# Optimality conditions for the calculus of variations with higher-order delta derivatives 

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#### Abstract

We prove the Euler-Lagrange delta-differential equations for problems of the calculus of variations on arbitrary time scales with delta-integral functionals depending on higher-order delta derivatives.


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## 1. Introduction

In recent years numerous works have been dedicated to the calculus of variations on time scales and their generalizations - see $7,12,13,18,21,22,23,24,26$ and the references therein. Most of them deal with delta or nabla derivatives of first-order $2,2,3,4,5,6,9,11,16,19,20]$, only a few with higher-order derivatives [10, 25]. Depending on the type of the functional being considered, different time scale Euler-Lagrange type equations are obtained. For variational problems of firstorder the Euler-Lagrange equations are valid for an arbitrary time scale $\mathbb{T}$, while for the problems with higher-order delta (or nabla) derivatives they are only valid in a certain class of time scales, more precisely, the ones for which the forward (or backward) jump operator is a polynomial of degree one [10, 25]. Here we consider variational problems involving Hilger derivatives of higher order, and prove a necessary optimality condition of the Euler-Lagrange type on an arbitrary time scale, i.e., without imposing any restriction to the jump operators.

## 2. Preliminaries

Here we recall some basic results and notation needed in the sequel. For the theory of time scales we refer the reader to $[1,8,14,15]$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The functions $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ and $\rho: \mathbb{T} \rightarrow \mathbb{T}$ are, respectively, the forward and backward jump operators: $\sigma(t)=\inf \{s \in \mathbb{T}: s>t\}$ with $\inf \emptyset=\sup \mathbb{T}$ (i.e., $\sigma(M)=M$ if $\mathbb{T}$ has a maximum $M$ ); $\rho(t)=$ $\sup \{s \in \mathbb{T}: s<t\}$ with $\sup \emptyset=\inf \mathbb{T}$ (i.e., $\rho(m)=m$ if $\mathbb{T}$ has a minimum $m$ ). The symbol $\emptyset$ denotes the empty set. The graininess function on $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$. For $\mathbb{T}=\mathbb{R}$ one has $\sigma(t)=t=\rho(t)$ and $\mu(t) \equiv 0$ for any $t \in \mathbb{R}$. For $\mathbb{T}=\mathbb{Z}$ one has $\sigma(t)=t+1, \rho(t)=t-1$,

[^0]and $\mu(t) \equiv 1$ for every $t \in \mathbb{Z}$. A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense, or left-scattered, if $\sigma(t)=t, \sigma(t)>t, \rho(t)=t$, or $\rho(t)<t$, respectively.

Let $\mathbb{T}=[a, b] \cap \mathbb{T}_{0}$ with $a<b$ and $\mathbb{T}_{0}$ a time scale. We define $\mathbb{T}^{\kappa}:=\mathbb{T} \backslash(\rho(b), b]$, and $\mathbb{T}^{\kappa^{0}}:=\mathbb{T}$, $\mathbb{T}^{\kappa^{n}}:=\left(\mathbb{T}^{\kappa^{n-1}}\right)^{\kappa}$ for $n \in \mathbb{N}$. The following standard notation is used for $\sigma$ (and $\rho$ ): $\sigma^{0}(t)=t$, $\sigma^{n}(t)=\left(\sigma \circ \sigma^{n-1}\right)(t), n \in \mathbb{N}$.

