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Abstract

Degree elevation is a typical corner-cutting algorithm. It refers to the process transforming control polygons when embedding a polynomial space of some degree into any polynomial space of higher degree. Dimension elevation similarly refers to the transformation of control polygons when embedding an Extended Chebyshev space possessing a Bernstein basis into another one, of higher dimension. Unlike degree elevation, this cannot always be split into successive (corner-cutting) steps elevating the dimension by one. What happens when it is not possible is investigated here. We shall see that the new control points can even be located outside the initial control polygons, giving evidence that dimension elevation is not always corner-cutting.

Keywords: Extended Chebyshev spaces; Bernstein bases; dimension elevation; critical length; shape preservation; corner cutting.

1. The problematic

Throughout the paper, we work on a given closed bounded interval $[a, b]$, $a < b$. An $(n+1)$-dimensional space $\mathbb{E}_n \subset C^n([a, b])$ is said to be a *W-space on $[a, b]$* if any Taylor interpolation problem in $(n+1)$ data is unisolvent in $\mathbb{E}_n$. It is said to be an *Extended Chebyshev space* (for short, EC-space) *on $[a, b]$* if any Hermite interpolation problem in $(n+1)$ data is unisolvent in $\mathbb{E}_n$. There is a strong link between the two classes of W- and EC-spaces on $[a, b]$. Of course, an EC-space on $[a, b]$ is a W-space on $[a, b]$, but the converse is not true, except for $n = 0$. Nevertheless, given an $(n+1)$-dimensional W-space $\mathbb{E}_n$ on $[a, b]$, its *critical length (on $[a, b]$)* is the supremum of all positive h such that $\mathbb{E}_n$ is an EC-space on any interval $J \subset [a, b]$ of length h. Because we are working on a closed bounded interval, it is a positive number $\ell \leqslant b - a$, and $\ell = b - a$ means that $\mathbb{E}_n$ is an EC-space on $[a, b]$, see [11]. The following crucial result too is specifically due to the fact that we are working on a closed bounded interval, see [8].

Theorem 1.1. *Let $\mathbb{E}_n$ be an $(n+1)$-dimensional W-space on $[a, b]$. Then, $\mathbb{E}_n$ is an EC-space on $[a, b]$ iff there exists a nested sequence*

$$\mathbb{E}_0 \subset \mathbb{E}_1 \subset \cdots \subset \mathbb{E}_{n-1} \subset \mathbb{E}_n, \tag{1}$$

where, for $i = 0, \ldots, n-1$, $\mathbb{E}_i$ is an $(i+1)$-dimensional W-space on $[a, b]$.

Given a nested sequence (1) of W-spaces on $[a, b]$, select any sequence $(U_0, \ldots, U_n)$ such that $U_i \in \mathbb{E}_i \backslash \mathbb{E}_{i-1}$ for $i = 0, \ldots, n$, with $\mathbb{E}_{-1} := \{0\}$. Then, it is usual to introduce the non-vanishing functions

$$w_i := \frac{W(U_0, ..., U_{i-2})\, W(U_0, ..., U_i)}{W(U_0, ..., U_{i-1})^2}\ , \quad 0 \leqslant i \leqslant n, \tag{2}$$

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where $W(U_0, ..., U_i)$ denotes the Wronskian of the sequence $(U_0, \ldots, U_i)$ with the convention that $W(\emptyset) = \mathbb{1}$. With this sequence of functions, it is classical to associate a sequence of differential operators $L_0, \ldots, L_n$ recursively defined on $C^n([a, b])$ by

$$L_0 F := \frac{F}{w_0} , \quad L_i F := \frac{1}{w_i} D L_{i-1} F, \quad 1 \leqslant i \leqslant n, \tag{3}$$

where D denotes the ordinary differentiation. As is well known, the space $\mathbb{E}_n$ is then composed of all $F \in C^n([a, b])$ such that $L_n F$ is constant on $[a, b]$. It is called the EC-space associated with the sequence $(w_0, \ldots, w_n)$ and denoted as $\mathbb{E}_n = EC(w_0, \ldots, w_n)$.

A *Bernstein basis relative to* (a, b) is a normalised sequence $(B_0, \ldots, B_n)$ in $C^n([a, b])$ such that, for $i = 0, \ldots, n$, the function B_i is positive on $]a, b[$ and it vanishes exactly i times at a and exactly $(n - i)$ times at b. The normalisation requirement means that $\sum_{i=0}^{n} B_i = \mathbb{1}$, where $\mathbb{1}$ denotes the constant function $\mathbb{1}(x) = 1$ for all $x \in [a, b]$.

