

ASYMPTOTICALLY LINEAR MAGNETIC FRACTIONAL PROBLEMS

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ABSTRACT. The aim of this paper is investigating the existence and multiplicity of weak solutions to non-local equations involving the *magnetic fractional Laplacian*, when the nonlinearity is subcritical and asymptotically linear at infinity. We prove existence and multiplicity results by using variational tools, extending to the magnetic local and non-local setting some known results for the classical and the fractional Laplace operators.

1. INTRODUCTION

Existence and multiplicity results for solutions of elliptic problems involving non-local operators have been faced by a large number of authors by using variational and topological methods also in view of applications; see, among others, the monograph [20] and the references therein.

From a probabilistic point of view non-local operators can be seen as the infinitesimal generators of Lévy stable diffusion processes. Moreover, fractional operators allow us to model unusual diffusion processes in turbulent fluid motions and material transports in fractured media.

In particular, the fractional Laplacian appears in generalizations of quantum mechanics and in the description of the motion of a chain or an array of particles that are connected by elastic springs (see [1, 16, 21]).

Motivated by this wide interest in the current literature and by the meaning that the non-local operators can have in the applications, we are interested here in a nonlinear fractional problem involving an asymptotically linear term at infinity and the *fractional magnetic Laplacian*. Indeed, the main results (see Theorems 1.1 and 1.2 below) give under quite general assumptions a non-local magnetic version of some previous results, already present in the current literature, that are valid for different classes of differential problems.

More precisely, given $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ we look for solutions of the problem

$$(P) \quad \begin{cases} (-\Delta)_A^s u = g(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where Ω is an open bounded subset of \mathbb{R}^N with Lipschitz boundary $\partial\Omega$, $N > 2s$, $s \in]0, 1[$ and $(-\Delta)_A^s$ is the *fractional magnetic Laplacian* defined in [9], generalizing an operator introduced in [14] (see also [15]), as follows

$$(1.1) \quad (-\Delta)_A^s u(x) := c_{N,s} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)}{|x-y|^{N+2s}} dy \quad x \in \mathbb{R}^N$$

with $u \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$, $B_\varepsilon(x)$ ball of center x and radius ε and

$$c_{N,s} = s^{2s} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}.$$

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If $N = 3$, $B := \nabla \times A$ physically represents an external magnetic field acting on a charged particle.

When $A \equiv 0$ and $u \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$, $(-\Delta)_A^s$ agrees with the standard fractional Laplacian $(-\Delta)^s$ defined as principal value integral

$$(-\Delta)^s u(x) = c_{N,s} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \quad x \in \mathbb{R}^N,$$

see e.g. [10, 20].

Moreover, as observed in [9, 14], the operator defined in (1.1), can be seen as the fractional counterpart of the well known *magnetic Laplacian* $(\nabla - iA)^2$, see e.g. [2, 19, 23], which is the Schrödinger operator for a particle in the presence of an external magnetic field, playing a fundamental role in Quantum Mechanics in the description of the dynamics of a particle in a non-relativistic setting. In addition, the magnetic Laplacian turns out to be the limiting case for $s \rightarrow 1^-$ of the magnetic fractional Laplacian (see [25]), just as it happens for the fractional Laplacian and the classical Laplace operator, in the spirit of Bourgain, Brezis, and Mironescu [8].

As far as it concerns the nonlinearity g , here we suppose that there exist $\beta_\infty \in \mathbb{R}$ and $f : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$g(x, t) := \beta_\infty t + f(x, t^2)t \quad \text{a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R},$$

hence problem (P) takes the form

$$(P_{A,\infty}) \quad \begin{cases} (-\Delta)_A^s u = \beta_\infty u + f(x, |u|^2)u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Now, problem $(P_{A,\infty})$ is a perturbation of the eigenvalue problem and, following [12], in Subection 2.2 we recall some features about the spectrum of the integro-differential operator $(-\Delta)_A^s$, which are very closed to the well known ones concerning the classic Laplace operator and the fractional Laplacian (see e.g. [24]).

Hereafter we denote respectively by $\sigma((-\Delta)_A^s)$ and $(\beta_m^s)_m$ the spectrum and the non-decreasing, diverging sequence of the eigenvalues of the operator $(-\Delta)_A^s$, repeated according to their multiplicity (see [12]).

