^{-q/x}, \quad p=1$: $h^1=arepsilon e^{q\arctan x}$.

Additionally we can assume $q = -1 \mod G^{\sim}$ if p = 0.

Some cases from Table 1 are equivalent with respect to point transformations which obviously do not belong to G^{\sim} . These transformations are called *additional equivalence transformations* and lead to simplification of further application of group classification results. The pairs of point-equivalent extension cases and the corresponding additional equivalence transformations are

$$6_{p=0} \to 5_{\tilde{n}=-4/3} \colon \quad \tilde{t} = t, \ \tilde{x} = x^{-1}, \ \tilde{u} = x^{3}u;$$

$$6_{p=-1} \to 4_{\tilde{n}=-4/3, \, \tilde{q}=q/2} \colon \quad \tilde{t} = t, \ \tilde{x} = \frac{x-1}{x+1}, \ \tilde{u} = 2^{-3/2}(x+1)^{3}u;$$

$$11_{\alpha \neq 0} \to 11_{\alpha=0}, \ 13_{\alpha \neq 0} \to 13_{\alpha=0} \ (n = -\frac{4}{3}) \colon \quad \tilde{t} = t, \ \tilde{x} = x, \ \tilde{u} = u + \alpha;$$

$$10 \to 11_{\alpha=0}, \ 12 \to 13_{\alpha=0} \ (n = -\frac{4}{3}) \colon \quad \tilde{t} = \frac{1}{\varepsilon n} e^{\varepsilon nt}, \ \tilde{x} = x, \ \tilde{u} = e^{-\varepsilon t}u.$$

The latter transformation was adduced e.g. in [5]. Case 6 with p=1 is reduced to Case 4 only over the complex field. Note also that Case 8 is reduced by the similar transformation

$$\tilde{t} = -\frac{1}{\varepsilon}e^{-\varepsilon t}, \ \tilde{x} = x, \ \tilde{u} = e^{-\varepsilon t}(u+1)$$

to the equation $\tilde{u}_{\tilde{t}} = (\tilde{u}^{-1}\tilde{u}_{\tilde{x}})_{\tilde{x}} - \varepsilon$ which is not in the class under consideration.

There are no other additional equivalence transformations in class (1). Therefore, up to point equivalence, possible cases of extension of maximal Lie invariance algebras are exhausted by Cases 1–5, $6_{p=1}$, 7–9, $11_{\alpha=0}$ and $13_{\alpha=0}$.

The singularity of the diffusion coefficient $D=u^{-4/3}$ with a number of different values of h admitting extensions of Lie invariance algebra can be explained in the framework of conditional equivalence groups. The equivalence group is extended under the condition $D=u^{-4/3}$. More precisely, the equivalence group G_1^{\sim} of the subclass of equations (1) with $D=u^{-4/3}$ is formed by the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \frac{\delta_3 x + \delta_4}{\delta_5 x + \delta_6}, \quad \tilde{u} = \pm \delta_1 (\delta_5 x + \delta_6)^3 u, \quad \tilde{h} = \delta_1^{-1} h,$$

where δ_i , i = 1, ..., 6, are arbitrary constants, $\delta_1 > 0$ and $\delta_3 \delta_6 - \delta_4 \delta_5 = \pm 1$. G_1^{\sim} is a non-trivial conditional equivalence group of class (1). Two first additional equivalence transformations belong to G_1^{\sim} .

Another example of a conditional equivalence group in class (1) arises under the condition h=0. The equivalence group G^{\sim} of the whole class is then extended with translations with respect to u, i.e. the complete equivalence group G_2^{\sim} of nonlinear diffusion equations (h=0) is formed by the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4, \quad \tilde{u} = \delta_5 u + \delta_6, \quad \tilde{D} = \delta_1^{-1} \delta_3^2 D,$$

where δ_i , i = 1, ..., 6, are arbitrary constants, $\delta_1 \delta_3 \delta_5 \neq 0$. The third additional equivalence transformation belongs to G_2^{\sim} .

The subclass of equations (1) with h being a constant admits an extension of generalized equivalence group. The prefix "generalized" means that transformations of the variables t, x and u can depend on arbitrary elements [8]. The associated generalized equivalence group G_3^{\sim} is generated by transformations from G^{\sim} and the latter kind of additional equivalence transformations, where ε is replaced by h.

Knowledge on conditional equivalence groups allows us to describe the set of admissible (from-preserving) transformations in class (1) completely. See e.g. [15] and references therein.

Note also that the subclass of equations (1) possessing nontrivial local conservation laws is exhausted by ones with h is a constant. Then the corresponding space CL of conservation laws is two-dimensional. The conserved vectors and the characteristics of basis elements of CL are

$$(xe^{-ht}u, e^{-ht}(-xDu_x + \int D)), xe^{-ht}$$
 and $(e^{-ht}u, -e^{-ht}Du_x), e^{-ht}$.

