On the refinement matrix mask of interpolating Hermite splines

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Abstract

We propose a new computational approach for constructing the refinement matrix mask of interpolating Hermite splines of any order and with general dilation factor. Our strategy exploits the refinability properties of cardinal B-splines with simple knots and simplifies the constructive procedures proposed so far.

Keywords: Cardinal Hermite Spline Interpolation; Cardinal B-Splines; Vector Subdivision; Refinement Matrix Mask; Arbitrary Dilation Factor

1. Introduction

Hermite subdivision schemes with dilation factor $m \in \mathbb{N}\setminus\{1\}$ form a special subclass of vector subdivision schemes which started to be investigated in the early 90's (see, e.g., [3, 4, 7, 8]) but is still subject of recent research [1, 2, 5, 6, 10]. The distinctive feature of Hermite schemes is the fact that the sequence of vector data $\mathbf{p}^{[j+1]} = \{\ \mathbf{p}_k^{[j+1]}\ \}_{k\in\mathbb{Z}}$, generated at refinement level $j+1$ ($j \in \mathbb{N}_0$) by the application of a *subdivision matrix* $\mathbf{A}^{[j]}$ to the previous data $\mathbf{p}^{[j]}$ so that $\mathbf{p}^{[j+1]} = \mathbf{A}^{[j]}\mathbf{p}^{[j]}$, contains vectors $\mathbf{p}_k^{[j+1]} = [\ p_{k,0}^{[j+1]}, \ldots, p_{k,n}^{[j+1]}\ ]^T \in \mathbb{R}^{n+1}$ ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) whose components correspond to the evaluation at $m^{-(j+1)}\mathbb{Z}$ of a certain function and its derivatives up to order n (see, e.g., Figure 1).

In the present paper we investigate Hermite subdivision schemes capable of generating, in the limit, interpolating Hermite splines. Precisely, for any given initial sequence of vector data $\{\mathbf{p}_k^{[0]} \in \mathbb{R}^{n+1}\}_{k\in\mathbb{Z}}$, the vector-valued limit function $\mathbf{f} = [f_0, \ldots, f_n]^T \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{n+1})$, linear combination of the shifts of the basic limit functions $\{\phi_{0,s}^{[0]}\}_{s=0}^n$ of the scheme, is formed by a first scalar-valued function $f_0 \in \mathcal{C}^n(\mathbb{R})$ which is a polynomial spline of degree $2n+1$ satisfying $f_s(k) = f_0^{(s)}(k) = p_{k,s}^{[0]}$, for all $k \in \mathbb{Z}$ and $s \in \{0, \ldots, n\}$. For any arbitrary dilation factor m and order n, such a Hermite scheme is described by the finite matrix sequence $\{\mathbf{A}_k \in \mathbb{R}^{(n+1)\times(n+1)}\}_{k=1-m}^{m-1}$ which is called the *refinement matrix mask* and is the building block to define the subdivision matrices $\{\mathbf{A}^{[j]}\}_{j\in\mathbb{N}_0}$.

The construction of such a refinement matrix mask is already known in the literature for $m = 2$, $n \in \mathbb{N}_0$. The authors in [7] constructed the matrices $\{\mathbf{A}_k\}_{k=-1,0,1}$ by introducing the transformation matrices $\mathbf{H}_L$ and $\mathbf{H}_R$ yielding $\{\mathbf{A}_k = \mathbf{H}_L\widetilde{\mathbf{A}}_k\mathbf{H}_R\}_{k=-1,0,1}$ with $\{\widetilde{\mathbf{A}}_k\}_{k=-1,0,1}$ the refinement matrix mask of degree-$(2n+1)$ B-splines with knots of multiplicity $n+1$ derived in [9]. An alternative approach, proposed more recently in [10], constructs explicitly the degree-$(2n+1)$ polynomials defining the basic limit functions $\{\phi_{0,s}^{[0]}\}_{s=0}^n$ on the intervals $[-1, 0]$ and $[0, 1]$ which cover their supports. The refinement mask $\{\mathbf{A}_k\}_{k=-1,0,1}$ is then obtained from the refinement equation, exploiting the evaluation of those polynomials.