Now we state our main results, referring to [22, Theorem 4.12], [3, Theorems 0.1 and 0.3] and [4, Theorem 3.1] for the case of the classic Laplace operator.

Theorem 1.1. *Let $s \in]0, 1[$, $N > 2s$, Ω be an open bounded subset of \mathbb{R}^N with Lipschitz boundary. Assume $\beta_\infty \notin \sigma((-\Delta)_A^s)$ and that*

- (f₁) *f is a Carathéodory function and $\sup_{|t| \leq a} |f(\cdot, t^2)t| \in L^\infty(\Omega)$ for all $a > 0$;*
- (f₂) *there exists $\lim_{t \rightarrow +\infty} f(x, t) = 0$ uniformly with respect to a.e. $x \in \Omega$.*

Then, problem $(P_{A,\infty})$ has at least a weak solution.

Theorem 1.2. *Under the assumptions of Theorem 1.1, assume also that*

- (f₃) *$\lim_{t \rightarrow 0} f(x, t) = \beta_0 \in \mathbb{R} \setminus \{0\}$ uniformly with respect to a.e. $x \in \Omega$;*
- (β) *there exist $h, k \in \mathbb{N}$, with $k \geq h$, such that $\beta_0 + \beta_\infty < \beta_h^s \leq \beta_k^s < \beta_\infty$.*

Then, problem $(P_{A,\infty})$ has at least $k - h + 1$ distinct pairs of non-trivial weak solutions.

We will show that Theorem 1.1 is a direct consequence of the Saddle Point Theorem (see [22, Theorem 4.6]), while the proof of Theorem 1.2 is based on the application of an abstract critical point theorem in [3, Theorem 2.9] that we recall in Section 2.3 for the reader convenience.

Remark 1.3. Observe that

- (i) the statement in Theorem 1.2 holds with slight changes in the proof also if, instead of condition (β), we require $\beta_\infty < \beta_h^s \leq \beta_k^s < \beta_0 + \beta_\infty$ (see [4, Theorem 3.1]);

- (ii) if β_0 in (f_3) belongs to $\{\pm\infty\}$, then we can reason as in [4, Remark 3.3];
- (iii) we remind to [4, Remarks 1.5 and 3.2] for some remarks about the case $\beta_0 = 0$;
- (iv) in case of resonance, i.e. if $\beta_\infty \in \sigma((-\Delta)_A^s)$, we can proceed as in [4, Theorem 1.2], up to add further assumptions;
- (v) the statement of Theorem 1.1 is a particular case of [11, Theorem 1] (see also [5, Remark 3.2]);
- (vi) for $A \equiv 0$ we refer to [5, Theorems 1.2, 1.4] (see also [6] and references therein for further related results).

Actually, our results are new also for the *local* magnetic Laplacian $(\nabla - iA)^2$. Indeed, denoting by $(\beta_m)_m$ the sequence of its eigenvalues (see Subsection 2.1 for details), arguing as for the *nonlocal* operator, we can prove

Theorem 1.4. *Let Ω be an open bounded subset of \mathbb{R}^N with Lipschitz boundary. Assume $\beta_\infty \notin \sigma((\nabla - iA)^2)$ and that (f_1) and (f_2) hold. Then, problem*

$$(P_{A,\infty}^{\text{loc}}) \quad \begin{cases} (\nabla - iA)^2 u = \beta_\infty u + f(x, |u|^2)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

has at least a weak solution.

Moreover, assume that (f_3) holds and (β) holds with β_h^s and β_k^s replaced respectively by β_h and β_k . Then, problem $(P_{A,\infty}^{\text{loc}})$ has at least $k - h + 1$ distinct pairs of non-trivial weak solutions.

This paper is organized as follows: in Section 2 we recall some properties about the spectrum of the (local) magnetic Laplacian (Subsection 2.1), depict the main aspects of our non-local setting (Subsection 2.2) and present some abstract tools (Subsection 2.3); then, in Section 3 we prove Theorems 1.1 and 1.2.

