# ON THE SOLVABILITY OF A PARAMETER-DEPENDENT CANTILEVER-TYPE BVP 

GENNARO INFANTE


#### Abstract

We discuss the solvability of a parameter dependent cantilever-type boundary value problem. We provide an existence and localization result for the positive solutions via a Birkhoff-Kellogg type theorem. We also obtain, under additional growth conditions, upper and lower bounds for the involved parameters. An example is presented in order to illustrate the theoretical results.


## 1. Introduction

Differential equations have been utilized to model the steady states of deflections of elastic beams; for example the fourth order ordinary differential equation

$$
\begin{equation*}
u^{(4)}(t)=f(t, u(t)), t \in(0,1), \tag{1.1}
\end{equation*}
$$

subject to the homogeneous boundary conditions (BCs)

$$
\begin{equation*}
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=u^{\prime \prime \prime}(1)=0, \tag{1.2}
\end{equation*}
$$

can be used as a model for the so-called cantilever bar. The boundary value problem (BVP) (1.1)-(1.2) describes a bar of length 1 which is clamped on the left end and is free to move at the right end, with vanishing bending moment and shearing force, see for example [1, 16, 27].

Under a mechanical point of view, some interesting cases appear when the shearing force at the right side of the beam does not vanish (see for example [12]):

- $u^{\prime \prime \prime}(1)+k_{0}=0$ models a force acting in 1 ,
- $u^{\prime \prime \prime}(1)+k_{1} u(1)=0$ describes a spring in 1 ,
- $u^{\prime \prime \prime}(1)+g(u(1))=0$ models a spring with a strongly nonlinear rigidity,
- $u^{\prime \prime \prime}(1)+g(u(\eta))=0$ describes a feedback mechanism, where the spring reacts to the displacement registered in a point $\eta$ of the beam,
- $u^{\prime \prime \prime}(1)+g\left(u\left(\eta_{1}\right), u^{\prime}\left(\eta_{2}\right), u^{\prime \prime}\left(\eta_{3}\right)\right)=0$ describes the case where the spring reacts to the displacement registered in the point $\eta_{1}$, the angular attitude registered in the point $\eta_{2}$ and the bending moment in the point $\eta_{3}$.

[^0]Of course, a different configuration of the beam may lead to more complicated BCs than the ones illustrated above. It is therefore not surprising that the case of non-homogeneous BCs has received attention by researchers. By means of critical point theory, Cabada and Terzian [4] and Bonanno, Chinnì and Terzian [2] and Yang, Chen and Yang [26] studied the parameter-dependent BVP

$$
\left\{\begin{array}{l}
u^{(4)}(t)=\lambda f(t, u(t)), t \in(0,1) \\
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=0 \\
u^{\prime \prime \prime}(1)+\lambda g(u(1))=0
\end{array}\right.
$$

while the case of $\lambda=1$ has been investigated in an earlier paper by Ma [17].
By classical fixed point index, Cianciaruso, Infante and Pietramala [5] studied the BVP

$$
\left\{\begin{array}{l}
u^{(4)}(t)=f(t, u(t)), t \in(0,1) \\
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=0 \\
u^{\prime \prime \prime}(1)+\hat{H}(u)=0
\end{array}\right.
$$

where $\hat{H}$ is a suitable functional (not necessarily linear) on $C[0,1]$. The functional approach for the BCs adopted in 5 fits within the interesting framework of nonlinear and nonlocal BCs; these are widely studied objects, we refer the reader to the reviews [3, 6, 18, 20, 19, 21, 25] and the manuscripts [7, 13, 23].

Regarding the higher order dependence in the forcing term, the ODE

$$
u^{(4)}(t)=f\left(t, u(t), u^{\prime}(t), u^{\prime \prime}(t), u^{\prime \prime \prime}(t)\right), \quad t \in[0,1]
$$

under the homogeneous BCs

$$
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=u^{\prime \prime \prime}(1)=0
$$

has been studied by Li [16] via fixed point index. The non-homogeneous case

$$
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=0, u^{\prime \prime \prime}(1)+g(u(1))=0
$$

has been studied, with the lower and upper solutions method, by Wei, Li and Li [24], while the case

$$
u(0)=u^{\prime}(0)=\int_{0}^{1} p(t) u(t) d t, u^{\prime \prime}(1)=u^{\prime \prime \prime}(1)=\int_{0}^{1} q(t) u^{\prime \prime}(t) d t
$$