The goal of our work is to provide an easier construction of the refinement matrix mask of interpolating Hermite splines for any arbitrary order and dilation factor, by exploiting the refinability properties of degree-$(2n+1)$ cardinal B-splines with simple knots. In Section 2 we recall basic facts about vector

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subdivision schemes, in Section 3 we review the main properties of the subclass of interpolating Hermite spline subdivision schemes and finally, in Section 4, we present our novel construction.

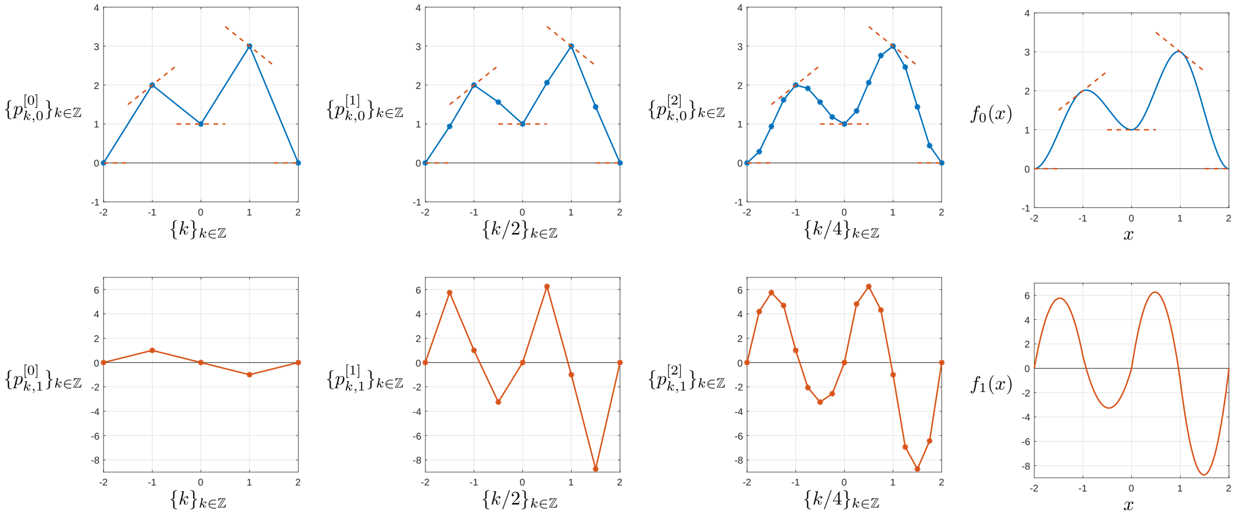

Figure 1: The application of the Hermite spline scheme in Example 4.1 ($m = 2$, $n = 1$) to the initial data $\mathbf{p}^{[0]} = \{\mathbf{p}_k^{[0]} \in \mathbb{R}^2\}_{k\in\mathbb{Z}}$ with $\mathbf{p}_{-1}^{[0]} = [2,1]^T$, $\mathbf{p}_0^{[0]} = [1,0]^T$, $\mathbf{p}_1^{[0]} = [3,-1]^T$ and $\mathbf{p}_k^{[0]} = \mathbf{0}$ for $k \notin \{-1,0,1\}$. The first column represents the initial data $\mathbf{p}^{[0]}$ over $\mathbb{Z}$. The second and the third columns represent the data after one subdivision step (i.e. $\mathbf{p}^{[1]}$ over $\mathbb{Z}/2$) and after two subdivision steps (i.e. $\mathbf{p}^{[2]}$ over $\mathbb{Z}/4$) respectively. The last column shows the two components of the vector-valued limit function $\mathbf{f}(x) = [f_0(x), f_1(x)]^T = \sum_{s=0,1}\sum_{k\in\mathbb{Z}} p_{k,s}^{[0]}\phi_{0,s}^{[0]}(x-k)$, where f_0 is a $\mathcal{C}^1$ cubic spline and $f_1 = f_0'$ is a $\mathcal{C}^0$ quadratic spline.