Notations

- $B_r(x)$ is the ball in \mathbb{R}^N of center x and radius r ;
- $\mathcal{R}z$, \bar{z} and $|z|$ are respectively the real part, the complex conjugate and the modulus of a given $z \in \mathbb{C}$;
- $L^2(\Omega, \mathbb{C})$ denotes the Lebesgue space of measurable functions $u : \Omega \rightarrow \mathbb{C}$ such that

$$|u|_2^2 = \int_{\Omega} |u(x)|^2 dx < +\infty,$$

being $|\cdot|$ the Euclidean norm in \mathbb{C} , endowed with the real scalar product

$$\langle u, v \rangle_2 := \mathcal{R} \int_{\Omega} u \bar{v} dx \quad \text{for all } u, v \in L^2(\Omega, \mathbb{C});$$

- the standard norm of L^p spaces is denoted by $|\cdot|_p$;
- $H_0^1(\Omega)$ denotes the Sobolev space $W_0^{1,2}(\Omega, \mathbb{C})$;
- $(o_m(1))_m$ denotes any infinitesimal sequence.

2. TOOLS AND FUNCTIONAL FRAMEWORK

In this section we introduce our functional setting and some tools needed in the proofs of Theorems 1.1 and 1.2.

2.1. (Local) Magnetic Laplacian. Given a L_{loc}^∞ -vector potential A , let us consider the magnetic Laplacian $(\nabla - iA)^2$ in Ω . By standard arguments, see e.g. [17], [18], it can be proved that, considering the zero Dirichlet boundary condition, there exists an orthonormal basis $(u_m)_m \subset H_0^1(\Omega)$ of eigenfunctions of the magnetic Laplacian with associated sequence of eigenvalues $(\beta_m)_m$ such that

$$0 < \beta_1 \leq \beta_2 \leq \dots \beta_m \leq \dots \quad \text{and} \quad \beta_m \rightarrow +\infty \text{ as } m \rightarrow +\infty,$$

with

$$\beta_m = \frac{\int_{\Omega} |(\nabla - iA)u_m|^2 dx}{\int_{\Omega} |u_m|^2 dx} \quad m \in \mathbb{N}.$$

Let us point out that the eigenvalues - unlike the eigenfunctions - do not change by *gauge invariance*, see e.g. [17, p. 46], [18, Appendix A] and that, denoted by λ_1 the first eigenvalue of the Laplace operator with zero Dirichlet boundary condition, it results $\beta_1 \geq \lambda_1$ (see [17, Theorem 10.4]). For any $A \in L_{\text{loc}}^{\infty}(\mathbb{R}^N \mathbb{R}^N)$, let us consider the semi-norm

$$[u]_{H_A^1(\Omega)}^2 = \int_{\Omega} |\nabla u - iA(x)u|^2 dx$$

and, as in [19], the space

$$H_A^1(\Omega) := \{u \in L^2(\Omega, \mathbb{C}) : [u]_{H_A^1(\Omega)} < +\infty\}$$

endowed with the norm

$$\|u\|_{H_A^1(\Omega)}^2 := |u|_2^2 + [u]_{H_A^1(\Omega)}^2.$$

2.2. (Non-local) Magnetic Laplacian. The fractional counterpart of Subsection 2.1 can be found in [24] for $A \equiv 0$ and in the general case in [12, Section 3]. Next we highlight the main features.

As for the classical definition of $H^s(\Omega)$, for any $s \in (0, 1)$, let us consider the space

$$H_A^s(\Omega) := \{u \in L^2(\Omega, \mathbb{C}) : [u]_{H_A^s(\Omega)} < +\infty\},$$

where

$$[u]_{H_A^s(\Omega)} = \left(\frac{c_{N,s}}{2} \iint_{\Omega \times \Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2},$$

endowed with the norm

$$(2.1) \quad \|u\|_{H_A^s(\Omega)} := \left(|u|_2^2 + [u]_{H_A^s(\Omega)}^2 \right)^{1/2}.$$

Moreover, denoted by $H_A^s(\mathbb{R}^N)$ the closure of $C_0^{\infty}(\mathbb{R}^N)$ with respect to the norm (2.1), following [12] (see also [24]), let us consider the functional space

$$X_{0,A} := \{u \in H_A^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

and define as in [9] the scalar product

$$(2.2) \quad \langle u, v \rangle_{X_{0,A}} := \frac{c_{N,s}}{2} \mathcal{R} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left(u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y) \right) \overline{\left(v(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} v(y) \right)}}{|x - y|^{N+2s}} dx dy.$$

The norm $\|u\|_{X_{0,A}} := \sqrt{\langle u, u \rangle_{X_{0,A}}}$ is equivalent to (2.1) in $H_A^s(\mathbb{R}^N)$ (see [13, Lemma 2.1]) and $(X_{0,A}, \langle \cdot, \cdot \rangle_{X_{0,A}})$ is a real separable Hilbert space (see [23, Lemma 7]).

