On the clique number of integral circulant graphs [*]

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Abstract

The concept of gcd-graphs is introduced by Klotz and Sander, which arises as a generalization of unitary Cayley graphs. The gcd-graph $X_n(d_1, ..., d_k)$ has vertices $0, 1, \ldots, n-1$, and two vertices x and y are adjacent iff $\gcd(x-y, n) \in D = \{d_1, d_2, ..., d_k\}$. These graphs are exactly the same as circulant graphs with integral eigenvalues characterized by So. In this paper we deal with the clique number of integral circulant graphs and investigate the conjecture proposed in [6] that clique number divides the number of vertices in the graph $X_n(D)$. We completely solve the problem of finding clique number for integral circulant graphs with exactly one and two divisors. For $k \geqslant 3$, we construct a family of counterexamples and disprove the conjecture in this case.

1 Introduction

Integral circulant graphs have been proposed as potential candidates for modeling quantum spin networks that might enable the perfect state transfer between antipodal sites in a network. Motivated by this, Saxena, Severini and Shraplinski [7] studied some properties of integral circulant graphs — bounds for number of vertices and diameter, bipartiteness and perfect state transfer. Stevanović, Petković and Bašić [8] improved the previous upper bound for diameter and showed that the diameter of these graphs is at most $O(\ln \ln n)$. Circulant graphs are important class of interconnection networks in parallel and distributed computing (see [4]).

Various properties of unitary Cayley graphs as a subclass of integral circulant graphs were investigated in some recent papers. In the work of Berrizbeitia and Giudici [1] and in the later paper of Fuchs [2], some lower and upper bounds for the longest induced cycles were given. Stevanović, Petković and Bašić [9] established a characterization of integral circulant graphs which allows perfect state transfer and proved that there is no perfect state transfer in the class of unitary Cayle graphs except for hypercubes K_2 and C_4. Klotz and Sander [6] determined the diameter, clique number, chromatic number and eigenvalues of unitary Cayley graphs. The latter group of authors proposed a generalization of unitary Cayley graphs named *gcd-graphs* and proved that they have to be integral. Integral circulant graphs were characterized by So [10] — a circulant graph is integral if and only if it is a gcd-graph. This is the solution to the second proposed question in [6].

Motivated by the third concluding problem in [6], we investigate the clique number of integral circulant graphs $X_n(D)$, where $D = \{d_1, d_2, \ldots, d_k\}$ and the numbers d_i are proper divisors of n. In Section 2 we extend the result of clique number and chromatic number for unitary Cayley graphs that are not connected. In Section 3 we completely characterize the clique number for integral circulant graphs with exactly two divisors $X_2(d_1, d_2)$. In previous cases when $k = 1$ or $k = 2$, the conjecture that the clique number of a graph $X_n(d_1, d_2, \ldots, d_k)$ must divide n is supported by Lemma 2.4 and Theorem 3.6. In Section 4 we refute the conjecture for $k \geqslant 3$ by constructing a class of counterexamples for $k = 3$ and $k = 4$. In Section 5 we propose a simple lower and upper bound for $\omega(X_n(d_1, d_2, \ldots, d_k))$, where k is an arbitrary natural number.

2 Preliminaries

Let us recall that for a positive integer n and subset $S \subseteq \{0, 1, 2, \ldots, n-1\}$, the circulant graph $G(n, S)$ is the graph with n vertices, labeled with integers modulo n, such that each vertex i is adjacent to $|S|$ other vertices $\{i + s \pmod{n} \mid s \in S\}$. The set S is called a symbol of $G(n, S)$. As we will consider only undirected graphs, we assume that $s \in S$ if and only if $n - s \in S$, and therefore the vertex i is adjacent to vertices $i \pm s \pmod{n}$ for each $s \in S$.

Recently, So [10] has characterized integral circulant graphs. Let

$$G_n(d) = \{k \mid gcd(k, n) = d, \ 1 \le k < n\}$$

be the set of all positive integers less than n having the same greatest common divisor d with n. Let D_n be the set of positive divisors d of n, with $d \le \frac{n}{2}$.

Theorem 2.1 ([10]) *A circulant graph $G(n, S)$ is integral if and only if*

$$S = \bigcup_{d \in D} G_n(d)$$

for some set of divisors $D \subseteq D_n$.

