# The energy method for high-order invariants in shallow water wave equations

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#### **Abstract**

Third order dispersive evolution equations are widely adopted to model one-dimensional long waves and have extensive applications in fluid mechanics, plasma physics and nonlinear optics. Among them are the KdV equation, the Camassa–Holm equation and the Degasperis–Procesi equation. They share many common features such as complete integrability, Lax pairs and bi-Hamiltonian structure. In this paper we revisit high-order invariants for these three types of shallow water wave equations by the energy method in combination of a skew-adjoint operator  $(1 - \partial_{xx})^{-1}$ . Several applications to seek high-order invariants of the Benjamin-Bona-Mahony equation, the regularized long wave equation and the Rosenau equation are also presented.

Keywords: Energy method; High-order invariant; Shallow water wave equation

# 1. Introduction

A family of third order dispersive evolution equations of the form

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \quad x \in \mathbb{R}, \ t > 0$$
(1.1)

frequently appeared in the simulation of the shallow water waves, see e.g., [1], where  $\alpha$ ,  $\gamma$  and  $c_i$  (i = 0, 1, 2, 3) are real constants; u denotes a horizontal velocity field with the independent spatial variable x and temporal variable t.

A typical such equation (1.1) with  $\alpha^2 = c_0 = c_2 = c_3 = 0$ ,  $c_1 = 2$ ,  $\gamma = -2$  is the KdV equation

$$u_t - 4uu_x - 2u_{xxx} = 0, \quad x \in R, \ t > 0,$$
 (1.2)

which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. The first four invariants of (1.2) are respectively as (see e.g., [2], although there is a minor typo in the coefficient of the fourth invariant, it does not affect the reading of this classic review)

$$M_1 = \int_R u dx$$
,  $M_2 = \int_R u^2 dx$ ,  $M_3 = \int_R \left(u_x^2 - \frac{2}{3}u^3\right) dx$ ,  $M_4 = \int_R \left(u_{xx}^2 - \frac{10}{3}uu_x^2 + \frac{5}{9}u^4\right) dx$ .

Taking  $\alpha^2 = c_3 = 1$ ,  $\gamma = c_0 = 0$ ,  $c_1 = -\frac{3}{2}$ ,  $c_2 = \frac{1}{2}$ , we have another example called *the Camassa–Holm equation* [3]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad x \in R, \ t > 0, \tag{1.3}$$

which models the unidirectional propagation of shallow water waves over a flat bottom. The first three invariants are listed as follows

$$E_1 = \int_R (u - u_{xx}) dx$$
,  $E_2 = \frac{1}{2} \int_R (u^2 + u_x^2) dx$ ,  $E_3 = \frac{1}{2} \int_R u(u^2 + u_x^2) dx$ .

The third example by assigning  $\alpha^2 = c_2 = c_3 = 1$ ,  $\gamma = c_0 = 0$ ,  $c_1 = -2$  is called the Degasperis-Procesi equation

$$u_t - u_{xyt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in R, \ t > 0,$$
 (1.4)

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which can be regarded as a model for nonlinear shallow water dynamics [4]. The frequently discussed invariants are

$$H_1 = \int_R (u - u_{xx}) dx$$
,  $H_2 = \int_R (u - u_{xx}) v dx$ ,  $H_3 = \int_R u^3 dx$ ,

where  $4v - v_{xx} = u$ .

Up to now, there have been thousands of papers focusing on the theoretical and numerical studies on these three equations. It is worth mentioning that the invariant-preserving property is a key index of the success for numerical methods. However, high-order invariants are usually difficult to preserve numerically. Liu *et al.* also pointed out "it appears a rather difficult task to preserve all three conservation laws" in [5]. In this work, higher-order invariants of these equations will be re-derived in view of the energy method, which may be possible to provide some thoughts for invariant-preserving numerical methods. Actually, the energy method originated from conservation laws in physics was first proposed in 1928 by Courant, Friedrichs and Lewy [6]. From then on, it has been widely applied to the mathematical and numerical analysis of nonlinear evolution equations. We trust the readers with [7] instead of a long list of references to relevant works.

The rest of the paper is arranged as follows. In Section 2, combining the energy method and a skew-adjoint operator, we show the high-order invariants for the KdV equation, the Camassa–Holm equation and the Degasperis–Procesi equation, respectively. Then we list several applications for seeking some high-order invariants of other types of the shallow water wave equations in Section 3.

