ON THE SOLVABILITY OF A PARAMETER-DEPENDENT CANTILEVER-TYPE BVP

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ABSTRACT. We discuss the solvability of a parameter dependent cantilever-type boundary value problem. We provide an existence and localization result for the positive solutions via a Birkhoff-Kellogg type theorem. We also obtain, under additional growth conditions, upper and lower bounds for the involved parameters. An example is presented in order to illustrate the theoretical results.

1. INTRODUCTION

Differential equations have been utilized to model the steady states of deflections of elastic beams; for example the fourth order ordinary differential equation

(1.1)
$$u^{(4)}(t) = f(t, u(t)), \ t \in (0, 1),$$

subject to the homogeneous boundary conditions (BCs)

(1.2)
$$u(0) = u'(0) = u''(1) = u'''(1) = 0,$$

can be used as a model for the so-called cantilever bar. The boundary value problem (BVP) (1.1)-(1.2) describes a bar of length 1 which is clamped on the left end and is free to move at the right end, with vanishing bending moment and shearing force, see for example [1, 16, 27].

Under a mechanical point of view, some interesting cases appear when the shearing force at the right side of the beam does not vanish (see for example [12]):

- $u'''(1) + k_0 = 0$ models a force acting in 1,
- $u'''(1) + k_1 u(1) = 0$ describes a spring in 1,
- u'''(1) + g(u(1)) = 0 models a spring with a strongly nonlinear rigidity,
- $u'''(1) + g(u(\eta)) = 0$ describes a feedback mechanism, where the spring reacts to the displacement registered in a point η of the beam,
- $u'''(1) + g(u(\eta_1), u'(\eta_2), u''(\eta_3)) = 0$ describes the case where the spring reacts to the displacement registered in the point η_1 , the angular attitude registered in the point η_2 and the bending moment in the point η_3 .

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Of course, a different configuration of the beam may lead to more complicated BCs than the ones illustrated above. It is therefore not surprising that the case of non-homogeneous BCs has received attention by researchers. By means of critical point theory, Cabada and Terzian [4] and Bonanno, Chinnì and Terzian [2] and Yang, Chen and Yang [26] studied the parameter-dependent BVP

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \ t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + \lambda g(u(1)) = 0, \end{cases}$$

while the case of $\lambda = 1$ has been investigated in an earlier paper by Ma [17].

By classical fixed point index, Cianciaruso, Infante and Pietramala [5] studied the BVP

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \ t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + \hat{H}(u) = 0, \end{cases}$$

where \hat{H} is a suitable functional (not necessarily linear) on C[0, 1]. The functional approach for the BCs adopted in [5] fits within the interesting framework of nonlinear and nonlocal BCs; these are widely studied objects, we refer the reader to the reviews [3, 6, 18, 20, 19, 21, 25] and the manuscripts [7, 13, 23].

Regarding the higher order dependence in the forcing term, the ODE

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, 1]$$

under the homogeneous BCs

$$u(0) = u'(0) = u''(1) = u'''(1) = 0$$

has been studied by Li [16] via fixed point index. The non-homogeneous case

$$u(0) = u'(0) = u''(1) = 0, u'''(1) + g(u(1)) = 0$$

has been studied, with the lower and upper solutions method, by Wei, Li and Li [24], while the case

$$u(0) = u'(0) = \int_0^1 p(t)u(t) \, dt, \\ u''(1) = u'''(1) = \int_0^1 q(t)u''(t) \, dt,$$

has been investigated by Khanfer and Bougoffa [14] via the Schauder fixed point theorem.

Here we study the solvability of the parameter-dependent BVP

(1.3)
$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t), u'(t), u''(t), u'''(t)), \ t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + \lambda H[u] = 0, \end{cases}$$

where f is a continuous function, H is a suitable compact functional in the space $C^3[0, 1]$ (this allows higher order dependence in the BCs) and λ is a non-negative parameter. For the existence result we adapt an approach used by the author [11], in the context of elliptic systems, that relies on a Birkhoff-Kellogg type theorem in cones due to Krasnosel'skii and Ladyženskii [15]. We also provide, under additional growth conditions, a localization result for the parameter λ . The results are new and complement the ones present in the papers [2, 4, 5, 14, 16, 17, 24, 26]. We also complement the results in [10], by obtaining additional qualitative properties (such as monotonicity and localization) of the solution. We illustrate the applicability of our theoretical results in an example.