Being Ω open and bounded, $X_{0,A} \hookrightarrow H^s(\Omega)$, and, since $\partial\Omega$ is Lipschitz, $X_{0,A} \hookrightarrow L^p(\Omega, \mathbb{C})$ for any $p \in [1, 2_s^*)$, with $2_s^* := \frac{2N}{N-2s}$ (see [13, Lemma 2.2]).

A function $u \in X_{0,A}$ is a *weak solution* of $(P_{A,\infty})$ if and only if, for all $\varphi \in X_{0,A}$,

$$\langle u, \varphi \rangle_{X_{0,A}} = \beta_{\infty} \mathcal{R} \int_{\Omega} u \bar{\varphi} dx + \mathcal{R} \int_{\Omega} f(x, |u|^2) u \bar{\varphi} dx.$$

Following [12], we call *variational Dirichlet eigenvalue*, or simply *eigenvalue*, a value $\beta \in \mathbb{R}$ for which there exists a nontrivial weak solution $u \in X_{0,A}$, called *eigenfunction*, of

$$(2.3) \quad \begin{cases} (-\Delta)_A^s u = \beta u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

In [12] it is proved that eigenfunctions of (2.3) corresponding to different eigenvalues are orthogonal with respect to (2.2) (see [12, Lemma 3.2]) and that, proceeding by induction, it is possible to show that there exists a sequence $(\beta_m^s)_m \subset \mathbb{R}$ of eigenvalues of (2.3) and a sequence $(f_m)_m \subset X_{0,A}$ of associated eigenfunctions such that

$$(2.4) \quad \beta_1^s = \min_{u \in X_{0,A} \setminus \{0\}} \frac{\|u\|_{X_{0,A}}^2}{|u|_2^2} \quad \text{and} \quad \beta_{m+1}^s = \min_{u \in \mathbb{E}_{m+1} \setminus \{0\}} \frac{\|u\|_{X_{0,A}}^2}{|u|_2^2} \quad \text{for any } m \in \mathbb{N}^*,$$

where $\mathbb{E}_1 = X_{0,A}$,

$$\mathbb{E}_{m+1} := \{u \in X_{0,A} : \langle u, f_j \rangle_{X_{0,A}} = 0 \text{ for every } j = 1, \dots, m\}$$

and $f_1 \in X_{0,A}$, $f_{m+1} \in \mathbb{E}_{m+1}$ for $m \geq 1$ attain the minima in (2.4) (see [12, Proposition 3.3]).

Moreover, the eigenfunctions f_m are orthogonal also with respect to the real L^2 -scalar product (see [12, Proposition 3.4]) and the eigenvalues β_m^s satisfy

$$(2.5) \quad 0 < \beta_1^s \leq \beta_2^s \leq \dots \leq \beta_m^s \leq \dots \quad \text{and} \quad \beta_m^s \rightarrow +\infty \text{ as } m \rightarrow +\infty$$

(see [12, Proposition 3.5]). Furthermore $(f_m)_m$ is an orthonormal basis of $L^2(\Omega, \mathbb{C})$ and an orthogonal one of $X_{0,A}$ ([12, Proposition 3.7]) and β_m^s is an eigenvalue with finite multiplicity for each $m \in \mathbb{N}$ (see [12, Proposition 3.8]).

Denoting for any $m \in \mathbb{N}^*$ by $\mathbb{H}_m := \text{span}\{f_1, \dots, f_m\}$, it results (with respect to (2.2))

$$X_{0,A} = \mathbb{H}_m \oplus \mathbb{E}_{m+1} \quad \text{and} \quad \mathbb{E}_{m+1} = \mathbb{H}_m^\perp = \overline{\text{span}\{f_j : j \geq m+1\}}.$$

Moreover, for any $m \in \mathbb{N}^*$ the m -eigenvalue can be characterized as

$$\beta_m^s = \max_{u \in \mathbb{H}_m \setminus \{0\}} \frac{\|u\|_{X_{0,A}}^2}{|u|_2^2}$$

following [20, Chapter 8].