has been investigated by Khanfer and Bougoffa [14] via the Schauder fixed point theorem.
Here we study the solvability of the parameter-dependent BVP

$$
\left\{\begin{array}{l}
u^{(4)}(t)=\lambda f\left(t, u(t), u^{\prime}(t), u^{\prime \prime}(t), u^{\prime \prime \prime}(t)\right), t \in(0,1)  \tag{1.3}\\
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=0 \\
u^{\prime \prime \prime}(1)+\lambda H[u]=0
\end{array}\right.
$$

where $f$ is a continuous function, $H$ is a suitable compact functional in the space $C^{3}[0,1]$ (this allows higher order dependence in the BCs ) and $\lambda$ is a non-negative parameter. For the existence result we adapt an approach used by the author [11], in the context of elliptic systems, that relies on a Birkhoff-Kellogg type theorem in cones due to Krasnosel'skiĕ and Ladyženskiĭ [15]. We also provide, under additional growth conditions, a localization result for the parameter $\lambda$. The results are new and complement the ones present in the papers [2, 4, 5, 14, 16, 17, 24, 26]. We also complement the results in [10], by obtaining additional qualitative properties (such as monotonicity and localization) of the solution. We illustrate the applicability of our theoretical results in an example.

## 2. Existence and localization of the eigenvalues

First of all we associate to the BVP (1.3) a perturbed Hammerstein integral equation of the form

$$
\begin{equation*}
u(t)=\lambda\left(\gamma(t) H[u]+\int_{0}^{1} k(t, s) f\left(s, u(s), u^{\prime}(s), u^{\prime \prime}(s), u^{\prime \prime \prime}(s)\right) d s\right) \tag{2.1}
\end{equation*}
$$

where the Green's function $k$ and the function $\gamma$ need to be determined; this is done by considering two auxiliary BVPs, a procedure found to be particularly useful in the case of nonlinear BCs, see for example [9] and references therein.

Regarding $k$ it is known (see for example Lemma 2.1 and Lemma 2.2 of [16]) that for $h \in C[0,1]$ the unique solution of the linear BVP

$$
\left\{\begin{array}{l}
u^{(4)}(t)=h(t), t \in(0,1) \\
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=u^{\prime \prime \prime}(1)=0
\end{array}\right.
$$

is given by

$$
u(t)=\int_{0}^{1} k(t, s) h(s) d s
$$

where

$$
k(t, s)= \begin{cases}\frac{1}{6}\left(3 t^{2} s-t^{3}\right), & s \geq t \\ \frac{1}{6}\left(3 s^{2} t-s^{3}\right), & s \leq t\end{cases}
$$

Note that the function $k$ has the following properties

$$
k(t, s), \frac{\partial k}{\partial t}(t, s), \frac{\partial^{2} k}{\partial t^{2}}(t, s) \geq 0 \text { on }[0,1] \times[0,1]
$$

and

$$
\left.\frac{\partial^{3} k}{\partial t^{3}}(t, s) \leq 0 \text { on }[0,1]^{2} \backslash\{(t, s) \mid t=s)\right\} .
$$

Regarding the function $\gamma$, note that (see for example [12])

$$
\gamma(t)=\frac{1}{6}\left(3 t^{2}-t^{3}\right)
$$

is the unique solution of the BVP

$$
\gamma^{(4)}(t)=0, \gamma(0)=\gamma^{\prime}(0)=\gamma^{\prime \prime}(1)=0, \gamma^{\prime \prime \prime}(1)+1=0 .
$$

By direct calculation, it can be observed that

$$
\gamma(t), \gamma^{\prime}(t), \gamma^{\prime \prime}(t),-\gamma^{\prime \prime \prime}(t) \geq 0 \text { on }[0,1] .
$$

With the above ingredients at our disposal, we can work in the space $C^{3}[0,1]$ endowed with the norm

$$
\|u\|_{3}:=\max _{j=0, \ldots, 3}\left\{\left\|u^{(j)}\right\|_{\infty}\right\}, \text { where }\|w\|_{\infty}=\sup _{t \in[0,1]}|w(t)| .
$$

Definition 2.1. We say that $\lambda$ is an eigenvalue of the BVP (1.3) with a corresponding eigenfunction $u \in C^{3}[0,1]$ with $\|u\|_{3}>0$ if the pair $(u, \lambda)$ satisfies the perturbed Hammerstein integral equation (2.1).