# 2. Main results

In what follows, we directly show that  $M_i$  (i = 1, 2, 3, 4),  $E_i$  (i = 1, 2, 3) and  $H_i$  (i = 1, 2, 3) are invariants of (1.2), (1.3) and (1.4) subjected to the periodic boundary conditions based on the energy method, respectively.

#### 2.1. Invariants of the KdV equation

**Proof:** (I) Multiplying by 1, u and  $(u^2 + u_{xx})$ , respectively, with (1.2), we have  $M_i$  (i = 1, 2, 3). In what follows, we show the fourth invariant  $M_4$  of the KdV equation by the energy method.

Multiplying both sides of (1.2) by  $2u_{xxxx} + \frac{10}{3}u_x^2 + \frac{20}{3}uu_{xx} + \frac{20}{9}u^3$  and integrating the result, we have

$$0 = \int_{R} \left( 2u_{xxxx} + \frac{10}{3}u_{x}^{2} + \frac{20}{3}uu_{xx} + \frac{20}{9}u^{3} \right) \cdot u_{t} dx$$

$$- \int_{R} \left( 2u_{xxxx} + \frac{10}{3}u_{x}^{2} + \frac{20}{3}uu_{xx} + \frac{20}{9}u^{3} \right) \cdot (4uu_{x} + 2u_{xxx}) dx$$

$$= \int_{R} \left[ 2u_{xx}u_{xxt} - \frac{10}{3}(u_{t}u_{x}^{2} + 2uu_{x}u_{xt}) + \frac{20}{9}u^{3}u_{t} \right] dx$$

$$- \int_{R} \left( 2u_{xxxx} + \frac{10}{3}u_{x}^{2} + \frac{20}{3}uu_{xx} + \frac{20}{9}u^{3} \right) \cdot (4uu_{x} + 2u_{xxx}) dx$$

$$= \frac{d}{dt}M_{4} - 8 \int_{R} uu_{x}u_{xxxx} dx - \frac{40}{3} \int_{R} uu_{x}^{3} dx - \frac{80}{3} \int_{R} u^{2}u_{x}u_{xx} dx - \frac{80}{9} \int_{R} u^{4}u_{x} dx$$

$$- 4 \int_{R} u_{xxx}u_{xxxx} dx - \frac{20}{3} \int_{R} u_{xxx}u_{x}^{2} dx - \frac{40}{3} \int_{R} uu_{xx}u_{xxx} dx - \frac{40}{9} \int_{R} u^{3}u_{xxx} dx. \tag{2.1}$$

It remains to check that the sum of all the integral terms in the above equation is zero. Calculating each term in (2.1) using the integration by parts, we have

$$-8 \int_{R} u u_{x} u_{xxxx} dx = -20 \int_{R} u_{x} u_{xx}^{2} dx,$$
 (2.2)

$$-\frac{80}{3} \int_{R} u^2 u_x u_{xx} dx = \frac{80}{3} \int_{R} u u_x^3 dx,$$
 (2.3)

$$-\frac{80}{9} \int_{R} u^4 u_x \mathrm{d}x = 0, \tag{2.4}$$

$$-4\int_{R}u_{xxx}u_{xxxx}\mathrm{d}x=0,$$
(2.5)

$$-\frac{20}{3} \int_{R} u_{xxx} u_{x}^{2} dx = \frac{40}{3} \int_{R} u_{x} u_{xx}^{2} dx,$$
 (2.6)

$$-\frac{40}{3} \int_{R} u u_{xx} u_{xxx} dx = \frac{20}{3} \int_{R} u_{x} u_{xx}^{2} dx, \tag{2.7}$$

$$-\frac{40}{9} \int_{R} u^{3} u_{xxx} dx = -\frac{40}{3} \int_{R} u u_{x}^{3} dx.$$
 (2.8)

Substituting (2.2)–(2.8) into (2.1), we have  $\frac{d}{dt}M_4 = 0$ , which completes the proof.