2. EXISTENCE AND LOCALIZATION OF THE EIGENVALUES

First of all we associate to the BVP (1.3) a perturbed Hammerstein integral equation of the form

(2.1)
$$u(t) = \lambda \Big(\gamma(t) H[u] + \int_0^1 k(t,s) f(s,u(s),u'(s),u''(s),u''(s)) \, ds \Big),$$

where the Green's function k and the function γ need to be determined; this is done by considering two auxiliary BVPs, a procedure found to be particularly useful in the case of nonlinear BCs, see for example [9] and references therein.

Regarding k it is known (see for example Lemma 2.1 and Lemma 2.2 of [16]) that for $h \in C[0, 1]$ the unique solution of the linear BVP

$$\begin{cases} u^{(4)}(t) = h(t), \ t \in (0, 1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

is given by

$$u(t) = \int_0^1 k(t,s)h(s) \, ds$$

where

$$k(t,s) = \begin{cases} \frac{1}{6}(3t^2s - t^3), & s \ge t, \\ \frac{1}{6}(3s^2t - s^3), & s \le t. \end{cases}$$

Note that the function k has the following properties

$$k(t,s), \frac{\partial k}{\partial t}(t,s), \frac{\partial^2 k}{\partial t^2}(t,s) \ge 0 \text{ on } [0,1] \times [0,1],$$

and

$$\frac{\partial^3 k}{\partial t^3}(t,s) \le 0 \text{ on } [0,1]^2 \setminus \{(t,s)|t=s)\}.$$

Regarding the function γ , note that (see for example [12])

$$\gamma(t) = \frac{1}{6} (3t^2 - t^3)$$

is the unique solution of the BVP

$$\gamma^{(4)}(t) = 0, \ \gamma(0) = \gamma'(0) = \gamma''(1) = 0, \ \gamma'''(1) + 1 = 0.$$

By direct calculation, it can be observed that

$$\gamma(t), \gamma'(t), \gamma''(t), -\gamma'''(t) \ge 0 \text{ on } [0, 1].$$

With the above ingredients at our disposal, we can work in the space $C^{3}[0, 1]$ endowed with the norm

$$||u||_3 := \max_{j=0,\dots,3} \{ ||u^{(j)}||_\infty \}, \text{ where } ||w||_\infty = \sup_{t \in [0,1]} |w(t)|.$$

Definition 2.1. We say that λ is an *eigenvalue* of the BVP (1.3) with a corresponding eigenfunction $u \in C^3[0,1]$ with $||u||_3 > 0$ if the pair (u, λ) satisfies the perturbed Hammerstein integral equation (2.1).

We make use of the following Birkhoff-Kellogg type theorem in order to seek the eigenfunctions of the BVP (1.3). We recall that a cone \mathcal{K} of a real Banach space (X, || ||) is a closed set with $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$, $\mu \mathcal{K} \subset \mathcal{K}$ for all $\mu \geq 0$ and $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

Theorem 2.2 (Theorem 2.3.6, [8]). Let (X, || ||) be a real Banach space, $U \subset X$ be an open bounded set with $0 \in U$, $\mathcal{K} \subset X$ be a cone, $T : \mathcal{K} \cap \overline{U} \to \mathcal{K}$ be compact and suppose that

$$\inf_{x \in \mathcal{K} \cap \partial U} \|Tx\| > 0.$$

Then there exist $\lambda_0 \in (0, +\infty)$ and $x_0 \in \mathcal{K} \cap \partial U$ such that $x_0 = \lambda_0 T x_0$.

We apply the Theorem 2.2 in the cone

(2.2)
$$K := \left\{ u \in C^3[0,1] : u(t), u'(t), u''(t), -u'''(t) \ge 0, \text{ for every } t \in [0,1] \right\}.$$

The cone (2.2) is a smaller cone than the one of positive functions used in [10], but larger than the one used in [16]. We consider the sets

$$K_{\rho} := \{ u \in K : ||u||_{3} < \rho \}, \ \overline{K}_{\rho} := \{ u \in K : ||u||_{3} \le \rho \},$$
$$\partial K_{\rho} := \{ u \in K : ||u||_{3} = \rho \},$$

where $\rho \in (0, +\infty)$.

The following Theorem provides an existence result for an eigenfunction possessing a fixed norm and a corresponding positive eigenvalue.

Theorem 2.3. Let $\rho \in (0, +\infty)$ and assume the following conditions hold.