We make use of the following Birkhoff-Kellogg type theorem in order to seek the eigenfunctions of the BVP (1.3). We recall that a cone $\mathcal{K}$ of a real Banach space $(X,\| \|)$ is a closed set with $\mathcal{K}+\mathcal{K} \subset \mathcal{K}, \mu \mathcal{K} \subset \mathcal{K}$ for all $\mu \geq 0$ and $\mathcal{K} \cap(-\mathcal{K})=\{0\}$.

Theorem 2.2 (Theorem 2.3.6, [8]). Let $(X,\| \|)$ be a real Banach space, $U \subset X$ be an open bounded set with $0 \in U, \mathcal{K} \subset X$ be a cone, $T: \mathcal{K} \cap \bar{U} \rightarrow \mathcal{K}$ be compact and suppose that

$$
\inf _{x \in \mathcal{K} \cap \partial U}\|T x\|>0
$$

Then there exist $\lambda_{0} \in(0,+\infty)$ and $x_{0} \in \mathcal{K} \cap \partial U$ such that $x_{0}=\lambda_{0} T x_{0}$.
We apply the Theorem 2.2 in the cone

$$
\begin{equation*}
K:=\left\{u \in C^{3}[0,1]: u(t), u^{\prime}(t), u^{\prime \prime}(t),-u^{\prime \prime \prime}(t) \geq 0, \text { for every } t \in[0,1]\right\} . \tag{2.2}
\end{equation*}
$$

The cone (2.2) is a smaller cone than the one of positive functions used in [10], but larger than the one used in [16]. We consider the sets

$$
\begin{gathered}
K_{\rho}:=\left\{u \in K:\|u\|_{3}<\rho\right\}, \bar{K}_{\rho}:=\left\{u \in K:\|u\|_{3} \leq \rho\right\}, \\
\partial K_{\rho}:=\left\{u \in K:\|u\|_{3}=\rho\right\},
\end{gathered}
$$

where $\rho \in(0,+\infty)$.
The following Theorem provides an existence result for an eigenfunction possessing a fixed norm and a corresponding positive eigenvalue.

Theorem 2.3. Let $\rho \in(0,+\infty)$ and assume the following conditions hold.
(a) $f \in C\left(\Pi_{\rho}, \mathbb{R}\right)$ and there exist $\underline{\delta}_{\rho} \in C\left([0,1], \mathbb{R}_{+}\right)$such that

$$
f(t, u, v, w, z) \geq \underline{\delta}_{\rho}(t), \text { for every }(t, u, v, w, z) \in \Pi_{\rho}
$$

where

$$
\Pi_{\rho}:[0,1] \times[0, \rho]^{3} \times[-\rho, 0]
$$

(b) $H: \bar{K}_{\rho} \rightarrow \mathbb{R}$ is continuous and bounded. Let $\underline{\eta}_{\rho} \in[0,+\infty)$ be such that

$$
H[u] \geq \underline{\eta}_{\rho}, \text { for every } u \in \partial K_{\rho} .
$$

(c) The inequality

$$
\begin{equation*}
\underline{\eta}_{\rho}+\int_{0}^{1} \underline{\delta}_{\rho}(s) d s>0 \tag{2.3}
\end{equation*}
$$

holds.
Then the BVP (1.3) has a positive eigenvalue $\lambda_{\rho}$ with an associated eigenfunction $u_{\rho} \in \partial K_{\rho}$.
Proof. Let $F u(t):=\int_{0}^{1} k(t, s) f\left(s, u(s), u^{\prime}(s), u^{\prime \prime}(s), u^{\prime \prime \prime}(s)\right) d s$ and $\Gamma u(t):=\gamma(t) H[u]$. Note that, due to the assumptions above, the operator $T=F+\Gamma$ maps $\bar{K}_{\rho}$ into $K$ and is compact; the compactness of $F$ follows from a careful use of the Arzelà-Ascoli theorem (see [22]) and $\Gamma$ is a finite rank operator.