2.3. An abstract critical point theorem. Now, let $(X, \|\cdot\|_X)$ be a Banach space, $(X', \|\cdot\|_{X'})$ its dual, I a C^1 -functional on X , I' its differential. The functional I satisfies the *Palais–Smale condition at level c* ($c \in \mathbb{R}$) if any sequence $(u_m)_m \subseteq X$ such that

$$(2.6) \quad \lim_{m \rightarrow +\infty} I(u_m) = c \quad \text{and} \quad \lim_{m \rightarrow +\infty} \|I'(u_m)\|_{X'} = 0$$

converges in X , up to subsequences. If $-\infty \leq a < b \leq +\infty$, I satisfies the Palais–Smale condition in $]a, b[$ if so is at each level $c \in]a, b[$.

We will use the following abstract critical point theorem whose proof is based on the pseudo-index related to the genus (see [7] for more details).

Theorem 2.1 ([3], Theorem 2.9). *Let $I \in C^1(X, \mathbb{R})$ and assume that:*

- (1) *I is even;*
- (2) *I satisfies the Palais–Smale condition in \mathbb{R} ;*
- (3) *there exist two closed subspaces $V, W \subset X$ such that $\dim V < +\infty$, $\text{codim } W < +\infty$ and two constants c_0, c_∞ , such that $c_\infty > c_0$, verifying the following assumptions:*
 - *$I(u) \geq c_0$ on $S_\rho \cap W$ (resp. on $S_\rho \cap V$), where $S_\rho = \{u \in X : \|u\|_X = \rho\}$;*
 - *$I(u) \leq c_\infty$ on V (resp. on W).*

If, moreover, $\dim V > \text{codim } W$, then I has at least $\dim V - \text{codim } W$ distinct pairs of critical points whose corresponding critical values belong to $[c_0, c_\infty]$.

3. PROOF OF THEOREM 1.1

By (f₁) and (f₂), for all $\varepsilon > 0$ there exists $a_\varepsilon > 0$ such that

$$(3.1) \quad |f(x, t^2)t| \leq \varepsilon|t| + a_\varepsilon, \quad \text{for a.e. } x \in \Omega, \text{ for all } t \in \mathbb{R}.$$

The weak solutions of problem ($P_{A,\infty}$) are the critical points of the C^1 -functional

$$*J_A(u) := \|u\|_{X_{0,A}}^2 - \beta_\infty \|u\|_2^2 - \int_\Omega F(x, |u|^2) dx,$$

defined in $X_{0,A}$, with $F(x, t) := \int_0^t f(x, s) ds$ and, for every $u, \varphi \in X_{0,A}$,

$$J'_A(u)[\varphi] = \langle u, \varphi \rangle_{X_0} - \beta_\infty \mathcal{R} \int_\Omega u \bar{\varphi} dx - \mathcal{R} \int_\Omega f(x, |u|^2) u \bar{\varphi} dx.$$

Under the assumptions of Theorems 1.1 and 1.2, the functional J_A satisfies the following compactness property.

Lemma 3.1. *Assume that (f₁) and (f₂) hold. Then, if $\beta_\infty \notin \sigma((-\Delta)_A^s)$, the functional J_A satisfies the Palais–Smale condition in \mathbb{R} .*

Proof. Let $c \in \mathbb{R}$ and $(u_m)_m$ be a sequence in $X_{0,A}$ such that (2.6) holds.

To prove that $(u_m)_m$ is bounded in $X_{0,A}$, arguing by contradiction, we assume that $\|u_m\|_{X_{0,A}} \rightarrow +\infty$ as $m \rightarrow +\infty$. Since $(w_m)_m := (u_m/\|u_m\|_{X_{0,A}})_m$ is bounded in $X_{0,A}$, there exists $w \in X_{0,A}$ such that, up to a subsequence, $(w_m)_m$ converges to w weakly in $X_{0,A}$ and strongly in $L^2(\Omega, \mathbb{C})$.

