# REGULARITY AND ENERGY TRANSFER FOR A NONLINEAR BEAM EQUATION 

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#### Abstract

In this paper we study some key effects of a discontinuous forcing term in a fourth order wave equation on a bounded domain, modeling the adhesion of an elastic beam with a substrate through an elastic-breakable interaction. By using a spectral decomposition method we show that the main effects induced by the nonlinearity at the transition from attached to detached states can be traced in a loss of regularity of the solution and in a migration of the total energy through the scales.


## 1. Introduction

The essential mechanism underlying the manifestation of adhesion phenomena in nature relies on the possible intermittency between the two states in which the material bodies experience partial contact or separation and an interesting mathematical problem consists in understanding the effects of this occurrence on the dynamical evolution problem. More precisely, the tricky question asks for a precise quantification of the nonlinearity injected into the system through a discontinuous forcing term.

In this paper we deal with the initial boundary value problem ruled by the elastic beam equation complemented by a source term undergoing a sharp discontinuity when a threshold on the displacement is reached. The analysis performed in this note suggests that the transition between the attacheddetached states entails two main effects, namely the loss of regularity in the velocity field and a migration of energy through the scales. We believe that this result suggests an interesting perspective (as in [6]) in the dynamical analysis of a large class of problems as the ones studied, for instance, in [2, 5].

The evolution of an elastic beam according to the Bernoulli-Navier model interacting with a rigid substrate through an elastic-breakable forcing term leads to the following semilinear initial boundary value problem in the unknown function $u(t, x)$ denoting the displacement at the time $t \geq 0$ of the material point located in the reference configuration at $x \in[0, L]$ :

$$
\begin{cases}\partial_{t t}^{2} u=-\kappa_{1}^{2} \partial_{x x x x}^{4} u-\Phi^{\prime}(u) & t>0,0<x<L  \tag{1.1}\\ \partial_{x} u(t, 0)=\partial_{x} u(t, L)=0 & t>0 \\ \partial_{x x x}^{3} u(t, 0)=\partial_{x x x}^{3} u(t, L)=0 & t>0 \\ u(0, x)=v_{0}(x) & 0<x<L \\ \partial_{t} u(0, x)=v_{1}(x) & 0<x<L\end{cases}
$$

where $\kappa_{1}$ is a positive constant representing the flexural stiffness of the beam. We shall assume that

$$
\begin{equation*}
v_{0} \in H^{2}(0, L), \quad v_{1} \in L^{2}(0, L) \tag{1.2}
\end{equation*}
$$

[^0]The forcing term $\Phi^{\prime}$ is thought to model the adhesive like interaction of the beam with the substrate, allowing the debonding after a given threshold is attained. This behavior is the source of a localized nonlinearity which confers to the evolution problem a nontrivial peculiarity. Then we assume $\Phi$ is the following function

$$
\Phi(u)= \begin{cases}\kappa_{2}^{2} u^{2} / 2 & \text { if }|u| \leq 1  \tag{1.3}\\ \kappa_{2}^{2} / 2 & \text { if }|u|>1\end{cases}
$$

In particular we have for all $u \neq \pm 1$

$$
\Phi^{\prime}(u)= \begin{cases}\kappa_{2}^{2} u & \text { if }|u|<1  \tag{1.4}\\ 0 & \text { if }|u|>1\end{cases}
$$

The natural energy associated to (1.1) (i.e. to any solution $u$ to (1.1)), is given at time $t$ by the quantity

$$
\begin{equation*}
E[u](t)=\int_{0}^{L}\left(\frac{\left(\partial_{t} u(t, x)\right)^{2}+\kappa_{1}^{2}\left(\partial_{x x}^{2} u(t, x)\right)^{2}}{2}+\Phi(u(t, x))\right) d x \tag{1.5}
\end{equation*}
$$