Let Γ be a multiplicative group with identity e. For $S \subset \Gamma$, $e \notin S$ and $S^{-1} = \{s^{-1} \mid s \in S\} = S$, Cayley graph $X = Cay(\Gamma, S)$ is the undirected graph having vertex set $V(X) = \Gamma$ and edge set $E(X) = \{\{a, b\} \mid ab^{-1} \in S\}$. For a positive integer $n > 1$ the unitary Cayley graph $X_n = Cay(Z_n, U_n)$ is defined by the additive group of the ring Z_n of integers modulo n and the multiplicative group $U_n = Z_n^*$ of its units.

Let D be a set of positive, proper divisors of the integer $n > 1$. Define the gcd-graph $X_n(D)$ to have vertex set $Z_n = \{0, 1, \ldots, n-1\}$ and edge set

$$E(X_n(D)) = \{\{a, b\} \mid a, b \in Z_n, \ gcd(a - b, n) \in D\}.$$

If $D = \{d_1, d_2, \ldots, d_k\}$, then we also write $X_n(D) = X_n(d_1, d_2, \ldots, d_k)$; in particular $X_n(1) = X_n$. Throughout the paper, we let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$ are distinct primes, and $\alpha_i \ge 1$. Also $f(n)$ represents the smallest prime divisor of n. By Theorem 2.1 we obtain that integral circulant graphs are Cayley graphs of the additive group of Z_n with respect to the Cayley set $S = \bigcup_{d \in D} G_n(d)$ and thus they are gcd-graphs. From Corollary 4.2 in [4], the graph $X_n(D)$ is connected if and only if $gcd(d_1, d_2, \ldots, d_k) = 1$.

Theorem 2.2 *If $d_1, d_2, \ldots, d_k$ are divisors of n, such that the greatest common divisor $gcd(d_1, d_2, \ldots, d_k)$ equals d, then the graph $X_n(d_1, d_2, \ldots, d_k)$ has exactly d isomorphic connected components of the form $X_{n/d}(\frac{d_1}{d}, \frac{d_2}{d}, \ldots, \frac{d_k}{d})$.*

Proof: If $gcd(d_1, d_2, \ldots, d_k) = d > 1$, in the graph $X_n(D)$ there are at least d connected components. We will prove that the subgraph induced by vertices $\{r, d + r, 2d + r, \ldots, (\frac{n}{d} - 1) \cdot d + r\}$ is connected, by constructing a path from vertex r to every other vertex in this component.

The Bézout's identity states that for integers a and b one can find integers x and y, such that $ax + by = gcd(a, b)$. By induction, we will prove that there are integers $x_1, x_2, \ldots, x_k$ such that $x_1 \cdot d_1 + x_2 \cdot d_2 + \ldots + x_k \cdot d_k = d$. For $k > 2$, we can find integers $y_1, y_2, \ldots y_{k-1}$ such that $y_1 \cdot d_1 + y_2 \cdot d_2 + \ldots + y_{k-1} \cdot d_{k-1} = gcd(d_1, d_2, \ldots, d_{k-1})$. Applying Bézout's identity on numbers $gcd(d_1, d_2, \ldots, d_{k-1})$ and d_k, it follows that

$$d = x \cdot gcd(d_1, d_2, \ldots, d_{k-1}) + y \cdot d_k = xy_1 \cdot d_1 + xy_2 \cdot d_2 + \ldots + xy_{k-1} \cdot d_{k-1} + y \cdot d_k.$$

Furthermore, for $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, n - 1$, let

$$H_n(d_i) = \{h \mid 0 \le h < n, \ h \equiv 0 \pmod{d_i}\} \subseteq Z_n,$$

and let $j + H_n(d_i)$ denote the subgraph of $X_n(D)$ with the vertex set $\{j + h \mid h \in H_n(d_i)\}$. Two vertices $j + h_1$ and $j + h_2$ are adjacent if $h_2 - h_1 \in G_n(d_i)$. Thus, from vertex r we can walk to every vertex with label $r + k \cdot d$, where $0 \le k \le \frac{n}{d}$, passing through subgraphs $H_n(d_1), H_n(d_2), \ldots, H_n(d_k)$ consecutively. $\square$

In [6] authors proved the following result for unitary Cayley graphs.

Theorem 2.3 *If $D = \{1\}$ then $\chi(X_n) = \omega(X_n) = f(n)$.*

Consider the set $D = \{d\}$, where $d \ge 1$ is a divisor of n. The graph $X_n(d)$ has d connected components - the residue classes modulo d in $Z_n = \{0, 1, 2, \ldots, n-1\}$. The degree of every vertex is $\phi(\frac{n}{d})$ where $\phi(n)$ denotes the Euler phi function.