Using the boundedness of $(w_m)_m$, by (2.6) we get

$$(3.2) \quad \begin{aligned} o_m(1) &= J'_A(u_m)[(w_m - w)/\|u_m\|_{X_{0,A}}] \\ &= \langle w_m, w_m - w \rangle_{X_{0,A}} - \beta_\infty \mathcal{R} \int_\Omega w_m \overline{(w_m - w)} dx - \mathcal{R} \int_\Omega \frac{f(x, |u_m|^2) u_m}{\|u_m\|_{X_{0,A}}} \overline{(w_m - w)} dx. \end{aligned}$$

Moreover, by the convergence of $(w_m)_m$ to w in $L^2(\Omega, \mathbb{C})$, it follows that

$$\left| \int_\Omega w_m \overline{(w_m - w)} dx \right| \leq \int_\Omega |w_m| |w_m - w| dx \leq \|w_m\|_2 \|w_m - w\|_2 = o_m(1)$$

and, using also (3.1), we infer that

$$\left| \int_\Omega \frac{f(x, |u_m|^2) u_m}{\|u_m\|_{X_{0,A}}} \overline{(w_m - w)} dx \right| \leq \varepsilon \|w_m\|_2 \|w_m - w\|_2 + \frac{a_\varepsilon \|w_m - w\|_1}{\|u_m\|_{X_{0,A}}} = o_m(1).$$

Then, by (3.2), it follows that $\langle w_m, w_m - w \rangle_{X_{0,A}} = o_m(1)$, therefore $w_m \rightarrow w$ in $X_{0,A}$ and $w \neq 0$.

Now, reasoning as in (3.2), for all $\varphi \in X_{0,A}$

$$(3.3) \quad \begin{aligned} o_m(1) &= J'_A(u_m)[\varphi/\|u_m\|_{X_{0,A}}] \\ &= \langle w_m, \varphi \rangle_{X_{0,A}} - \beta_\infty \mathcal{R} \int_\Omega w_m \bar{\varphi} dx - \mathcal{R} \int_\Omega \frac{f(x, |u_m|^2) u_m}{\|u_m\|_{X_{0,A}}} \bar{\varphi} dx. \end{aligned}$$

Moreover, for every $\varepsilon > 0$ let $\tilde{m}_\varepsilon \in \mathbb{N}$ be such that, for every $m \geq \tilde{m}_\varepsilon$, $\|u_m\|_{X_{0,A}} > a_\varepsilon/\varepsilon$, where $a_\varepsilon > 0$ is given in (3.1). Then, using (3.1), for every $\varepsilon > 0$ and $m \geq \tilde{m}_\varepsilon$,

$$\left| \int_\Omega \frac{f(x, |u_m|^2) u_m}{\|u_m\|_{X_{0,A}}} \bar{\varphi} dx \right| \leq \varepsilon \|w_m\|_2 \|\varphi\|_2 + \frac{a_\varepsilon \|\varphi\|_1}{\|u_m\|_{X_{0,A}}} \leq C\varepsilon,$$

for a suitable $C > 0$, so that, for every $\varphi \in X_{0,A}$,

$$\lim_{m \rightarrow +\infty} \mathcal{R} \int_\Omega \frac{f(x, |u_m|^2) u_m}{\|u_m\|_{X_{0,A}}} \bar{\varphi} dx = 0.$$

Thus, passing to the limit in (3.3), we get that, for every $\varphi \in X_{0,A}$,

$$\langle w, \varphi \rangle_{X_{0,A}} = \beta_\infty \mathcal{R} \int_\Omega w \overline{\varphi} dx,$$

namely $\beta_\infty \in \sigma((-\Delta)_A^s)$, against our assumption.

Hence $(u_m)_m$ is bounded in $X_{0,A}$ and, due to the reflexivity of our space, there exists $u_0 \in X_{0,A}$ such that, up to a subsequence, $u_m \rightharpoonup u_0$ in $X_{0,A}$, $u_m \rightarrow u_0$ in $L^2(\Omega, \mathbb{C})$,

$$\langle J'_A(u_m), u_m - u_0 \rangle \rightarrow 0 \quad \text{as } m \rightarrow +\infty$$

and, by (3.1),

$$\left| \int_\Omega f(x, |u_m|^2) u_m \overline{(u_m - u_0)} dx \right| \rightarrow 0 \quad \text{as } m \rightarrow +\infty.$$

Therefore, reasoning as before, $u_m \rightarrow u_0$ in $X_{0,A}$ and the proof is complete. \square

Now we are ready to prove our main results.

