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A reduction formula for the Kampé de Fériet function

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ABSTRACT

A generalization is provided for a reduction formula for the Kampé de Fériet function due to Cvijović.

1. Introduction

Recently, the authors have derived several new summation formulas for hypergeometric-type series containing the digamma or psi function $\psi(z)$. The summation formula [1]

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\frac{(\alpha)_n}{(\lambda)_n}z^n=\frac{\alpha z}{\lambda^2(1-z)^{\alpha+1}}\,{}_3F_2\left(\left.\begin{matrix}\alpha+1,\lambda,\lambda\\ \lambda+1,\lambda+1\end{matrix}\right|\frac{z}{z-1}\right), \tag{1.1}$$

where z lies in the domain $|z|<1$, $\Re(z)<1/2$, was employed to obtain a reduction formula for the Kampé de Fériet (hereafter KdF) function that we shall in Section 2 extend to a much more general result. In Eq. (1.1) and below, all parameters and variables are complex unless otherwise noted or it is obvious from the context. The Pochhammer symbol $(\alpha)_n$, where n is an integer (positive, negative or zero) is defined simply by $(\alpha)_n\equiv\Gamma(\alpha+n)/\Gamma(\alpha)$.

In the sequel, the sequence $(\alpha_1,\ldots,\alpha_p)$ is denoted simply by (α_p) and the product of p Pochhammer symbols $((\alpha_p))$ is defined by $((\alpha_p))_n\equiv(\alpha_1)_n\cdots(\alpha_p)_n$, where an empty product $p=0$ reduces to unity. The KdF function is a generalized hypergeometric function in two variables defined by the double series

$$F^{p:r;u}_{q:s;v}\left[\left.\begin{matrix}(a_p):(c_r);(f_u)\\ (b_q):(d_s);(g_v)\end{matrix}\right|x,y\right]\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a_p))_{m+n}}{((b_q))_{m+n}}\frac{((c_r))_m}{((d_s))_m}\frac{((f_u))_n}{((g_v))_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{1.2}$$

See for example [2,3] for an introduction to the KdF function and its properties including its convergence criteria.

In [1,4] we showed that

$$\sum_{n=0}^{\infty}[\psi(\lambda+n+1)-\psi(\lambda)]\frac{((\mu_p))_n}{((\nu_q))_n}z^n=\frac{1}{\lambda}F^{p:2;1}_{q:1;0}\left[\left.\begin{matrix}(\mu_p):\lambda,\quad 1;1\\ (\nu_q):\lambda+1;-\end{matrix}\right|z,z\right]. \tag{1.3}$$

This result shows that a generalized hypergeometric-type series containing the digamma function may essentially be represented by a specialization of the KdF function in two equal variables.

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2. Reduction formula

As a byproduct of efficiently deriving closed form representations for certain series due to Miller [4] Cvijović [1, equation (3.3)] employed Eqs. (1.1) and (1.3) to obtain the reduction formula for the KdF function

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix}\alpha & : & \beta-1, 1 & ; & 1\\ \beta & : & \beta & ; & -\end{matrix}\middle| z, z\right] = \frac{1}{(1-z)^{\alpha}}\, {}_3F_2\left(\begin{matrix}\alpha, \beta-1, & \beta-1\\ \beta, & \beta\end{matrix}\middle|\frac{z}{z-1}\right), \tag{2.1}$$

where z lies in the domain $|z| < 1$, and $\Re(z) < 1/2$.

The KdF function has proved of practical utility in representing solutions to a wide range of problems in pure and applied mathematics and mathematical physics. See the books by Exton [2,5]; for additional examples of applications, see [6,7]. Reduction formulas such as Eq. (2.1) essentially represent the KdF function as a generalized hypergeometric function of lower order or some other function in one variable. Obviously, identifying such reductions have great value in simplifying solutions. Thus, compilations of them such as [3, pp. 28–32] and [8] are especially important, since there is no *a priori* way of knowing their existence.

It is the purpose here to augment the known results intimated above by showing that the formula (2.1) is a specialization of the much more general result

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix}\alpha & : & \beta-\epsilon, \gamma & ; & \epsilon\\ \beta & : & \delta & ; & -\end{matrix}\middle| z, z\right] = \frac{1}{(1-z)^{\alpha}}\, {}_3F_2\left(\begin{matrix}\alpha, \beta-\epsilon, & \delta-\gamma\\ \beta, & \delta\end{matrix}\middle|\frac{z}{z-1}\right) \tag{2.2}$$

which we derive below. Clearly, when $\gamma = 1$, $\epsilon = 1$ and $\delta = \beta$, Eq. (2.2) reduces to Eq. (2.1). Convergence for the KdF function occurs when $|z| < 1$ (see [3, p. 27]); convergence for ${}_3F_2$ function obviously occurs when $|z/(z-1)| < 1$. Thus, Eqs. (2.1) and (2.2) are valid for z in the domain $|z| < 1$, $\Re(z) < 1/2$.

