# A Caffarelli-Kohn-Nirenberg type inequality on Riemannian Manifolds

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15th June 2009

### Abstract

We establish a generalization to Riemannian manifolds of the Caffarelli-Kohn-Nirenberg inequality. The applied method is based on the use of conformal Killing vector fields and Enzo Mitidieri's approach to Hardy inequalities.

2000 AMS Mathematics Classification numbers: 58E35, 26D10

Key words: Caffarelli-Kohn-Nirenberg Inequality, conformal Killing vector fields

## 1 Introduction

It is well-known that Hardy type inequalities have been widely used in the theory of differential equations. An inequality of this kind is the celebrated general interpolation inequality with weights obtained by Caffarelli, Kohn and Nirenberg in [4]. The latter can be stated in the form of the following

**Theorem 1.** (Caffarelli-Kohn-Nirenberg [4]) For all  $a, b \in \mathbb{R}$  and  $u \in C_0^{\infty}(\mathbb{R}^n \setminus \{0\})$  the inequality

$$C \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{a+b+1}} \, dx \le \left( \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^{2b}} \, dx \right)^{1/2} \tag{1}$$

holds with sharp constant C = C(a, b) = |n - (a + b + 1)|/2.

(We note that the presented above Theorem 1 is the version of the Caffarelli-Kohn-Nirenberg Inequality discussed in [8].)

There is a very big number of recent papers dedicated to generalizations of the Hardy, Sobolev and Caffarelli-Kohn-Nirenberg inequalities in various contexts. For this reason we merely cite just a few of them (e.g. [1, 2, 5, 6, 7, 8, 9, 10]) directing the interested reader to use an internet search instrument for further references.

Since most of the problems in differential geometry can be reduced to problems in differential equations on Riemannian manifolds, it is important to have in this case investigation tools similar to those successfully used in the Euclidean case, in particular, integral inequalities for functions in weighted Sobolev type spaces. Hence the purpose of this work is to obtain another generalization of (1) to Riemannian manifolds.

Let M be a complete n-dimensional manifold,  $n \geq 3$ , endowed with a Riemannian metric  $g = (g_{ij})$  given in local coordinates  $x^i$ , i = 1, 2, ..., n, around some point. We suppose that on M exist (nontrivial) conformal Killing vector fields  $h = h^i \frac{\partial}{\partial x^i}$ :

$$\nabla^{i}h^{j} + \nabla^{j}h^{i} = \frac{2}{n}(div \ h) \ g^{ij} = \mu \ g^{ij}.$$
 (2)

Here  $\nabla^i$  is the covariant derivative corresponding to the Levi-Civita connection, uniquely determined by g, div h is the covariant divergence operator and summation from 1 to n over repeated Latin indices is understood.

Now we state the main result of this paper:

**Theorem 2.** Let p > 1 and  $u \in W^{1,p}(M)$ . Suppose that M admits a  $C^1$  conformal Killing vector field  $h = h^i \frac{\partial}{\partial x^i}$  such that div h > 0. Then

$$C \int_{M} \frac{div \ h}{|h|^{a+b+1}} \ |u|^{p} \ dV \le \left( \int_{M} \frac{div \ h}{|h|^{aq}} \ |u|^{p} \ dV \right)^{1/q} \left( \int_{M} \frac{(div \ h)^{1-p}}{|h|^{bp}} \ |\nabla u|^{p} \ dV \right)^{1/p} \tag{3}$$

where dV is the volume form of M,  $|h|^2 = g_{ij}h^ih^j$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  and the constant C = |n - (a + b + 1)|/(pn) is sharp.

The proof of this result is based on the idea of the E. Mitidieri's vector field approach to Hardy inequalities, proposed in [9], which is applied here to conformal Killing vector fields on Riemannian manifolds.

Note that in Theorem 2 we do not impose additional conditions neither on the gradient of the weight function nor on its second derivatives (e.g. certain kind of super-harmonicity).

