Corners and collapse: Some simple observations concerning critical masses and boundary blow-up in the fully parabolic Keller–Segel system

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Abstract

Our main result shows that the mass 2π is critical for the minimal Keller–Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{$\star$}$$

considered in a quarter disc $\Omega = \{\, (x_1, x_2) \in \mathbb{R} : x_1 > 0, x_2 > 0, x_1^2 + x_2^2 < R^2 \,\}$, $R > 0$, in the following sense: For all reasonably smooth nonnegative initial data u_0, v_0 with $\int_\Omega u_0 < 2\pi$, there exists a global classical solution to the Neumann initial boundary value problem associated to ($\star$), while for all $m > 2\pi$ there exist nonnegative initial data u_0, v_0 with $\int_\Omega u_0 = m$ so that the corresponding classical solution of this problem blows up in finite time.

At the same time, this gives an example of boundary blow-up in ($\star$).

Up to now, precise values of critical masses had been observed in spaces of radially symmetric functions or for parabolic–elliptic simplifications of ($\star$) only.

Key words: Chemotaxis, critical mass, global existence, blow-up

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1 Introduction

We study critical mass phenomena of solutions to the minimal fully parabolic Keller–Segel model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, T), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0, T), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0, v(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

in bounded domains Ω. This system has been suggested in [17] to model aggregation behavior of certain slime mold amoebae and has, along with many variants, since then been extensively studied, see for instance the surveys [13], [1] and [19].

Before discussing qualitative properties of solutions of (1.1), we note that (1.1) is locally well-posed also in non-smooth domains. While for a four-component system introduced in [17], a well-posedness result has recently been obtained in [14], for (1.1) it mainly suffices to reference the classical work [10]. That is done in Section 2, where we prove

Proposition 1.1. *Suppose that*

$$\begin{aligned} &\Omega \textit{ is a piecewise } C^{1+\alpha} \textit{ bounded domain in } \mathbb{R}^2 \textit{ for some } \alpha \in (0,1) \\ &\quad \textit{with a finite number of vertices and with nonvanishing interior angles} \end{aligned} \tag{1.2}$$

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(meaning that Ω is a bounded planar domain of class $\Sigma^{1,\alpha}$ for some $\alpha \in (0,1)$ in the sense of [7, Definition 2.1]) and

$$0 \le u_0 \in C^0(\overline{\Omega}) \quad \textit{as well as} \quad 0 \le v_0 \in \bigcup_{p>2} W^{1,p}(\Omega). \tag{1.3}$$

Then there exist $T_{\max} = T_{\max}(u_0, v_0) \in (0, \infty]$ and a pair of functions (u, v) solving (1.1) in $\overline{\Omega} \times [0, T_{\max})$ both weakly and classically in the sense of Definition 2.1 with the property that $T_{\max}$ is maximal, meaning that this solution cannot be extended beyond $T_{\max}$.

In order to describe the critical mass phenomena in more detail, we henceforth let $T_{\max}(u_0, v_0)$ be as given by Proposition 1.1 and for Ω as in (1.2) introduce the values

$$M_\star(\Omega) \coloneqq \sup\left\{ M > 0 \;\middle|\; \forall m \in (0, M]\ \forall u_0, v_0 \text{ fulfilling (1.3) and } \int_\Omega u_0 = m : T_{\max}(u_0, v_0) = \infty \right\} \tag{1.4}$$

as well as

$$M^\star(\Omega) \coloneqq \inf\left\{ M > 0 \;\middle|\; \forall m \ge M\ \exists u_0, v_0 \text{ fulfilling (1.3) and } \int_\Omega u_0 = m : T_{\max}(u_0, v_0) < \infty \right\}. \tag{1.5}$$

That is, for all masses smaller than $M_\star(\Omega)$, all solutions exist globally, while for all masses larger than $M^\star(\Omega)$, initial data exist whose corresponding solutions blow up in finite time. This leaves open the possibility that $M_\star(\Omega) < M^\star(\Omega)$. In this case, there would be an intermediate regime in which for some masses all solutions exist globally and for some masses they do not (cf. [8]).