We say that a function $f: \mathbb{T} \rightarrow \mathbb{R}$ is delta-differentiable at $t \in \mathbb{T}^{\kappa}$ if there is a number $f^{\Delta}(t)$ such that for all $\varepsilon>0$ there exists a neighborhood $U$ of $t$ such that

$$
\left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right| \leq \varepsilon|\sigma(t)-s|, \quad \text { for all } s \in U
$$

We call $f^{\Delta}(t)$ the delta-derivative of $f$ at $t$. We note that if the number $f^{\Delta}(t)$ exists then it is unique in $\mathbb{T}^{\kappa}$ (see [14, 15]). In the special cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}, f^{\Delta}$ reduces to the standard derivative $f^{\prime}(t)$ and the forward difference $\Delta f(t)=f(t+1)-f(t)$, respectively. Whenever $f^{\Delta}$ exists, the following formula holds: $f^{\sigma}(t)=f(t)+\mu(t) f^{\Delta}(t)$, where we abbreviate $f \circ \sigma$ by $f^{\sigma}$. Let $f^{\Delta^{0}}=f$. We define the $r$ th-delta derivative of $f: \mathbb{T}^{\kappa^{r}} \rightarrow \mathbb{R}, r \in \mathbb{N}$, to be the function $\left(f^{\Delta^{r-1}}\right)^{\Delta}$, provided $f^{\Delta^{r-1}}$ is delta differentiable on $\mathbb{T}^{\kappa^{r}}$.

A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous if it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$. A function $f: \mathbb{T} \rightarrow \mathbb{R}^{n}$ is rd-continuous if all its components are rd-continuous. The set of all rd-continuous functions is denoted by $C_{r d}$. Similarly, $C_{r d}^{r}$ will denote the set of functions with delta derivatives up to order $r$ belonging to $C_{r d}$. A function $f$ is of class $f \in C_{p r d}^{r}$ if $f^{\Delta^{i}}$ is continuous for $i=0, \ldots, r-1$, and $f^{\Delta^{r}}$ exists and is rd-continuous for all, except possibly at finitely many $t \in \mathbb{T}^{\kappa^{r}}$.

A piecewise rd-continuous function $f: \mathbb{T} \rightarrow \mathbb{R}$ possess an antiderivative $F^{\Delta}=f$, and in this case the delta integral is defined by $\int_{c}^{d} f(t) \Delta t=F(d)-F(c)$ for all $c, d \in \mathbb{T}$. It satisfies

$$
\int_{t}^{\sigma(t)} f(\tau) \Delta \tau=\mu(t) f(t)
$$

If $\mathbb{T}=\mathbb{R}$, then $\int_{a}^{b} f(t) \Delta t=\int_{a}^{b} f(t) d t$, where the integral on the right hand side is the usual Riemann integral; if $\mathbb{T}=\mathbb{Z}$ and $a<b$, then $\int_{a}^{b} f(t) \Delta t=\sum_{k=a}^{b-1} f(k)$.

## 3. Main results

Consider the following higher-order problem of the calculus of variations up to order $r, r \geq 1$ :

$$
\begin{equation*}
\mathcal{L}(y(\cdot))=\int_{a}^{\rho^{r-1}(b)} L\left(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^{r}}(t)\right) \Delta t \longrightarrow \min \tag{1}
\end{equation*}
$$

subject to boundary conditions

$$
\begin{equation*}
y(a)=y_{a}^{0}, \quad y\left(\rho^{r-1}(b)\right)=y_{b}^{0}, \cdots, y^{\Delta^{r-1}}(a)=y_{a}^{r-1}, \quad y^{\Delta^{r-1}}\left(\rho^{r-1}(b)\right)=y_{b}^{r-1} \tag{2}
\end{equation*}
$$

where $\mathbb{T}$ is a bounded time scale with $a:=\min \mathbb{T}$ and $b:=\max \mathbb{T}, L:\left[a, \rho^{r}(b)\right]_{\mathbb{T}} \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ is a given function, where we use the notation $[c, d]_{\mathbb{T}}:=[c, d] \cap \mathbb{T}$, and $y_{a}^{i}, y_{b}^{i} \in \mathbb{R}, i=0, \ldots, r-1$. The results of the paper are trivially generalized for functions $y:[a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n}$, but for simplicity of presentation we restrict ourselves to the scalar case $n=1$.