Remark 1. Suppose the general form of the KdV equation is

$$u_t - auu_x - bu_{xxx} = 0$$
,

and the corresponding high-order invariant

$$M(t) = \int_{R} (u_{xx}^2 - Auu_x^2 + Bu^4) dx.$$

Using the same method above, we could derive

$$\begin{cases} 5a = 3Ab, \\ 12Bb = Aa, \end{cases}$$

which can be rewritten as

$$\frac{a}{b} = \frac{3A}{5} = \frac{12B}{A}.$$

Therefore, it follows

$$A^2 = 20B.$$

For instance, when a = -6, b = -1, we have A = 10, B = 5, which deduces to the KdV equation as

$$u_t + 6uu_x + u_{xxx} = 0,$$

with a fourth-order invariant

$$M(t) = \int_{R} (u_{xx}^2 - 10uu_x^2 + 5u^4) dx.$$

# 2.2. Invariants of the Camassa-Holm equation

**Proof:** Multiplying by 1 and u on both sides of (1.3), respectively, and then integrating the results, which implies  $E_1$  and  $E_2$  through the integration by parts. Below, we prove  $E_3$  by the energy method. Firstly, noticing that (1.3) can be written with a skew-adjoint operator  $(1 - \partial_{xx})^{-1}$  as

$$u_t + uu_x + \partial_x (1 - \partial_{xx})^{-1} \left(u^2 + \frac{1}{2}u_x^2\right) = 0.$$

Let  $g = (1 - \partial_{xx})^{-1} \left(u^2 + \frac{1}{2}u_x^2\right)$ . Then we see from the above equation that (1.3) is equivalent to

$$\begin{cases} u_t + uu_x + g_x = 0, \\ g - g_{xx} = u^2 + \frac{1}{2}u_x^2. \end{cases}$$
 (2.9)

Multiplying (2.9) by  $3u^2 + u_x^2 - 2(uu_x)_x$  and integrating the result on both sides, we have

$$0 = \int_{R} (u_{t} + uu_{x} + g_{x}) \cdot (3u^{2} + u_{x}^{2} - 2(uu_{x})_{x}) dx$$

$$= \int_{R} u_{t} \cdot (3u^{2} + u_{x}^{2} - 2(uu_{x})_{x}) dx + \int_{R} (uu_{x} + g_{x}) \cdot (3u^{2} + u_{x}^{2} - 2(uu_{x})_{x}) dx$$

$$\triangleq A + B.$$
(2.11)

Calculating each term derives that

$$A = \int_{R} u_{t} \cdot (3u^{2} + u_{x}^{2} - 2(uu_{x})_{x}) dx$$

$$= \int_{R} u_{t} \cdot (3u^{2} + u_{x}^{2}) dx + \int_{R} 2uu_{x} \cdot u_{xt} dx$$

$$= \int_{R} u_{t} \cdot 3u^{2} dx + \int_{R} u_{t} \cdot u_{x}^{2} dx + \int_{R} u \cdot (u_{x}^{2})_{t} dx$$

$$= \int_{R} (u^{3})_{t} dx + \int_{R} (u \cdot u_{x}^{2})_{t} dx$$

$$= \frac{d}{dt} \int_{R} (u^{3} + uu_{x}^{2}) dx$$
(2.12)

and

$$B = \int_{R} (uu_{x} + g_{x}) \cdot (3u^{2} + u_{x}^{2} - 2(uu_{x})_{x}) dx$$

$$= \int_{R} u \cdot u_{x}^{3} dx + \int_{R} g_{x} \cdot (3u^{2} + u_{x}^{2}) dx - \int_{R} g_{x} \cdot 2(uu_{x})_{x} dx$$

$$= \int_{R} u \cdot u_{x}^{3} dx + \int_{R} g_{x} \cdot (3u^{2} + u_{x}^{2}) dx + 2 \int_{R} g_{xx} \cdot uu_{x} dx$$

$$= \int_{R} u \cdot u_{x}^{3} dx + \int_{R} g_{x} \cdot (3u^{2} + u_{x}^{2}) dx + 2 \int_{R} (g - u^{2} - \frac{1}{2}u_{x}^{2}) \cdot uu_{x} dx$$

$$= \int_{R} g_{x} \cdot (3u^{2} + u_{x}^{2}) dx + 2 \int_{R} g \cdot uu_{x} dx$$

$$= \int_{R} g_{x} \cdot (3u^{2} + u_{x}^{2}) dx - \int_{R} g_{x} \cdot u^{2} dx$$

$$= \int_{R} g_{x} \cdot (2u^{2} + u_{x}^{2}) dx - \int_{R} g_{x} \cdot u^{2} dx$$

$$= 2 \int_{R} g_{x} \cdot (g - g_{xx}) dx = 0.$$
(2.13)

Substituting (2.12) and (2.13) into (2.11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{R} (u^3 + uu_x^2) \mathrm{d}x = 0,$$

which implies  $E_3$ .