3 Similarity solutions

Cases 7–13 of Table 1 are presented by 'constant coefficient' diffusion–reaction equations. Moreover, all of these cases either are usual nonlinear diffusion equations or can be reduced to them by additional equivalence transformations. Exact solutions of 'constant coefficient' diffusion– reaction equations have been already investigated intensively. See for example, [3, 4, 5, 14]. That is why we choose Cases 4–6 as representatives among truly nonlinear variable-coefficient fin equations, which are most interesting for Lie reduction.

As shown in the previous section, the equation

$$u_t = (u^{-4/3}u_x)_x + h^1(x)u (2)$$

(Case 6 of Table 1) admits the two-dimensional (non-commutative) Lie invariance algebra $\mathfrak g$ generated by the operators

$$X_1 = \partial_t$$
, $X_2 = -4qt\partial_t + 4(x^2 + p)\partial_x - 3(4x + q)u\partial_u$.

A complete list of inequivalent non-zero subalgebras of \mathfrak{g} is exhausted by the algebras $\langle X_1 \rangle$, $\langle X_2 \rangle$ and $\langle X_1, X_2 \rangle$.

Lie reduction of equation (2) to an algebraic equation can be made with the two-dimensional subalgebra $\langle X_1, X_2 \rangle$ which coincides with the whole algebra \mathfrak{g} . The associated ansatz and the reduced algebraic equation have the form

6.0.
$$\langle X_1, X_2 \rangle$$
: $u = C(x^2 + p)^{-3/2} (h^1(x))^{-3/4}, \quad C^{4/3} = \frac{3}{16} (q^2 + 16p).$

Substituting the solution $C=\pm\frac{3^{3/4}}{8}(q^2+16p)^{3/4}$ of the reduced algebraic equation into the ansatz, we construct the exact solution

$$u = \pm \frac{3^{3/4}}{8} (q^2 + 16p)^{3/4} (x^2 + p)^{-3/2} (h^1(x))^{-3/4}$$

of equation (2).

The ansatzes and reduced equations corresponding to the one-dimensional subalgebras from the optimal system are following:

6.1.
$$\langle X_1 \rangle$$
: $u = (\varphi(\omega))^{-3}, \ \omega = x; \ 3\varphi_{\omega\omega} = h^1(\omega)\varphi^{-3};$

6.2.
$$\langle X_2 \rangle$$
: $u = ((x^2 + p)^{1/2} (h^1(x))^{1/4} \varphi(\omega))^{-3}, \ \omega = th^1(x);$
 $3q^2 \omega^2 \varphi_{\omega\omega} + \frac{9}{2} q^2 \omega \varphi_{\omega} - 3\varphi^{-4} \varphi_{\omega} + \frac{3}{16} (q^2 + 16p)\varphi - \varepsilon \varphi^{-3} = 0.$

The obtained reduced equations obviously have partial exact solutions which lead to the above exact solution of equation (2) and can be constructed via reduction to algebraic equations. The problem is to find some different solutions. We are only able to reduce the order of equation 6.1. Namely, in the variables

$$y = (\omega^2 + p)^{-1/2} (h^1(\omega))^{-1/4} \varphi, \quad \psi = (\omega^2 + p)^{-1/2} (h^1(\omega))^{-1/4} ((\omega^2 + p)\varphi_\omega - \omega\varphi)$$

constructed with the induced symmetry operator $4(\omega^2 + p)\partial_{\omega} + (4\omega + q)\varphi\partial_{\varphi}$ equation 6.1 takes the form $(4\psi - qy)\psi_y + q\psi + 4py = \frac{4}{3}\varepsilon y^{-3}$. A better way for construction of exact solutions for equations of Case 6 with $p \leq 0$ is to map them to Cases 4 and 5 with additional equivalence transformations and then study the latter cases.

Let us review results on Lie reduction of Cases 4 and 6. For each from these cases we denote the basis symmetry operators adduced in Table 1 by X_1 and X_2 . Structure and list of

inequivalent subalgebras of the Lie invariance algebras are the same as ones in Case 6. The associated ansatzes and reduced equations have the form $(\varepsilon' = \operatorname{sign} t)$:

4.0.
$$\langle X_1, X_2 \rangle$$
: $u = Cx^{\frac{q+2}{n}}, (q+2)(nq+n+q+2)C^{n+1} + \varepsilon n^2C = 0;$