Take $u \in \partial K_{\rho}$, then we have

$$
\begin{align*}
& \|T u\|_{3} \geq\left\|(T u)^{\prime \prime \prime}\right\|_{\infty} \geq\left|(T u)^{\prime \prime \prime}(0)\right|  \tag{2.4}\\
& =\left|-H[u]-\int_{0}^{1} f\left(s, u(s), u^{\prime}(s), u^{\prime \prime}(s), u^{\prime \prime \prime}(s)\right) d s\right| \\
& \\
& =H[u]+\int_{0}^{1} f\left(s, u(s), u^{\prime}(s), u^{\prime \prime}(s), u^{\prime \prime \prime}(s)\right) d s \geq \underline{\eta}_{\rho}+\int_{0}^{1} \underline{\delta}_{\rho}(s) d s
\end{align*}
$$

Note that the RHS of (2.4) does not depend on the particular $u$ chosen. Therefore we have

$$
\inf _{u \in \partial K_{\rho}}\|T u\|_{3} \geq \underline{\eta}_{\rho}+\int_{0}^{1} \underline{\delta}_{\rho}(s) d s>0
$$

and the result follows by Theorem 2.2.
The following Corollary provides an existence result for the existence of uncountably many couples of eigenvalues-eigenfunctions.

Corollary 2.4. In addition to the hypotheses of Theorem 2.3, assume that $\rho$ can be chosen arbitrarily in $(0,+\infty)$. Then for every $\rho$ there exists a non-negative eigenfunction $u_{\rho} \in \partial K_{\rho}$ of the BVP 1.3) to which corresponds a $\lambda_{\rho} \in(0,+\infty)$.

The next result provides some upper and lower bounds on the eigenvalues.

Theorem 2.5. In addition to the hypotheses of Theorem 2.3 assume the following conditions hold.
(d) There exist $\bar{\delta}_{\rho} \in C\left([0,1], \mathbb{R}_{+}\right)$such that

$$
f(t, u, v, w, z) \leq \bar{\delta}_{\rho}(t), \text { for every }(t, u, v, w, z) \in \Pi_{\rho}
$$

(e) Let $\bar{\eta}_{\rho} \in[0,+\infty)$ be such that

$$
H[u] \leq \bar{\eta}_{\rho}, \text { for every } u \in \partial K_{\rho}
$$

Then $\lambda_{\rho}$ satisfies the following estimates

$$
\frac{\rho}{\left(\bar{\eta}_{\rho}+\int_{0}^{1} \bar{\delta}_{\rho}(s) d s\right)} \leq \lambda_{\rho} \leq \frac{\rho}{\left(\underline{\eta}_{\rho}+\int_{0}^{1} \underline{\delta}_{\rho}(s) d s\right)}
$$

Proof. By Theorem 2.3 there exist $u_{\rho} \in \partial K_{\rho}$ and $\lambda_{\rho}$ such that

$$
\begin{equation*}
u_{\rho}(t)=\lambda_{\rho}\left(\gamma(t) H\left[u_{\rho}\right]+\int_{0}^{1} k(t, s) f\left(s, u_{\rho}(s), \ldots, u_{\rho}^{\prime \prime \prime}(s)\right) d s\right) \tag{2.5}
\end{equation*}
$$

By differentiating (2.5) we obtain

$$
\begin{aligned}
u_{\rho}^{\prime}(t) & =\lambda_{\rho}\left(\gamma^{\prime}(t) H\left[u_{\rho}\right]+\int_{0}^{1} \frac{\partial k}{\partial t}(t, s) f\left(s, u_{\rho}(s), \ldots, u_{\rho}^{\prime \prime \prime}(s)\right) d s\right) \\
u_{\rho}^{\prime \prime}(t) & =\lambda_{\rho}\left(\gamma^{\prime \prime}(t) H\left[u_{\rho}\right]+\int_{0}^{1} \frac{\partial^{2} k}{\partial t^{2}}(t, s) f\left(s, u_{\rho}(s), \ldots, u_{\rho}^{\prime \prime \prime}(s)\right) d s\right), \\
u_{\rho}^{\prime \prime \prime}(t) & =\lambda_{\rho}\left(-H\left[u_{\rho}\right]-\int_{t}^{1} f\left(s, u_{\rho}(s), u_{\rho}^{\prime}(s), u_{\rho}^{\prime \prime}(s), u_{\rho}^{\prime \prime \prime}(s)\right) d s\right),
\end{aligned}
$$

which implies

$$
\begin{equation*}
\left\|u_{\rho}^{\prime \prime \prime}\right\|_{\infty}=\lambda_{\rho}\left(H\left[u_{\rho}\right]+\int_{0}^{1} f\left(s, u_{\rho}(s), \ldots, u_{\rho}^{\prime \prime \prime}(s)\right) d s\right) \tag{2.6}
\end{equation*}
$$