(a) $f \in C(\Pi_{\rho}, \mathbb{R})$ and there exist $\underline{\delta}_{\rho} \in C([0, 1], \mathbb{R}_{+})$ such that

 $f(t, u, v, w, z) \ge \underline{\delta}_{\rho}(t), \text{ for every } (t, u, v, w, z) \in \Pi_{\rho},$

where

$$\Pi_{\rho}: [0,1] \times [0,\rho]^3 \times [-\rho,0]$$

(b) $H: \overline{K}_{\rho} \to \mathbb{R}$ is continuous and bounded. Let $\underline{\eta}_{\rho} \in [0, +\infty)$ be such that

$$H[u] \geq \underline{\eta}_{\rho}, \text{ for every } u \in \partial K_{\rho}$$

(c) The inequality

(2.3)
$$\underline{\eta}_{\rho} + \int_{0}^{1} \underline{\delta}_{\rho}(s) \, ds > 0$$

holds.

Then the BVP (1.3) has a positive eigenvalue λ_{ρ} with an associated eigenfunction $u_{\rho} \in \partial K_{\rho}$.

Proof. Let $Fu(t) := \int_0^1 k(t,s) f(s, u(s), u'(s), u''(s), u''(s)) ds$ and $\Gamma u(t) := \gamma(t) H[u]$. Note that, due to the assumptions above, the operator $T = F + \Gamma$ maps \overline{K}_{ρ} into K and is compact; the compactness of F follows from a careful use of the Arzelà-Ascoli theorem (see [22]) and Γ is a finite rank operator.

Take $u \in \partial K_{\rho}$, then we have

$$(2.4) ||Tu||_{3} \ge ||(Tu)'''||_{\infty} \ge |(Tu)'''(0)|$$

= $|-H[u] - \int_{0}^{1} f(s, u(s), u'(s), u''(s), u'''(s)) ds|$
= $H[u] + \int_{0}^{1} f(s, u(s), u'(s), u''(s), u'''(s)) ds \ge \underline{\eta}_{\rho} + \int_{0}^{1} \underline{\delta}_{\rho}(s) ds.$

Note that the RHS of (2.4) does not depend on the particular u chosen. Therefore we have

$$\inf_{u\in\partial K_{\rho}}\|Tu\|_{3} \ge \underline{\eta}_{\rho} + \int_{0}^{1} \underline{\delta}_{\rho}(s) \, ds > 0,$$

and the result follows by Theorem 2.2.

The following Corollary provides an existence result for the existence of uncountably many couples of eigenvalues–eigenfunctions.

Corollary 2.4. In addition to the hypotheses of Theorem 2.3, assume that ρ can be chosen arbitrarily in $(0, +\infty)$. Then for every ρ there exists a non-negative eigenfunction $u_{\rho} \in \partial K_{\rho}$ of the BVP (1.3) to which corresponds a $\lambda_{\rho} \in (0, +\infty)$.

The next result provides some upper and lower bounds on the eigenvalues.

Theorem 2.5. In addition to the hypotheses of Theorem 2.3 assume the following conditions hold.

(d) There exist $\overline{\delta}_{\rho} \in C([0,1],\mathbb{R}_+)$ such that

$$f(t, u, v, w, z) \leq \overline{\delta}_{\rho}(t), \text{ for every } (t, u, v, w, z) \in \Pi_{\rho}.$$

(e) Let $\overline{\eta}_{\rho} \in [0, +\infty)$ be such that

$$H[u] \leq \overline{\eta}_{\rho}, \text{ for every } u \in \partial K_{\rho}.$$

Then λ_{ρ} satisfies the following estimates

$$\frac{\rho}{\left(\overline{\eta}_{\rho} + \int_{0}^{1} \overline{\delta}_{\rho}(s) \, ds\right)} \leq \lambda_{\rho} \leq \frac{\rho}{\left(\underline{\eta}_{\rho} + \int_{0}^{1} \underline{\delta}_{\rho}(s) \, ds\right)}.$$

Proof. By Theorem 2.3 there exist $u_{\rho} \in \partial K_{\rho}$ and λ_{ρ} such that

(2.5)
$$u_{\rho}(t) = \lambda_{\rho} \Big(\gamma(t) H[u_{\rho}] + \int_{0}^{1} k(t,s) f(s, u_{\rho}(s), \dots, u_{\rho}''(s)) \, ds \Big).$$