Further we assume that M admits a conformal Killing vector field  $h = h^i \frac{\partial}{\partial x^i}$  such that

$$\nabla^i h^j + \nabla^j h^i = c \ g^{ij} = \frac{2}{n} \ (div \ h) \ g^{ij}, \tag{4}$$

where  $c \neq 0$  is a constant. This means we suppose that M admits a homothety ('homothetic motion') which is not an infinitesimal isometry of M. Actually we may choose c = 2 in (4) and hence  $div \ h = n$  without loss of generality. (Otherwise, since  $c \neq 0$ , we could consider 2h/c instead of h.) Such a special conformal vector field will be used to obtain the following *exact* generalization of the Caffarelli-Kohn-Nirenberg Inequality in  $\mathbb{R}^n$ .

**Corollary 1.** If div h = n and p = q = 2, then (3) becomes

$$C\int_{M} \frac{|u|^2}{|h|^{a+b+1}} \, dV \le \left(\int_{M} \frac{|u|^2}{|h|^{2a}} \, dV\right)^{1/2} \left(\int_{M} \frac{|\nabla u|^2}{|h|^{2b}} \, dV\right)^{1/2} \tag{5}$$

with sharp constant C = C(a, b) = |n - (a + b + 1)|/2.

Note that for  $M = \mathbb{R}^n$ ,  $g^{ij} = \delta^{ij}$ —the Euclidean metric of  $\mathbb{R}^n$  and  $h = x^i \frac{\partial}{\partial x^i}$ , the radial vector field corresponding to a dilational transformation in  $\mathbb{R}^n$ , the inequality (5) reduces to the Caffarelli-Kohn-Nirenberg Inequality (1).

If a = p - 1, b = 0 in (3), then we can get easily the sharp Hardy type inequality established in [3]. Namely:

**Corollary 2.** ([3]) Let p > 1 and  $u \in W^{1,p}(M)$ . Suppose that M admits a  $C^1$  conformal Killing vector field h such that div h > 0. Then  $(\operatorname{div} h)|u|^p/|h|^p \in L^1(M)$  and

$$\left(\frac{|n-p|}{np}\right)^p \int_M \frac{div \ h}{|h|^p} \ |u|^p \ dV \le \int_M (div \ h)^{1-p} \ |\nabla u|^p \ dV.$$

We observe that the Caffarelli-Kohn-Nirenberg Inequality (1) with b = 0 and a = -1 represents the classical Uncertainity Principle of quantum mechanics:

$$\frac{n^2}{4} \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^2 \le \left( \int_{\mathbb{R}^n} |x|^2 |u|^2 dx \right) \left( \int_{\mathbb{R}^n} |\nabla u|^2 dx \right).$$

Then Theorem 2 immediately implies its Riemannian analog:

**Corollary 3.** Let p > 1, 1/p+1/q = 1 and  $u \in C_0^{\infty}(M \setminus \{\text{zeros of } h\})$ . Suppose that M admits a  $C^1$  conformal Killing vector field h such that div h > 0. Then the following Uncertainty Principle inequality holds:

$$\frac{1}{p} \int_{M} (div \ h) \ |u|^{p} \ dV \le \left( \int_{M} (div \ h) \ |h|^{q} \ |u|^{p} \ dV \right)^{1/q} \left( \int_{M} (div \ h)^{1-p} \ |\nabla u|^{p} \ dV \right)^{1/p}.$$

