Applied Mathematics Letters 22 (2009) 906–909

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Closed-form formulae for the derivatives of trigonometric functions at rational multiples of π

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ARTICLE INFO

Article history:
Received 27 May 2008
Received in revised form 1 July 2008
Accepted 2 July 2008

Keywords:
Trigonometric functions
Hurwitz zeta function
Legendre chi function
Lerch zeta function
Bernoulli polynomials
Euler polynomials

ABSTRACT

In this sequel to our recent note [D. Cvijović, Values of the derivatives of the cotangent at rational multiples of π, Appl. Math. Lett. http://dx.doi.org/10.1016/j.aml.2008.03.013] it is shown, in a unified manner, by making use of some basic properties of certain special functions, such as the Hurwitz zeta function, Lerch zeta function and Legendre chi function, that the values of all derivatives of four trigonometric functions at rational multiples of π can be expressed in closed form as simple finite sums involving the Bernoulli and Euler polynomials. In addition, some particular cases are considered.

1. Introduction

Recently, Adamchik [1, p. 4, Eq. 26] solved completely a long-standing problem of finding a closed-form expression for the higher derivatives of the cotangent function [1–3]. In a recent note [4], we have deduced closed-form expressions for the values of the higher derivatives of the cotangent function at rational multiples of π, which are considerably simpler than those found by Kölbig [3, Theorem 4].

In this sequel, by using different arguments, we provide another more direct and shorter proof of this result (Theorem 1(i)) and also derive corresponding (and previously unknown) simple closed-form formulae for the cosecant, tangent and secant functions (Theorems 1(ii) and 2). More specifically, we show in a unified manner (by making use of the properties of certain special functions, such as the Hurwitz zeta function, Lerch zeta function and Legendre chi function) that the values of all derivatives of these trigonometric functions at rational multiples of π can be expressed as finite sums involving the Bernoulli and Euler polynomials.

2. Statement and proof of the results

In what follows, we set $\imath := \sqrt{-1}$, an empty sum is interpreted as zero, and the classical Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$, are defined by [5, pp. 59 and 63]:

$$\frac{t\,\mathrm{e}^{tx}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi) \quad \text{and} \quad \frac{2\,\mathrm{e}^{tx}}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi) \tag{2.1}$$
$$(n \in \mathbb{N}_0 := \mathbb{N} \cup 0;\ \mathbb{N} := 1, 2, 3, \ldots).$$

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doi:10.1016/j.aml.2008.07.019

Our main results are as follows.

Theorem 1. *If $n, p, q \in \mathbb{N}$ and p and q are such that $1 \le p < q$, then, in terms of the Bernoulli polynomials $B_n(x)$, we have:*

(i) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \cot(\pi x)\Big|_{x=\frac{p}{q}} = \imath^{n+1} \, \frac{2\,(2\pi q)^n}{n+1} \sum_{\alpha=1}^{q} \mathrm{e}^{\frac{2\pi \imath \alpha p}{q}} B_{n+1}\left(\frac{\alpha}{q}\right);$$

(ii) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \csc(\pi x)\Big|_{x=\frac{p}{q}} = \imath^{n+1} \, \frac{2\,(2\pi q)^n}{n+1} \sum_{\alpha=1}^{q} \mathrm{e}^{\frac{\pi \imath (2\alpha-1) p}{q}} B_{n+1}\left(\frac{2\alpha-1}{2\,q}\right).$$

Theorem 2. *If $n, p \in \mathbb{N}, q \in \mathbb{N} \setminus \{1, 2\}$ and p and q are such that $1 \le p < q/2$, then, in terms of the Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$, we have:*