By differentiating (2.5) we obtain

$$u_{\rho}'(t) = \lambda_{\rho} \Big(\gamma'(t) H[u_{\rho}] + \int_{0}^{1} \frac{\partial k}{\partial t}(t,s) f(s, u_{\rho}(s), \dots, u_{\rho}''(s)) \, ds \Big),$$

$$u_{\rho}''(t) = \lambda_{\rho} \Big(\gamma''(t) H[u_{\rho}] + \int_{0}^{1} \frac{\partial^{2} k}{\partial t^{2}}(t,s) f(s, u_{\rho}(s), \dots, u_{\rho}'''(s)) \, ds \Big),$$

$$u_{\rho}'''(t) = \lambda_{\rho} \Big(-H[u_{\rho}] - \int_{t}^{1} f(s, u_{\rho}(s), u_{\rho}'(s), u_{\rho}''(s)) \, ds \Big),$$

which implies

(2.6)
$$\|u_{\rho}^{\prime\prime\prime}\|_{\infty} = \lambda_{\rho} \Big(H[u_{\rho}] + \int_{0}^{1} f(s, u_{\rho}(s), \dots, u_{\rho}^{\prime\prime\prime}(s)) \, ds \Big).$$

Furthermore note that

$$0 \le k(t,s), \frac{\partial k}{\partial t}(t,s), \frac{\partial^2 k}{\partial t^2}(t,s) \le 1, \text{ on } [0,1] \times [0,1],$$

and

$$0 \le \gamma(t), \gamma'(t), \gamma''(t) \le 1 \text{ on } [0, 1],$$

which yield

$$\rho = \|u_{\rho}\|_{3} = \|u_{\rho}^{\prime\prime\prime}\|_{\infty}.$$

From (2.6) and the estimates (d) and (e) we obtain

$$\rho = \lambda_{\rho} \Big(H[u_{\rho}] + \int_0^1 f(s, u_{\rho}(s), \dots, u_{\rho}''(s)) \, ds \Big) \le \lambda_{\rho} \big(\overline{\eta}_{\rho} + \int_0^1 \overline{\delta}_{\rho}(s) \, ds \big),$$

and

$$\rho = \lambda_{\rho} \Big(H[u_{\rho}] + \int_{0}^{1} f(s, u_{\rho}(s), \dots, u_{\rho}^{\prime\prime\prime}(s)) \, ds \Big) \ge \lambda_{\rho} \Big(\underline{\eta}_{\rho} + \int_{0}^{1} \underline{\delta}_{\rho}(s) \, ds \Big),$$

which proves the result.

We conclude with an example that illustrates the applicability of the previous theoretical results.

Example 2.6. Consider the BVP

(2.7)
$$\begin{cases} u^{(4)}(t) = \lambda t e^{u(t)} (1 + (u''(t))^2), \ t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + \lambda \left(\frac{1}{1 + (u(\frac{1}{2}))^2} + \int_0^1 t^3 u''(t) \, dt\right) = 0. \end{cases}$$

Fix $\rho \in (0, +\infty)$. Thus we may take

$$\underline{\eta}_{\rho}(t) = \frac{1}{1+\rho^2}, \ \overline{\eta}_{\rho}(t) = 1 + \frac{\rho}{4}, \ \underline{\delta}_{\rho}(t) = t, \ \overline{\delta}_{\rho}(t) = te^{\rho}(1+\rho^2).$$

Thus we have

$$\underline{\eta}_{\rho} + \int_0^1 \underline{\delta}_{\rho}(s) \, ds = \frac{1}{1+\rho^2} + \int_0^1 s \, ds \ge \frac{1}{2},$$

which implies that (2.3) is satisfied for every $\rho \in (0, +\infty)$.

Thus we can apply Corollary 2.4 and Theorem 2.5, obtaining uncountably many pairs of positive eigenvalues and eigenfunctions $(u_{\rho}, \lambda_{\rho})$ for the BVP 2.7, where $||u_{\rho}||_{3} = ||u_{\rho}'''||_{\infty} = \rho$ and

$$\frac{4\rho}{2e^{\rho}\rho^{2} + 2e^{\rho} + \rho + 4} \le \lambda_{\rho} \le \frac{2\rho\left(\rho^{2} + 1\right)}{\rho^{2} + 3}.$$

The Figure 1 (produced with the program MAPLE) illustrates the region of localization of the $(u_{\rho}, \lambda_{\rho})$ pairs.