A function $y(\cdot) \in C_{p r d}^{r}$ is said to be admissible if it is satisfies condition (21). An admissible $y(\cdot)$ is a weak local minimizer for (1)-(2) if there exists $\delta>0$ such that $\mathcal{L}(y(\cdot)) \leq \mathcal{L}(\bar{y}(\cdot))$ for any admissible $\bar{y} \in \mathrm{C}_{p r d}^{r}$ with $\|y-\bar{y}\|_{r, \infty}<\delta$, where

$$
\|y\|_{r, \infty}:=\sum_{i=0}^{r}\left\|y^{\Delta^{i}}\right\|_{\infty}
$$

$y^{\Delta^{0}}=y$ and $\|y\|_{\infty}:=\sup _{t \in\left[a, \rho^{r}(b)\right]_{\mathbb{T}}}|y(t)|$. For simplicity of notation we introduce the operator [y] defined by $[y](t)=\left(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^{r}}(t)\right)$. Then, functional (1) can be written as

$$
\mathcal{L}(y(\cdot))=\int_{a}^{\rho^{r-1}(b)} L[y](t) \Delta t
$$

We assume that $\left(t, u_{1}, \ldots, u_{r+1}\right) \rightarrow L\left(t, u_{1}, \ldots, u_{r+1}\right)$ has continuous partial derivatives $\frac{\partial L}{\partial u_{i}}$ for all $t \in\left[a, \rho^{r}(b)\right]_{\mathbb{T}}, i=1, \ldots, r+1$, and $t \rightarrow L[y](t)$ and $t \rightarrow \frac{\partial L}{\partial u_{i}}[y](t), i=1, \ldots, r+1$, are piecewise rd-continuous for all admissible functions $y(\cdot)$.

### 3.1. The higher-order Euler-Lagrange equation

We now prove the Euler-Lagrange equation for problem (11)-(21).
Remark 1. In order for the problem to be nontrivial we require the time scale $\mathbb{T}$ to have at least $2 r+1$ points. Indeed, if the time scale has only $2 r$ points, then it can be written as $\mathbb{T}=\left\{a, \sigma(a), \ldots, \sigma^{2 r-1}(a)\right\}$ and

$$
\begin{align*}
& \int_{a}^{\rho^{r-1}(b)} L\left(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^{r}}(t)\right) \Delta t \\
& =\int_{a}^{\sigma^{r}(a)} L\left(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^{r}}(t)\right) \Delta t=\sum_{i=0}^{r-1} \int_{\sigma^{i}(a)}^{\sigma^{i+1}(a)} L\left(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^{r}}(t)\right) \Delta t \\
&  \tag{3}\\
& =\sum_{i=0}^{r-1}\left(\sigma^{i+1}(a)-\sigma^{i}(a)\right) L\left(\sigma^{i}(a), y\left(\sigma^{i}(a)\right), y^{\Delta}\left(\sigma^{i}(a)\right), \ldots, y^{\Delta^{r}}\left(\sigma^{i}(a)\right)\right)
\end{align*}
$$

Having in mind the boundary conditions and the formula $f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$, we can conclude that the sum in (3) is constant for every admissible function $y(\cdot)$.

Theorem 1. If $y(\cdot)$ is a weak local minimizer for the problem (11)-(2), then $y(\cdot)$ satisfies the Euler-Lagrange equation

