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A new hypergeometric transformation of the Rathie–Rakha type

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ABSTRACT

A general transformation involving generalized hypergeometric functions has been recently found by Rathie and Rakha using simple arguments and exploiting Gauss's summation theorem. In this sequel to the work of Rathie and Rakha, a new hypergeometric transformation formula is derived by their method and by appealing to Gauss's second summation theorem. In addition, it is shown that the method fails to give similar hypergeometric transformations in the cases of the classical summation theorems of Kummer, Bailey, Watson and Dixon.

1. Introduction

In a recent paper by Rathie and Rakha [1, Eq. (2.1)], the following general transformation involving generalized hypergeometric functions (see Section 2):

$$ {}_{p+2}F_{q+2}\left[\begin{matrix} a, & b, & a_1,\ldots,a_p \\ c-a+i, & c-b+i+j, & b_1,\ldots,b_q \end{matrix}\middle| z\right] = \frac{\Gamma(c-a+i)\,\Gamma(c-b+i+j)}{\Gamma(c+i)\Gamma(c-a-b+i+j)} \sum_{n=0}^{\infty} \frac{(a)_n(b-j)_n}{(c+i)_n n!} $$
$$ \times {}_{p+2}F_{q+2}\left[\begin{matrix} -n, & b, & a_1,\ldots,a_p \\ b-j, & c+i+n, & b_1,\ldots,b_q \end{matrix}\middle| -z\right] \quad (i,j=0,1,2,\ldots) \tag{1.1} $$

has been established using simple arguments and by exploiting the well-known Gauss summation theorem for ${}_2F_1$ [2, p. 243, Eq. (III.3)]:

$$ {}_2F_1\left[\begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix}\middle| 1\right] = \frac{\Gamma(\gamma)\;\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\;\Gamma(\gamma-\beta)} \quad (\Re(\gamma-\alpha-\beta)>0). \tag{1.2} $$

Numerous (new and known) hypergeometric transformations and identities are then easily deduced as its special cases or simple consequences.

In this sequel to the work of Rathie and Rakha, it is shown that an additional new hypergeometric transformation formula of the same type:

$$ {}_{p+2}F_{q+2}\left[\begin{matrix} \frac{1}{2}(a+b)+\frac{1}{4}, & \frac{1}{2}(a+b)+\frac{3}{4}, & a_1,\ldots,a_p \\ a+\frac{1}{2}, & b+\frac{1}{2}, & b_1,\ldots,b_q \end{matrix}\middle| z\right] = \frac{\Gamma\left(a+\frac{1}{2}\right)\,\Gamma\left(b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(a+b+\frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{(2a)_n(2b)_n}{\left(a+b+\frac{1}{2}\right)_n n!} $$
$$ \times {}_{p+5}F_{q+5}\left[\begin{matrix} -n, & 2a+n, & 2b+n, & \frac{1}{2}(a+b)+\frac{1}{4}, & \frac{1}{2}(a+b)+\frac{3}{4}, & a_1,\ldots,a_p \\ a+b+\frac{1}{2}+n, & a, & b, & a+\frac{1}{2}, & b+\frac{1}{2}, & b_1,\ldots,b_q \end{matrix}\middle| -\frac{z}{8}\right] \tag{1.3} $$

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can be derived by similar arguments and by appealing to *Gauss's second summation theorem* [2, p. 243, Eq. (III.6)]:

$$ {}_2F_1\left[\begin{matrix}\alpha, & \beta \\ \frac{1}{2}(\alpha+\beta+1) & \end{matrix}\middle|\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}. \tag{1.4} $$

2. Preliminaries and proof of the main result

For nonnegative integers p and q, the generalized hypergeometric function in a variable (argument) z with p numerator parameters $a_1,\ldots,a_p$ and q denominator parameters $b_1,\ldots,b_q$ is, as usual, defined by means of the hypergeometric series (see [2, Chapts. 1 and 2], [3, pp. 44–55] and [4, Chapt. 7])

$$ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{matrix}\middle| z\right] = \sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{z^m}{m!} \tag{2.1} $$

whenever this series converges and elsewhere by analytic continuation. Here $(\alpha)_m$ stands for the Pochhammer symbol defined by (see [3, pp. 6–8] and [4, p. 758])

$$ (\alpha)_n = \begin{cases} 1 & (n=0;\,\alpha\neq 0) \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (n\in\mathbb{N}) \end{cases} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \tag{2.2} $$

Γ being the familiar Gamma function, which is meromorphic in the whole complex plane with poles at $z = 0, -1, -2, \ldots$.