Proof of Theorem 1.1. If $\beta_1^s < \beta_\infty$, the statement follows by a standard application of the Saddle Point Theorem (see [22, Theorem 4.6]). In particular, due to Lemma 3.1, we only need to check its *geometrical* assumptions.

First of all observe that, for every $m \in \mathbb{N}^*$,

$$(3.4) \quad \|u\|_{X_{0,A}}^2 \leq \beta_m^s |u|_2^2 \text{ for every } u \in \mathbb{H}_m$$

and

$$(3.5) \quad \|u\|_{X_{0,A}}^2 \geq \beta_m^s |u|_2^2 \text{ for every } u \in \mathbb{E}_m.$$

Then, using (2.5), let us consider $\nu \in \mathbb{N}^*$ such that $\beta_\nu^s < \beta_\infty < \beta_{\nu+1}^s$ and

$$(3.6) \quad \varepsilon \in (0, \min\{\beta_\infty - \beta_\nu^s, \beta_{\nu+1}^s - \beta_\infty\}).$$

Thus, by (3.1) and (3.5), for every ε there exists $C_\varepsilon > 0$ such that for every $u \in \mathbb{E}_{\nu+1}$

$$J_A(u) \geq \|u\|_{X_{0,A}}^2 - \beta_\infty |u|_2^2 - \varepsilon |u|_2^2 - C_\varepsilon \|u\|_{X_{0,A}} \geq \left(1 - \frac{\beta_\infty + \varepsilon}{\beta_{\nu+1}^s}\right) \|u\|_{X_{0,A}}^2 - C_\varepsilon \|u\|_{X_{0,A}}$$

so that, using (3.6), we obtain that there exists $\alpha_1 > 0$ such that, for all $u \in \mathbb{E}_{\nu+1}$, $J_A(u) \geq -\alpha_1$, namely [22, (I₄) of Theorem 4.6].

Moreover, by (3.1), (3.4) and (3.6), for all $u \in \mathbb{H}_\nu$

$$J_A(u) \leq \|u\|_{X_{0,A}}^2 - \beta_\infty |u|_2^2 + \varepsilon |u|_2^2 + C_\varepsilon \|u\|_{X_{0,A}} \leq \left(1 - \frac{\beta_\infty - \varepsilon}{\beta_\nu^s}\right) \|u\|_{X_{0,A}}^2 + C_\varepsilon \|u\|_{X_{0,A}} \rightarrow -\infty$$

as $\|u\|_{X_{0,A}} \rightarrow +\infty$, and so, for $\rho > 0$ large enough, that $J_A(u) \leq -\alpha_2$ for all $u \in \mathbb{H}_\nu \cap S_\rho$ where $S_\rho := \{u \in X_{0,A} : \|u\|_{X_{0,A}} = \rho\}$, with $\alpha_2 > \alpha_1$, getting [22, (I₃) of Theorem 4.6].

On the other hand, by (2.4), if $0 < \beta_\infty < \beta_1^s$,

$$J_A(u) \geq \|u\|_{X_{0,A}}^2 - \beta_\infty |u|_2^2 - \varepsilon |u|_2^2 - C_\varepsilon \|u\|_{X_{0,A}} \geq \left(1 - \frac{\beta_\infty + \varepsilon}{\beta_1^s}\right) \|u\|_{X_{0,A}}^2 - C_\varepsilon \|u\|_{X_{0,A}}$$

and, if $\beta_\infty \leq 0$,

$$J_A(u) \geq \|u\|_{X_{0,A}}^2 - \beta_\infty |u|_2^2 - \varepsilon |u|_2^2 - C_\varepsilon \|u\|_{X_{0,A}} \geq \left(1 - \frac{\varepsilon}{\beta_1^s}\right) \|u\|_{X_{0,A}}^2 - C_\varepsilon \|u\|_{X_{0,A}}$$

for all $u \in X_{0,A}$. Thus, in both cases, if $\varepsilon \in (0, \beta_1^s - \beta_\infty)$ and $\varepsilon \in (0, \beta_1^s)$ respectively, we have that the functional J_A is bounded from below and so we get a weak solution for problem $(P_{A,\infty})$ by a direct minimization argument. \square