2. Basic facts about vector subdivision schemes

A univariate (shift-invariant) *vector subdivision scheme of order* $n \in \mathbb{N}_0$ takes as input an initial sequence of vector data (*control points*)

$$\mathbf{p}^{[0]} := \left[\cdots \quad \left(\mathbf{p}_{-1}^{[0]}\right)^T, \quad \left(\mathbf{p}_{0}^{[0]}\right)^T, \quad \left(\mathbf{p}_{1}^{[0]}\right)^T, \quad \cdots\right]^T, \qquad \mathbf{p}_k^{[0]} := \begin{bmatrix} p_{k,0}^{[0]} \\ \vdots \\ p_{k,n}^{[0]} \end{bmatrix} \in \mathbb{R}^{n+1}, \qquad k \in \mathbb{Z},$$

and, at each *refinement level* $j \in \mathbb{N}_0$, computes from $\mathbf{p}^{[j]}$ the refined sequence of vector data $\mathbf{p}^{[j+1]}$ by applying the rules

$$\mathbf{p}_h^{[j+1]} = \sum_{k\in\mathbb{Z}} \mathbf{A}_{h-mk}^{[j]}\,\mathbf{p}_k^{[j]}, \qquad j \in \mathbb{N}_0,\ h \in \mathbb{Z}, \tag{1}$$

for some *dilation factor* $m \in \mathbb{N}\setminus\{1\}$ and *refinement mask* $\{\mathbf{A}_k^{[j]} \in \mathbb{R}^{(n+1)\times(n+1)}\}_{k\in\mathbb{Z}}$. Equation (1) can be rewritten in matrix form as

$$\mathbf{p}^{[j+1]} = \mathbf{A}^{[j]}\,\mathbf{p}^{[j]} \tag{2}$$

where the involved matrices $\mathbf{A}^{[j]}$ (called *subdivision matrices*) are block m-slanted matrices defined by

$$\mathbf{A}^{[j]} := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ & \mathbf{A}_{m-1}^{[j]} & \mathbf{A}_{-1}^{[j]} & \mathbf{A}_{-m-1}^{[j]} & \\ & \mathbf{A}_{m}^{[j]} & \mathbf{A}_{0}^{[j]} & \mathbf{A}_{-m}^{[j]} & \\ & \mathbf{A}_{m+1}^{[j]} & \mathbf{A}_{1}^{[j]} & \mathbf{A}_{1-m}^{[j]} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3}$$

The vector scheme in (1) is said to be *convergent* if and only if there exist continuous vector-valued functions

$$\begin{array}{rcll} \phi_{k,s}^{[j]} : \mathbb{R} & \longrightarrow & \mathbb{R}^{n+1} & \\ x & \longmapsto & \begin{bmatrix} \varphi_{0,s}^{[j]}(m^j x - k) \\ \vdots \\ \varphi_{n,s}^{[j]}(m^j x - k) \end{bmatrix}, & s \in \{0, \dots, n\},\ j \in \mathbb{N}_0,\ k \in \mathbb{Z}, \end{array}$$

that, by introducing the matrix notation $\Phi_k^{[j]}(x) := \left[\phi_{k,0}^{[j]}(x),\ \dots,\ \phi_{k,n}^{[j]}(x)\right]$, satisfy the *refinement equation*

$$\Phi_h^{[j]}(x) \;=\; \sum_{k\in\mathbb{Z}} \Phi_k^{[j+1]}(x)\, \mathbf{A}_{k-mh}^{[j]}, \qquad h \in \mathbb{Z}. \tag{4}$$

For any $j \in \mathbb{N}_0$, (4) can be rewritten in matrix form as

$$\Phi^{[j]}(x) \;=\; \Phi^{[j+1]}(x)\, \mathbf{A}^{[j]} \qquad \text{with} \qquad \Phi^{[j]}(x) \;:=\; \Big[\ \dots,\ \Phi_{-1}^{[j]}(x),\ \Phi_0^{[j]}(x),\ \Phi_1^{[j]}(x),\ \dots\ \Big]. \tag{5}$$

For any input vector sequence $\mathbf{p}^{[0]}$, a convergent vector scheme generates in the limit a continuous vector-valued function $\mathbf{f} = [f_0, \dots, f_n]^T$ that, in light of (2) and (5), fulfils $\mathbf{f}(x) = \Phi^{[j]}(x)\, \mathbf{p}^{[j]}$, for all $j \in \mathbb{N}_0$. In particular, for $j = 0$

$$\mathbf{f}(x) = \sum_{k\in\mathbb{Z}} \Phi_k^{[0]}(x)\, \mathbf{p}_k^{[0]} \;=\; \sum_{s=0}^{n} \sum_{k\in\mathbb{Z}} p_{k,s}^{[0]}\, \phi_{k,s}^{[0]}(x) \quad \text{with} \quad f_r(x) = \sum_{s=0}^{n} \sum_{k\in\mathbb{Z}} p_{k,s}^{[0]}\, \varphi_{r,s}^{[0]}(x-k)\,, \quad r = 0, \dots, n. \tag{6}$$