If the space $\mathbb{E}_n$ is an EC-space on $[a, b]$ containing $\mathbb{1}$, then the (n-dimensional) space $D\mathbb{E}_n$ is a W-space on $[a, b]$, but not necessarily an EC-space on $[a, b]$. The following result was proved in [10].

Theorem 1.2. *Given $n \geqslant 1$, let $\mathbb{E}_n$ be an $(n + 1)$-dimensional EC-space on $[a, b]$, containing the constants. Then, $\mathbb{E}_n$ possesses a Bernstein basis relative to (a, b) iff the space $D\mathbb{E}_n$ is an EC-space.*

An EC-space $\mathbb{E}_n$ on $[a, b]$ which possesses a Bernstein basis $(B_0, \ldots, B_n)$ relative to (a, b) is said to be *good for design on* $[a, b]$. In such a case, any points $P_0, \ldots, P_n \in \mathbb{R}^d$, $d \geqslant 1$, are called *the Bézier points (or control points) relative to* (a, b) of the function $F \in \mathbb{E}_n{}^d$ defined by

$$F(x) := \sum_{i=0}^{n} B_i(x) P_i, \quad x \in [a, b].$$

Clearly, an $(n + 1)$-dimensional W-space $\mathbb{E}_n$ on $[a, b]$ is an EC-space good for design on $[a, b]$ iff it contains a nested sequence (1) with $\mathbb{E}_0 = \text{span}(\mathbb{1})$, that is, iff it is of the form $\mathbb{E}_n = EC(\mathbb{1}, w_1, \ldots, w_n)$.

Theorem 1.3. *Let $\mathbb{E}_n \subset \mathbb{E}_{n+1}$ be two W-spaces on $[a, b]$, of dimension $(n + 1)$ and $(n + 2)$, respectively. Assume that $\mathbb{E}_n$ is an EC-space good for design on $[a, b]$. Then, so is $\mathbb{E}_{n+1}$. Moreover, there exists $\alpha_1, \ldots, \alpha_n \in]0, 1[$ such that, for any $F \in \mathbb{E}_n{}^d$, the Bézier points $P_0^\star, \ldots, P_{n+1}^\star \in \mathbb{R}^d$ of F considered as an element of $\mathbb{E}_{n+1}{}^d$, are obtained from its initial Bézier points $P_0, \ldots, P_n \in \mathbb{R}^d$ as follows*

$$P_0^\star = P_0, \quad P_i^\star = (1 - \alpha_i) P_{i-1} + \alpha_i P_i \text{ for } i = 1, \ldots, n, \quad P_{n+1}^\star = P_n. \tag{4}$$

The first claim is an obvious consequence of Th. 1.1. For the rest, see [13], [8]. We refer to the situation addressed in Theorem 1.3 as an *elementary step of dimension elevation*, and to (4) as elementary dimension elevation formulæ. The passage from $P_0, \ldots, P_n$ to $P_0^\star, \ldots, P_{n+1}^\star$ is *corner-cutting*. As a consequence, it is *shape preserving*: a planar polygon $[P_0, \ldots, P_n]$ (control polygon) which is monotone in one direction (resp. convex) is transformed into a polygon having the same property.

The general context we want to address in the present note is the following one:

$$\mathbb{E}_n \subset \mathbb{E}_{n+r}, \quad \text{for some } r \geqslant 2, \tag{5}$$

where $\mathbb{E}_n$ is an $(n + 1)$-dimensional EC-space good for design on $[a, b]$, while $\mathbb{E}_{n+r}$ is only known to be an $(n + r)$-dimensional W-space on $[a, b]$. How does Th. 1.3 extend to this situation? The first natural question to consider is

(Q$_0$)   *Does this situation automatically imply that $\mathbb{E}_{n+r}$ is an EC-space on $[a, b]$?*

The trivial example of the space $\mathbb{E}_3$ spanned on $[a, b]$ by the functions $1, x, \cos x, \sin x$, clearly shows that the answer to (Q$_0$) is negative: indeed, for $\mathbb{E}_3$ to be an EC-space (or an EC-space good for design) on $[a, b]$, we have to require that $b - a < 2\pi$, while there is no limitation for the space $\mathbb{E}_1$ of all affine functions on $[a, b]$.

