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ON THE PERIODIC ORBITS OF THE THIRD-ORDER DIFFERENTIAL EQUATION $x''' - \mu x'' + x' - \mu x = \varepsilon F(x, x', x'')$

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Abstract. In this paper we study the periodic orbits of the third-order differential equation $x''' - \mu x'' + x' - \mu x = \varepsilon F(x, x', x'')$, where ε is a small parameter and the function F is of class $\mathcal{C}^2$.

1. Introduction and statement of the main results

In the qualitative theory of differential equations one of the main problems is the study of their periodic orbits, their existence, their number and their stability. A *limit cycle* of a differential equation is a periodic orbit isolated in the set of all periodic orbits of the differential equation.

In this paper we deal with the third-order differential equation

$$x''' - \mu x'' + x' - \mu x = \varepsilon F(x, x', x''), \tag{1}$$

Here the variables x and t, and the parameters μ and ε are real, moreover ε is a small real parameter and the function $F : \Omega \to \mathbb{R}$ is of class $\mathcal{C}^2$. Here Ω is an open subset of $\mathbb{R}^3$. The prime denotes derivative with respect to an independent variable t. The objective is to study the periodic solutions of this differential equation.

There are many papers studying the periodic orbits of third–order differential equations. In particular our class of equations (1) is not far from the ones studied in [13] and [2]. But our main tool for studying the periodic orbits of equation (1) is completely different to the tools of the mentioned papers. We shall use the *averaging theory*, more precisely the Theorems 3 and 4 of the appendix. Many of the papers dealing with the periodic orbits of third–order differential equations use Schauder's or Leray–Schauder's fixed point theorem, see for instance [4, 8, 9], or the nonlocal reduction method see [1], and others [5]. The non-autonomous case of the differential equation (1) was studied in [6] with $\mu \neq 0$. As in [6], our main tool for study the periodic orbits of equation (1), was the averaging theory. But in [6] they only need to use Theorem 3, and here we shall use Theorem 3 when $\mu \neq 0$ and Theorem 4 when $\mu = 0$.

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We recall that a *simple zero* r_0^* of a function $\mathcal{F}(r_0)$ is defined by $\mathcal{F}(r_0^*) = 0$ and $(d\mathcal{F}(r_0^*)/dr_0) \neq 0$.

The main results on the periodic solutions of the third-order differential equation (1) are the following two theorems.

Theorem 1. *Consider $\mu \neq 0$ into the differential equation* (1)*. For $\varepsilon \neq 0$ sufficiently small and every positive simple zero r_0^* of the function*

$$\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} F(A, B, C)\cos\theta\, d\theta,$$

where

$$A = -\frac{r_0(\cos\theta + \mu\sin\theta)}{1+\mu^2},\ B = \frac{r_0(\sin\theta - \mu\cos\theta)}{1+\mu^2},\ C = \frac{r_0(\cos\theta + \mu\sin\theta)}{1+\mu^2},$$

the differential equation (1) *has a periodic solution $x_\varepsilon(t)$ tending to the periodic solution*

$$x(t) = -\frac{r_0^*(\cos t + \mu\sin t)}{1+\mu^2},$$

of $x''' - \mu x'' + x' - \mu x = 0$ when $\varepsilon \to 0$.

In order to state the next result we need the following definitions

$$f_1(r_0, Z_0) = \frac{1}{2\pi}\int_0^{2\pi} F(\alpha, \beta, \gamma)\cos\theta\, d\theta,$$
$$f_2(r_0, Z_0) = \frac{1}{2\pi}\int_0^{2\pi} F(\alpha, \beta, \gamma) d\theta,$$

where

$$\alpha = -Z_0 - r_0\cos\theta, \quad \beta = r_0\sin\theta, \quad \gamma = r_0\cos\theta.$$

Theorem 2. *Consider $\mu = 0$ into the differential equation* (1)*. If there exists (r_0, Z_0) such that*

$$f_1(r_0, Z_0) = 0, \quad f_2(r_0, Z_0) = 0 \quad \text{and} \quad \det\left(\frac{\partial(f_1, f_2)}{\partial(r, Z)}\right)(r_0, Z_0) \neq 0,$$

then the differential equation (1) *with $\mu = 0$ has a periodic solution $x_\varepsilon(t)$ tending to the periodic solution*

$$x(t) = -r_0\cos t - Z_0,$$

of $x''' + x' = 0$ when $\varepsilon \to 0$.