$\mathbf{f}$ has the regularity of the least regular function among its components $f_0, \dots, f_n$. Due to (6), the $n+1$ vector-valued functions $\phi_{0,s}^{[0]}(x) = [\varphi_{0,s}^{[0]}(x), \dots, \varphi_{n,s}^{[0]}(x)]^T$, $s \in \{0, \dots, n\}$ are called the *basic limit functions* of the vector scheme and each of them is obtained by refining the initial sequence of vector data defined by $p_{k,r}^{[0]} = \delta_{k,0}\delta_{r,s}$, $r \in \{0, \dots, n\}$, $k \in \mathbb{Z}$.
Note that, when $n = 0$, the above description recovers the known results on scalar subdivision with dilation factor m.

3. A short review of the subclass of interpolating Hermite spline subdivision schemes

The interpolating Hermite spline scheme of order $n \in \mathbb{N}_0$ is a special instance of convergent vector subdivision scheme that, for any choice of $m \in \mathbb{N} \setminus \{1\}$, generates interpolating Hermite splines of degree $2n+1$. The following properties [5, 7] make it a special member of the class of vector subdivision schemes.

i) For all $j \in \mathbb{N}_0$, each matrix in the jth level refinement mask $\{\mathbf{A}_k^{[j]} \in \mathbb{R}^{(n+1)\times(n+1)}\}_{k\in\mathbb{Z}}$ satisfies

$$\mathbf{A}_k^{[j]} \;=\; \mathbf{D}^{-j-1}\, \mathbf{A}_k\, \mathbf{D}^j \quad \text{with} \quad \mathbf{D} \;=\; \begin{bmatrix} 1 & & & \\ & m^{-1} & & \\ & & \ddots & \\ & & & m^{-n} \end{bmatrix};$$

moreover, the refinement mask is compactly supported on $[1-m, m-1] \cap \mathbb{Z}$, i.e. $\mathbf{A}_k$ is the zero matrix for $k > m-1$ and $k < 1-m$.

ii) For all $r, s \in \{0, \dots, n\}$,

$$\varphi_{r,s}^{[0]}(x) \;=\; \frac{d^r}{dx^r}\, \varphi_{0,s}^{[0]}(x) \qquad \text{and} \qquad \varphi_{r,s}^{[j]}(k) \;=\; \delta_{k,0}\, \delta_{r,s}, \qquad \forall j \in \mathbb{N}_0,\ k \in \mathbb{Z}.$$

In particular, the latter requires, for every $j \in \mathbb{N}_0$, $\mathbf{A}_0^{[j]}$ to be the $(n+1)$-dimensional identity matrix which implies $\mathbf{A}_0 \;=\; \mathbf{D}$ due to i).

iii) For all $s \in \{0,\dots,n\}$, $\varphi_{0,s}^{[0]} \in \mathcal{C}^n(\mathbb{R})$ is a polynomial spline of degree $d = 2n+1$.

Remark 3.1. Condition i) implies,

$$\mathrm{supp}(\varphi_{r,s}^{[0]}) \subseteq [-1,1], \qquad \forall r,s \in \{0,\dots,n\}. \tag{7}$$

Indeed, consider, for every $j \in \mathbb{N}_0$, $k_L^{[j]}, k_R^{[j]} \in \mathbb{Z}$ such that

$$\mathbf{p}_{k_L^{[j]}}^{[j]},\ \mathbf{p}_{k_R^{[j]}}^{[j]} \neq \mathbf{0} \qquad \text{and} \qquad \mathbf{p}_k^{[j]} = \mathbf{0} \quad \text{for} \quad k < k_L^{[j]} \ \vee\ k > k_R^{[j]}.$$

From (1) and i) one gets, for every $j \in \mathbb{N}_0$,

$$m\,(\,k_L^{[j]} - 1\,) + 1 \leq k_L^{[j+1]} \qquad \text{and} \qquad k_R^{[j+1]} \leq m\,(\,k_R^{[j]} + 1\,) - 1. \tag{8}$$