From now on, assume that $\mathbb{E}_{n+r}$ **too is an EC-space good for design on** $[a, b]$. What can we say about the dimension elevation process associated with (5), that is, as previously, the passage from the Bézier points $P_0, \ldots, P_n \in \mathbb{R}^d$ (relative to (a, b)) of a given function $F \in \mathbb{E}_n{}^d$, to its Bézier points $P_0^\star, \ldots, P_{n+r}^\star \in \mathbb{R}^d$ (relative to (a, b)) when viewing F as an element of $\mathbb{E}_{n+r}{}^d$? Let us write $\mathbb{E}_n$ as $\mathbb{E}_n = EC(\mathbb{1}, w_1, \ldots, w_n)$ and let L_n be the associated order $(n + 1)$ differential operator. Then, due to (5), the space $\mathbb{F}_r := DL_n\mathbb{E}_{n+r}$ is r-dimensional and it is a W-space on $[a, b]$, but not necessarily an EC-space on $[a, b]$.

Proposition 1.4. [9] *The space* $\mathbb{F}_r$ *defined above is an EC-space on* $[a, b]$ *iff we can go from* $\mathbb{E}_n$ *to* $\mathbb{E}_{n+r}$ *through* r *elementary dimension elevation steps.*

1)- Suppose that $\mathbb{F}_r$ is an EC-space on $[a, b]$. According to Prop. 1.4, among all possible nested sequences

$$\mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \cdots \subset \mathbb{E}_{n+r-1} \subset \mathbb{E}_{n+r}, \quad \text{with } \dim \mathbb{E}_{n+i} = n + i + 1, \ i = 1, \ldots, r - 1, \tag{6}$$

we can choose one (and in fact, infinitely many) so that each intermediate space $\mathbb{E}_{n+i}$, $i = 1, \ldots, r - 1$, is a W-space on $[a, b]$. For such a sequence, apply the first part of Th. 1.3 successively to each elementary dimension elevation step $\mathbb{E}_{n+i} \subset \mathbb{E}_{n+i+1}$, $i = 0, \ldots, r - 1$. This ensures that $\mathbb{E}_{n+r}$ is in turn an EC-space good for design on $[a, b]$. Moreover, repeated application of elementary dimension elevation formulæ shows that the passage from the initial to the final Bézier points can be summarised as follows:

$$\text{for } i = 0, \ldots, n + r, \ P_i^\star \text{ is a strictly convex combination of the points } P_{\max(0, i-r)}, \ldots, P_{\min(i,n)}, \tag{7}$$

with coefficients independent of F. Being the result of an r step corner cutting algorithm, this process is shape preserving.

2)- We now suppose that $\mathbb{F}_r$ is not an EC-space on $[a, b]$. Given any nested sequence (6), there exists at least one integer $i \in \{1, \ldots, r - 1\}$ such that $\mathbb{E}_{n+i}$ is not a W-space on $[a, b]$. Let $\ell < b - a$ be the critical length on $[a, b]$ of the W-space $\mathbb{F}_r$. For each selected subinterval $[c, d] \subset [a, b]$ such that $0 < d - c < \ell$, we are in the previously examined situation, but after restriction to $[c, d]$. However, restriction to small intervals is not what we are interested in here. The problem we want to tackle is: what can we say on the global interval $[a, b]$ itself? It is easily checked that each final Bézier point $P_i^\star$ is automatically an affine combination of the initial Bézier points $P_{\max(0, i-r)}, \ldots, P_{\min(i,n)}$, with coefficients independent of F. Now, on $[a, b]$, can we always answer affirmatively the natural questions below (from the least to the most demanding):

(Q_1) *Is this affine combination a strictly convex one, i.e., is* (7) *valid?*

(Q_2) *Is dimension elevation shape preserving?*   (8)

(Q_3) *Does it result from an* r-*step corner cutting algorithm?*

We will see in next section that even the weakest among these properties is not always satisfied.