Moreover, when Ω is a disc, we may ask whether these values change when the situation is restricted to radially symmetric settings, i.e. require that

$$u_0, v_0 \text{ are radially symmetric.} \tag{1.6}$$

To that end, we also set

$$M_\star(\Omega, \mathrm{rad}) \coloneqq \sup\left\{ M > 0 \;\middle|\; \forall m \in (0, M]\ \forall u_0, v_0 \text{ fulfilling (1.3), (1.6) and } \int_\Omega u_0 = m : T_{\max}(u_0, v_0) = \infty \right\}$$

and

$$M^\star(\Omega, \mathrm{rad}) \coloneqq \inf\left\{ M > 0 \;\middle|\; \forall m \ge M\ \exists u_0, v_0 \text{ fulfilling (1.3), (1.6) and } \int_\Omega u_0 = m : T_{\max}(u_0, v_0) < \infty \right\}.$$

Several partial results for the size of these values are available. If Ω is a smooth, bounded, planar domain, then $M_\star(\Omega) \ge 4\pi$ and $M_\star(\Omega, \mathrm{rad}) \ge 8\pi$, cf. [25]. For such domains and all masses larger than 4π and not equaling an integer multiple of 4π, unbounded solutions are constructed in [15] (and also more recently in a different way in [9]) – these, however, may potentially exist globally in time so that this result has no direct influence on $M^\star(\Omega)$. If $\Omega = B_R(0) \subseteq \mathbb{R}^2$, $R > 0$, then $M^\star(\Omega, \mathrm{rad}) \le 8\pi$ (and hence also $M^\star(\Omega) \le 8\pi$), cf. [21] (see also the earlier work [12]). Thus, if Ω is a disc, then $M_\star(\Omega, \mathrm{rad}) = 8\pi = M^\star(\Omega, \mathrm{rad})$, while for arbitrary smooth, bounded, planar domains Ω it is up to now only known that $4\pi \le M_\star(\Omega) \le M^\star(\Omega) \le 8\pi$. In [24], the critical mass identity $M_\star(\Omega) = 4\pi = M^\star(\Omega)$ is conjectured for smooth, bounded, planar domains Ω. Moreover, [24, Theorem 1] implies that blow-up has to occur at a single point at the boundary (or not at all) for masses between 4π and 8π. The actual occurrence of such boundary blow-up has, up to now, not been shown for (1.1).

As usual, more is known for parabolic–elliptic simplifications of (1.1), where the second equation is replaced by $0 = \Delta v - v + u$. For such systems, namely, [27, Theorem 2] and [23, Theorem 3.2] show $M_\star(\Omega) \ge 4\pi$ and $M^\star(\Omega) \le 4\pi$, respectively, for all smooth, bounded domains $\Omega \subseteq \mathbb{R}^2$, where the latter result additionally assumes that $\partial\Omega$ contains a line segment. (For an early blow-up result in a related system, see also [16].) Together, they imply $M_\star(\Omega) = 4\pi = M^\star(\Omega)$ for such domains. Moreover, as in the fully parabolic case, $M_\star(\Omega, \mathrm{rad}) = 8\pi = M^\star(\Omega, \mathrm{rad})$ holds whenever Ω is a disc (cf. [22, Theorem 2.1 and Corollary 3.1]). For a detailed discussion of the parabolic–elliptic system, we refer to [28]. Let us also briefly mention that critical mass phenomena (with slightly different flavors) have also been detected if $\Omega = \mathbb{R}^2$ ([5], [31]), if fluid-interaction is accounted for ([11, 32]), if the signal is produced indirectly ([30]) or if different boundary conditions are imposed on v ([3], [8]).