(i) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \tan(\pi x)\Big|_{x=\frac{p}{q}} = \imath^{n+1}\,(2\pi q)^n \sum_{\alpha=1}^{q} (-1)^{\alpha-1} \mathrm{e}^{\frac{2\pi \imath \alpha p}{q}} \begin{cases} -E_n\left(\dfrac{\alpha}{q}\right) & q \text{ is odd} \\ \dfrac{2}{n+1}\, B_{n+1}\left(\dfrac{\alpha}{q}\right) & q \text{ is even}; \end{cases}$$

(ii) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \sec(\pi x)\Big|_{x=\frac{p}{q}} = \imath^{n}(2\pi q)^n \cdot \sum_{\alpha=1}^{q} (-1)^{\alpha-1} \mathrm{e}^{\frac{\pi \imath (2\alpha-1) p}{q}} \begin{cases} E_n\left(\dfrac{2\alpha-1}{2\,q}\right) & q \text{ is odd} \\ -\dfrac{2}{n+1}\, B_{n+1}\left(\dfrac{2\alpha-1}{2\,q}\right) & q \text{ is even}. \end{cases}$$

Remark 1. As the following examples show, the above formulae can be rewritten in a somewhat different form:

$$\csc(\pi x)^{(2n-1)}\big|_{x=\frac{p}{q}} = \frac{(-1)^n (2\pi q)^{2n-1}}{n} \sum_{\alpha=1}^{q} B_{2n}\left(\frac{2\alpha-1}{2\,q}\right) \cos\left(\frac{\pi\,(2\alpha-1)p}{q}\right),$$
$$\csc(\pi x)^{(2n)}\big|_{x=\frac{p}{q}} = \frac{(-1)^{n-1}\, 2(2\pi q)^{2n}}{2n+1} \sum_{\alpha=1}^{q} B_{2n+1}\left(\frac{2\alpha-1}{2\,q}\right) \sin\left(\frac{\pi\,(2\alpha-1)p}{q}\right).$$

Further, it should be noted that a number of simpler expressions could be obtained by specializing the results for $x = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$. For instance:

$$\csc(\pi x)^{(2n-1)}\big|_{x=\frac{1}{2}} = 0, \qquad \csc(\pi x)^{(2n)}\big|_{x=\frac{1}{2}} = (-1)^{n-1} \frac{4(4\pi)^{2n}}{2n+1} B_{2n+1}\left(\frac{1}{4}\right);$$
$$\cot(\pi x)^{(2n-1)}\big|_{x=\frac{1}{4}} = \cot(\pi x)^{(2n-1)}\big|_{x=\frac{3}{4}} = (-1)^n (4\pi)^{2n-1} (2^{2n}-1) \frac{B_{2n}(0)}{n};$$
$$\cot(\pi x)^{(2n)}\big|_{x=\frac{1}{4}} = -\cot(\pi x)^{(2n)}\big|_{x=\frac{3}{4}} = (-1)^{n-1} \frac{4(8\pi)^{2n}}{2n+1} B_{2n+1}\left(\frac{1}{4}\right).$$

Let $\zeta(s, a)$ and $\zeta^*(s, a)$ denote, respectively, the Hurwitz (or generalized) zeta function defined by [5, p. 96]:

$$\zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (a \notin \mathbb{Z}_0^- := \{0, -1, -2, -3, \ldots\};\ \Re(s) > 1) \tag{2.2}$$

and its alternating counterpart defined by (see e.g. [6, p. 761]):

$$\zeta^*(s, a) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s} \quad (a \notin \mathbb{Z}_0^-;\ \Re(s) > 0), \tag{2.3}$$

and observe that the following relationships hold [5, p. 85, Eq. 17]:

$$\zeta(1-n, a) = -\frac{B_n(a)}{n} \quad (n \in \mathbb{N}) \tag{2.4}$$

and [6, p. 761, Eq. 2.3]

$$\zeta^*(-n, a) = \frac{1}{2}\, E_n(a) \quad (n \in \mathbb{N}_0). \tag{2.5}$$

Proof of Theorem 1. Our proof requires the use of the Lerch zeta function and the Legendre chi function which are defined, respectively, by the series [5, p. 89, Eq. 7]:

$$\ell_s(\xi) := \sum_{k=1}^{\infty} \frac{e^{2\pi\imath k\xi}}{k^s} \quad (\xi \in \mathbb{R};\, \Re(s) > 1), \tag{2.6}$$

and (see, for instance, [7]):

$$\chi_s(z) := \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^s} \quad (|z| \le 1;\, \Re(s) > 1), \tag{2.7}$$

and their meromorphic continuations over the whole s-plane. First, we shall show that

$$\ell_0(\xi) = -\frac{1}{2} + \frac{\imath}{2}\cot(\pi\xi), \qquad \ell_{1-n}(\xi) = \frac{\imath}{2(2\pi\imath)^{n-1}} \frac{d^{n-1}}{d\xi^{n-1}} \cot(\pi\xi) \quad (\xi \in \mathbb{R}\setminus\mathbb{Z};\ n \in \mathbb{N}\setminus\{1\}) \tag{2.8}$$

and

$$\chi_{1-n}(e^{\pi\imath\xi}) = \frac{\imath}{2(\pi\imath)^{n-1}} \frac{d^{n-1}}{d\xi^{n-1}} \csc(\pi\xi) \quad (\xi \in \mathbb{R}\setminus\mathbb{Z};\ n \in \mathbb{N}). \tag{2.9}$$

To prove (2.8) note that

$$\frac{\partial}{\partial\xi}\,\ell_s(\xi) = 2\pi\imath\,\ell_{s-1}(\xi), \tag{2.10}$$

which, in turn, follows from (2.6) for $\Re(s) > 2$ and by analytic continuation for all s. The definition in (2.6) also yields $\ell_1(\xi) = -\log\left(1 - e^{2\pi\imath\xi}\right)$ $(\xi \in \mathbb{R}\setminus\mathbb{Z})$ and from this we obtain $\ell_0(\xi)$ by (2.10). Using (2.10) repeatedly with initial value $\ell_0(\xi)$ leads to $\ell_{1-n}(\xi)$ given by (2.8). Likewise, we have (2.9) by making use of

$$\frac{\partial}{\partial\xi}\,\chi_s(e^{\pi\imath\xi}) = \pi\imath\,\chi_{s-1}(e^{\pi\imath\xi}) \tag{2.11}$$

and

$$\chi_0(e^{\pi\imath\xi}) = \frac{\imath}{2}\csc(\pi\xi) \quad (\xi \in \mathbb{R}\setminus\mathbb{Z}). \tag{2.12}$$

Second, from (2.6), we obtain (*cf.* [8, p. 1531])

$$\ell_s\left(\frac{p}{q}\right) = \sum_{k=0}^{\infty} \frac{e^{2\pi\imath(k+1)p/q}}{(k+1)^s} = \sum_{\alpha=0}^{q-1}\sum_{k=0}^{\infty} \frac{e^{2\pi\imath kp}\, e^{2\pi\imath(\alpha+1)p/q}}{q^s(k+(\alpha+1)/q)^s},$$

so that, in view of the definition in (2.2), we have (see [8, p. 1530, Eq. 8(a)])

$$\ell_s\left(\frac{p}{q}\right) = \frac{1}{q^s}\sum_{\alpha=1}^{q} \zeta\left(s, \frac{\alpha}{q}\right) e^{\frac{2\pi\imath\alpha p}{q}} \quad (p, q \in \mathbb{N};\, p = 1, \ldots, q). \tag{2.13}$$

Similarly, starting from (2.7), we find that

$$\chi_s\left(e^{\frac{\pi\imath p}{q}}\right) = \frac{1}{(2q)^s}\sum_{\alpha=1}^{q} \zeta\left(s, \frac{2\alpha-1}{2q}\right) e^{\frac{\pi\imath(2\alpha-1)p}{q}} \quad (p, q \in \mathbb{N};\, p = 1, \ldots, q). \tag{2.14}$$

It should be noted that (2.13) and (2.14) are derived for $\Re(s) > 1$ but, by the principle of analytic continuation, they hold true for any complex s, $s \ne 1$.