Another result which can be handled by the technique developed in this paper is one of the results obtained in [10]. In that work C. Xia proves, among other things, the following sharp Caffarelli-Kohn-Nirenberg inequality for any  $u \in C_0^{\infty}(\mathbb{R}^n)$ :

$$\int_{\mathbb{R}^n} |x|^{\gamma r} |u|^r \, dx \le \frac{r}{n+\gamma r} \left( \int_{\mathbb{R}^n} |x|^{\alpha p} |\nabla u|^p \, dx \right)^{1/p} \left( \int_{\mathbb{R}^n} |x|^{\beta} |u|^{\frac{p(r-1)}{p-1}} \, dx \right)^{(p-1)/p},$$

where  $n \ge 2$ ,  $1 , <math>\alpha > 0$ ,  $\beta > 0$  satisfy

$$\frac{1}{p} + \frac{\alpha}{n} > 0, \quad \frac{p-1}{p(r-1)} + \frac{\beta}{n} > 0, \quad \frac{1}{r} + \frac{\gamma}{n} > 0 \quad \gamma = \frac{\alpha-1}{r} + \frac{(p-1)\beta}{pr}.$$
 (6)

The method used in the present paper permits to obtain its generalization:

**Corollary 4.** Suppose that the parameters  $\alpha, \beta, \gamma, r, p$  satisfy the condition (6) and the function  $u \in C_0^\infty(M)$ . In addition suppose that M admits a  $C^1$  conformal Killing vector field h such that div h > 0. Then

$$\int_{M} (div \ h) |h|^{\gamma r} \ |u|^{r} \ dV \leq$$

$$\leq \frac{rn}{n + \gamma r} \left( \int_{M} (div \ h)^{1-p} \ |h|^{\alpha p} \ |\nabla u|^{p} \ dV \right)^{1/p} \left( \int_{M} (div \ h) \ |h|^{\beta} \ |u|^{(r-1)q} \ dV \right)^{1/q},$$

$$re \ q = p/(p-1).$$
(7)

where

We shall prove Theorem 2 and Corollary 4 in the next section. In section 3 we briefly present comments and concluding remarks.

#### $\mathbf{2}$ An application of the vector fields method

We begin with a preliminary

**Lemma.** If h is a conformal Killing vector field satisfying (2) then

$$div\left(\frac{h}{|h|^k}\right) = \frac{n-k}{2}\frac{\mu}{|h|^k},\tag{8}$$

where  $k \in \mathbb{R}$ . If k > 0, the relation (8) holds on  $M \setminus \{zeros \ of \ h\}$ ; otherwise, it holds on the whole of M.

**Proof.** From (2) we have that the covariant divergence

$$div\left(\frac{h}{|h|^k}\right) = \frac{n}{2}\frac{\mu}{|h|^k} - k|h|^{-k-2}h_jh^i\nabla_ih^j.$$
(9)

Further we interchange the indices i and j and use (2) in the following way:

$$h_j h^i \nabla_i h^j = h_j h_i \nabla^i h^j = h_j h_i (-\nabla^j h^i + \mu g^{ij}) = -h_j h^i \nabla_i h^j + \mu |h|^2.$$

Hence

$$h_j h^k \nabla_k h^j = \frac{\mu}{2} |h|^2. \tag{10}$$

Thus the relation (8) follows from (9) and (10).

After this little preparatory work we are now ready for the

**Proof of Theorem 4.** As usual, we may assume that  $u \in C_0^{\infty}(M)$  without loss of generality. Then integrating by parts and using (8) with k = a + b + 1 we obtain that

$$\int_{M} \frac{|u|^{p-1}(h, \nabla u)}{|h|^{a+b+1}} \, dV = -\frac{n - (a+b+1)}{2p} \int_{M} \frac{\mu}{|h|^{a+b+1}} |u|^{p} \, dV.$$