Since the vectors $\{\mathbf{p}_k^{[j]}\}_{k\in\mathbb{Z}}$ are attached to $m^{-j}\mathbb{Z}$, using (8) we have that, for $\mathbf{f}$ in (6),

$$\bigcup_{r=0}^{n} \mathrm{supp}(f_r) = \lim_{j\to\infty} \left[\frac{k_L^{[j+1]}-1}{m^{j+1}},\ \frac{k_R^{[j+1]}+1}{m^{j+1}} \right] \subseteq \lim_{j\to\infty} \left[\frac{k_L^{[j]}-1}{m^{j}},\ \frac{k_R^{[j]}+1}{m^{j}} \right] \subseteq \left[k_L^{[0]}-1,\ k_R^{[0]}+1 \right].$$

As previously observed, for every $s \in \{0,\dots,n\}$, $\mathbf{f} = \phi_{0,s}^{[0]}$ is obtained by $\mathbf{p}^{[0]}$ satisfying $p_{k,r}^{[0]} = \delta_{k,0}\delta_{r,s}$, $r \in \{0,\dots,n\}$, $k \in \mathbb{Z}$, for which $k_L^{[0]} = k_R^{[0]} = 0$.

Remark 3.2. In light of ii) and iii), for any initial data $\mathbf{p}^{[0]}$, f_0 in (6) is a $\mathcal{C}^n$ spline of degree d and, for all $r \in \{0,\dots,n\}$,

$$f_r(x) = \frac{d^r}{dx^r} f_0(x) \qquad \text{with} \qquad f_r(m^{-j}k) = p_{k,r}^{[j]}, \qquad j \in \mathbb{N}_0,\ k \in \mathbb{Z}. \tag{9}$$

4. Constructing the mask of Hermite splines from cardinal B-splines with simple knots

Let $N_d \in \mathcal{C}^{d-1}(\mathbb{R})$ be the cardinal B-spline of degree $d = 2n+1$ supported on $[-n-1, n+1]$ and defined over the knot vector $\{-n-1, -n, \dots, 0, \dots, n, n+1\}$ with all simple knots. N_d satisfies the refinement equation (see, e.g., [11])

$$N_d(x) = \sum_{k=-(m-1)(n+1)}^{(m-1)(n+1)} b_k\, N_d(mx-k) \tag{10}$$

$$\text{where} \qquad \frac{1}{m} \sum_{k=-(m-1)(n+1)}^{(m-1)(n+1)} b_k\, z^k = z^{-(m-1)(n+1)} \left(\frac{1-z^m}{m(1-z)} \right)^{d+1}, \qquad z \in \mathbb{C},\ |z| = 1.$$

In light of (10), for any given sequence of control points $\mathbf{c}^{[0]} = \{c_k^{[0]} \in \mathbb{R}\}_{k\in\mathbb{Z}}$, we can express the associated spline as

$$f(x) = \sum_{k\in\mathbb{Z}} c_k^{[0]}\, N_d(x-k) = \sum_{k\in\mathbb{Z}} c_k^{[j]}\, N_d(m^j x - k), \qquad j \in \mathbb{N}_0,$$

$$\text{where} \qquad c_h^{[j+1]} = \sum_{k\in\mathbb{Z}} b_{h-mk}\, c_k^{[j]}, \qquad h \in \mathbb{Z}. \tag{11}$$

Thus, for all $j \in \mathbb{N}_0$, $s \in \{0,\dots,n\}$, $y \in \mathbb{R}$,

$$\frac{1}{m^{sj}}\, f^{(s)}(m^{-j}y) = \frac{1}{m^{sj}} \frac{d^s}{dx^s} f(x)\Big|_{x=m^{-j}y} = \sum_{k\in\mathbb{Z}} c_k^{[j]}\, N_d^{(s)}(y-k),$$

and, in particular,

$$\frac{1}{m^{sj}}\, f^{(s)}(m^{-j}h) \;=\; \sum_{k\in\mathbb{Z}} c_k^{[j]}\, N_d^{(s)}(h-k), \qquad h\in\mathbb{Z}. \tag{12}$$