Furthermore note that

$$
0 \leq k(t, s), \frac{\partial k}{\partial t}(t, s), \frac{\partial^{2} k}{\partial t^{2}}(t, s) \leq 1, \text { on }[0,1] \times[0,1]
$$

and

$$
0 \leq \gamma(t), \gamma^{\prime}(t), \gamma^{\prime \prime}(t) \leq 1 \text { on }[0,1]
$$

which yield

$$
\rho=\left\|u_{\rho}\right\|_{3}=\left\|u_{\rho}^{\prime \prime \prime}\right\|_{\infty} .
$$

From (2.6) and the estimates $(d)$ and (e) we obtain

$$
\rho=\lambda_{\rho}\left(H\left[u_{\rho}\right]+\int_{0}^{1} f\left(s, u_{\rho}(s), \ldots, u_{\rho}^{\prime \prime \prime}(s)\right) d s\right) \leq \lambda_{\rho}\left(\bar{\eta}_{\rho}+\int_{0}^{1} \bar{\delta}_{\rho}(s) d s\right)
$$

and

$$
\rho=\lambda_{\rho}\left(H\left[u_{\rho}\right]+\int_{0}^{1} f\left(s, u_{\rho}(s), \ldots, u_{\rho}^{\prime \prime \prime}(s)\right) d s\right) \geq \lambda_{\rho}\left(\underline{\eta}_{\rho}+\int_{0}^{1} \underline{\delta}_{\rho}(s) d s\right)
$$

which proves the result.
We conclude with an example that illustrates the applicability of the previous theoretical results.

Example 2.6. Consider the BVP

$$
\left\{\begin{array}{l}
u^{(4)}(t)=\lambda t e^{u(t)}\left(1+\left(u^{\prime \prime \prime}(t)\right)^{2}\right), t \in(0,1)  \tag{2.7}\\
u(0)=u^{\prime}(0)=u^{\prime \prime}(1)=0 \\
u^{\prime \prime \prime}(1)+\lambda\left(\frac{1}{1+\left(u\left(\frac{1}{2}\right)\right)^{2}}+\int_{0}^{1} t^{3} u^{\prime \prime}(t) d t\right)=0
\end{array}\right.
$$

Fix $\rho \in(0,+\infty)$. Thus we may take

$$
\underline{\eta}_{\rho}(t)=\frac{1}{1+\rho^{2}}, \bar{\eta}_{\rho}(t)=1+\frac{\rho}{4}, \underline{\delta}_{\rho}(t)=t, \bar{\delta}_{\rho}(t)=t e^{\rho}\left(1+\rho^{2}\right) .
$$

Thus we have

$$
\underline{\eta}_{\rho}+\int_{0}^{1} \underline{\delta}_{\rho}(s) d s=\frac{1}{1+\rho^{2}}+\int_{0}^{1} s d s \geq \frac{1}{2}
$$

which implies that 2.3 is satisfied for every $\rho \in(0,+\infty)$.
Thus we can apply Corollary 2.4 and Theorem 2.5, obtaining uncountably many pairs of positive eigenvalues and eigenfunctions $\left(u_{\rho}, \lambda_{\rho}\right)$ for the BVP 2.7, where $\left\|u_{\rho}\right\|_{3}=\left\|u_{\rho}^{\prime \prime \prime}\right\|_{\infty}=\rho$ and

$$
\frac{4 \rho}{2 \mathrm{e}^{\rho} \rho^{2}+2 \mathrm{e}^{\rho}+\rho+4} \leq \lambda_{\rho} \leq \frac{2 \rho\left(\rho^{2}+1\right)}{\rho^{2}+3}
$$

The Figure 1 (produced with the program MAPLE) illustrates the region of localization of the $\left(u_{\rho}, \lambda_{\rho}\right)$ pairs.


Figure 1. Localization of $\left(u_{\rho}, \lambda_{\rho}\right)$

## Acknowledgement

G. Infante was partially supported by G.N.A.M.P.A. - INdAM (Italy). This research has been accomplished within the UMI Group TAA "Approximation Theory and Applications".