The ${}_pF_q$ function is symmetric in its numerator parameters, and likewise in its denominator parameters, and, in general, the variable z, the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ take on complex values, provided that no denominator parameter is allowed to be zero or a negative integer. The series defining ${}_pF_q$ converges for all values of z when $p \le q$. If $p = q+1$ the series converges when $|z| < 1$, when $z = 1$ if $\Re(b_1+\cdots+b_q-a_1-\cdots-a_p) > 0$ and when $z = -1$ if $\Re(b_1+\cdots+b_q-a_1-\cdots-a_p) > -1$.

In order to derive (1.3), we proceed as follows. We first recall the well-known identity ([3, p. 7, Eq. (44)] and [4, p. 758])

$$ (\alpha)_{n+m} = (\alpha)_m(\alpha+m)_n. \tag{2.3} $$

Starting with the left-hand side of (1.3) and calling it $S(z)$, upon recalling the definition of the hypergeometric function ${}_pF_q$ in (2.1), we have

$$ S(z) = \sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}(a+b)+\frac{1}{4}\right)_m\left(\frac{1}{2}(a+b)+\frac{3}{4}\right)_m}{\left(a+\frac{1}{2}\right)_m\left(b+\frac{1}{2}\right)_m}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{z^m}{m!}. $$

However, by making use of (see [3, p. 7, Eq. (48)] and [4, p. 758])

$$ (\alpha)_{2m} = 2^{2m}\left(\frac{\alpha}{2}\right)_m\left(\frac{\alpha+1}{2}\right)_m, \tag{2.4} $$

which when applied on $\left(\frac{1}{2}(a+b)+\frac{1}{4}\right)_m\left(\frac{1}{2}(a+b)+\frac{3}{4}\right)_m$ gives $2^{-2m}\left(a+b+\frac{1}{2}\right)_{2m}$, as well as the relation given by (2.2), $S(z)$ may be rewritten as

$$ S(z) = M_{a,b}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(a+b+\frac{1}{2}+2m\right)}{\Gamma\left(a+\frac{1}{2}+m\right)\Gamma\left(b+\frac{1}{2}+m\right)}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{1}{m!}\left(\frac{z}{4}\right)^m \tag{2.5} $$

with

$$ M_{a,b} = \frac{\Gamma\left(a+\frac{1}{2}\right)\,\Gamma\left(b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(a+b+\frac{1}{2}\right)}. $$

Next, by applying Gauss's second summation theorem (1.4) with

$$ \alpha = 2a+2m \quad\text{and}\quad \beta = 2b+2m, \tag{2.6} $$

we obtain

$$ S(z) = M_{a,b}\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{1}{m!}\left(\frac{z}{4}\right)^m {}_2F_1\left[\begin{matrix}2a+2m,\ 2b+2m \\ a+b+\frac{1}{2}+2m\end{matrix}\middle|\frac{1}{2}\right], \tag{2.7} $$

provided that $a, b, a+b \neq -\frac{1}{2}(2m-1)$, $m \in \mathbb{N}$ (see Section 3), so, upon expressing ${}_2F_1$ as a hypergeometric series (with an index n) by (2.1) and using the identity (2.3), we have

$$\begin{aligned} S(z) &= M_{a,b} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2a+2m)_n\,(2b+2m)_n}{\left(a+b+\frac{1}{2}+2m\right)_n} \frac{1}{n!}\left(\frac{1}{2}\right)^n \frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{1}{m!}\left(\frac{z}{4}\right)^m \\ &= M_{a,b} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2a)_{n+2m}(2b)_{n+2m}}{\left(a+b+\frac{1}{2}\right)_{n+2m}} \frac{\left(\frac{1}{2}\right)^n}{n!} \frac{\left(a+b+\frac{1}{2}\right)_{2m}}{(2a)_{2m}(2b)_{2m}} \frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{1}{m!}\left(\frac{z}{4}\right)^m. \end{aligned}$$