$$
\begin{align*}
& \frac{\partial L}{\partial y^{\Delta^{r}}}[y](t)-\int_{a}^{\sigma(t)} \frac{\partial L}{\partial y^{\Delta^{r-1}}}[y]\left(\tau_{r}\right) \Delta \tau_{r} \\
& \quad+\sum_{i=0}^{r-3}(-1)^{i} \int_{a}^{\sigma(t)} \int_{a}^{\sigma\left(\tau_{r}\right)} \cdots \int_{a}^{\sigma\left(\tau_{r-i}\right)} \frac{\partial L}{\partial y^{\Delta^{r-2-i}}}[y]\left(\tau_{r-1-i}\right) \Delta \tau_{r-1-i} \cdots \Delta \tau_{r-1} \Delta \tau_{r} \\
& (-1)^{r} \int_{a}^{\sigma(t)}\left\{\int_{a}^{\sigma\left(\tau_{r}\right)}\left[\cdots \int_{a}^{\sigma\left(\tau_{2}\right)} \frac{\partial L}{\partial y}[y]\left(\tau_{1}\right) \Delta \tau_{1}+c_{1} \cdots\right] \Delta \tau_{r-1}-(-1)^{r-1} c_{r-1}\right\} \Delta \tau_{r}-c_{r}=0 \tag{4}
\end{align*}
$$

for some constants $c_{1}, \ldots, c_{r}$ and all $t \in\left[a, \rho^{r}(b)\right]_{\mathbb{T}}$.
Proof. We first introduce some notation: $y_{0}(t)=y(t), y_{1}(t)=y^{\Delta}(t), \ldots, y_{r-1}(t)=y^{\Delta^{r-1}}(t)$, $u(t)=y^{\Delta^{r}}(t)$. Then problem (11)-(2) takes the following form:

$$
\begin{gathered}
\mathcal{L}[y(\cdot)]=\int_{a}^{\rho^{r-1}(b)} L\left(t, y_{0}(t), y_{1}(t), \ldots, y_{r-1}(t), u(t)\right) \Delta t \longrightarrow \min \\
\left\{\begin{array}{l}
y_{i}^{\Delta}(t)=y^{i+1}(t), \quad i=0, \ldots, r-2, \\
y_{r-1}^{\Delta}(t)=u(t),
\end{array}\right. \\
y^{j}(a)=y_{a}^{j}, y^{j}\left(\rho^{r-1}(b)\right)=y_{b}^{j}, j=0, \ldots, r-1
\end{gathered}
$$

With the notation $x=\left(y_{0}, y_{1}, \ldots, y_{r-1}\right)$, our problem (1)-(2) can be written as the optimal control problem

$$
\begin{gather*}
\mathcal{L}[x(\cdot)]=\int_{a}^{\rho^{r-1}(b)} L(t, x(t), u(t)) \Delta t \longrightarrow \min \\
x^{\Delta}(t)=A x(t)+B u(t)  \tag{5}\\
\varphi\left(x(a), x\left(\rho^{r-1}(b)\right)=\left[\begin{array}{l}
x(a)-x_{a} \\
x\left(\rho^{r-1}(b)\right)-x_{b}
\end{array}\right]=0\right.
\end{gather*}
$$

where

$$
A=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{array}\right), \quad B=\left(\begin{array}{c}
0 \\
\vdots \\
1
\end{array}\right)
$$

Note that assumption (A1) of [17, Theorem 9.4] holds: matrix $I+\mu(t) A$ is invertible, and the matrix $\nabla \varphi\left(x(a), x\left(\rho^{r-1}(b)\right)\right.$ has full rank. Therefore, if $(x(\cdot), u(\cdot))$ is a weak local minimum for (5), then there exists a constant $\lambda$ and a function $p:\left[a, \rho^{r-1}(b)\right]_{\mathbb{T}} \rightarrow \mathbb{R}^{r}, p \in C_{p r d}^{1}$, such that $(\lambda, p(\cdot)) \neq 0$ and the following conditions hold:

$$
\begin{gather*}
-p^{\Delta}(t)=A^{T} p^{\sigma}(t)+\lambda\left[\frac{\partial L}{\partial x}(t, x(t), u(t))\right]^{T} \\
B^{T} p^{\sigma}(t)+\lambda \frac{\partial L}{\partial u}(t, x(t), u(t))=0 \tag{6}
\end{gather*}
$$

for all $t \in\left[a, \rho^{r}(b)\right]_{\mathbb{T}}$. Consequently, if $y(\cdot)$ is a weak local minimizer for (11)-(2), then