## References

[1] D. R. Anderson and J. Hoffacker, Existence of solutions for a cantilever beam problem, J. Math. Anal. Appl., 323 (2006), 958-973.
[2] G. Bonanno, A. Chinnì and S. Tersian, Existence results for a two point boundary value problem involving a fourth-order equation, Electron. J. Qual .Theory Differ. Equ., 33 (2015), 9pp.
[3] A. Cabada, An overview of the lower and upper solutions method with nonlinear boundary value conditions, Bound. Value Probl. (2011), Art. ID 893753, 18 pp.
[4] A. Cabada and S. Tersian, Multiplicity of solutions of a two point boundary value problem for a fourthorder equation, Appl. Math. Comput., 219 (2013), 5261-5267.
[5] F. Cianciaruso, G. Infante and P. Pietramala, Solutions of perturbed Hammerstein integral equations with applications, Nonlinear Anal. Real World Appl., 33 (2017), 317-347.
[6] R. Conti, Recent trends in the theory of boundary value problems for ordinary differential equations, Boll. Un. Mat. Ital., 22 (1967), 135-178.
[7] C. S. Goodrich, Pointwise conditions for perturbed Hammerstein integral equations with monotone nonlinear, nonlocal elements, Banach J. Math. Anal., 14 (2020), 290-312.
[8] D. Guo and V. Lakshmikantham, Nonlinear problems in abstract cones, Academic Press, Boston, 1988.
[9] G. Infante, A short course on positive solutions of systems of ODEs via fixed point index, in Lecture Notes in Nonlinear Analysis (LNNA), 16 (2017), 93-140.
[10] G. Infante, Positive solutions of systems of perturbed Hammerstein integral equations with arbitrary order dependence, Philos. Trans. Roy. Soc. A, 379 (2021), no. 2191, Paper No. 20190376, 10 pp.
[11] G. Infante, Eigenvalues of elliptic functional differential systems via a Birkhoff-Kellogg type theorem, Mathematics, 9 (2021), n. 4.
[12] G. Infante and P. Pietramala, A cantilever equation with nonlinear boundary conditions, Electron. J. Qual. Theory Differ. Equ., Spec. Ed. I, 15 (2009), 1-14.
[13] G. L. Karakostas and P. Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem, Topol. Methods Nonlinear Anal., 19 (2002), 109-121.
[14] A. Khanfer and L. Bougoffa, A cantilever beam problem with small deflections and perturbed boundary data, J. Funct. Spaces, 2021, Article ID 9081623, 9 p. (2021).
[15] M. A. Krasnosel'skii and L. A. Ladyženskiŭ, The structure of the spectrum of positive nonhomogeneous operators, Trudy Moskov. Mat. Obšč, 3 (1954), 321-346.
[16] Y. Li, Existence of positive solutions for the cantilever beam equations with fully nonlinear terms, Nonlinear Anal. Real World Appl., 27 (2016), 221-237.
[17] T. F. Ma, Positive solutions for a beam equation on a nonlinear elastic foundation, Math. Comput. Modelling, 39 (2004), 1195-1201.
[18] R. Ma, A survey on nonlocal boundary value problems, Appl. Math. E-Notes, 7 (2007), 257-279.
[19] S. K. Ntouyas, Nonlocal initial and boundary value problems: a survey, Handbook of differential equations: ordinary differential equations. Vol. II, Elsevier B. V., Amsterdam, (2005), 461-557.
[20] M. Picone, Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 10 (1908), 1-95.
[21] A. Štikonas, A survey on stationary problems, Green's functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions, Nonlinear Anal. Model. Control, 19 (2014), 301-334.
[22] J. R. L. Webb, Compactness of nonlinear integral operators with discontinuous and with singular kernels, J. Math. Anal. Appl., 509 (2022), Paper No. 126000, 17 pp.
[23] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, J. London Math. Soc., 74 (2006), 673-693.
[24] M. Wei, Y. Li and G. Li, Lower and upper solutions method to the fully elastic cantilever beam equation with support, Adv. Difference Equ., 2021, 301 (2021).
[25] W. M. Whyburn, Differential equations with general boundary conditions, Bull. Amer. Math. Soc., 48 (1942), 692-704.
[26] L. Yang, H. Chen and X. Yang, The multiplicity of solutions for fourth-order equations generated from a boundary condition, Appl. Math. Lett., 24 (2011), 1599-1603.
[27] Q. Yao, Monotonically iterative method of nonlinear cantilever beam equations, Appl. Math. Comput., 205 (2008), 432-437.

Gennaro Infante, Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

Email address: gennaro.infante@unical.it


[^0]:    2020 Mathematics Subject Classification. Primary 34B08, secondary 34B10, 34B18, 47H30.
    Key words and phrases. Positive solution, functional boundary condition, cone, Birkhoff-Kellogg type theorem.