# 2.3. Invariants of the Degasperis-Procesi equation

**Proof:** Integrating on both sides of (1.4), it easily obtains  $H_1$ . Then we show invariants  $H_2$  and  $H_3$  of (1.4), respectively. Firstly let  $g = (1 - \partial_{xx})^{-1} (\frac{3}{2}u^2)$ , then (1.4) is equivalent to

$$\begin{cases} u_t + uu_x + g_x = 0, \\ g - g_{xx} = \frac{3}{2}u^2. \end{cases}$$
 (2.14)

Multiplying by 2u - 6v on both sides of (2.14) and then integrating the result, we have

$$0 = \int_{R} (u_{t} + uu_{x} + g_{x}) \cdot (2u - 6v) dx$$

$$= \int_{R} u_{t} \cdot (2u - 6v) dx + \int_{R} uu_{x} \cdot (2u - 6v) dx + \int_{R} g_{x} \cdot (2u - 6v) dx$$

$$\triangleq C + D. \tag{2.16}$$

The each term in the above identity is estimated as

$$C = \int_{R} u_{t} \cdot (2u - 6v) dx = 2 \int_{R} u_{t} \cdot u dx - 6 \int_{R} u_{t} \cdot v dx = 2 \int_{R} u_{t} \cdot u dx - 6 \int_{R} (4v_{t} - v_{xxt}) \cdot v dx$$

$$= 2 \int_{R} u_{t} \cdot u dx - 24 \int_{R} v_{t} \cdot v dx - 6 \int_{R} v_{xt} \cdot v_{x} dx = \frac{d}{dt} \int_{R} (u^{2} - 12v^{2} - 3v_{x}^{2}) dx$$

$$= \frac{d}{dt} \int_{R} (u^{2} - 3(4v - v_{xx}) \cdot v) dx = \frac{d}{dt} \int_{R} (u^{2} - 3uv) dx = \frac{d}{dt} \int_{R} u \cdot (u - 3v) dx$$

$$= \frac{d}{dt} \int_{R} u \cdot (v - v_{xx}) dx = \frac{d}{dt} \int_{R} (u - u_{xx}) \cdot v dx$$
(2.17)

and

$$D = \int_{R} u u_{x} \cdot (2u - 6v) dx + \int_{R} g_{x} \cdot (2u - 6v) dx$$

$$= -6 \int_{R} u u_{x} \cdot v dx + \int_{R} g_{x} \cdot (2u - 6v) dx$$

$$= 3 \int_{R} u^{2} \cdot v_{x} dx + \int_{R} g_{x} \cdot (2u - 6v) dx$$

$$= 2 \int_{R} (g - g_{xx}) \cdot v_{x} dx + \int_{R} g_{x} \cdot (2u - 6v) dx$$

$$= 2 \int_{R} g \cdot v_{x} dx - 2 \int_{R} g_{xx} \cdot v_{x} dx + \int_{R} g_{x} \cdot (2v - 2v_{xx}) dx$$

$$= 2 \int_{R} g \cdot v_{x} dx + 2 \int_{R} g_{x} \cdot v dx - 2 \int_{R} g_{xx} \cdot v_{x} dx - 2 \int_{R} g_{x} \cdot v_{xx} dx$$

$$= 2 \int_{R} (gv)_{x} dx - 2 \int_{R} (g_{x} \cdot v_{x})_{x} dx = 0.$$
(2.18)

Substituting (2.17) and (2.18) into (2.16), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{R} (u - u_{xx}) \cdot v \mathrm{d}x = 0,$$

which implies  $H_2$ .

Finally, we show  $H_3$ . Multiplying (2.14) on both sides by  $u^2$  and integrating the result, it yields by noting (2.15)

$$0 = \int_{R} (u_t + uu_x + g_x) \cdot u^2 dx$$

$$= \int_{R} u_t \cdot u^2 dx + \int_{R} u^3 \cdot u_x dx + \int_{R} g_x \cdot u^2 dx$$

$$= \int_{R} \left(\frac{1}{3}u^3\right)_t dx + \frac{2}{3} \int g_x \cdot (g - g_{xx}) dx$$

$$= \frac{1}{3} \frac{d}{dt} \int_{R} u^3 dx,$$

which implies the invariant  $H_3$ .