Finally, in view of the identity [5, p. 56, Eq. (1)]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m)$$

and by appealing to (2.4) and [3, p. 7, Eq. (47)]

$$(-n)_m = \frac{(-1)^m n!}{(n-m)!} \quad (0 \leq m \leq n),$$

we get

$$\begin{aligned} S(z) &= M_{a,b} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(2a)_{n+m}(2b)_{n+m}}{\left(a+b+\frac{1}{2}\right)_{n+m}} \frac{\left(\frac{1}{2}\right)^{n-m}}{(n-m)!} \frac{\left(a+b+\frac{1}{2}\right)_{2m}}{(2a)_{2m}(2b)_{2m}} \frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{1}{m!}\left(\frac{z}{4}\right)^m \\ &= M_{a,b} \sum_{n=0}^{\infty} \frac{(2a)_n(2b)_n}{\left(a+b+\frac{1}{2}\right)_n}\frac{1}{n!}\left(\frac{1}{2}\right)^n \sum_{m=0}^{n} \frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{1}{m!}\left(-\frac{z}{8}\right)^m \\ &\quad \times \frac{(-n)_m(2a+n)_m(2b+n)_m\left(\frac{1}{2}(a+b)+\frac{1}{4}\right)_m\left(\frac{1}{2}(a+b)+\frac{3}{4}\right)_m}{\left(a+b+\frac{1}{2}+n\right)_m(a)_m(b)_m\left(a+\frac{1}{2}\right)_m\left(b+\frac{1}{2}\right)_m} \end{aligned}$$

and it is clear that the last expression is, in fact, the right-hand side of the proposed result in (1.3). This completes the proof of (1.3).

3. Concluding remarks

Note that one step in the derivation of the hypergeometric transformation (1.3) given in the previous section needs to be explained and justified in more detail. Namely, the transition from Eq. (2.5) to Eq. (2.7) involves Gauss's second summation formula (1.4) and the validity of its application should be considered in connection with the values of the parameters involved. Gauss's second summation formula itself has no restrictions on the values of parameters and only the restrictions due to the definitions of the gamma and hypergeometric ${}_2F_1$ functions are present. Having this in mind, it follows by (2.5) and (2.6) that Gauss's formula can be applied provided that the following conditions are satisfied: $a, b, a+b \neq -\frac{1}{2}(2m-1)$, $m \in \mathbb{N}$.

Rathie and Rakha [1] did not analyze the values of the parameters while deriving their hypergeometric transformation (1.1) but it can readily be shown that in this derivation Gauss's summation formula (1.2) is validly applied on condition that

$$\Re(c-a-b+i+j) > 0 \quad (i,j = 0,1,2,\ldots). \tag{3.1}$$

Indeed, starting with the left-hand side of (1.1), we have that (see [1, p. 6] or use (2.1) and (2.2))

$$\begin{aligned} {}_{p+2}F_{q+2}\left[\begin{matrix} a, & b, & a_1,\ldots,a_p \\ c-a+i, & c-b+i+j, & b_1,\ldots,b_q \end{matrix}\middle| z\right] &= \frac{\Gamma(c-a+i)\,\Gamma(c-b+i+j)}{\Gamma(c+i)\,\Gamma(c-a-b+i+j)} \\ &\times \sum_{m=0}^{\infty} \frac{\Gamma(c+i+2m)\,\Gamma(c-a-b+i+j)}{\Gamma(c-a+i+m)\,\Gamma(c-b+i+j+m)} \frac{(a)_m(b)_m}{(c+i)_{2m}} \frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{z^m}{m!}, \end{aligned}$$

and thus, (3.1) follows from this and (1.2) with $\alpha = a+m$, $\beta = b+m-j$ and $\gamma = c+i+2m$.