2. Dimension elevation beyond nested sequences of W-spaces

Kernels of linear differential operators with constant coefficients are classical examples of W-spaces on the whole real line. They are invariant under translation, and therefore they are closed under differentiation. They will serve to illustrate dimension elevation, in particular in the absence of a nested sequence (6) of W-spaces in order to try to answer the questions (8). Given such an operator $\mathcal{L}_n = p_n(D)$, where the characteristic polynomial p_n of degree $(n + 1)$ has unit leading coefficient, the critical length of $\mathbb{E}_n := \ker \mathcal{L}_n$ is defined as in the previous section, but now on the whole real line [5, 11]. It is the real number $\ell_n \in]0, +\infty]$ such that $\mathbb{E}_n$ is an EC-space on $[0, h]$ iff if $h < \ell_n$, and, from now on, we will always be working on some $[a, b] = [0, h]$. Moreover, $\ell_n = +\infty$ iff all roots of p_n are real. The space $\mathbb{E}_n$ contains the constants iff if $p_n(0) = 0$. If so, the critical length $\widetilde{\ell}_n$ of the space $D\mathbb{E}_n$ is called *the critical length for design* of $\mathbb{E}_n$. We thus know that $\mathbb{E}_n$ is an EC-space good for design on $[0, h]$ iff if $h < \widetilde{\ell}_n$ (see Th. 1.2). The numerical test to determine finite critical lengths, built in [3] (see also [2]), will be of crucial help below.
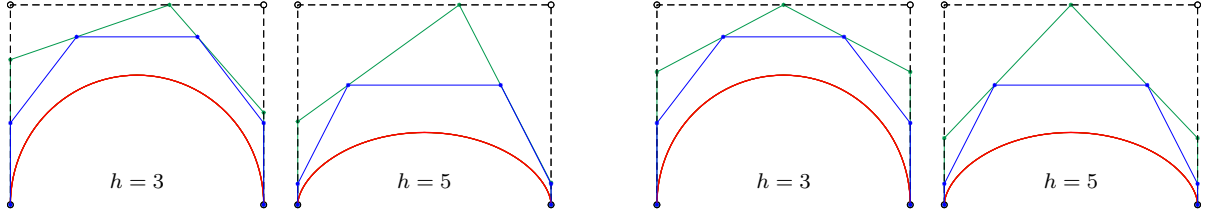
Figure 1: Corner cutting algorithms on $[0, h]$, obtained by inserting $\mathbb{E}_4 = \mathrm{span}(\mathbb{E}_3, U)$ between $\mathbb{E}_3$ and $\mathbb{E}_5$, respectively spanned by $1, x, \cos x, \sin x$ and $1, x, \cos x, \sin x, \cosh x, \sinh x$. Left: $U(x) := e^x$. Right: $U(x) := \sinh x + \sinh(h - x)$. See Ex. 2.2.

If $\mathcal{L}_{n+r} = p_{n+r}(D)$ is another operator, of order $(n + r + 1)$, with $r \geqslant 1$, the inclusion $\mathbb{E}_n \subset \mathbb{E}_{n+r}$ is satisfied iff if p_n is a divisor of p_{n+r}. This is why, in order to illustrate the problematic stated in Section 1, throughout the rest of the section we assume that

$$p_n(0) = 0, \quad p_{n+r} = p_n q_r, \quad \text{where } q_r \text{ is a real polynomial of degree } r. \tag{9}$$

The simplest situation is described in the proposition below.

Proposition 2.1. *Suppose that the polynomial q_r in (9) has only real roots. Then, the corresponding critical lengths for design satisfy $\widetilde{\ell}_n \leqslant \widetilde{\ell}_{n+r}$ and, for any $h < \widetilde{\ell}_n$ there exists an r step corner cutting algorithm transforming the Bézier points (relative to $[0, h]$) of any $F \in \mathbb{E}_n{}^d$ into its Bézier points (relative to $[0, h]$) in $\mathbb{E}_{n+r}{}^d$.*

Proof. Let us write q_r as $q_r(x) = \prod_{k=1}^r (x - a_k)$, with $a_1, \ldots, a_r \in \mathbb{R}$. Then, for $i = 1, \ldots, r - 1$, consider the polynomial $p_{n+i}(x) := p_n(x) \prod_{k=1}^i (x - a_k)$, and denote by $\mathbb{E}_{n+i}$ the kernel of the operator $\mathcal{L}_{n+i}$ with characteristic polynomial p_{n+i}. This yields the nested sequence $\mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \cdots \subset \mathbb{E}_{n+r-1} \subset \mathbb{E}_{n+r}$. On $[0, h]$, with $h < \widetilde{\ell}_n$, it represents r elementary dimension elevation steps. The claims follow from Th. 1.3. $\square$