Definition 1.1. We say that a function $u:[0, \infty) \times[0, L] \rightarrow \mathbb{R}$ is a dissipative solution of (1.1) if
(i) $u \in C([0, \infty) \times[0, L])$;
(ii) $\partial_{t} u, \partial_{x x}^{2} u \in L^{\infty}\left(0, \infty ; L^{2}(0, L)\right)$ and $u \in L^{\infty}\left(0, T ; H^{2}(0, L)\right)$ for every $T>0$;
(iii) $\partial_{x} u(t, 0)=\partial_{x} u(t, L)=0$ for almost every $t>0$;
(iv) $u$ is $a$ weak solution to (1.1), i.e. for every test function $\varphi \in C^{\infty}\left(\mathbb{R}^{2}\right)$ with compact support such that $\partial_{x} \varphi(\cdot, 0)=\partial_{x} \varphi(\cdot, L)=0$

$$
\begin{align*}
\int_{0}^{\infty} \int_{0}^{L} & \left(u \partial_{t t}^{2} \varphi+\kappa_{1}^{2} \partial_{x x}^{2} u \partial_{x x}^{2} \varphi+h_{u} \varphi\right) d t d x  \tag{1.6}\\
& -\int_{0}^{L} v_{1}(x) \varphi(0, x) d x+\int_{0}^{L} v_{0}(x) \partial_{t} \varphi(0, x) d x=0
\end{align*}
$$

where $h_{u} \in \partial \Phi^{\prime}(u)$, that is the subdifferential of $\Phi^{\prime}(u)$;
(v) $u$ may dissipate energy, i.e. for almost every $t>0: E[u](t) \leq E[u](0)$, i.e. by taking into account (1.5),

$$
\begin{align*}
& \int_{0}^{L}\left(\frac{\left(\partial_{t} u(t, x)\right)^{2}+\kappa_{1}^{2}\left(\partial_{x x}^{2} u(t, x)\right)^{2}}{2}+\Phi(u(t, x))\right) d x  \tag{1.7}\\
& \leq \int_{0}^{L}\left(\frac{\left(v_{1}(x)\right)^{2}+\kappa_{1}^{2}\left(\partial_{x x}^{2} v_{0}(x)\right)^{2}}{2}+\Phi\left(u_{0}(x)\right)\right) d x
\end{align*}
$$

In the paper [3] the well-posedness of (1.1) is studied in detail (in the same spirit of the work [4] concerning the adhesive interaction for the second order wave operator) and existence of solutions in the sense of Definition 1.1 is proved, moreover some counterxamples to uniqueness of solutions are provided.

The main result of this paper is the following.
Theorem 1.1. Let $u$ be a solution of (1.1) in the sense of Definition 1.1. If
$v_{0}$ and $v_{1}$ are constants with $\left|v_{0}\right|<1 ;$
there exists a time $\bar{t}>0$ such that $|u(\bar{t}, x)|>1$ for every $x \in[0, L]$
$u \in C^{1}([0, \infty) \times[0, L])$
$u$ is energy preserving, i.e. $E[u](t)=E[u](0)$
then

$$
u(t, \cdot) \text { is constant for every } t \geq \bar{t}
$$

Remark 1.1. The interesting consequence of the previous theorem relies in enlightening the effect of the nonlinearity hidden in the transition between the two regimes ruled by the conditions $|u|<1$ and $|u|>1$ corresponding to attached and detached states. It allows to state that a solution experiencing the transition in a region strictly contained in $(0, L)$ fails to be in $C^{1}([0, \infty) \times[0, L])$.

## 2. Spectral Decomposition

We consider the set of eigenvalues $\left\{\mu_{n}\right\}_{n \in \mathbb{N}}$ and eigenvectors $\left\{u_{n}\right\}_{n \in \mathbb{N}}$ of the differential operator $\partial_{x x}^{2}$ with homogeneous Neumann boundary conditions such that

$$
\left\{\begin{array}{l}
\partial_{x x}^{2} u_{n}=\mu_{n} u_{n}, \text { in }(0, L), \\
\partial_{x} u_{n}(0)=\partial_{x} u_{n}(L)=0,
\end{array} \quad \text { and } \quad \int_{0}^{L} u_{n} u_{m} d x=\delta_{n m}\right.
$$

Since $\partial_{x x}^{2}$ with homogeneous Neumann boundary conditions is a negative operator (see [1]), the eigenvalues are nonpositive, so we set $\mu_{n}=-\lambda_{n}^{2}$ and get the identity

$$
\begin{equation*}
\partial_{x x}^{2} u_{n}=-\lambda_{n}^{2} u_{n}, \quad \text { in }(0, L) \tag{2.1}
\end{equation*}
$$