Now, in view of (9), we can interpret $f^{(s)}(m^{-j}h)$ as the control point $p_{h,s}^{[j]}$ for our Hermite scheme. Thus, after left-multiplying both sides of (1) by $\mathbf{D}^{j+1}$, due to i) we obtain

$$\begin{bmatrix} f^{(0)}(m^{-j-1}h) \\ \dfrac{f^{(1)}(m^{-j-1}h)}{m^{j+1}} \\ \vdots \\ \dfrac{f^{(n)}(m^{-j-1}h)}{m^{n(j+1)}} \end{bmatrix} = \mathbf{D}^{j+1}\mathbf{p}_h^{[j+1]} \;=\; \sum_{k=\lceil\frac{h-m+1}{m}\rceil}^{\lfloor\frac{h+m-1}{m}\rfloor} \mathbf{D}^{j+1}\mathbf{A}_{h-mk}^{[j]}\mathbf{D}^{-j}\mathbf{D}^{j}\mathbf{p}_k^{[j]} = \sum_{k=\lceil\frac{h-m+1}{m}\rceil}^{\lfloor\frac{h+m-1}{m}\rfloor} \mathbf{A}_{h-mk} \begin{bmatrix} f^{(0)}(m^{-j}k) \\ \dfrac{f^{(1)}(m^{-j}k)}{m^{j}} \\ \vdots \\ \dfrac{f^{(n)}(m^{-j}k)}{m^{nj}} \end{bmatrix}. \tag{13}$$

Now, the vectors with the samples of $f^{(s)}$, $s=0,\dots,n$ can be expressed via (12) with respect to the integer samples of N_d and its derivatives. Moreover, on the left-hand side, we can use (11) to express $\mathbf{c}^{[j+1]}$ with respect to $\mathbf{c}^{[j]} = \{c_k^{[j]}\}_{k\in\mathbb{Z}}$. Thus (13) can be stated in the equivalent matrix form

$$\mathbf{N}\,\mathbf{B}\,\mathbf{c}^{[j]} \;=\; \mathbf{N}\,\mathbf{c}^{[j+1]} \;=\; \begin{bmatrix} \ddots & \vdots & & & \\ & \mathbf{D} & & & \\ & \mathbf{A}_1 & \mathbf{A}_{1-m} & & \\ & \vdots & \vdots & & \\ & \mathbf{A}_{m-1} & \mathbf{A}_{-1} & & \\ & & \boxed{\mathbf{D}} & & \\ & & \mathbf{A}_1 & \mathbf{A}_{1-m} & \\ & & \vdots & \vdots & \\ & & \mathbf{A}_{m-1} & \mathbf{A}_{-1} & \\ & & & \mathbf{D} & \\ & & & \vdots & \ddots \end{bmatrix} \mathbf{N}\,\mathbf{c}^{[j]}, \tag{14}$$

where the framed $\mathbf{D}$ indicates the row indices from 0 to n and the column indices from 0 to n, and the matrices $\mathbf{B}$, $\mathbf{N}$ are defined entry-wise, for $h,k\in\mathbb{Z}$, by

$$\mathbf{B}(h,k) \;=\; b_{h-mk} \quad\text{and}\quad \mathbf{N}(h,k) \;=\; N_d^{(s)}(\ell-k) \quad\text{for}\quad h \;=\; (n+1)\ell+s,\ s\in\{0,\dots,n\}. \tag{15}$$

Since we want (14) to hold for every choice of $\mathbf{c}^{[j]}$, the next result follows.

Theorem 4.1. *Let $n\in\mathbb{N}_0$, $d=2n+1$ and $m\in\mathbb{N}\setminus\{1\}$. The refinement mask $\{\mathbf{A}_k\}_{k=1-m}^{m-1}$ of the interpolating Hermite spline scheme of order n and dilation factor m that fulfills i), ii) and iii) is uniquely determined by the equations*

$$\mathbf{A}_0 = \mathbf{D}\in\mathbb{R}^{(n+1)\times(n+1)} \qquad \textit{and} \qquad \overline{\mathbf{A}} := \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{1-m} \\ \vdots & \vdots \\ \mathbf{A}_{m-1} & \mathbf{A}_{-1} \end{bmatrix} = \mathbf{N}_L\mathbf{B}_L\mathbf{N}_R^{-1} \in \mathbb{R}^{(m-1)(n+1)\times 2(n+1)}, \tag{16}$$

$$\textit{where}\qquad \begin{aligned} \mathbf{N}_L \;&=\; \mathbf{N}(\,n+1:n+(n+1)(m-1),\ 1-n:m+n-1\,) \;\in\; \mathbb{R}^{(m-1)(n+1)\times(m+d-2)}, \\ \mathbf{B}_L \;&=\; \mathbf{B}(\,1-n:m+n-1,\ -n:n+1\,) \;\in\; \mathbb{R}^{(m+d-2)\times 2(n+1)}, \\ \mathbf{N}_R \;&=\; \mathbf{N}(\,0:d,\ -n:n+1\,) \;\in\; \mathbb{R}^{2(n+1)\times 2(n+1)}, \end{aligned}$$

are finite portions of the bi-infinite matrices in (14). Moreover, by construction, the interpolating Hermite spline scheme reproduces polynomials up to degree d and cardinal B-splines of degree d.