# 3. Applications to other periodic nonlinear dispersive waves

# 3.1. Benjamin-Bona-Mahony equation

Consider the Benjamin-Bona-Mahony equation [8] of the form

$$u_t - u_{xxt} + u_x + \varepsilon u u_x = 0, \quad x \in R. \tag{3.1}$$

It can be written as

$$u_t + \partial_x (1 - \partial_{xx})^{-1} \left( u + \frac{\varepsilon}{2} u^2 \right) = 0, \quad x \in \mathbb{R}.$$

Let  $g = (1 - \partial_{xx})^{-1} \left( u + \frac{\varepsilon}{2} u^2 \right)$ , then the equation (3.1) turns out to be

$$\begin{cases} u_t + g_x = 0, \\ g - g_{xx} = u + \frac{\varepsilon}{2}u^2. \end{cases}$$
 (3.2)

Multiplying both sides of (3.2) by  $u^2$  and integrating the result, and then using (3.3), we have

$$0 = \int_{R} (u_t + g_x) \cdot u^2 dx = \int_{R} u_t \cdot u^2 dx + \int_{R} g_x \cdot u^2 dx$$

$$= \int_{R} u_t \cdot u^2 dx + \frac{2}{\varepsilon} \int_{R} g_x \cdot (g - g_{xx} - u) dx = \int_{R} u_t \cdot u^2 dx - \frac{2}{\varepsilon} \int_{R} g_x \cdot u dx$$

$$= \int_{R} u_t \cdot u^2 dx + \frac{2}{\varepsilon} \int_{R} u_t \cdot u dx = \frac{d}{dt} \int_{R} \left(\frac{1}{3}u^3 + \frac{1}{\varepsilon}u^2\right) dx,$$

which indicates

$$\int_{R} \frac{1}{3} \left( u^3 + \frac{1}{\varepsilon} u^2 \right) dx$$

is a three-order invariant for (3.1).

#### 3.2. Regularized long wave equation

Consider the regularized long wave equation [9] of the form

$$u_t - \mu u_{xxt} + u_x + u^p u_x = 0, (3.4)$$

where  $\mu > 0$  is a positive constant. When p = 2, it is called modified regularized long wave equation; when  $p \ge 3$ , it is called generalized regularized long wave equation. Similar to the foregoing argument, (3.4) can be written as an equivalent form of

$$\begin{cases} u_t + g_x = 0, \\ g - \mu g_{xx} = u + \frac{1}{p+1} u^{p+1}. \end{cases}$$
 (3.5)

Multiplying both sides of (3.5) by  $u^{p+1}$ , integrating the result, and then using (3.6), we have

$$0 = \int_{R} (u_{t} + g_{x}) \cdot u^{p+1} dx = \int_{R} u_{t} \cdot u^{p+1} dx + \int_{R} g_{x} \cdot u^{p+1} dx$$

$$= \int_{R} u_{t} \cdot u^{p+1} dx + (p+1) \int_{R} g_{x} \cdot (g - \mu g_{xx} - u) dx$$

$$= \int_{R} u_{t} \cdot u^{p+1} dx - (p+1) \int_{R} g_{x} \cdot u dx$$

$$= \int_{R} u_{t} \cdot u^{p+1} dx + (p+1) \int_{R} u_{t} \cdot u dx$$

$$= \frac{d}{dt} \int_{R} \left( \frac{1}{p+2} u^{p+2} + \frac{p+1}{2} u^{2} \right) dx,$$

which indicates

$$\int_{R} \left( \frac{1}{p+2} u^{p+2} + \frac{p+1}{2} u^{2} \right) dx$$

is a high-order invariant for (3.4). This corrects an invariant  $I_3$  in Example 4 appeared in [10] (pp. 492).

### 3.3. Rosenau equation

Consider the Rosenau equation [11]

$$u_t + u_{xxxxt} + u_x + uu_x = 0, (3.7)$$

which is equivalent to

$$\begin{cases} u_t + g_x = 0, \\ g + g_{xxxx} = u + \frac{1}{2}u^2. \end{cases}$$
 (3.8)

Multiplying both sides of (3.8) by  $u^2$  and noticing (3.9), similar to the argument in the above, we have a third-order invariant for (3.7) of the form

$$\int_{R} \left(\frac{1}{3}u^3 + u^2\right) dx.$$

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