Lemma 2.4 *For the gcd-graph $X_n(d)$ it holds that:*

$$\chi(X_n(d)) = \omega(X_n(d)) = f\left(\frac{n}{d}\right).$$

Proof: Let $p = f(\frac{n}{d})$ be the smallest prime divisor of $\frac{n}{d}$. The vertices $0, d, 2d, \ldots, (p-1)d$ induce a clique in the graph $X_n(d)$, because the greatest common divisor of $d \cdot (a-b)$ and n equals d, for $0 < a, b < p$. Therefore, we have inequality $\chi(X_n(d)) \geqslant \omega(X_n(d)) \geqslant p$.

On the other hand, consider the component with the vertices $r, d+r, 2d+r, \ldots, (\frac{n}{d}-1)d+r$, for some $r \in Z_d$. Two vertices $d \cdot a + r$ and $d \cdot b + r$ are adjacent if and only if $gcd(a-b, \frac{n}{d}) = 1$, which is evidently true because $|a-b| < \frac{n}{d}$. From Theorem 2.3, we get that chromatic number in such component is the least prime dividing $\frac{n}{d}$. The same observation holds for every residue class modulo d, by Theorem 2.2. Thus, the chromatic number and the clique number in $X_n(d)$ are equal to $f(\frac{n}{d})$. $\qquad\square$

3 Clique number for $k = 2$

Let D be a two element set $D = \{d_1, d_2\}$, where $d_1 > d_2$. Let Q be the set of all prime divisors of n that does not divide d. The main result of this section is following theorem.

Theorem 3.1 *In the graph $X_n(d_1, d_2)$ we have:*

$$\omega\left(X_n(d_1, d_2)\right) = \begin{cases} \min\left(\min_{p \in Q} p, f(n) \cdot f\left(\frac{n}{d}\right)\right), & \text{if } d_2 = 1, \\ \omega(X_{\frac{n}{d_2}}(1, \frac{d_1}{d_2})), & \text{if } d_2 \mid d_1 \text{ and } d_2 > 1, \\ \max\left(f\left(\frac{n}{d_1}\right), f\left(\frac{n}{d_2}\right)\right), & \text{otherwise.} \end{cases}$$

According to the definition, the edge set of $X_n(d_1, d_2)$ is the union of the edge sets of graphs $X_n(d_1)$ and $X_n(d_2)$. We color the edges of the graph $X_n(d_1, d_2)$ with two colors: edge $\{a, b\}$ is blue if $gcd(a-b, n) = d_1$ and red if $gcd(a-b, n) = d_2$. Therefore by Lemma 2.4,

$$\omega\left(X_n(d_1, d_2)\right) \geqslant \max\left(f\left(\frac{n}{d_1}\right), f\left(\frac{n}{d_2}\right)\right). \tag{1}$$

3.1 Case $1 \in D$

Assume that $D = \{1, d\}$, where $d = p_1^{\beta_1} p_2^{\beta_2} \cdot \ldots \cdot p_k^{\beta_k}$. Let i be the first index such that $\beta_i < \alpha_i$ i.e. $f(\frac{n}{d}) = p_i$. By (1), we know that $\omega(X_n(1, d)) \geqslant p_i$.

Lemma 3.2 *In the graph $X_n(1, d)$ we have*

$$\omega(X_n(1, d)) \leqslant f(n) \cdot f\left(\frac{n}{d}\right).$$

Proof: Color the edges of the graph $X_n(1, d)$ with two colors. Let blue edges be those with $gcd(a-b, n) = d$ and red edges those with $gcd(a-b, n) = 1$. If we have two adjacent blue edges (a, b) and (a, c) then edge joining the vertices b and c must be blue, if exists. This follows from the fact that if $d \mid a-b$ and $d \mid a-c$, then d must divide $gcd(b-c, n)$.

Thus, any two maximal cliques composed only of blue edges are vertex disjoint. Now let $K_1, K_2, \ldots, K_x$ be all the maximal cliques of blue edges in a maximal clique C^* of $X_n(1, d)$. Then any edge joining two vertices in different cliques K_i and K_j is red. Furthermore, any vertex in $V(C^*) \setminus (V(K_1) \cup V(K_2) \cup \ldots \cup V(K_x))$ does not belong to any blue clique, which implies that $V(C^*) \setminus (V(K_1) \cup V(K_2) \cup \ldots \cup V(K_x))$ induces a clique composed only of red edges and is denoted by C. By Lemma 2.4 the order of K_i ($i \in \{1, 2, \ldots, x\}$) is at most $p_i = f(\frac{n}{d})$ and the order of C is at most p_1.