Proof. $\mathbf{A}_0 = \mathbf{D}$ is already asked in condition ii). For the remaining matrices, we observe that the block $\overline{\mathbf{A}}$ can be extracted from the bigger matrix in (14) with row indices from $n+1$ to $n+(n+1)(m-1)$ and column indices from 0 to $2n+1=d$. Thus, on the right-hand side, we need only to consider the rows of $\mathbf{N}$ with indices from 0 to d. Since, for every $s \in \{0,\dots,n\}$, $\operatorname{supp}(N_d^{(s)}) = [-n-1, n+1]$, due to (15), we can also cut the columns of $\mathbf{N}$ considering only indices from $-n$ to $n+1$, leading to

$$\mathbf{N}_R = \left[\begin{array}{c|c|c|c|c} N_d(-n) & N_d(1-n) & \cdots & N_d(n) & 0 \\ N_d^{(1)}(-n) & N_d^{(1)}(1-n) & \cdots & N_d^{(1)}(n) & 0 \\ \vdots & \vdots & & \vdots & 0 \\ N_d^{(n)}(-n) & N_d^{(n)}(1-n) & \cdots & N_d^{(n)}(n) & 0 \\ \hline 0 & N_d(-n) & \cdots & N_d(n-1) & N_d(n) \\ 0 & N_d^{(1)}(-n) & \cdots & N_d^{(1)}(n-1) & N_d^{(1)}(n) \\ 0 & \vdots & & \vdots & \vdots \\ 0 & N_d^{(n)}(-n) & \cdots & N_d^{(n)}(n-1) & N_d^{(n)}(n) \end{array}\right] \in \mathbb{R}^{(d+1)\times(d+1)}. \tag{17}$$

In a similar fashion we can cut the matrices on the left-hand side considering only the rows of $\mathbf{N}$ with indices from $n+1$ to $n+(n+1)(m-1)$ and the columns of $\mathbf{B}$ with indices from $-n$ to $n+1$. The only non-zero entries of $\mathbf{N}$ contained in those rows are the ones with column index between $1-n$ and $m+n-1$, extrema included. For $\mathbf{B}$ the only non-zero entries contained in the considered columns lie on the rows with indices from $-(m-1)(n+1)-mn$ to $(m-1)(n+1)+m(n+1)$. Thus the useful indices for the cut are the ones between α and β where

$$\begin{aligned} \alpha &= \max_{m\geq 2, n\geq 0} \{\, 1-n,\ -(m-1)(n+1)-mn \,\} \;=\; 1-n, \\ \beta &= \min_{m\geq 2, n\geq 0} \{\, m+n-1,\ (m-1)(n+1)+m(n+1) \,\} \;=\; m+n-1. \end{aligned}$$

From (17), it is clear that $\det(\mathbf{N}_R) \neq 0$. Thus $\mathbf{N}_R^{-1}$ in (16) is well-defined as well as the matrices $\{\mathbf{A}_k\}_{k=1-m}^{m-1}$. Since all the matrices involved in (14) are properly slanted, (14) holds for every choice of the sequence $\mathbf{c}^{[j]}$.

Now we only have to prove that for every initial sequence $\mathbf{p}^{[0]}$, the scheme produces a $\mathcal{C}^n$ polynomial spline of degree d and its derivatives. In view of (2) and (14), we observe that, after a subdivision step, each of the added new control vectors depends only on two consecutive control vectors from the previous step. Since the refinement rules have been chosen to reproduce degree-d cardinal B-splines and thus polynomials up to degree d, the first component of the vector-valued limit function will be a polynomial of degree d on every unitary interval with integer extrema. The fact that we are interpolating the first n derivatives yields $\mathcal{C}^n$ regularity, so concluding the proof. □

Remark 4.2. When $m=3$ one has that $\mathbf{N}_L = \mathbf{N}_R$, for every $n \in \mathbb{N}_0$.