4.1.
$$\langle X_1 \rangle$$
: $u = (\varphi(\omega))^{\frac{1}{n+1}}, \ \omega = x, \quad \varphi_{\omega\omega} + \varepsilon(n+1)\omega^q \varphi^{\frac{1}{n+1}} = 0 \quad \text{if } n \neq -1;$
 $u = \exp(\varphi(\omega)), \ \omega = x, \quad \varphi_{\omega\omega} + \varepsilon\omega^q e^{\varphi} = 0 \quad \text{if } n = -1;$

4.2.
$$\langle X_2 \rangle$$
: $u = |t|^{-\frac{q+2}{nq}} \varphi(\omega), \ \omega = |t|^{\frac{1}{q}} x, \quad (\varphi^n \varphi_\omega)_\omega + \varepsilon \omega^q \varphi + \varepsilon' \frac{q+2}{nq} \varphi - \varepsilon' \frac{1}{q} \omega \varphi_\omega = 0;$

5.0.
$$\langle X_1, X_2 \rangle$$
: $u = Ce^{\frac{x}{n}}, (n+1)C^{n+1} + \varepsilon n^2 C = 0;$

5.1.
$$\langle X_1 \rangle$$
: $u = (\varphi(\omega))^{\frac{1}{n+1}}, \ \omega = x, \quad \varphi_{\omega\omega} + \varepsilon(n+1)e^{\omega}\varphi^{\frac{1}{n+1}} = 0 \quad \text{if } n \neq -1;$
 $u = \exp(\varphi(\omega)), \ \omega = x, \quad \varphi_{\omega\omega} + \varepsilon e^{\varphi + \omega} = 0 \quad \text{if } n = -1;$

5.2.
$$\langle X_2 \rangle$$
: $u = |t|^{-\frac{1}{n}} \varphi(\omega), \ \omega = x + \ln|t|, \ (\varphi^n \varphi_\omega)_\omega + \varepsilon e^\omega \varphi + \varepsilon' n^{-1} \varphi - \varepsilon' \varphi_\omega = 0.$

Reduction to algebraic equations gives the following solutions of the initial equations:

4.
$$u = \left(-\frac{q+2}{\varepsilon n^2}(nq+n+q+2)\right)^{-\frac{1}{n}} x^{\frac{q+2}{n}};$$

5.
$$u = \left(-\frac{n+1}{\varepsilon n^2}\right)^{-\frac{1}{n}} e^{\frac{x}{n}}$$
.

There are Emden–Fowler and Lane–Emden equations and their different modifications among the reduced ordinary differential equations. Solutions of these equations are known for a number of parameter values (see e.g. [13]). As a result, classes of exact solutions can be constructed for fin equations corresponding to Cases 4 and 5 of Table 1 for a wide set of the parameters n and q.

4 On nonclassical symmetries

We also study conditional (nonclassical) symmetries of equations from class (1). As well-known, the operators with the vanishing coefficient of ∂_t gives so-called 'no-go' case in study of conditional symmetries of an arbitrary (1 + 1)-dimensional evolution equation since the problem on their finding is reduced to a single equation which is equivalent to the initial one (see e.g. [16]). Since the determining equation has more independent variables and, therefore, more freedom degrees, it is more convenient often to guess a simple solution or a simple ansatz for the determining equation, which can give a parametric set of complicated solutions of the initial equation. For example, the fin equation

$$u_t = (u^{-1}u_x)_x + xu \tag{3}$$

is conditionally invariant with respect to the operator $\partial_x + tu\partial_u$. The associated ansatz $u = e^{tx}\varphi(\omega)$, $\omega = t$, reduces equation (3) to the equation $\varphi_\omega = 0$, i.e. $u = Ce^{tx}$ is its non-Lie exact solution which can be additionally extended with symmetry transformations.

It is known that non-linear diffusion equations (i.e. equations (1) with h = 0) possess conditional symmetry operators which have non-vanishing coefficients of ∂_t and are inequivalent to Lie invariance operators only in case of exponential diffusion coefficients (Case 9 of Table 1 and equivalent equations). Moreover, solutions associated with such conditional symmetry operators are still Lie invariant. The opposite situation is for $h \neq 0$. There exist conditional symmetry operators of equations (1), which have non-vanishing coefficients of ∂_t , are inequivalent to Lie

invariance operators and even lead to truly non-Lie exact solutions. Thus, consider again equation (3). It admits also the conditional symmetry operator $\partial_t + xu\partial_u$. The associated ansatz $u = e^{tx}\varphi(\omega)$, $\omega = x$, reduces equation (3) to the equation $(\varphi^{-1}\varphi_{\omega})_{\omega} = 0$. The general solution $\varphi = C_1 e^{C_2 x}$ of the reduced equations gives a solution of (3), which is simplified to the above constructed one with symmetry transformations.

Exhaustive description of nonclassical symmetry operators of equations (1) will be a subject of a forthcoming paper.

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