Let y be the size of clique C. If we choose one vertex from each clique with blue edges, then these x vertices with y vertices from clique C form a clique with red edges in the graph $X_n(1, d)$. Therefore, $x + y \leqslant p_1$. The size of every blue clique is bounded by p_i. The number of vertices in the maximal clique of $X_n(1, d)$ is

$$\sum_{j=1}^{x} |K_j| + |C| \leq x \cdot p_i + y = x \cdot (p_i - 1) + (x + y) \leq p_1 \cdot (p_i - 1) + p_1 = p_i \cdot p_1,$$

which means that the size of maximal clique is less than or equal to $f(n) \cdot f(\frac{n}{d})$. $\qquad\square$

Let R be the set of all prime divisors of d which also divide $\frac{n}{d}$. Let q be the least prime number from the set Q, if exists. Let number M be the product of all primes in both sets R and Q.
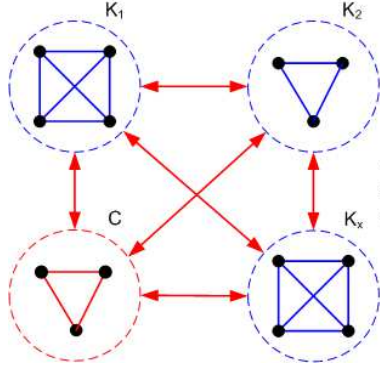
Figure 1: *The maximal clique in graph $X_n(1, d)$*

Lemma 3.3 *For an arbitrary divisor d of n, the following inequality holds:*

$$\omega(X_n(1, d)) \leqslant \min_{p \in Q} p = q.$$

Proof: Let p be an arbitrary prime number of n which does not divide d. If we assume that maximal clique has more than p vertices, then there must be two vertices a and b with the same residue modulo p. This means that $gcd(a - b, n)$ is divisible by p, and therefore is equal to neither 1 nor d. Therefore, we have $\omega(X_n(1, d)) \leqslant p$. □

Theorem 3.4 *If Q is an empty set or $q > p_1 \cdot p_i$, then*

$$\omega(X_n(1, d)) = p_1 \cdot p_i.$$

Proof: According to Lemma 3.2 it is enough to construct a clique of size $p_1 \cdot p_i$ with vertices $x_{rs} = a_s \cdot d + r$, where $0 \leqslant s < p_i$ and $0 \leqslant r < p_1$. We choose numbers a_s as solutions of the following congruence equations:

$$
\begin{aligned}
a_s &\equiv s &&(\text{mod } p) &&\text{for every } p \in R \\
a_s \cdot d &\equiv s \cdot p_1 &&(\text{mod } p) &&\text{for every } p \in Q
\end{aligned}
$$

This linear congruence system has a solution if and only if $gcd(d, p) \mid s \cdot p_1$ for $p \in Q$. The last relation is trivially satisfied since d and $p \in Q$ are relatively prime. Therefore, using Chinese reminder theorem we can uniquely determinate numbers a_s modulo M.

Consider an arbitrary difference $\Delta = x_{rs} - x_{r's'} = d \cdot (a_s - a_{s'}) + (r - r')$. Assume first that $r \neq r'$. For every prime divisor p of d (and therefore for every prime $p \in R$), the number Δ cannot be divisible by p because $0 < |r - r'| < p_1$. If $p \in Q$, we have $x_{rs} - x_{r's'} \equiv (s - s') \cdot p_1 + (r - r') \pmod{p}$. The residue $|(s - s') \cdot p_1 + (r - r')| \leqslant (p_i - 1) \cdot p_1 + (p_1 - 1) < p_i \cdot p_1 < q$ is less than p and never equal to zero - which means that the greatest common divisor of Δ and n equals 1. In other case, we have $r = r'$. Again, for an arbitrary prime $p \in Q$, the residue of Δ modulo p is $(s - s') \cdot p_1$, which is never equal to 0. When p is a prime number from R, by definition the difference $a_s - a_{s'} \equiv (s - s') \pmod{p}$ can not be divisible by p according to $0 < |s - s'| < p_i \leqslant p$. Therefore, in this case we have $gcd(\Delta, n) = d$.