$$
\begin{equation*}
p_{r-1}^{\sigma}(t)=-\lambda \frac{\partial L}{\partial u}[y](t) \tag{7}
\end{equation*}
$$

holds for all $t \in\left[a, \rho^{r}(b)\right]_{\mathbb{T}}$, where $p_{r-1}^{\sigma}(t)$ is defined recursively by

$$
\begin{align*}
p_{0}^{\sigma}(t) & =-\int_{a}^{\sigma(t)} \lambda \frac{\partial L}{\partial y_{0}}[y]\left(\tau_{1}\right) \Delta \tau_{1}-c_{1}  \tag{8}\\
p_{i}^{\sigma}(t) & =-\int_{a}^{\sigma(t)}\left[\lambda \frac{\partial L}{\partial y_{i}}[y]\left(\tau_{i+1}\right)+p_{i-1}^{\sigma}\left(\tau_{i+1}\right)\right] \Delta \tau_{i+1}-c_{i-1}, i=1, \ldots, r-1 \tag{9}
\end{align*}
$$

with $c_{i}, i=0, \ldots, r-1$, constants. From (7)-(9) we obtain that equation

$$
\begin{align*}
& \lambda \frac{\partial L}{\partial u}[y](t)-\int_{a}^{\sigma(t)} \lambda \frac{\partial L}{\partial y_{r-1}}[y]\left(\tau_{r}\right) \Delta \tau_{r} \\
& \quad+\sum_{i=0}^{r-3}(-1)^{i} \int_{a}^{\sigma(t)} \int_{a}^{\sigma\left(\tau_{r}\right)} \cdots \int_{a}^{\sigma\left(\tau_{r-i}\right)} \lambda \frac{\partial L}{\partial y_{r-2-i}}[y]\left(\tau_{r-1-i}\right) \Delta \tau_{r-1-i} \cdots \Delta \tau_{r-1} \Delta \tau_{r} \\
& (-1)^{r} \int_{a}^{\sigma(t)}\left\{\int_{a}^{\sigma\left(\tau_{r}\right)}\left[\cdots \int_{a}^{\sigma\left(\tau_{2}\right)} \lambda \frac{\partial L}{\partial y_{0}}[y]\left(\tau_{1}\right) \Delta \tau_{1}+c_{1} \cdots\right] \Delta \tau_{r-1}-(-1)^{r-1} c_{r-1}\right\} \Delta \tau_{r}-c_{r}=0 \tag{10}
\end{align*}
$$

holds for all $t \in\left[a, \rho^{r}(b)\right]_{\mathbb{T}}$. We show next that $\lambda \neq 0$. First observe that if $f \in C_{p r d}^{1}$ and $f^{\sigma}(t)=0$ for all $t \in[a, b]_{\mathbb{T}}^{\kappa}$, then $f(t)=0$ for all $t \in[\sigma(a), b]_{\mathbb{T}}$. Suppose, contrary to our claim, that $\lambda=0$ in equation (6) and (77). Then, we can write the system of equations

$$
\begin{cases}p_{0}^{\Delta}(t) & =0,  \tag{11}\\ p_{i}^{\Delta}(t) & =-p_{i-1}^{\sigma}(t), \quad i=1, \ldots, r-1 \\ p_{r-1}^{\sigma}(t) & =0,\end{cases}
$$