Example 2.2. Take $p_3(x) := x(x^2 + 1)$ and $p_5(x) := x(x^2 + 1)(x^2 - 1)$. The spaces $\mathbb{E}_3$ and $\mathbb{E}_5$ are respectively spanned by the four functions $1, x, \cos x, \sin x$, and the six functions $1, x, \cos x, \sin x, \cosh x, \sinh x$. Their critical lengths for design are $\widetilde{\ell}_3 = 2\pi$, $\widetilde{\ell}_5 \approx 7.853$, see [3]. Between $\mathbb{E}_3$ and $\mathbb{E}_5$ we can insert the space $\mathbb{E}_4$, with characteristic polynomial either $p_4(x) = p_3(x)(x - 1)$ or $p_4(x) = p_3(x)(x + 1)$. The two step corner cutting algorithm corresponding to the first case is shown in Fig. 1, with $h = 3$ and $h = 5$, the second case being obtained symmetrically. Note that the value $h = 5$ is already too close to $\widetilde{\ell}_3$ to permit clear visualization of the "strict" corner cutting. These are the only two possibilities if we want to keep within the framework of kernels of linear differential operators with constant coefficients. However, for each $h \leqslant \widetilde{\ell}_3$ there are infinitely many possible ways to define intermediate five-dimensional EC-spaces (or, equivalently, W-spaces) $\mathbb{E}_4$ on $[0, h]$, namely all spaces spanned by the functions $1, x, \cos x, \sin x, \alpha \sinh x + \beta \sinh(h - x)$, where α, β are any positive numbers. For $\alpha = \beta = 1$, the space $\mathbb{E}_4$ is symmetric, and it yields the symmetric corner cutting algorithm shown in Fig. 1, right.

Factorising q_r on $\mathbb{R}$, we can repeatedly elevate the dimension by one as in Prop. 2.1, as many times as the number of real roots of q_r, and then elevate it two by two. This is the reason why we subsequently focus on the case $r = 2$, the degree two polynomial q_2 having no real root.

Proposition 2.3. *Let $a \pm ib$, $a \in \mathbb{R}$, $b > 0$, be the roots of q_2. Then, we can go from $\mathbb{E}_n$ to $\mathbb{E}_{n+2}$ through two elementary dimension elevation steps iff if $h < \pi/b$.*

Proof. We know that $\mathcal{L}_n \mathbb{E}_{n+2} = \ker q_2(D)$. Let us use the notations introduced for Prop. 1.4. Due to (3) we have $\mathcal{L}_n = (\prod_{i=1}^n w_i) D L_n$. Accordingly, $\ker q_2(D) = (\prod_{i=1}^n w_i) F_2$. This space is thus an EC-space on $[0, h]$ iff so is $\mathbb{F}_2$. Now, $\ker q_2(D)$ being spanned by the two functions $e^{ax} \cos(bx), e^{ax} \sin(bx)$, its critical length is π/b. Accordingly, the claim results from Prop. 1.4. $\square$
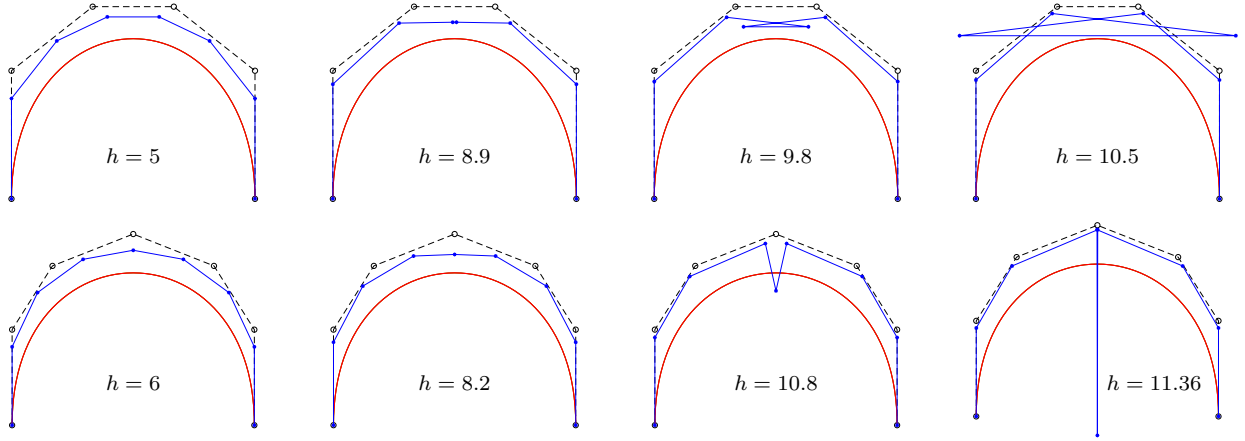
4

Figure 2: Dimension elevation $\mathbb{E}_n \subset \mathbb{E}_{n+2}$ on $[0, h]$: $\mathbb{E}_n$ is the degree n polynomial space, $\mathbb{E}_{n+2}$ is spanned by $\mathbb{E}_n$ and $\cos, \sin$, for $n = 5, 6$. Loss of shape preservation and convex combinations as h increases in $]\pi, \widetilde{\ell}_{n+2}[=]\pi, 11.5268[$. See Ex. 2.4.