Hence

$$\frac{|n - (a + b + 1)|}{2p} \int_{M} \frac{\mu}{|h|^{a + b + 1}} |u|^{p} \, dV \le \int_{M} \frac{|u|^{p - 1} |\nabla u|}{|h|^{a + b}} \, dV$$
$$= \int_{M} \frac{|u|^{p/q} \mu^{1/q}}{|h|^{a}} \cdot \frac{|\nabla u| \mu^{-1/q}}{|h|^{b}} \, dV$$
$$\le \left(\int_{M} \frac{\mu}{|h|^{aq}} |u|^{p} \, dV\right)^{1/q} \left(\int_{M} \frac{\mu^{1 - p}}{|h|^{bp}} |\nabla u|^{p} \, dV\right)^{1/p}$$

by the Hölder inequality with q = p/(p-1). Thus

$$\frac{|n-(a+b+1)|}{2p} \int_{M} \frac{\mu}{|h|^{a+b+1}} |u|^{p} \, dV \le \left(\int_{M} \frac{\mu}{|h|^{aq}} |u|^{p} \, dV\right)^{1/q} \left(\int_{M} \frac{\mu^{1-p}}{|h|^{bp}} |\nabla u|^{p} \, dV\right)^{1/p}.$$

Then (3) follows from the latter inequality with  $\mu = 2(div h)/n$  (see (2)). It is clear that the constant C is sharp [8, 9].

**Proof of Corollary 4.** By (8) with  $k = -\gamma r$  we have

$$\int_M |h|^{\gamma r} |u|^{r-1} (h, \nabla u) \, dV = -\frac{n+\gamma r}{2r} \int_M \mu |h|^{\gamma r} |u|^r \, dV$$

after integration by parts. Then

$$\begin{split} &\int_M \mu |h|^{\gamma r} |u|^r \ dV \leq \frac{2r}{n+\gamma r} \int_M |h|^{\gamma r+1} |u|^{r-1} |\nabla u| \ dV \\ &= \frac{2r}{n+\gamma r} \int_M |h|^{\alpha} |\nabla u| \mu^{-1/q} . \mu^{1/q} |h|^{\gamma r+1-\alpha} |u|^{r-1} \ dV \end{split}$$

$$\leq \frac{2r}{n+\gamma r} \left( \int_{M} \mu^{1-p} |h|^{\alpha p} |\nabla u|^{p} dV \right)^{1/p} \left( \int_{M} \mu |h|^{(\gamma r+1-\alpha)q} |u|^{(r-1)q} dV \right)^{1/q}$$

by the Hölder inequality with q = p/(p-1). Hence we obtain (7) setting  $\beta = (\gamma r + 1 - \alpha)p/(p-1)$  which is equivalent to the last relation in (6).

### 3 Conclusion

We have proved a Riemannian analog of the Caffarelli-Kohn-Nirenberg Inequality using the Enzo Mitidieri's approach to Hardy inequalities, devised and developed in [9], applied to conformal Killing vector fields. We would also like to observe that the presented proof of Theorem 2 can be considered as another proof of the Caffarelli-Kohn-Nirenberg Inequality in  $\mathbb{R}^n$  simply taking  $h = x^i \frac{\partial}{\partial x^i}$ , the radial vector field which has been used in [8, 9].

We point out that the quadratic inequality method used by David Costa in his elegant paper [8] applies to Riemannian manifolds as well. Indeed, if  $h = h^i \frac{\partial}{\partial x^i}$  is a conformal Killing vector field such that div h = n, consider another vector field

$$W = \frac{\nabla u}{|h|^b} + t \ u \frac{h}{|h|^{a+1}} = \left(\frac{1}{|h|^b} g^{ij} u \frac{\partial u}{\partial x^i} + t \ u \frac{h^i}{|h|^{a+1}}\right) \frac{\partial}{\partial x^i}$$

where  $t \in \mathbb{R}$  and  $g^{ij}$  is the inverse matrix of  $g_{ij}$ . From  $g(W, W) \geq 0$ , that is, from  $g_{ij}W^iW^j \geq 0$ , and the identity (8), it is obvious that one can prove (5) following the argument in [8], p. 313.

### Acknowledgements

We would like to thank Professors Enzo Mitidieri and David Costa for their encouragement. We would also like to thank CNPq, Brasil, for partial financial support.

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