for all $t \in\left[a, \rho^{r}(b)\right]_{\mathbb{T}}$. From the last equation we have $p_{r-1}(t)=0, \forall t \in\left[\sigma(a), \rho^{r-1}(b)\right]_{\mathbb{T}}$. This implies that $p_{r-1}^{\Delta}(t)=0, \forall t \in\left[\sigma(a), \rho^{r}(b)\right]_{\mathbb{T}}$, and consequently $p_{r-2}^{\sigma}(t)=0, \forall t \in\left[\sigma(a), \rho^{r}(b)\right]_{\mathbb{T}}$. Therefore, $p_{r-2}(t)=0, \forall t \in\left[\sigma^{2}(a), \rho^{r-1}(b)\right]_{\mathbb{T}}$. Repeating this procedure we have $p_{1}(t)=0$ for all $t \in\left[\sigma^{r-1}(a), \rho^{r-1}(b)\right]_{\mathbb{T}}$. Hence, $0=p_{1}^{\Delta}(t)=-p_{0}^{\sigma}(t)=-p_{0}^{\Delta}(t) \mu(t)-p_{0}(t)=-p_{0}(t)$ for all $t \in\left[\sigma^{r-1}(a), \rho^{r}(b)\right]_{\mathbb{T}}$. Note that the first equation of (11) implies $p_{0}(t)=c$ for some constant $c$ and all $t \in\left[a, \rho^{r-1}(b)\right]_{\mathbb{T}}$. Since the time scale has at least $2 r+1$ points (see Remark (1), the set $t \in\left[\sigma^{r-1}(a), \rho^{r-1}(b)\right]_{\mathbb{T}}$ is nonempty and we conclude that $p_{0}(t)=0$ for all $t \in\left[a, \rho^{r-1}(b)\right]_{\mathbb{T}}$. Substituting this into the second equation we get $p_{1}^{\Delta}(t)=d$ for some constant $d$ and all $t \in$ $\left[a, \rho^{r-1}(b)\right]_{\mathbb{T}}$. Having in mind that $p_{1}\left(t_{0}\right)=0$ for some $t_{0} \in\left[a, \rho^{r-1}(b)\right]_{\mathbb{T}}$ we obtain $p_{1}(t)=0$ for all $t \in\left[a, \rho^{r-1}(b)\right]_{\mathbb{T}}$. Repeating this procedure we conclude that $p_{i}(t)=0, i=1, \ldots, r-1$, for all $t \in\left[a, \rho^{r-1}(b)\right]_{\mathbb{T}}$. This contradicts the fact that $(\lambda, p(\cdot)) \neq 0$. Hence, equation (10) can be divided by $\lambda$ and (4) is proved.

### 3.2. Corollaries

For illustrating purposes we consider now the two simplest situations, i.e., $r=1$ and $r=2$.
Corollary 1 (cf. [6, 16]). If $y(\cdot)$ is a weak local minimizer for the problem

$$
\mathcal{L}(y(\cdot))=\int_{a}^{b} L\left(t, y(t), y^{\Delta}(t)\right) \Delta t \longrightarrow \min
$$

subject to boundary conditions $y(a)=y_{a}$ and $y(b)=y_{b}$, then $y(\cdot)$ satisfies the Euler-Lagrange equation

$$
\frac{\partial L}{\partial y^{\Delta}}\left(t, y(t), y^{\Delta}(t)\right)=\int_{a}^{\sigma(t)} \frac{\partial L}{\partial y}\left(\tau, y(\tau), y^{\Delta}(\tau)\right) \Delta \tau+c_{1}
$$

for some constant $c_{1}$ and all $t \in[a, b]_{\mathbb{T}}^{\mathcal{K}}$.
Corollary 2 (cf. [10, 25]). If $y(\cdot)$ is a weak local minimizer for the problem

$$
\mathcal{L}(y(\cdot))=\int_{a}^{\rho(b)} L\left(t, y(t), y^{\Delta}(t), y^{\Delta \Delta}\right) \Delta t \longrightarrow \min
$$

subject to boundary conditions $y(a)=y_{a}^{0}, y(\rho(b))=y_{b}, y^{\Delta}(a)=y_{a}^{1}$, and $y^{\Delta}(\rho(b))=y_{b}^{1}$, then $y(\cdot)$ satisfies the Euler-Lagrange equation