Example 2.4. Consider the inclusion $\mathbb{E}_n \subset \mathbb{E}_{n+2}$, $n \geqslant 2$, where $\mathbb{E}_n := \mathbb{P}_n$ is the degree n polynomial space and $\mathbb{E}_{n+2}$ is the cycloidal space spanned by $\mathbb{P}_n$ and the two functions $\cos$ and $\sin$. In terms of characteristic polynomials we thus have $p_{n+2}(x) = p_n(x)(x^2 + 1) = x^{n+1}(x^2 + 1)$. The critical lengths of cycloidal spaces $\mathbb{E}_n$ are now well known, see [6, 1]. Prop. 2.3 ensures that, on $[0, h]$, we can go from $\mathbb{E}_n$ to $\mathbb{E}_{n+2}$ through two elementary dimension elevation steps for any positive $h < \pi$. By contrast, for $h \geqslant \pi$, no $(n + 2)$-dimensional space inserted between $\mathbb{E}_n$ and $\mathbb{E}_{n+2}$ is a W-space on $[0, h]$. We thus have numerically investigated the passage from given initial Bézier points $P_0, \ldots, P_n$ relative to $(0, h)$ to the corresponding final Bézier points $P_0^\star, \ldots, P_{n+2}^\star$, for increasing values of $h \in]\pi, \widetilde{\ell}_{n+2}[$ and increasing values of $n \geqslant 2$. As h increases, we successively lose shape preservation (negative answer to (Q_2) and therefore to (Q_3)), then the strict convex combinations (7) (negative answer to (Q_1)). This is illustrated in Fig. 2 with $n = 5$ and $n = 6$. When approaching $\widetilde{\ell}_{n+2}$, the central (two) point(s) even explode(s) much further away from the initial control polygon than shown in Fig. 2, this not being due to numerical problems. The behaviour is similar for any other value of the integer $n \geqslant 2$, depending on its parity. To provide the reader with a brief analysis of this phenomenon, consider the function $\Phi^\star := \big(S, S' \ldots, S^{(n+1)}\big)$, where $S \in \mathbb{E}_{n+2}$ satisfies $S(0) = S'(0) = \cdots = S^{(n+1)}(0) = 0$, $S^{(n+2)}(0) = 1$. Taking account of the invariance of the W-space $\mathbb{E}_{n+2}$ under translation, we know from Th. 2.2 in [7] that $\widetilde{\ell}_{n+2}$ is the supremum of all positive h such that the Bézier points $\Pi_i^\star(0, h)$, $i = 1, \ldots, n+1$, of $\Phi^\star$ relative to $(0, h)$ exist, obtained by intersecting convenient osculating flats of $\Phi^\star$ at 0 and h. In other words, the functions

$$\Delta_{i,j}(x) := \det\left(\Phi^{\star\prime}(x), \ldots, \Phi^{\star(i)}(x), \Phi^{\star\prime}(0), \ldots, \Phi^{\star(j)}(0)\right), \quad i, j > 0, \ i + j = n + 2,$$

do not vanish on $]0, \widetilde{\ell}_{n+2}[$, and there is at least one such pair (i, j) for which $\Delta_{i,j}(x)(\widetilde{\ell}_{n+2}) = 0$. From the results in [6, 1], we can more precisely say that, if $n = 2k$, only $\Delta_{k+1,k+1}$ vanishes at $\widetilde{\ell}_{n+2}$, which means that the central Bézier point $\Pi_{k+1}^\star(0, \widetilde{\ell}_{n+2})$ do not exist, or is located at infinity. Through affine images, this explains the second row in Fig.2. The same holds true for odd values of n, but now with the two central final Bézier points.