Lastly, set $s = 1 - n$ $(n \in \mathbb{N}\setminus\{1\})$. Part (i) now follows at once by applying (2.13) in conjunction with (2.8) and (2.4). Also, Eq. (2.14), in conjunction with (2.9) and (2.4), gives part (ii). □

Proof of Theorem 2. Instead of the functions $\ell_s(\xi)$ and $\chi_s(z)$ used above, we now introduce the functions $\ell_s^*(\xi)$ and $\chi_s^*(z)$ by means of the series

$$\ell_s^*(\xi) := \sum_{k=1}^{\infty} (-1)^{k-1}\frac{e^{2\pi\imath k\xi}}{k^s} \quad (\xi \in \mathbb{R};\, \Re(s) > 0) \tag{2.15}$$

and

$$\chi_s^*(z) := \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)^s} \quad (|z| \leq 1; \Re(s) > 0), \tag{2.16}$$

and their meromorphic continuations over the whole s-plane.

The proof follows precisely along the same lines as that of Theorem 1: We first show that

$$\ell_0^*(\xi) = \frac{1}{2} + \frac{\imath}{2}\tan(\pi\xi), \qquad \ell_{1-n}^*(\xi) = \frac{\imath}{2(2\pi\imath)^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\tan(\pi\xi)$$
$$\left(\xi \in \mathbb{R} \setminus \left\{(2k+1)\frac{1}{2} \,|k \in \mathbb{Z}\right\};\ n \in \mathbb{N} \setminus \{1\}\right) \tag{2.17}$$

and

$$\chi_{1-n}^*(\mathrm{e}^{\pi\imath\xi}) = \frac{\imath}{2(\pi\imath)^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\sec(\pi\xi) \quad \left(\xi \in \mathbb{R} \setminus \left\{(2k+1)\frac{1}{2}|k \in \mathbb{Z}\right\};\ n \in \mathbb{N}\right). \tag{2.18}$$

We next establish the following formulae valid for $p, q \in \mathbb{N}$ $(p = 1, \ldots, q)$:

$$\ell_s^*\left(\frac{p}{q}\right) = \frac{1}{q^s}\sum_{\alpha=1}^{q}(-1)^{\alpha-1}\mathrm{e}^{\frac{2\pi\imath\alpha p}{q}} \begin{cases} \zeta^*\left(s, \dfrac{\alpha}{q}\right) & q \text{ is odd} \\ \zeta\left(s, \dfrac{\alpha}{q}\right) & q \text{ is even}, \end{cases} \tag{2.19}$$

$$\chi_s^*\left(\frac{p}{q}\right) = \frac{1}{q^s}\sum_{\alpha=1}^{q}(-1)^{\alpha-1}\mathrm{e}^{\frac{\pi\imath(2\alpha-1)p}{q}} \begin{cases} \zeta^*\left(s, \dfrac{2\alpha-1}{2q}\right) & q \text{ is odd} \\ \zeta\left(s, \dfrac{2\alpha-1}{2q}\right) & q \text{ is even} \end{cases} \tag{2.20}$$

and finally set $s = 1 - n$ $(n \in \mathbb{N} \setminus \{1\})$. It is straightforward to see that part (i) is obtained by (2.19) together with (2.15) and either (2.4) or (2.5). Also, Eq. (2.20), together with (2.16) and either (2.4) or (2.5), yields part (ii). □

Acknowledgements

The author would like to thank the Associate Editor Nicolas Papamichael and the two anonymous referees of this journal for their helpful comments and valuable suggestions. This work was financially supported by the Ministry of Science of the Republic of Serbia under research project number 144004.

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