These results naturally lead to the question whether there is a domain Ω with $M_\star(\Omega) = M^\star(\Omega)$; that is, whether there is a critical mass distinguishing between global existence and finite-time blow-up also for the fully parabolic system (1.1) in non-radially symmetric settings (and if there is, what its precise value is). Our main results states that for certain domains Ω, we can indeed guarantee that $M_\star(\Omega) = M^\star(\Omega)$. For instance, 2π is the critical mass for quarter discs.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a circular sector with central angle $\theta \in (0, \frac{\pi}{2}]$ (and arbitrary positive finite radius). Then*

$$M_\star(\Omega) = 4\theta = M^\star(\Omega).$$

The proof of Theorem 1.2 is split into two parts. We first show in Section 3 that the solution (u, v) constructed in Proposition 1.1 based on [10] is global whenever the initial mass is sufficiently small. This is achieved by adapting the proof in [25] to non-smooth domains and thereby by crucially relying on the Trudinger–Moser inequality (cf. [6, Proposition 2.3]) whose constants depend on the minimal interior angle of the domain.

Theorem 1.3. *Suppose* (1.2) *and denote the minimal interior angle of Ω by θ (and set $\theta = \pi$ if there are no corners). Then*

$$M_\star(\Omega) \geq 4\theta.$$

Next, in Section 4, if Ω is a circular sector and given sufficiently large initial mass, we construct initial data such that the corresponding solutions blow up in finite time. This is achieved by restricting finite-time blow-up solutions on discs (constructed in [21]) to Ω.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a circular sector with central angle $\theta \in (0, 2\pi)$ (and arbitrary positive finite radius). Then*

$$M^\star(\Omega) \leq 4\theta.$$

Moreover, for each $m > 4\theta$, one can choose u_0, v_0 fulfilling (1.3), *$\int_\Omega u_0 = m$ and $T_{\max}(u_0, v_0) < \infty$ such that blow-up occurs at $0 \in \partial\Omega$ in the sense that there are $(x_k)_{k\in\mathbb{N}} \subseteq \overline{\Omega}$ and $(t_k)_{k\in\mathbb{N}} \subseteq [0, T_{\max}(u_0, v_0))$ such that $x_k \to 0$, $t_k \nearrow T_{\max}$ and $u(x_k, t_k) \to \infty$ as $k \to \infty$.*

Theorem 1.4 appears to be the first result for (1.1) (or any fully parabolic chemotaxis system considered on a bounded domain), where blow-up is shown to take place on the boundary. For most choices of $\theta \in (0, 2\pi)$, Theorem 1.4 shows that corners of Ω may be blow-up points, but the choice $\theta = \pi$ makes it clear that blow-up can also happen at smooth parts of the boundary.

We also note that our technique fails if one considers (1.1) with (no-flux boundary conditions for u and) homogeneous Dirichlet boundary conditions for v. However, for such a system blow-up at a boundary point probably should not be expected at all: At least for certain parabolic–elliptic simplifications all blow-up points have been proven to be contained in the interior of the domain ([29]).

2 Local existence: Proof of Proposition 1.1

Throughout this section, we fix a domain Ω fulfilling (1.2) and u_0, v_0 satisfying (1.3).

Definition 2.1. Let $T \in (0, \infty]$.

- A pair of nonnegative functions (u, v) is called a *weak solution* of (1.1) (in $\overline{\Omega} \times [0, T)$) if

$$u \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times [0, T)) \cap L^2_{\mathrm{loc}}([0, T); W^{1,2}(\Omega)), \tag{2.1}$$
$$v \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times [0, T)) \cap C^0([0, T); W^{1,2}(\Omega)), \tag{2.2}$$

  the function $(0, T) \ni t \mapsto \|\nabla v(t)\|^2_{L^2(\Omega)}$ is absolutely continuous,

$$(u_t, v_t) \in L^2_{\mathrm{loc}}([0, T); (W^{1,2}(\Omega))^\star) \times L^2_{\mathrm{loc}}([0, T); L^2(\Omega)) \tag{2.3}$$

  and u, v fulfill $u(\cdot, 0) = u_0$ and $v(\cdot, 0) = v_0$ a.e. in Ω as well as

$$\begin{aligned} &\int_0^t \int_\Omega u_t\varphi = -\int_0^t \int_\Omega (\nabla u - u\nabla v) \cdot \nabla\varphi \quad \text{and} \\ &\int_0^t \int_\Omega v_t\varphi = -\int_0^t \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^t \int_\Omega v\varphi + \int_0^t \int_\Omega u\varphi \end{aligned} \tag{2.4}$$

  for all $\varphi \in L^2((0, T); W^{1,2}(\Omega))$ and all $t \in (0, T)$.