Proof of Theorem 1.2. To get the statement, first observe that, by (f₂) and (f₃),

$$\lim_{|t| \rightarrow +\infty} \frac{F(x, t^2)}{t^2} = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{F(x, t^2)}{t^2} = \beta_0,$$

uniformly with respect to a.e. $x \in \Omega$. Therefore, for every $\varepsilon > 0$ there exist $r_\varepsilon > \delta_\varepsilon > 0$ such that

$$|F(x, t^2)| \leq \varepsilon t^2, \text{ if } |t| > r_\varepsilon \quad \text{and} \quad |F(x, t^2) - \beta_0 t^2| \leq \varepsilon t^2, \text{ if } |t| < \delta_\varepsilon,$$

for a.e. $x \in \Omega$.

Moreover, by (f₁), taking any $q \in]0, 4s/(N - 2s)[$, there exists $c_{r_\varepsilon} > 0$ such that

$$|F(x, t^2)| \leq c_{r_\varepsilon} |t|^{q+2}, \quad \text{if } \delta_\varepsilon \leq |t| \leq r_\varepsilon \text{ and for a.e. } x \in \Omega.$$

Thus, for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$F(x, t^2) \leq (\beta_0 + \varepsilon)t^2 + c_\varepsilon |t|^{q+2}, \quad \text{for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R},$$

so that, for all $u \in X_{0,A}$, using Sobolev inequalities,

$$J_A(u) \geq \|u\|_{X_{0,A}}^2 - (\beta_\infty + \beta_0 + \varepsilon) \|u\|_2^2 - c'_\varepsilon \|u\|_{X_{0,A}}^{q+2} \quad \text{for all } u \in X_{0,A},$$

for a suitable $c'_\varepsilon > 0$.

Therefore, given h as in (β), by (3.5) we obtain that, for $\varepsilon \in (0, \beta_h^s - (\beta_\infty + \beta_0))$, there exists $\tilde{c}_\varepsilon > 0$ such that, for every $u \in \mathbb{E}_h$,

$$J_A(u) \geq \left(1 - \frac{\beta_\infty + \beta_0 + \varepsilon}{\beta_h^s}\right) \|u\|_{X_{0,A}}^2 - c'_\varepsilon \|u\|_{X_{0,A}}^{q+2} \geq \tilde{c}_\varepsilon \|u\|_{X_{0,A}}^2 - c'_\varepsilon \|u\|_{X_{0,A}}^{q+2}.$$

Hence we can conclude that, if ρ is small enough, there exists $c_0 > 0$ such that

$$(3.7) \quad J_A(u) \geq c_0 \quad \text{for all } u \in S_\rho \cap \mathbb{E}_h,$$

with S_ρ as in the proof of Theorem 1.1.

Moreover, taking k as in (β), $\varepsilon \in (0, (\beta_\infty - \beta_k^s)/\beta_k^s)$, and using (3.1) and (3.4), we have that there exist $C_\varepsilon, c_\infty > 0$ such that for all $u \in \mathbb{H}_k$,

$$J_A(u) \leq \left(1 + \varepsilon - \frac{\beta_\infty}{\beta_k^s}\right) \|u\|_{X_{0,A}}^2 + C_\varepsilon \|u\|_{X_{0,A}} \leq c_\infty.$$

Finally, to have $c_0 < c_\infty$ it is enough to take $\varepsilon \in (0, \min\{\beta_h^s - (\beta_\infty + \beta_0), (\beta_\infty - \beta_k^s)/\beta_k^s\})$ and ρ sufficiently small in (3.7).

Now we can conclude the proof. Indeed, J_A is even and by Lemma 3.1 it satisfies the Palais–Smale condition in \mathbb{R} ; moreover taking $W = \mathbb{E}_h$ and $V = \mathbb{H}_k$, Theorem 2.1 applies and J_A has at least $k - h + 1$ distinct pairs of critical points. □

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