When Q is an empty set, we can use the same construction to get a clique of size $p_1 \cdot p_i$. □

Theorem 3.5 *If $q < p_1 \cdot p_i$, then*

$$\omega(X_n(1, d)) = q.$$

Proof: According to Lemma 3.5 it is enough to find q vertices that form clique in the graph $X_n(1, d)$. Define numbers $x_k = a_k \cdot d + b_k$ for $k = 0, 1, \ldots, q - 1$, where b_k is the residue of k modulo p_1 and the following conditions are satisfied:

$$
\begin{aligned}
a_k \cdot d + b_k &\equiv k &&(\text{mod } p) &&\text{for every } p \in Q \\
a_k &\equiv \lfloor k/p_1 \rfloor &&(\text{mod } p) &&\text{for every } p \in R
\end{aligned}
$$

Numbers a_k can be uniquely determined using Chinese Reminder Theorem modulo M, because d and p are relatively prime, for every prime number from Q. We will prove that the greatest common divisor of $x_k - x_{k'}$ and n is always equal to d or 1, which would complete the proof. For every $p \in Q$, we have that $x_k - x_{k'} \equiv k - k' \pmod{p}$. Since $|k - k'| < q$, $x_k - x_{k'}$ can not be divisible by p.

Next, consider the case when k and k' have the same residue modulo p_1 and $k \neq k'$. If $a_k - a_{k'}$ is divisible by some $p \in R$, this means that we have also $\lfloor k/p_1 \rfloor \equiv \lfloor k'/p_1 \rfloor \pmod{p}$. Since k is less than $p_1 \cdot p_i$, we conclude that the

integer parts of numbers $\frac{k}{p_1}$ and $\frac{k'}{p_1}$ are equal. Together with the assumption that fractional parts of these numbers are the same, we get that $k = k'$. This is a contradiction and therefore $gcd(x_k - x_{k'}, n) = d$. In the second case, we have that number $x_k - x_{k'}$ is not divisible by any $p \in R$, because d is divisible by p and $0 < |b_k - b_{k'}| < p_1$. Thus, we have $gcd(x_k - x_{k'}, n) = 1$ which completes the proof. $\qquad\square$

Finally we reach the following main result of this subsection:

Theorem 3.6 *For any divisor d of n, there holds:*

$$\omega(X_n(1, d)) = \min\ \left(\min_{p \in Q} p, f(n) \cdot f\left(\frac{n}{d}\right)\right).$$

3.2 Case $1 \notin D$

Theorem 3.7 *Let $X_n(d_1, d_2)$ be a gcd-graph with both divisors greater than one. Then the following equality holds:*

$$\omega\left(X_n(d_1, d_2)\right) = \begin{cases} \omega(X_{\frac{n}{d_2}}(1, \frac{d_1}{d_2})), & \text{if } d_2 \mid d_1, \\ \max\left(f\left(\frac{n}{d_1}\right), f\left(\frac{n}{d_2}\right)\right), & \text{otherwise.} \end{cases}$$

Proof: If a maximal clique has edges of both colors, then there exists a non-monochromatic triangle. Therefore, we can find vertices a, b, c such that:

$$gcd(a - b, n) = d_1, \quad gcd(a - c, n) = d_2, \quad gcd(b - c, n) = d_2$$

By subtraction, we get that $d_2 \mid (a - c) - (b - c)$ and finally d_2 divides d_1. We excluded the case with two blue edges, because than d_1 would divide d_2, which is impossible. Therefore, the graph X_n is disconnected according to Theorem 2.2 and we obtained an equivalent problem for the divisor set $D' = \{1, \frac{d_1}{d_2}\}$ and gcd-graph $X_{n/d_2}(D)$.

In the other case (d_2 does not divide d_1), the maximal clique is monochromatic and by Theorem 2.3 we completely determine $\omega(X_n(d_1, d_2))$. $\qquad\square$

4 Counterexamples

In order to test the conjecture proposed in [6] for integral circulant graphs with more then two divisors, we implemented *Backtrack Algorithm with pruning* [5] for finding the clique number. For $k = 3$ and $k = 4$, we construct infinite families of integral circulant graphs, such that clique number does not divide n. For example, we obtain that $\omega(X_{20}(1, 4, 10)) = 6$ and $\omega(X_{30}(1, 2, 6, 15)) = 7$ which is verified by an exhausted search algorithm. The next proposition in theoretic way disproves the conjecture.