We recall the identity

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n. \tag{2.3}$$

For brevity calling the left side of Eq. (2.2) $F(z, z)$, we have upon recalling the definition of the KdF function (see Eq. (1.2)) and utilizing (2.3)

$$\begin{aligned}F(z,z) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\beta)_{m+n}}\frac{(\beta-\epsilon)_m(\gamma)_m}{(\delta)_m}(\epsilon)_n\frac{z^m}{m!}\frac{z^n}{n!}\\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\beta)_m}\frac{(\beta-\epsilon)_m(\gamma)_m}{(\delta)_m}\frac{z^m}{m!}\sum_{n=0}^{\infty}\frac{(\alpha+m)_n(\epsilon)_n}{(\beta+m)_n}\frac{z^n}{n!}\\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\beta)_m}\frac{(\beta-\epsilon)_m(\gamma)_m}{(\delta)_m}\frac{z^m}{m!}\,{}_2F_1\left(\begin{matrix}\alpha+m, \epsilon\\ \beta+m\end{matrix}\middle| z\right).\end{aligned}$$

Now, applying Euler's transformation (see e.g. [9, p. 33, equation (19)])

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| z\right) = \frac{1}{(1-z)^a}\,{}_2F_1\left(\begin{matrix}a, c-b\\ c\end{matrix}\middle|\frac{z}{z-1}\right)$$

with $a = \alpha + m$, $b = \epsilon$, and $c = \beta + m$ we see upon again utilizing Eq. (2.3) that

$$\begin{aligned}F(z,z) &= \frac{1}{(1-z)^{\alpha}}\sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\beta)_m}\frac{(\beta-\epsilon)_m(\gamma)_m}{m!(\delta)_m}\left(\frac{z}{1-z}\right)^m {}_2F_1\left(\begin{matrix}\alpha+m, \beta-\epsilon+m\\ \beta+m\end{matrix}\middle|\frac{z}{z-1}\right)\\ &= \frac{1}{(1-z)^{\alpha}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m}{(\beta)_m}\frac{(\beta-\epsilon)_m(\gamma)_m}{m!(\delta)_m}\left(\frac{z}{z-1}\right)^{m+n}\frac{(-1)^m(\alpha+m)_n(\beta-\epsilon+m)_n}{(\beta+m)_n n!}\\ &= \frac{1}{(1-z)^{\alpha}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta-\epsilon)_{m+n}}{(\beta)_{m+n}}\frac{(-1)^m(\gamma)_m}{m!n!(\delta)_m}\left(\frac{z}{z-1}\right)^{m+n}.\end{aligned}$$

Assuming absolutely convergent series for an arbitrary function $B(m, n)$ by invoking series rearrangement via

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}B(m, n-m)$$

we see from the latter result for $F(z, z)$ that

$$F(z,z) = \frac{1}{(1-z)^{\alpha}}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta-\epsilon)_n}{(\beta)_n}\left(\frac{z}{z-1}\right)^n\sum_{m=0}^{n}\frac{(-1)^m(\gamma)_m}{m!(\delta)_m(n-m)!}.$$

However, since $(-1)^m/(n-m)! = (-n)_m/n!$, we may write

$$F(z, z) = \frac{1}{(1-z)^\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta-\epsilon)_n}{n!(\beta)_n} \left(\frac{z}{z-1}\right)^n \sum_{m=0}^{n} \frac{(-n)_m(\gamma)_m}{m!(\delta)_m}.$$

Now, utilizing the Vandermonde–Chu identity (see, for instance, [9, p. 30, equation (8)])

$${}_2F_1\left(\left.\begin{matrix}-n, a\\ b\end{matrix}\right| 1\right) = \frac{(b-a)_n}{(b)_n},$$

since the latter finite m-summation may be written as

$$\sum_{m=0}^{n} \frac{(-n)_m(\gamma)_m}{m!(\delta)_m} = {}_2F_1\left(\left.\begin{matrix}-n, \gamma\\ \delta\end{matrix}\right| 1\right)$$

we obtain finally

$$F(z, z) = \frac{1}{(1-z)^\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta-\epsilon)_n(\delta-\gamma)_n}{(\beta)_n(\delta)_n n!} \left(\frac{z}{z-1}\right)^n$$

which is a restatement of Eq. (2.2). This completes our proof of Eq. (2.2).

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