A simple bootstrap argument allows us to deduce that for every $n \in \mathbb{N}$,

$$
\begin{equation*}
u_{n} \in C^{\infty}([0, L]) \tag{2.2}
\end{equation*}
$$

and they are eigenvectors of the differential operator $\partial_{x x x x}^{4}$ satisfying the following conditions:

$$
\left\{\begin{array}{l}
\partial_{x x x x}^{4} u_{n}=\lambda_{n}^{4} u_{n}, \text { in }(0, L),  \tag{2.3}\\
\partial_{x} u_{n}(0)=\partial_{x} u_{n}(L)=0, \\
\partial_{x x x}^{3} u_{n}(0)=\partial_{x x x}^{3} u_{n}(L)=0,
\end{array} \quad \text { and } \quad \int_{0}^{L} u_{n} u_{m} d x=\delta_{n m}\right.
$$

A direct computation delivers

$$
u_{n}=\left\{\begin{array}{ll}
L^{-1 / 2}, & \text { if } n=0,  \tag{2.4}\\
\sqrt{\frac{2}{L}} \cos \left(\frac{n \pi}{L} x\right), & \text { if } n>0,
\end{array} \quad \text { and } \quad \lambda_{n}=\frac{n \pi}{L}\right.
$$

Let $u$ be a solution of (1.1) according to Definition 1.1. Since $\left\{u_{n}\right\}_{n \in \mathbb{N}}$ is an Hilbert basis in $H^{2}(0, L)$ we are allowed to decompose $u$ as follows

$$
\begin{equation*}
u(t, x)=\sum_{n=0}^{\infty} \alpha_{n}(t) u_{n}(x) \tag{2.5}
\end{equation*}
$$

Let us start our analysis by considering the two different regimes $|u|<1$ and $|u|>1$.
Lemma 2.1. If $|u|<1$, then

$$
\begin{equation*}
\alpha_{n}(t)=A_{n} \cos \left(\omega_{n} t+\varphi_{n}\right), \quad \omega_{n}=\left[\kappa_{1}^{2}\left(\frac{n \pi}{L}\right)^{4}+\kappa_{2}^{2}\right]^{1 / 2}, \quad E[u](t)=\sum_{n=0}^{\infty} \frac{1}{2} A_{n}^{2} \omega_{n}^{2} \tag{2.6}
\end{equation*}
$$

for almost every $t>0$, all $n \in \mathbb{N}$ and the constants $A_{n}, \varphi_{n}$ are implicitly defined through the initial conditions

$$
\begin{equation*}
v_{0}(x)=\sum_{n=0}^{\infty} A_{n} \cos \left(\varphi_{n}\right) u_{n}(x), \quad v_{1}(x)=-\sum_{n=0}^{\infty} A_{n} \omega_{n} \sin \left(\varphi_{n}\right) u_{n}(x) \tag{2.7}
\end{equation*}
$$

Proof. By using (2.3) and (2.5) in (1.1) we get

$$
\left\{\begin{array}{l}
\ddot{\alpha}_{n}+\left[\kappa_{1}^{2}\left(\frac{n \pi}{L}\right)^{4}+\kappa_{2}^{2}\right] \alpha_{n}=0 \\
\alpha_{n}(0)=A_{n} \cos \left(\varphi_{n}\right) \\
\dot{\alpha}_{n}(0)=-A_{n} \omega_{n} \sin \left(\varphi_{n}\right)
\end{array}\right.
$$

which returns the first two equations of (2.6), while the third one follows from

$$
\begin{aligned}
E[u](t) & =\sum_{n=0}^{\infty} \frac{1}{2}\left\{\left[\partial_{t}\left(A_{n} \cos \left(\omega_{n} t+\varphi_{n}\right)\right)\right]^{2}+\omega_{n}^{2} A_{n}^{2} \cos ^{2}\left(\omega_{n} t+\varphi_{n}\right)\right\} \\
& =\sum_{n=0}^{\infty} \frac{1}{2}\left\{A_{n}^{2} \omega_{n}^{2} \sin ^{2}\left(\omega_{n} t+\varphi_{n}\right)+\omega_{n}^{2} A_{n}^{2} \cos ^{2}\left(\omega_{n} t+\varphi_{n}\right)\right\}=\sum_{n=0}^{\infty} \frac{1}{2} A_{n}^{2} \omega_{n}^{2}
\end{aligned}
$$