Since the entries of $\mathbf{B}$ are given in (10) (see (15)) and the entries of $\mathbf{N}$ can be easily obtained by recalling the properties of cardinal B-splines and their derivatives, Theorem 4.1 gives us an easy-to-implement method for constructing the refinement mask $\{\mathbf{A}_k \in \mathbb{R}^{(n+1)\times(n+1)}\}_{k=1-m}^{m-1}$ for any order $n \in \mathbb{N}_0$ and dilation factor $m \in \mathbb{N}\setminus\{1\}$. We conclude providing some illustrative examples.

Example 4.1 ($m=2$, $n=1$, $d=3$)**.** Figure 1 shows the result of its application to some initial data.

$$\{b_k\}_{k=-2}^{2} = \{\, 1,\ 4,\ 6,\ 4,\ 1 \,\} \,/\, 8, \qquad \mathbf{N}_L = \left[\begin{array}{c|c|c} 1/6 & 2/3 & 1/6 \\ -1/2 & 0 & 1/2 \end{array}\right], \qquad \mathbf{N}_R = \left[\begin{array}{c|c} \mathbf{N}_L & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 \\ 0 \end{matrix} & \mathbf{N}_L \end{array}\right],$$

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{-1} \end{bmatrix} = \left[\begin{array}{cc|cc} 1/2 & 1/8 & 1/2 & -1/8 \\ -3/4 & -1/8 & 3/4 & -1/8 \end{array}\right].$$

Example 4.2 ($m = 2$, $n = 2$, $d = 5$)**.**

$$\{b_k\}_{k=-3}^{3} = \{\, 1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1 \,\} \ / \ 32,$$

$$\mathbf{N}_L = \left[\begin{array}{c|c|c|c|c} 1/120 & 13/60 & 11/20 & 13/60 & 1/120 \\ -1/24 & -5/12 & 0 & 5/12 & 1/24 \\ 1/6 & 1/3 & -1 & 1/3 & 1/6 \end{array}\right], \qquad \mathbf{N}_R = \left[\begin{array}{c|c} \mathbf{N}_L & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \mathbf{N}_L \end{array}\right],$$

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, \qquad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{-1} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1/2 & 5/32 & 1/64 & 1/2 & -5/32 & 1/64 \\ -15/16 & -7/32 & -1/64 & 15/16 & -7/32 & 1/64 \\ 0 & -3/8 & -1/16 & 0 & 3/8 & -1/16 \end{array}\right].$$

Example 4.3 ($m = 3$, $n = 2$, $d = 5$)**.**

$$\{b_k\}_{k=-6}^{6} = \{\, 1,\ 6,\ 21,\ 50,\ 90,\ 126,\ 141,\ 126,\ 90,\ 50,\ 21,\ 6,\ 1 \,\} \ / \ 243,$$

$$\mathbf{N}_L = \mathbf{N}_R = \left[\begin{array}{c|c|c|c|c|c} 1/120 & 13/60 & 11/20 & 13/60 & 1/120 & 0 \\ -1/24 & -5/12 & 0 & 5/12 & 1/24 & 0 \\ 1/6 & 1/3 & -1 & 1/3 & 1/6 & 0 \\ \hline 0 & 1/120 & 13/60 & 11/20 & 13/60 & 1/120 \\ 0 & -1/24 & -5/12 & 0 & 5/12 & 1/24 \\ 0 & 1/6 & 1/3 & -1 & 1/3 & 1/6 \end{array}\right],$$

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/9 \end{bmatrix}, \quad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{-2} \\ \mathbf{A}_2 & \mathbf{A}_{-1} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 64/81 & 16/81 & 4/243 & 17/81 & -2/27 & 2/243 \\ -40/81 & 0 & 2/243 & 40/81 & -13/81 & 4/243 \\ -40/81 & -32/81 & -10/243 & 40/81 & -8/81 & 1/243 \\ \hline 17/81 & 2/27 & 2/243 & 64/81 & -16/81 & 4/243 \\ -40/81 & -13/81 & -4/243 & 40/81 & 0 & -2/243 \\ 40/81 & 8/81 & 1/243 & -40/81 & 32/81 & -10/243 \end{array}\right].$$

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