- A pair of nonnegative functions (u,v) is called a *classical solution* of (1.1) if
$$u,v \in C^0((\overline{\Omega}\setminus V)\times[0,T))\cap C^{2,1}((\overline{\Omega}\setminus V)\times(0,T)),$$
where V denotes the set of vertices of Ω, and the equations in (1.1) are fulfilled pointwise in $(\overline{\Omega}\setminus V)\times[0,T)$.

Remark 2.2. Let $T\in(0,\infty)$ and let (u,v) be a weak solution of (1.1). Then (2.1)–(2.4) assert $\Delta v\in L^2(\Omega\times(0,T))$, hence (2.2) and (2.4) imply $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^2=-\int_\Omega v_t\Delta v$ a.e. in $(0,T)$.

The following local existence result and the extensibility criterion is essentially due to [10]. We note that a local existence result for less regular initial data has been proven in [2, Theorem 1].

Lemma 2.3. *There exist $T_{\max}\in(0,\infty]$ and a unique weak solution of* (1.1) *in $\overline{\Omega}\times[0,T_{\max})$ in the sense of Definition 2.1. Moreover, if $T_{\max}$ is chosen to be as large as possible, then*
$$T_{\max}<\infty \quad \textit{implies} \quad \limsup_{t\nearrow T_{\max}}\left(\int_\Omega(u\ln u)(\cdot,t)+\int_0^t\int_\Omega v_t^2\right)=\infty. \tag{2.5}$$

Proof. Let $p>2$ such that $v_0\in W^{1,p}(\Omega)$. By the construction in [10, Theorem 3.3], for sufficiently small $T>0$ functions $(u,v)\in(C([0,T];L^2(\Omega)))$ can be found which constitute a weak solution of (1.1) in $\overline{\Omega}\times[0,T)$ and which satisfy $u(\cdot,T)\in L^\infty(\Omega)$ and $v(\cdot,T)\in W^{1,p}(\Omega)$. With $(u,v)(\cdot,T)$ as initial data, the process can be repeated. We suppose that $T_{\max}$ is chosen maximally. From the construction in [10, part (iii) of the proof of Theorem 3.3], this means that either $T=\infty$ or one of
$$\int_0^t\|\mathrm{e}^{-v(\cdot,s)}u(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s,\quad \|v(\cdot,t)\|_{L^\infty(\Omega)},\quad \|\nabla v(\cdot,t)\|_{L^2(\Omega)}\quad\text{or}\quad \|\mathrm{e}^{-v(\cdot,t)}u(\cdot,t)\|_{L^2(\Omega)} \tag{2.6}$$
has to be unbounded as $t\nearrow T_{\max}$. In order to show (2.5), let us assume that $T_{\max}<\infty$ and that there exists $c_1>0$ such that
$$\int_\Omega(u\ln u)(\cdot,t)+\int_0^t\int_\Omega v_t^2\le c_2\qquad\text{for all }t\in(0,T_{\max}).$$
By [10, Lemma 4.10 and Lemma 4.11], both u and v are then bounded in $\Omega\times(0,T_{\max})$ and hence so are all quantities in (2.6). Finally, testing the first and second equation with u_- and v_-, respectively, yields nonnegativity of both u and v a.e. ☐

Next, we make use of parabolic regularity theory to show that the solution constructed in Lemma 2.3 is also a classical solution.