FIGURE 1. Localization of $(u_{\rho}, \lambda_{\rho})$

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References

- D. R. Anderson and J. Hoffacker, Existence of solutions for a cantilever beam problem, J. Math. Anal. Appl., 323 (2006), 958–973.
- [2] G. Bonanno, A. Chinnì and S. Tersian, Existence results for a two point boundary value problem involving a fourth-order equation, *Electron. J. Qual . Theory Differ. Equ.*, **33** (2015), 9pp.
- [3] A. Cabada, An overview of the lower and upper solutions method with nonlinear boundary value conditions, *Bound. Value Probl.* (2011), Art. ID 893753, 18 pp.
- [4] A. Cabada and S. Tersian, Multiplicity of solutions of a two point boundary value problem for a fourthorder equation, Appl. Math. Comput., 219 (2013), 5261–5267.
- [5] F. Cianciaruso, G. Infante and P. Pietramala, Solutions of perturbed Hammerstein integral equations with applications, Nonlinear Anal. Real World Appl., 33 (2017), 317–347.
- [6] R. Conti, Recent trends in the theory of boundary value problems for ordinary differential equations, Boll. Un. Mat. Ital., 22 (1967), 135–178.
- [7] C. S. Goodrich, Pointwise conditions for perturbed Hammerstein integral equations with monotone nonlinear, nonlocal elements, *Banach J. Math. Anal.*, 14 (2020), 290–312.
- [8] D. Guo and V. Lakshmikantham, Nonlinear problems in abstract cones, Academic Press, Boston, 1988.
- [9] G. Infante, A short course on positive solutions of systems of ODEs via fixed point index, in *Lecture Notes in Nonlinear Analysis (LNNA)*, 16 (2017), 93–140.
- [10] G. Infante, Positive solutions of systems of perturbed Hammerstein integral equations with arbitrary order dependence, *Philos. Trans. Roy. Soc. A*, **379** (2021), no. 2191, Paper No. 20190376, 10 pp.
- [11] G. Infante, Eigenvalues of elliptic functional differential systems via a Birkhoff-Kellogg type theorem, Mathematics, 9 (2021), n. 4.
- [12] G. Infante and P. Pietramala, A cantilever equation with nonlinear boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*, Spec. Ed. I, 15 (2009), 1–14.
- [13] G. L. Karakostas and P. Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem, *Topol. Methods Nonlinear Anal.*, **19** (2002), 109–121.
- [14] A. Khanfer and L. Bougoffa, A cantilever beam problem with small deflections and perturbed boundary data, J. Funct. Spaces, 2021, Article ID 9081623, 9 p. (2021).
- [15] M. A. Krasnosel'skii and L. A. Ladyženskii, The structure of the spectrum of positive nonhomogeneous operators, *Trudy Moskov. Mat. Obšč*, 3 (1954), 321–346.
- [16] Y. Li, Existence of positive solutions for the cantilever beam equations with fully nonlinear terms, Nonlinear Anal. Real World Appl., 27 (2016), 221–237.
- [17] T. F. Ma, Positive solutions for a beam equation on a nonlinear elastic foundation, Math. Comput. Modelling, 39 (2004), 1195–1201.
- [18] R. Ma, A survey on nonlocal boundary value problems, Appl. Math. E-Notes, 7 (2007), 257–279.
- [19] S. K. Ntouyas, Nonlocal initial and boundary value problems: a survey, Handbook of differential equations: ordinary differential equations. Vol. II, Elsevier B. V., Amsterdam, (2005), 461–557.

- [20] M. Picone, Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 10 (1908), 1–95.
- [21] A. Stikonas, A survey on stationary problems, Green's functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19** (2014), 301–334.
- [22] J. R. L. Webb, Compactness of nonlinear integral operators with discontinuous and with singular kernels, J. Math. Anal. Appl., 509 (2022), Paper No. 126000, 17 pp.
- [23] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, J. London Math. Soc., 74 (2006), 673–693.
- [24] M. Wei, Y. Li and G. Li, Lower and upper solutions method to the fully elastic cantilever beam equation with support, *Adv. Difference Equ.*, **2021**, 301 (2021).
- [25] W. M. Whyburn, Differential equations with general boundary conditions, Bull. Amer. Math. Soc., 48 (1942), 692–704.
- [26] L. Yang, H. Chen and X. Yang, The multiplicity of solutions for fourth-order equations generated from a boundary condition, *Appl. Math. Lett.*, **24** (2011), 1599–1603.
- [27] Q. Yao, Monotonically iterative method of nonlinear cantilever beam equations, Appl. Math. Comput., 205 (2008), 432–437.

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