DETERMINATION OF EQUILIBRIUM PARAMETERS OF THE MARLE MODEL FOR POLYATOMIC GASES

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Abstract. The BGK model is a relaxation-time approximation of the celebrated Boltzmann equation, and the Marle model is a direct extension of the BGK model in a relativistic framework. In this paper, we introduce the Marle model for polyatomic gases based on the Jüttner distribution devised in [Ann. Phys., 377, (2017), 414–445], and show the existence of a unique set of equilibrium parameters of the Jüttner distribution.

1. Introduction

1.1. **Relativistic extended thermodynamics of polyatomic molecules.** Recently, in 2017, a relativistic extended thermodynamics (RET) of rarefied polyatomic gases was discussed in [10], where the relativistic Maxwellian, also called the Jüttner distribution, was derived for polyatomic gases for the first time. To be precise, let $f \equiv f(x^\alpha, p^\beta, \mathcal{I})$ be the momentum distribution function of relativistic particles on the phase space point (x^α, p^β) with the internal energy $\mathcal{I} \geq 0$ due to the rotation and vibrations of particles, where $x^\alpha = (ct, x) \in \mathbb{R}^+ \times \mathbb{R}^3$ is the space-time coordinate, and $p^\beta = (\sqrt{(mc)^2 + |p|}, p) \in \mathbb{R}^+ \times \mathbb{R}^3$ is the four-momentum. Here c is the speed of light, m is the rest mass of a particle, and Greek indices run from 0 to 3. Throughout this paper, the metric tensor $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$ are given by

$$g_{\alpha\beta} = g^{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$$

and we use the raising and lowering indices as

$$g_{\alpha\mu} p^\mu = p_\alpha, \qquad g^{\alpha\mu} p_\mu = p^\alpha,$$

which implies $p_\alpha = (p^0, -p)$. Then it follows from the Einstein summation convention that

$$p^\mu q_\mu = p_\mu q^\mu = p^0 q^0 - \sum_{i=1}^{3} p^i q^i.$$

For $f \equiv f(t, x, p, \mathcal{I})$, macroscopic descriptions are given by the particle-particle flux V^μ and energy-momentum tensor $T^{\mu\nu}$ [10]:

$$V^\mu = mc \int_{\mathbb{R}^3} \int_0^\infty p^\mu f \phi(\mathcal{I})\, d\mathcal{I}\, \frac{dp}{p^0}, \qquad T^{\mu\nu} = \frac{1}{mc} \int_{\mathbb{R}^3} \int_0^\infty p^\mu p^\nu f \left(mc^2 + \mathcal{I}\right) \phi(\mathcal{I})\, d\mathcal{I}\, \frac{dp}{p^0}, \tag{1.1} \boxed{\texttt{VT}}$$

where $\phi(\mathcal{I}) \geq 0$ is the state density so that $\phi(\mathcal{I})\, d\mathcal{I}$ represents the number of the internal states of a molecule having the internal energy between $\mathcal{I}$ and $\mathcal{I} + d\mathcal{I}$. The form of $\phi(\mathcal{I})$ may be determined differently depending on the physical context, but it should be able to recover the state density for classical particles in the non-relativistic limit:

$$\lim_{c \to \infty} \phi(\mathcal{I}) = I^\sigma \qquad \text{with} \qquad \sigma = \frac{f^i - 2}{2},$$

where I denotes the variable representing the internal energy for classical particles and $f^i \geq 0$ is the internal degrees of freedom due to the internal motion of molecules (see [10, Sec 4.1] for details). For monatomic gases, $\sigma = -1$. In this paper, we choose $\phi(\mathcal{I})$ of the form

$$\phi(\mathcal{I}) = \mathcal{I}^\sigma \ (\sigma > -1),$$

which is employed in [11] to propose another type of relativistic BGK model for polyatomic gases. The entropy four-vector h^μ is given by

$$h^\mu = -k_B c \int_{\mathbb{R}^3} \int_0^\infty p^\mu f \ln f \phi(\mathcal{I})\, d\mathcal{I} \frac{dp}{p^0},$$

where k_B denotes the Boltzmann constant. Finally, we introduce the Jüttner distribution for polyatomic gases:

$$f_E \equiv f_E(n, U^\mu, \gamma) = \frac{n}{M(\gamma)} e^{-\left(1 + \frac{\mathcal{I}}{mc^2}\right) \frac{\gamma}{mc^2} U^\mu p_\mu} \qquad \text{with} \qquad M(\gamma) = \int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{\mathcal{I}}{mc^2}\right) \frac{\gamma}{mc} p^0} \phi(\mathcal{I})\, d\mathcal{I}\, dp,$$