Lemma 2.4. *Let (u,v) be the weak solution of* (1.1) *given by Lemma 2.3 with maximal existence time $T_{\max}$. Then (u,v) is also a classical solution in the sense of Definition 2.1.*

Proof. For every $\psi\in C^2(\overline{\Omega})$ with $\partial_\nu\psi=0$ on $\partial\Omega$ and every $\varphi\in C^1(\overline{\Omega})$, direct computations show that
$$\int_0^T\int_\Omega(\psi u)_t\varphi=-\int_0^T\int_\Omega(\nabla(\psi u)-2u\nabla\psi-\psi u\nabla v)\cdot\nabla\varphi+\int_0^T\int_\Omega(u\Delta\psi+u\nabla\psi\cdot\nabla v)\varphi\quad\text{and}$$
$$\int_0^T\int_\Omega(\psi v)_t\varphi=-\int_0^T\int_\Omega(\nabla(\psi v)-2v\nabla\psi)\cdot\nabla\varphi+\int_0^T\int_\Omega(v\Delta\psi-v\psi+u\psi)\varphi$$
hold for all $T\in(0,T_{\max})$. The main idea is now to fix a point $x\in\overline{\Omega}\setminus V$, where V denotes the set of vertices of Ω, and $0<\tau<T<T_{\max}$, choose a cut-off function $\psi\in C^2(\overline{\Omega})$ with $\partial_\nu\psi=0$ on $\partial\overline{\Omega}$ which equals 1 in a neighbourhood of x and vanishes on a neighbourhood of V (existence of such ψ can be shown by, e.g., following the proof of [4, Lemma 3.2]) and then to apply various parabolic regularity results to ψu and to ψv which solve certain parabolic equations on smooth domains containing $\operatorname{supp}\psi$. In each step, we may choose a new cut-off function $\tilde\psi$ with smaller support so that regularity information on, say, $\nabla(\psi v)$ translates to information on ∇v on all of $\operatorname{supp}\tilde\psi$.

Indeed, in this way [26, Theorem 1.3] first shows that v is Hölder continuous near $S_x\coloneqq\{x\}\times(\tau,T)$, whenceupon [20, Theorem 1.1] asserts Hölder continuity also of ∇v near S_x. Then we can again make use of [26, Theorem 1.3] and [20, Theorem 1.1] to first obtain that u and then also that ∇u is Hölder continuous near S_x. Thus, two applications of [18, Theorem IV.5.3] yield $u,v\in C^{2,1}((\overline{\Omega}\setminus V)\times(0,T_{\max}))$. As both u and v are also continuous in $[0,T_{\max})$ as functions with values in L^∞ by Lemma 2.3, they also belong to $C^0((\overline{\Omega}\setminus V)\times[0,T_{\max}))$. Finally, three testing procedures with test functions supported in $(0,T_{\max})\times\Omega$, near $(0,T_{\max})\times(\overline{\Omega}\setminus V)$ and near $\{0\}\times\Omega$, respectively, show that (u,v) also solves (1.1) classically in $(\overline{\Omega}\setminus V)\times[0,T_{\max})$. ☐

Proof of Proposition 1.1. In Lemma 2.3, a local weak solution of (1.1) has been constructed, which is also a classical solution by Lemma 2.4. ☐

3 Global existence: Proof of Theorem 1.3

Throughout this section, we assume (1.2) and denote the minimal interior angle of Ω by θ (and set $\theta = \pi$ if there are no corners). Moreover, we fix u_0, v_0 fulfilling (1.3) as well as the solution (u, v) of (1.1) and its maximal existence time $T_{\max}$ given by Proposition 1.1.

We shall show $T_{\max} = \infty$ for sufficiently small $\int_\Omega u_0$, which we will achieve by following the reasoning in [25] and [10]. (We also refer to [2] where these techniques have been adapted to systems with different boundary conditions for v.)

The approach rests on two crucial observations. The first one, which has been first noted in [25] and [10], is that (1.1) admits an energy-type functional.

Lemma 3.1. *For all $t \in (0, T_{\max})$, the estimate*

$$\begin{aligned}
&\int_\Omega u(t)\ln u(t) - \int_\Omega u(t)v(t) + \frac{1}{2}\int_\Omega v^2(t) + \frac{1}{2}\int_\Omega |\nabla v(t)|^2 + \int_0^t \int_\Omega u|\nabla(\ln u - v)|^2 + \int_0^t \int_\Omega v_t^2 \\
&\le \int_\Omega u_0 \ln u_0 - \int_\Omega u_0 v_0 + \frac{1}{2}\int_\Omega v_0^2 + \frac{1}{2}\int_\Omega |\nabla v_0|^2
\end{aligned}$$

holds.