Example 2.5. Some additional examples first inclined us to conjecture that the answer to (Q_1), (Q_2) was always positive iff $\widetilde{\ell}_n < \widetilde{\ell}_{n+2}$. However the conjecture was proved to be false by considering the simple inclusion $\mathbb{E}_2 \subset \mathbb{E}_4$, with $p_2(x) = x(x^2 + 1)$ and $p_4(x) = p_2(x)(x^2 + b^2)$, with $b > 1$. In that case $\widetilde{\ell}_2 = \pi$ and $\widetilde{\ell}_4$ is known, see [3] and references therein. In particular $\widetilde{\ell}_4 = \pi$ for $b = 2$ and $b = 3$. For $h = \pi^-$, the curve visually coincides with the segment $[P_0, P_2]$, and when we additionally have b close to 1, so does the final

5

control polygon. Therefore, a clear answer to (Q_1), (Q_2) requires the exact computation of the final Bézier points as affine combinations of the initial ones for $h < \min(\widetilde{\ell_2}, \widetilde{\ell_4})$. This is done when P_0, P_1, P_2 are the Bézier points relative to $(0, h)$ of the function $\Phi(x) = (\cos x, \sin x)$, its final ones, $P_0^\star, \ldots, P_4^\star$, being the first two components of the Bézier points $\Pi_0^\star, \ldots, \Pi_4^\star$ of the function $\Phi^\star(x) = \big(\Phi(x), \cos(bx), \sin(bx)\big)$. The latter being obtained as intersections of convenient osculating flats of $\Phi^\star$ at 0 and h, we know that (see [13, 8])

$$\Pi_1^\star - \Pi_0^\star = \lambda_{1,1}^\star \Phi^{\star\prime}(0), \quad \Pi_2^\star - \Pi_0^\star = \lambda_{2,1}^\star \Phi^{\star\prime}(0) + \lambda_{2,2}^\star \Phi^{\star\prime\prime}(0), \tag{10}$$

with positive $\lambda_{1,1}^\star, \lambda_{2,2}^\star$. Taking account of the symmetry of the space $\mathbb{E}_4$, we have

$$\lambda_{1,1}^\star = \frac{\det\big(\Phi^{\star\prime}(0), \Phi^{\star\prime\prime}(0), \Phi^{\star\prime\prime\prime}(0), \Phi^\star(h) - \Phi^\star(0)\big)}{\det\big(\Phi^{\star\prime}(0), \Phi^{\star\prime\prime}(0), \Phi^{\star\prime\prime\prime}(0), \Phi^{\star\prime}(h)\big)}, \quad \lambda_{2,2}^\star = \frac{\det\big(\Phi^{\star\prime}(0), \Phi^{\star\prime\prime}(0), \Phi^\star(h) - \Phi^\star(0), \Phi^{\star\prime}(h)\big)}{\det\big(\Phi^{\star\prime}(0), \Phi^{\star\prime\prime}(0), \Phi^{\star\prime}(h), \Phi^{\star\prime\prime}(h)\big)},$$

that is,

$$\lambda_{1,1}^\star = \frac{1}{b}\frac{b^2(1 - \cos h) + \cos(bh) - 1}{b \sin h - \sin(bh)},$$

$$\lambda_{2,2}^\star = \frac{1}{b}\frac{(b^2 + 1)\big[1 - \cos h \cos(bh)\big] + (b^2 - 1)\big[\cos(bh) - \cos h\big] - 2b \sin h \sin(bh)}{2b\big[1 - \cos h \cos(bh)\big] - (b^2 + 1)\sin h \sin(bh)}.$$

We similarly have $P_1 - P_0 = \lambda_{1,1}\Phi'(0)$ and $P_2 - P_0 = \lambda_{2,1}\Phi'(0) + \lambda_{2,2}\Phi''(0)$, with

$$\lambda_{1,1} = \frac{\det\big(\Phi'(0), \Phi(h) - \Phi(0)\big)}{\det\big(\Phi'(0), \Phi'(h)\big)} = \frac{1 - \cos h}{\sin h}, \quad \lambda_{2,2} = \frac{\det\big(\Phi'(0), \Phi(h) - \Phi(0)\big)}{\det\big(\Phi'(0), \Phi''(0)\big)} = 1 - \cos h.$$

Combining this with the relations obtained from (10) by projection onto the first two components and with the symmetry of $\mathbb{E}_4$, we obtain

$$P_1^\star = (1 - A)P_0 + AP_1, \quad P_2^\star = CP_0 + (1 - 2C)P_1 + CP_2, \quad \text{with } A := \frac{\lambda_{1,1}^\star}{\lambda_{1,1}} \text{ and } C := \frac{\lambda_{2,2}^\star}{\lambda_{2,2}}.$$