These two theorems are proved in the next section. The proofs are based on the averaging theory for computing periodic orbits, see the appendix.

2. Proof of the theorems

We start with some preliminaries. Indeed with some change of coordinates that will be used in the proofs of Theorems 1 and 2.

If $y = x'$ and $z = x''$, then we write the third-order differential equation (1) as a first-order differential system in the open subset $\Omega \subset \mathbb{R}^3$. Thus we have the differential system

$$
\begin{aligned}
x' &= y,\\
y' &= z,\\
z' &= -y + \mu(x+z) + \varepsilon F(x,y,z).
\end{aligned}
\tag{2}
$$

System (2) with $\varepsilon = 0$ will be called the *unperturbed system*, otherwise we have the *perturbed system*. The unperturbed system has a unique singular point, the origin with eigenvalues $i, -i, \mu$. Doing the linear change of variables $(X,Y,Z)^T = C(x,y,z)^T$ with

$$
C = \begin{pmatrix} 0 & -\mu & 1\\ -\mu & 1 & 0\\ -1 & 0 & -1 \end{pmatrix},
$$

we transform the differential system (2) into the next differential system having its linear part in the real Jordan normal form, i.e.

$$
\begin{aligned}
X' &= -Y + \varepsilon \widetilde{F}(X,Y,Z),\\
Y' &= X,\\
Z' &= \mu Z - \varepsilon \widetilde{F}(X,Y,Z),
\end{aligned}
\tag{3}
$$

where $\widetilde{F}(X,Y,Z) = F(A,B,C)$ being

$$
A = -\frac{X+Z+\mu Y}{1+\mu^2}, \quad B = \frac{Y-\mu(X+Z)}{1+\mu^2}, \quad C = \frac{X+\mu(Y-\mu Z)}{1+\mu^2}.
$$

Now we pass from the cartesian variables (X,Y,Z) to the cylindrical ones (r,θ,Z) of $\mathbb{R}^3$, where $X = r\cos\theta$ and $Y = r\sin\theta$. In these new variables the differential system (3) becomes

$$
\begin{aligned}
r' &= \varepsilon\, G(r,\theta,Z)\cos\theta,\\
\theta' &= 1 - \varepsilon \frac{G(r,\theta,Z)\sin\theta}{r},\\
Z' &= \mu Z - \varepsilon\, G(r,\theta,Z),
\end{aligned}
\tag{4}
$$

where $G(r,\theta,Z) = \widetilde{F}(r\cos\theta, r\sin\theta, Z)$.

Now we change the independent variable from t to θ, and denote the derivative with respect to θ by a dot. Therefore the differential system (4) becomes

$$
\begin{aligned}
\dot{r} &= \frac{dr}{d\theta} = \varepsilon\, G(r,\theta,Z)\cos\theta + O(\varepsilon^2),\\
\dot{Z} &= \frac{dZ}{d\theta} = \mu Z + \varepsilon \frac{\mu Z\sin\theta - r}{r} G(r,\theta,Z) + O(\varepsilon^2).
\end{aligned}
\tag{5}
$$

We must study the following cases separately: $\mu \neq 0$ and $\mu = 0$.

Case $\mu \neq 0$. We can write system (5) as follows

$$\dot{\mathbf{x}} = F_0(\theta, \mathbf{x}) + \varepsilon F_1(\theta, \mathbf{x}, \varepsilon) + O(\varepsilon^2), \tag{6}$$

where

$$\mathbf{x} = \begin{pmatrix} r \\ Z \end{pmatrix}, \; F_0(\theta, \mathbf{x}) = \begin{pmatrix} 0 \\ \mu Z \end{pmatrix}, \; F_1(\theta, \mathbf{x}, \varepsilon) = \begin{pmatrix} G(r, \theta, Z) \cos\theta \\ \\ \dfrac{\mu Z \sin\theta - r}{r} G(r, \theta, Z) \end{pmatrix}.$$

We shall study the periodic solutions of system (6) using the averaging theory, more precisely Theorem 3 of the appendix. First we look for the periodic solutions of the unperturbed system

$$\dot{\mathbf{x}} = F_0(\theta, \mathbf{x}).$$

Note that these periodic solutions are

$$(r(\theta), Z(\theta)) = (r_0, 0),$$

for every $r_0 > 0$, i.e. these periodic orbits are circles in the plane $Z = 0$ for system (4). Of course all these periodic solutions in the coordinates (r, Z) are $2\pi-$periodic in the variable θ.