$$
\begin{aligned}
\frac{\partial L}{\partial y^{\Delta \Delta}}\left(t, y(t), y^{\Delta}(t), y^{\Delta \Delta}(t)\right) & -\int_{a}^{\sigma(t)} \frac{\partial L}{\partial y^{\Delta}}\left(\tau_{2}, y\left(\tau_{2}\right), y^{\Delta}\left(\tau_{2}\right), y^{\Delta \Delta}\left(\tau_{2}\right)\right) \Delta \tau_{2} \\
& +\int_{a}^{\sigma(t)}\left[\int_{a}^{\sigma\left(\tau_{2}\right)} \frac{\partial L}{\partial y}\left(\tau_{1}, y\left(\tau_{1}\right), y^{\Delta}\left(\tau_{1}\right), y^{\Delta \Delta}\left(\tau_{1}\right)\right) \Delta \tau_{1}+c_{1}\right] \Delta \tau_{2}-c_{2}=0
\end{aligned}
$$

for some constants $c_{1}$ and $c_{2}$ and all $t \in[a, \rho(b)]_{\mathbb{T}}^{\kappa}$.

### 3.3. An example

Let $\mathbb{T}=[a, b] \cap h \mathbb{Z}$, where $h \mathbb{Z}:=\{h z \mid z \in \mathbb{Z}\}, h>0$. Then for any $f \in C_{p r d}^{r}$ we have
where $f^{\Delta^{i} \sigma^{j-i}}(t)$ stands for $f^{\Delta^{i}}\left(\sigma^{j-i}(t)\right)$. We will show this by induction. For $j=1$

$$
\int_{a}^{\sigma(t)} f(\xi) \Delta \xi=\int_{a}^{t} f(\xi) \Delta \xi+\int_{t}^{t+h} f(\xi) \Delta \xi=\int_{a}^{t} f(\xi) \Delta \xi+h f(t)
$$

and then $\left[\int_{a}^{\sigma(t)} f(\xi) \Delta \xi\right]^{\Delta}=f(t)+h f^{\Delta}(t)=f^{\sigma}$. Now assume that (12) is true for all $j=1, \ldots, k$. Then for $j=k+1$

$$
\begin{aligned}
& \underbrace{\left[\int_{a}^{\sigma(t)}\left(\int_{a}^{\sigma} \cdots \int_{a}^{\sigma} f\right) \Delta \tau\right]^{\Delta^{k+1}}}_{k+1-i \text { integrals }}=(\underbrace{\int_{a}^{t} \int_{a}^{\sigma} \cdots \int_{a}^{\sigma}}_{k+1-i} f \Delta \tau+h \underbrace{\int_{a}^{\sigma(t)} \cdots \int_{a}^{\sigma}}_{k-i} f \Delta \tau)^{\Delta^{k+1}} \\
& =(\underbrace{\int_{a}^{\sigma(t)} \cdots \int_{a}^{\sigma}}_{k-i} f \Delta \tau)^{\Delta^{k}}+[h(\underbrace{\int_{a}^{\sigma(t)} \cdots \int_{a}^{\sigma}}_{k-i} f \Delta \tau)^{\Delta^{k}}]^{\Delta}=f^{\Delta^{i} \sigma^{k-i}}+\left(h f^{\Delta^{i} \sigma^{k-i}}\right)^{\Delta}=f^{\Delta^{i} \sigma^{k+1-i}} .
\end{aligned}
$$

Delta differentiating $r$ times both sides of equation (4) and in view of (12), we obtain the $h$-EulerLagrange equation in delta differentiated form:

$$
L_{y^{\Delta^{r}}}^{\Delta^{r}}\left(t, y, y^{\Delta}, \ldots, y^{\Delta^{r}}\right)+\sum_{i=0}^{r-1}(-1)^{r-i} L_{y^{\Delta^{i}}}^{\Delta^{i} \sigma^{r-i}}\left(t, y, y^{\Delta}, \ldots, y^{\Delta^{r}}\right)=0
$$

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