Proposition 4.1 *The clique number of integral circulant graph $X_{20}(1, 4, 10)$ equals 6 or 7.*

Proof: The vertices $0, 1, 4, 8, 11, 12$ form a clique in considered graph, which is easy to check. We color edges in $X_{20}(1, 4, 10)$ with three colors: *red* if $gcd(a - b, 20) = 1$, *blue* if $gcd(a - b, 20) = 4$, and *green* if $gcd(a - b, 20) = 10$.

By Theorem 2.3 and Lemma 2.4, the maximal clique with red edges has 2 vertices, the maximal clique with blue edges has 5 vertices, and for green color it has 2 vertices. If a triangle has one blue and one green edge - it follows that the third edge cannot be red, because of parity. If the third edge is blue, than the absolute value of the difference on the green edge is divisible by 4, which is impossible. Likewise, if the third edge is green than the absolute value of the difference on the blue edge is divisible by 5. Therefore, there is no triangle in graph which contain both blue and green edges.

Assume that the maximal clique is two-colored. By previous consideration maximal clique can contain red and blue edges or red and green edges. In the first case our problem is to find the maximal clique in integral circulant graph $X_{20}(1, 4)$. Applying Theorem 3.5 we conclude that $\omega(X_{20}(1, 4)) = 5$, but we already found a clique of size 6. Analogously, the size of maximal clique with red and green edges is $\omega(X_{20}(1, 10)) = 4$ by Theorem 3.4. It means that maximal clique must contain all three colors.

Now, the maximal clique is three-colored and consists of x cliques with blue edges and y cliques with green edges. Using mentioned fact that there is no triangle with blue and green edges, we can easily notice that only red edges join these $x + y$ cliques. If we choose one vertex from each clique, we obtain a red edge clique with $x + y$ vertices. But, the maximal clique with red edges has only two vertices, implying that $x = y = 1$. So, the upper bound for the clique number is $2 + 5 = 7$ and the lower bound is 6, and neither of them is a divisor of 20. $\qquad\square$

Proposition 4.2 *Let $X_n(D)$ be the integral circulant graph with the set of divisors $D = \{d_1, d_2, \ldots, d_k\}$. If $N = n \cdot p$ where p is an arbitrary prime number greater then n, then the following equality holds*

$$\omega(X_N(D)) = \omega(X_n(D)).$$

Proof: Since $p > n$ and $gcd(a - b, n) = gcd(a - b, p \cdot n)$ for arbitrary vertices $a, b \in X_n(D)$, we have inequality $\omega(X_N(D)) \geqslant \omega(X_n(D))$. Now, assume that vertices $\{a_1, a_2, \ldots, a_c\}$ form a maximal clique in $X_N(D)$ and consider vertices $\{b_1, b_2, \ldots, b_c\}$ in graph $X_n(D)$, where b_i is the remainder of a_i modulo n. Prime number p cannot divide any of the numbers d_i and thus $a_i - a_j$ is not divisible by p for every $1 \leqslant i < j \leqslant c$. Now, we have

$$gcd(b_i - b_j, n) = gcd(a_i - a_j, n) = gcd(a_i - a_j, N) \in D.$$

This means that $\omega(X_N(D)) \leqslant \omega(X_n(D))$ and finally $\omega(X_N(D)) = \omega(X_n(D))$. $\qquad\square$

Using this proposition we obtain a class of counterexamples $X_{20p}(1, 4, 10)$ based on the graph $X_{20}(1, 4, 10)$, where p is a prime number greater then 20.

5 Concluding remarks

In this paper we moved a step towards describing the clique number of integral circulant graphs. We find an explicit formula for the clique number of $X(d_1, d_2, \ldots, d_k)$ when $k \leqslant 2$. This leads us to the main result of the paper - discarding a conjecture proposed in [6], that the clique number divides n. We constructed families of counterexamples for $k = 3$ and $k = 4$.

We also examine numerous examples for $k \geqslant 2$ divisors and obtain the following inequality:

$$\max_{d_i \in D} f\left(\frac{n}{d_i}\right) \leqslant \omega(X_n(D)) \leqslant \prod_{i=1}^{k} f\left(\frac{n}{d_i}\right).$$

The lower bound follows from Lemma 2.4, and the upper bound can be proven by induction on k. Namely, after adding edges $\{a, b\}$ such that $gcd(a - b, n) = d_i$ into the graph $X_n(d_1, d_2, \ldots, d_{i-1})$, we divide every color class in maximum $f(\frac{n}{d_i})$ independent parts. Therefore, the number of color classes is less than or equal to the product of numbers $f(\frac{n}{d_i})$ for all $d_i \in D$.

We leave for future study to see whether this bound can be improved.

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