We shall describe the different elements which appear in the statement of Theorem 3 in the particular case of the differential system (5). Thus we have that $k = 1$ and $n = 2$. Let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. Then we take the open bounded subset V of $\mathbb{R}$ as $V = (r_1, r_2)$, $\alpha = r_0 \in V$, $\beta : [r_1, r_2] \to \mathbb{R}$ is defined as $\beta(r_0) = 0$. The set $\mathcal{Z}$ is

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, 0), r_0 \in [r_1, r_2]\}.$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider that the $2\pi-$periodic solution $\mathbf{x}(\theta) = \mathbf{z}_\alpha = (r_0, 0)$.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear differential system (14) associated to the $2\pi-$periodic solution $\mathbf{z}_\alpha = (r_0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^2$, we get

$$M(\theta) = M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\mu\theta} \end{pmatrix}.$$

Note that the matrix $M(\theta)$ does not depend of the particular periodic orbit z_α. Since the matrix

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 - e^{-2\pi\mu} \end{pmatrix},$$

is not identically zero because $\mu \neq 0$, it satisfies the assumptions of statement (ii) of Theorem 3, and we can apply this theorem to system (5).

Now $\xi : \mathbb{R}^2 \to \mathbb{R}$ is $\xi(r, Z) = r$. We calculate the function (15)

$$\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} F(A, B, C)\cos\theta\, d\theta,$$

where A, B and C are defined in the statement of Theorem 1. Then by Theorem 3 we have that for every simple zero $r_0^* \in [r_1, r_2]$ of the function $\mathcal{F}(r_0)$ we have a periodic solution $(r_\varepsilon(\theta), Z_\varepsilon(\theta))$ of system (5) such that

$$(r_\varepsilon(0), Z_\varepsilon(0)) \to (r_0^*, 0) \quad \text{as} \quad \varepsilon \to 0.$$

Going back through the changes of coordinates we get a periodic solution $(r_\varepsilon(t), \theta_\varepsilon(t), Z_\varepsilon(t))$ of system (4) such that

$$(r_\varepsilon(t), \theta_\varepsilon(t), Z_\varepsilon(t)) \to (r_0^*, t, 0) \quad \text{as} \quad \varepsilon \to 0.$$

Consequently we obtain a periodic solution $(X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t))$ of system (3) such that

$$(X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t)) \to (r_0^*\cos t, r_0^*\sin t, 0) \quad \text{as} \quad \varepsilon \to 0.$$

Therefore since

$$x = -\frac{X + Z + \mu Y}{1 + \mu^2}, \tag{7}$$

we have a periodic solution $(x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t))$ of the system (2) such that

$$x_\varepsilon(t) \to -\frac{r_0^*(\cos t + \mu\sin t)}{1 + \mu^2} \quad \text{as} \quad \varepsilon \to 0.$$

Note that the previous expression provides a periodic solution of the linear differential equation $x''' - \mu x'' + x' - \mu x = 0$. Hence we have proved Theorem 1.

Case $\mu = 0$. System (5) with $\mu = 0$ becomes

$$\begin{aligned} \dot{r} &= \frac{dr}{d\theta} = \varepsilon\, G(r, \theta, Z)\cos\theta + O(\varepsilon^2), \\ \dot{Z} &= \frac{dZ}{d\theta} = -\varepsilon G(r, \theta, Z) + O(\varepsilon^2), \end{aligned} \tag{8}$$

with $(r, Z) \in D \subset \mathbb{R}^2$ and D an open subset in $\mathbb{R}^2$.