Lemma 2.2. If $|u|>1$, then

$$
\begin{align*}
\alpha_{n}(t) & =\left\{\begin{array}{ll}
B_{n} \cos \left(\nu_{n} t+\psi_{n}\right), & \text { if } n \neq 0, \\
C_{0}+C_{1} t, & \text { if } n=0,
\end{array} \quad \nu_{n}=\left|\kappa_{1}\right|\left(\frac{n \pi}{L}\right)^{2},\right.  \tag{2.8}\\
E[u](t) & =\frac{L}{2} \kappa_{2}^{2}+\frac{1}{2} C_{1}^{2}+\frac{1}{2} \sum_{n=1}^{\infty} B_{n}^{2} \nu_{n}^{2} \tag{2.9}
\end{align*}
$$

for almost every $t>0$, all $n \in \mathbb{N}$ and the constants $B_{n}, \psi_{n}$ are implicitly defined through the initial conditions

$$
v_{0}(x)=\sum_{n=1}^{\infty} B_{n} \cos \left(\psi_{n}\right) u_{n}(x)+C_{0} u_{0}(x), \quad v_{1}(x)=-\sum_{n=1}^{\infty} B_{n} \omega_{n} \sin \left(\psi_{n}\right) u_{n}(x)+C_{1} u_{0}(x)
$$

Proof. By using (2.3) and (2.5) in (1.1) we get

$$
\begin{aligned}
& \ddot{\alpha}_{n}+\kappa_{1}^{2}\left(\frac{n \pi}{L}\right)^{4} \alpha_{n}=0, \\
& \alpha_{n}(0)=\left\{\begin{array}{ll}
A_{n} \cos \left(\varphi_{n}\right), & \text { if } n \neq 0, \\
C_{0}, & \text { if } n=0,
\end{array} \quad \dot{\alpha}_{n}(0)= \begin{cases}-A_{n} \omega_{n} \sin \left(\varphi_{n}\right), & \text { if } n \neq 0 \\
C_{1}, & \text { if } n=0\end{cases} \right.
\end{aligned}
$$

which returns (2.8), while (2.9) follows from

$$
\begin{aligned}
E[u](t) & =\frac{L}{2} \kappa_{2}^{2}+\frac{1}{2} C_{1}^{2}+\sum_{n=1}^{\infty} \frac{1}{2}\left\{\left[\partial_{t}\left(B_{n} \cos \left(\nu_{n} t+\psi_{n}\right)\right)\right]^{2}+\nu_{n}^{2} B_{n}^{2} \cos ^{2}\left(\nu_{n} t+\psi_{n}\right)\right\} \\
& =\frac{L}{2} \kappa_{2}^{2}+\frac{1}{2} C_{1}^{2}+\sum_{n=1}^{\infty} \frac{1}{2}\left\{B_{n}^{2} \nu_{n}^{2} \sin ^{2}\left(\nu_{n} t+\psi_{n}\right)+\nu_{n}^{2} B_{n}^{2} \cos ^{2}\left(\nu_{n} t+\psi_{n}\right)\right\} \\
& =\frac{L}{2} \kappa_{2}^{2}+\frac{1}{2} C_{1}^{2}+\sum_{n=1}^{\infty} \frac{1}{2} B_{n}^{2} \nu_{n}^{2}
\end{aligned}
$$

Proof of Theorem 1.1. Due to (1.8) we decompose the initial data as in (2.7). Thanks to (2.5), and (2.6), we know that

$$
u(0, x)=\alpha_{0}(0) u_{0}, \quad u_{0}=L^{-1 / 2}, \quad E[u](0)=\frac{1}{2} A_{0}^{2} \omega_{0}^{2}=\frac{1}{2} A_{0}^{2} \kappa_{2}^{2}, \quad A_{n}=0, n \neq 0
$$

Moreover, as long as $|u(t, \cdot)|<1$, due to (2.6),

$$
\left|\alpha_{0}(t)\right| \leq A_{0} L^{-1 / 2}
$$

Let us distinguish three possible scenarios.
If

$$
A_{0}<L^{1 / 2}
$$

we have that $|u(t, x)| \leq\left|\alpha_{0}(t)\right| \leq A_{0} L^{-1 / 2}<1$. As a consequence (1.9) is never satisfied.
If

$$
A_{0} \geq L^{1 / 2}
$$

from (2.6) in Lemma [2.1, we have that $\left|\alpha_{0}\right|$ monotonically grows up to the value $|u(\bar{t}, x)| \geq 1$, for all $x$ in $[0, L]$. From [2.2, we have that $\partial_{t}\left|\alpha_{0}\right|(\bar{t}) \geq 0$ so, by a translation in time argument, we can deal with the solution evaluated in $t-\bar{t}$. Hence, we are in the hypotheses of Lemma 2.2, From here, we then get that

$$
E[u](\bar{t})=\frac{L}{2} \kappa_{2}^{2}+\frac{1}{2} C_{1}^{2}+\sum_{n=1}^{\infty} \frac{1}{2} B_{n}^{2} \nu_{n}^{2} .
$$