Proof. This follows by a direct calculation, see for instance [25, Lemma 3.3]. ☐

Second, we will rely on the following consequence of the Trudinger–Moser inequality.

Lemma 3.2. *Then there exists $C_\Omega > 0$ such that*

$$\int_\Omega \mathrm{e}^{|\varphi|} \le C_\Omega \exp\left(\frac{1}{8\theta}\int_\Omega |\nabla\varphi|^2 + \frac{1}{|\Omega|}\int_\Omega |\varphi|\right) \qquad \textit{for all } \varphi \in W^{1,2}(\Omega).$$

Proof. As shown for instance in [10, Corollary 2.7], this follows from the Trudinger–Moser inequality proved in [7, Theorem 1.2] (see also [6, Proposition 2.3]). ☐

Similarly as in [25, Lemma 3.4], these preparations allow us to show that solutions exist globally whenever the mass of the first component is sufficiently small.

Lemma 3.3. *Suppose $m := \int_\Omega u_0 < 4\theta$. Then $T_{\max} := T_{\max}(u_0, v_0) = \infty$.*

Proof. Since $m < 4\theta$, we can find $\eta > 0$ such that $\frac{(1+\eta)m}{8\theta} < \frac{1}{2}$. As integrating the first equation in (1.1) shows $\int_\Omega u = m$ in $(0, T_{\max})$, applying Jensen's inequality and Lemma 3.2 yields

$$\begin{aligned}
(1+\eta)\int_\Omega uv - \int_\Omega u \ln u &= m\int_\Omega \frac{u}{m}\ln\left(\frac{\mathrm{e}^{(1+\eta)v}}{u}\right) \le m\ln\left(\int_\Omega \frac{u}{m}\frac{\mathrm{e}^{(1+\eta)v}}{u}\right) = m\ln\left(\frac{1}{m}\int_\Omega \mathrm{e}^{(1+\eta)v}\right) \\
&\le m\ln\left(\frac{C_\Omega}{m}\exp\left(\frac{1+\eta}{8\theta}\int_\Omega |\nabla v|^2 + \frac{1+\eta}{|\Omega|}\int_\Omega v\right)\right) \\
&\le m\ln\left(\frac{C_\Omega}{m}\right) + \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{(1+\eta)m}{|\Omega|}\int_\Omega v \qquad \text{in } (0, T_{\max}),
\end{aligned}$$

where in the last step we have used $\frac{(1+\eta)m}{8\theta} < \frac{1}{2}$. As testing the second equation in (1.1) with 1 reveals $\int_\Omega v \le \max\{\int_\Omega v_0, m\}$, we thus conclude that there is $c_1 > 0$ such that

$$\int_\Omega uv \le \frac{1}{1+\eta}\int_\Omega u\ln u + \frac{1}{2(1+\eta)}\int_\Omega |\nabla v|^2 + c_1 \qquad \text{in } (0, T_{\max}).$$

Together with Lemma 3.1, this shows boundedness of $\int_\Omega u \ln u$ in $(0, T_{\max})$. As $\int_0^{T_{\max}}\int_\Omega v_t^2$ is also bounded by Lemma 3.1, we can conclude from (2.5) that $T_{\max} = \infty$. ☐

Proof of Theorem 1.3. According to the definition of $M_\star(\Omega)$ in (1.4), $M_\star(\Omega) \ge 4\theta$ immediately follows from Lemma 3.3. ☐

4 Finite-time blow-up: Proof of Theorem 1.4 and Theorem 1.2

In this final section, we prove Theorem 1.4. As crucial ingredient we use the known existence of blow-up solutions on a disk and to this aim recall [21, Proposition 1.3]:

Lemma 4.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^2$ with some $R > 0$, and let $m > 8\pi$. Then for all $T > 0$ and each $p > 1$ there exist $\varepsilon > 0$ and $(u_0, v_0) \in \mathcal{I} := \{(\hat{u}_0, \hat{v}_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \mid \hat{u}_0$ and $\hat{v}_0$ are radially symmetric and positive in $\overline{\Omega}\}$ such that $\int_\Omega u_0 = m$ and such that the solutions of (1.1) for all initial data $(\widetilde{u}_0, \widetilde{v}_0) \in \mathcal{I}$ satisfying $\|\widetilde{u}_0 - u_0\|_{L^p(\Omega)} + \|\widetilde{v}_0 - v_0\|_{W^{1,2}(\Omega)} < \varepsilon$ lead to solutions blowing up before time T.*

In restricting these solutions to circular sectors, we have to ensure that the boundary conditions are still satisfied, and note the following elementary fact:

Lemma 4.2. *Let $R > 0$, $\xi, \nu \in \partial B_1(0)$ with $\xi \cdot \nu = 0$ and $\varphi \in C^1(B_R(0))$ be radially symmetric. Then*

$$\nabla\varphi(x) \cdot \nu = 0 \qquad \text{for all } x \in L := \mathbb{R}\xi \cap B_R(0).$$

Proof. Without loss of generality, we may assume $\xi = (1, 0)$ and $\nu = (0, 1)$. Then $L = \{\, (x_1, x_2) \in B_R(0) : x_2 = 0 \,\}$ and the radial symmetry of φ implies $\varphi(x_1, x_2) = \varphi(x_1, -x_2)$ for all $(x_1, x_2) \in B_R(0)$; that is, $(0, \sqrt{R^2 - x_1^2}) \ni h \mapsto \varphi(x_1, h)$ is an even function for all $x_1 \in (0, R)$ and hence has derivative 0 at 0. $\square$

Lemma 4.3. *Let Ω be a circular sector with central angle $\theta \in (0, 2\pi)$ and radius $R > 0$ and let $m > 4\theta$. Then there exist u_0, v_0 satisfying (1.3) and $\int_\Omega u_0 = m$ such that the solution of (1.1) given by Proposition 1.1 blows up in finite time.*

Proof. As $\widetilde{m} := \frac{2\pi}{\theta} m > 8\pi$, based on Lemma 4.1 we let $(\widetilde{u}, \widetilde{v})$, with $(\widetilde{u}_0, \widetilde{v}_0) := (\widetilde{u}(\cdot, 0), \widetilde{v}(\cdot, 0))$ satisfying $\int_\Omega u_0 = \widetilde{m}$, be radially symmetric classical solutions of (1.1) on $\widetilde{\Omega} := B_R(0)$ blowing up at some finite time $T \in (0, \infty)$. Moreover, [24, Theorem 3] asserts that 0 is the only blow-up point of $\widetilde{u}$. We set $(u_0, v_0) := (\widetilde{u}_0, \widetilde{v}_0)|_{\overline{\Omega}}$ and note that due to the radial symmetry of u_0 we have

$$\int_\Omega u_0 = \frac{\theta}{2\pi} \int_{\tilde{\Omega}} \widetilde{u}_0 = \frac{\theta}{2\pi} \widetilde{m} = m$$

and that 0 is also a blow-up point of u. We next claim that $(u, v) := (\widetilde{u}, \widetilde{v})|_{\overline{\Omega} \times [0,T)}$ is a classical solution of (1.1): That the differential equations, the initial conditions and the boundary conditions for the circular arc are fulfilled follows immediately from the fact that (u, v) is a solution of (1.1) in $\widetilde{\Omega} \times [0, T)$. Finally, Lemma 4.2 shows that the boundary conditions are also fulfilled on the remaining (smooth part of the) boundary. Therefore, (u, v) is a solution of (1.1) in the sense of Definition 2.1 and due to the uniqueness statement in Lemma 2.3 thus has to coincide with the solution given by Proposition 1.1. $\square$

Proof of Theorem 1.4. This is a direct consequence of (1.5) and Lemma 4.3. $\square$

At last, we note that Theorem 1.3 and Theorem 1.4 entail Theorem 1.2.

Proof of Theorem 1.2. As the minimal interior angle of a circular sector with central angle θ is $\min\{\theta, \frac{\pi}{2}\}$, Theorem 1.2 results as a combination of Theorem 1.3 and Theorem 1.4. $\square$

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