Note that system (8) is in standard form of averaging theory for applying Theorem 4, see (16). Then the averaged differential system is

$$(\dot{r}, \dot{Z}) = \varepsilon g^0(r, Z), \tag{9}$$

where

$$g^0(r, Z) = \frac{1}{2\pi}\int_0^{2\pi} (G(r, s, Z)\cos s, G(r, s, Z))\, ds. \tag{10}$$

By Theorem 4 for every singular point $\mathbf{p} = (r_0, Z_0)$ of system (9) such that

$$\det\left(\frac{\partial g^0}{\partial(r, Z)}\right)\Bigg|_{(r,Z)=\mathbf{p}} \neq 0, \tag{11}$$

there exists a $2\pi-$periodic solution $(r_\varepsilon(\theta), Z_\varepsilon(\theta))$ of system (8) such that $(r_\varepsilon(0), Z_\varepsilon(0)) \to \mathbf{p}$ as $\varepsilon \to 0$. Then (11) is equivalent to

$$\det\left(\frac{\partial(f_1, f_2)}{\partial(r, Z)}\right)(r_0, Z_0) \neq 0,$$

where f_1 and f_2 are the functions defined just before the statement of Theorem 2.

Going back through the changes of coordinates the $2\pi-$periodic solution $(r_\varepsilon(\theta), Z_\varepsilon(\theta))$ of system (8) provides the periodic solution $(X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t))$ of system (2) with $\mu = 0$ such that

$$(X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t)) \to (r_0 \cos t, r_0 \sin t, Z_0) \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, from (7) we have a periodic solution $(x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t))$ of the system (2) such that

$$x_\varepsilon(t) \to -r_0 \cos t - Z_0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence we have proved Theorem 2.

3. Appendix

In this appendix we present the basic results from averaging theory that we need for proving the results of this paper.

We consider the problem of the bifurcation of $T-$periodic solutions from the differential system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}, \varepsilon) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{12}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, $T-$periodic in the variable t, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{13}$$

has a submanifold of periodic solutions. A solution of this problem is given using averaging theory. For a general introduction to averaging theory see the books of Sanders and Verhulst [11], and Verhulst [12].

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of unperturbed system (13) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z})$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}. \tag{14}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (14), and $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates, i.e., $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Theorem 3. *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)), \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that, for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of* (13) *is $T-$periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of* (14) *such that the matrix*

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$$

has in the right hand up corner the $k \times (n-k)$ zero matrix, and in the right hand down corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi \left(\frac{1}{T} \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right). \tag{15}$$

If there exists an $\mathrm{a} \in V$ *with* $\mathcal{F}(\mathrm{a}) = 0$ *and* $\det((d\mathcal{F}/d\alpha)(\mathrm{a})) \neq 0$, *then there is a $T-$periodic solution $\varphi(t, \varepsilon)$ of system* (12) *such that* $\varphi(0, \varepsilon) \to \mathrm{a}$ *as $\varepsilon \to 0$.*

Theorem 3 goes back to Malkin [7] and Roseau [10], for a shorter proof see [3].

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [12]

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon f(t, \mathbf{x}) + \varepsilon^2 g(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{16}$$

with $\mathbf{x} \in D \subset \mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $f(t, \mathbf{x})$ and $g(t, \mathbf{x}, \varepsilon)$ are $T-$periodic in t. Separately we consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f^0(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{17}$$

where

$$f^0(\mathbf{y}) = \frac{1}{T} \int_0^T f(t, \mathbf{y}) dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (16).

Theorem 4. *Consider the differential equation* (16) *and suppose that*

(i) *the functions f, g, $f_{\mathbf{x}}$, $g_{\mathbf{x}}$ and $f_{\mathbf{xx}}$ are defined, continuous and bounded by a constant independent of ε in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$;*

(ii) *f and g are $T-$periodic in t (T independent of ε).*

If p is a singular point of the averaged equation (17) *and*

$$\det \left(\frac{\partial f^0}{\partial \mathbf{y}} \right) |_{\mathbf{y}=p} \neq 0,$$

then there exists a $T-$periodic solution $\varphi(t, \varepsilon)$ of equation (16) *which is close to p such that $\varphi(0, \varepsilon) \to p$ as $\varepsilon \to 0$.*

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