Due to (1.11), $E[u](\bar{t})=E[u](0)=\frac{1}{2} A_{0}^{2} \kappa_{2}^{2}$.
Let us exploit the condition $\left|\alpha_{0}(\bar{t})\right|=L^{1 / 2}$. Due to the assumption (1.10), from the (2.6) of Lemma 2.1 and (2.8) of Lemma [2.2, we have that

$$
\begin{equation*}
A_{0} \cos \left(\left|\kappa_{2}\right| \bar{t}+\phi_{o}\right)=C_{0}+C_{1} \bar{t} \tag{2.10}
\end{equation*}
$$

Moreover, we also have that

$$
\begin{equation*}
-A_{0}\left|\kappa_{2}\right| \sin \left(\left|\kappa_{2}\right| \bar{t}+\phi_{0}\right) L^{-1 / 2}=C_{1} L^{-1 / 2} . \tag{2.11}
\end{equation*}
$$

By combining (2.10) and (2.11), we get

$$
\begin{equation*}
C_{1}^{2}=\kappa_{2}^{2}\left(A_{0}^{2}-L\right) . \tag{2.12}
\end{equation*}
$$

Therefore

$$
E[u](\bar{t})-E[u](0)=\sum_{n=1}^{\infty} \frac{1}{2} B_{n}^{2} \nu_{n}^{2}=0 .
$$

This implies that $B_{n}=0$, for all $n \geq 1$. Therefore, the only allowed solution is the constant one.
Eventually, the last case $A_{0} \geq L^{\overline{1 / 2}}$ and $\dot{\alpha}_{0}(0)=0$ is already in the assumptions of Lemma 2.2, the solution is

$$
\begin{equation*}
u_{0}=C_{0} L^{-1 / 2}=A_{0} L^{-1 / 2} \tag{2.13}
\end{equation*}
$$

where $C_{0}=A_{0}$ is due to the continuity and $C_{1}=0$ is due to the derivability in $t=0$.
Corollary 2.1. Assume that (1.8) and (1.9) hold. Let $u$ be a dissipative solution of (1.1) according to Definition 1.1. If (1.10) is not satisfied, then

$$
\begin{equation*}
\left|C_{1}\right| \leq\left|\kappa_{2}\right| \sqrt{A_{0}^{2}-L}, \quad B_{n} \lesssim n^{-5 / 2}, \forall n>1 . \tag{2.14}
\end{equation*}
$$

Proof. Due to (1.8) and (1.9), we have that $A_{0} \geq L^{1 / 2}$. Moreover, from (1.10), we have that

$$
\begin{equation*}
\frac{L}{2} \kappa_{2}^{2}+\frac{1}{2} C_{1}^{2}+\frac{1}{2} \sum_{n=1}^{\infty} B_{n}^{2} \nu_{n}^{2}-\frac{1}{2} A_{0}^{2} \kappa_{2}^{2}=E[u](t)-E[u](0) \leq 0 \tag{2.15}
\end{equation*}
$$

which leads to

$$
C_{1}^{2}-\kappa_{2}^{2}\left(A_{0}^{2}-L\right) \leq-\sum_{n=1}^{\infty} B_{n}^{2} \nu_{n}^{2} \leq 0,
$$

that is the first of (2.14).
On the other hand, from (2.15) and (2.8) of Lemma 2.2, we have that

$$
\begin{equation*}
0 \leq \sum_{n=1}^{\infty} B_{n}^{2} n^{4} \leq \frac{\kappa_{2}^{2}\left(A_{0}^{2}-L\right) L^{4}}{\pi^{4} \kappa_{1}^{2}}<\infty \tag{2.16}
\end{equation*}
$$

and then the second of (2.14).
Remark 2.1. The previous results lead to the following conclusion. A solution of (1.1) experiencing a transition from the attached to the detached regimes, according to the requirements to Corollary 2.1, exhibits energy migration through the scales ruled by the second of (2.14).

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