We conclude by remarking that, by utilizing Gauss's summation theorem, a general transformation involving generalized hypergeometric functions has been recently found by Rathie and Rakha [1]. In this work, by using their method and Gauss's second summation theorem, we have deduced a new hypergeometric transformation of the Rathie–Rakha type. However, we have failed to extend the method and deduce similar transformations by using Kummer's, Bailey's, Watson's and Dixon's theorems. The reason is that these theorems cannot be (rigorously) applied due to restrictions on the values of the parameters. As an illustration, consider the case of Kummer's summation theorem [2, p. 243, Eq. (III.5)]

$${}_2F_1\left[\begin{matrix} \alpha, & \beta \\ \alpha-\beta+1 & \end{matrix}\middle| -1\right] = \frac{\Gamma(\alpha-\beta+1)\,\Gamma\left(\frac{1}{2}\alpha+1\right)}{\Gamma\left(\frac{1}{2}\alpha-\beta+1\right)\,\Gamma(\alpha+1)}. \tag{3.2}$$

By a formal application of the method used by Rathie and Rakha and (3.2) we obtain

$$
{}_{p+1}F_{q+1}\left[\begin{matrix}2a-b+1, & a_1,\ldots,a_p\\ a+\frac{1}{2}, & b_1,\ldots,b_q\end{matrix}\middle| z\right] = \frac{\Gamma(2a+1)\,\Gamma(a-b+1)}{\Gamma(a+1)\,\Gamma(2a-b+1)}\sum_{n=0}^{\infty}(-1)^n\frac{(2a)_n(b)_n}{(2a-b+1)_n n!}\,{}_{p+3}F_{q+3}
$$

$$
\times\left[\begin{matrix}-n, & 2a+n, & 2a-b+1, & a_1,\ldots,a_p\\ a, & b, & a+\frac{1}{2}, & b_1,\ldots,b_q\end{matrix}\middle| z\right]. \tag{3.3}
$$

However, Kummer's formula (3.2) is valid when $\Re(\alpha-\beta+1-\alpha-\beta) > -1$ (see Section 2), *i.e.* when $\Re(\beta) < 1$, and it is impossible to apply it to the expression $\Gamma(a+1+m)\,\Gamma(2a-b+1+m)/[\Gamma(2a+1+2m)\,\Gamma(a-b+1)]$ in

$$
{}_{p+1}F_{q+1}\left[\begin{matrix}2a-b+1, & a_1,\ldots,a_p\\ a+\frac{1}{2}, & b_1,\ldots,b_q\end{matrix}\middle| z\right] = \frac{\Gamma(2a+1)\,\Gamma(a-b+1)}{\Gamma(a+1)\,\Gamma(2a-b+1)}
$$

$$
\times\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\,\frac{(4z)^m}{m!}\,\frac{\Gamma(a+1+m)\,\Gamma(2a-b+1+m)}{\Gamma(2a+1+2m)\,\Gamma(a-b+1)}, \tag{3.4}
$$

since $\alpha = 2a+2m$ and $\beta = b+m$ and then we have $\Re(\beta+m) < 1$, $m \in \mathbb{N}_0$, which obviously fails for m large enough. Observe that (3.4) is derived starting from the left-hand side of (3.3) by making use of (2.4) which is applied on $(a+1)_m\left(a+\frac{1}{2}\right)_m$ and gives $2^{-2m}(2a+1)_{2m}$, as well as the definitions of the Pochhammer symbol $(\alpha)_m$ in (2.2) and the hypergeometric function ${}_pF_q$ in (2.1). Similar reasoning shows that the method used here and in [1] fails to give hypergeometric transformations similar to (1.2) and (1.3) in the cases of the classical theorems of Bailey, Watson and Dixon.

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