It is easily checked that shape preservation is obtained iff $g(h) := 2C + A - 1 < 0$. Now, the various expressions above indicate that $g(h) = 0$ for $bh = \pi$. On the other hand, from Prop. 2.3 we know that $g(h) < 0$ for $h < \pi/b$. Accordingly, π/b is the first positive zero of the function g, i.e., $h < \pi/b$ is the necessary and sufficient condition for shape preservation. Whether $\widetilde{\ell_2} < \widetilde{\ell_4}$ (i.e., $b < 2$) or $\widetilde{\ell_2} = \widetilde{\ell_4}$ (i.e., $b = 2$ or $b = 3$), or $\widetilde{\ell_2} > \widetilde{\ell_4}$ (i.e., $b \in]2, +\infty[\backslash\{3\}$), we lose shape preservation when h approaches $\min(\widetilde{\ell_2}, \widetilde{\ell_4})$, namely above the blue hyperbola branch $h = \pi/b$ in Fig. 3, middle. The green curve shows the first positive zero of the function $g_1(h) := 2C - 1$, determined numerically. Above it, the answer to (Q_1) is negative. The previous comments are illustrated in Fig. 3, left and right, at two different points (h, b).

Let us now consider the limit case $b = 1$, where $\mathbb{E}_4$ is spanned by the functions $1, \cos x, \sin x, x\cos x, x\sin x$, with $\widetilde{\ell_4} \approx 4.4934$. Fig. 3, middle, suggests that shape preservation is obtained for any value of $h < \min(\widetilde{\ell_2}, \widetilde{\ell_4}) = \pi$. This is confirmed by calculating the corresponding quantities on $]0, \pi[$:

$$\lambda_{1,1}^\star = \frac{2(1 - \cos h) - h \sin h}{\sin h - h \cos h}, \quad \lambda_{2,2}^\star = \frac{h - \sin h}{h + \sin h}, \quad g(h) = \frac{-(1 + \cos h)(h - \sin h)^2}{(1 - \cos h)(h + \sin h)(\sinh - h \cos h)},$$

the function g being indeed negative on $]0, \pi[$.

Through the test built in [3] we could investigate numerically many more examples. In most cases, when h approaches $\min(\widetilde{\ell_n}, \widetilde{\ell_{n+2}})$, we loose shape preservation and even (7). Even though we are not able to conjecture in which examples this occurs, the previous few illustrations give indisputable evidence that the answer to any of the questions (Q_i), $i = 1, 2, 3$, is not always affirmative.

Readers interested in the subject are referred to [4] where additional examples are presented and commented. In particular, they will see why, in all situations $\widetilde{\ell_n} > \widetilde{\ell_{n+2}}$, we necessarily lose shape preservation when h approaches $\widetilde{\ell_{n+2}}^-$, which is consistent with the brief analysis in Ex. 2.4.
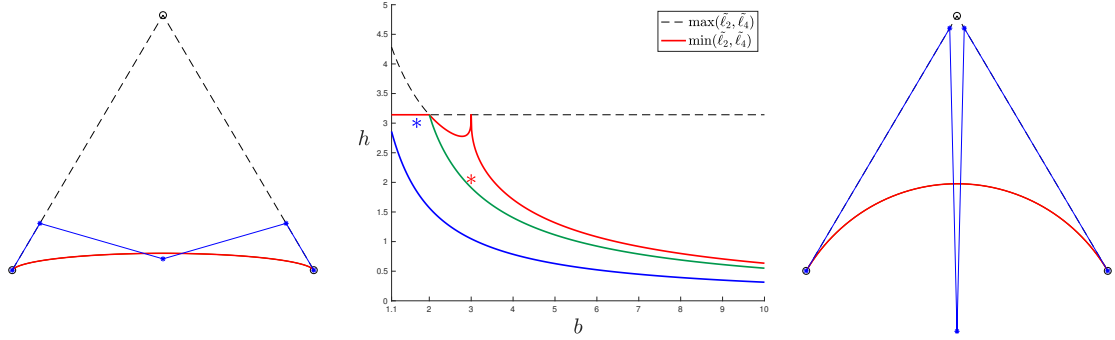
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Figure 3: $\mathbb{E}_2 \subset \mathbb{E}_4$, with $p_2(x) = x(x^2 + 1)$, $p_4(x) = p_2(x)(x^2 + b^2)$, $b > 1$. Middle: in function of b, $\min(\widetilde{\ell}_2, \widetilde{\ell}_4)$ (red); limit value of the length h for shape preservation (blue); limit value of h for convex combinations (green). Dimension elevation at: $*$ (left), $*$ (right). See Ex. 2.5.

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