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which maximizes the entropy $h := h^\alpha U_\alpha$ (for details, see [10, Sec.4]). Here the equilibrium parameters $n, U^\mu = (\sqrt{c^2+|U|^2}, U)$, and $\gamma := \frac{mc^2}{k_B T}$ are unknown functions of t and x representing the equilibrium density, four-velocity, and the ratio between the rest energy of a particle and the product of the Boltzmann constant and equilibrium temperature T respectively.

1.2. **Marle model for polyatomic gases.** The BGK model [2] is a relaxation-time approximation of the celebrated Boltzmann equation, which has been widely used in physics and engineering for practical purposes. For monatomic molecules, a direct extension of the BGK model in the relativistic framework was first proposed by Marle [8, 9] based on the Eckart frame [4]. In this paper, we present the direct application of RET [10] to the Marle model [8, 9], which reads

$$\partial_t f + \hat{p}\cdot\nabla_x f = \frac{cm}{\tau(1+\frac{\mathcal{I}}{mc^2})p^0}(f_E - f) =: Q(f) \tag{1.2}$$
PMarle

where $\hat{p} := cp/p^0$ is the normalized momentum and τ denotes the relaxation time in the rest frame. In the Eckart frame [4], the particle-particle flux V_f^μ of (1.1) can be expressed by the observable quantities as $V_f^\mu = mn_f U_f^\mu$ where mn_f denotes the number density and $U_f^\mu = (\sqrt{c^2+|U_f|^2}, U_f)$ the Eckart four-velocity defined by

$$\begin{aligned} n_f^2 &= \left(\int_{\mathbb{R}^3}\int_0^\infty f\phi(\mathcal{I})\,d\mathcal{I}\,dp\right)^2 - \sum_{i=1}^3 \left(\int_{\mathbb{R}^3}\int_0^\infty p^i f\phi(\mathcal{I})\,d\mathcal{I}\,\frac{dp}{p^0}\right)^2, \\ U_f^\mu &= \frac{c}{n_f}\int_{\mathbb{R}^3}\int_0^\infty p^\mu f\phi(\mathcal{I})\,d\mathcal{I}\,\frac{dp}{p^0} \end{aligned} \tag{1.3}$$
eckart

respectively. Since U_f^μ has a constant length, i.e. $U_f^\mu U_{f\mu} = c^2$, the number density n_f can be written with respect to U_f^μ as

$$n_f = \frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty U_{f\mu}p^\mu f\phi(\mathcal{I})\,d\mathcal{I}\,\frac{dp}{p^0}. \tag{1.4}$$
n

Note that the head term $(1+\frac{\mathcal{I}}{mc^2})^{-1}$ on the right-hand side of (1.2) is considered for consistency with the particle-particle flux V_f^μ in the application of the Eckart frame (see (2.2)).

1.3. **Main result.** In relativistic BGK-type models, the equilibrium parameters n, U^μ and γ of the Jüttner distribution are determined in a way that the equation satisfies the conservation laws of V^μ and $T^{\mu\nu}$. In this process, γ is often defined through a nonlinear relation due to the relativistic nature. Therefore, in order to study the equations rigorously, it is necessary to show that γ can be uniquely determined by the distribution f, see [1, 5, 6, 7] for similar problems. We also refer to [3, 12] for the hydrodynamic limit of the relativistic Boltzmann equation regarding the range of γ. The aim of this paper is to prove that the equilibrium parameters of f_E in the polyatomic Marle model (1.2) can be uniquely determined in a way that (1.2) satisfies the fundamental properties of the Boltzmann equation, the conservation laws and H-theorem. Our main result is as follows.

⟨main⟩ **Theorem 1.1.** *Let f be non-negative and not trivially zero so that V_f^μ exists. Assume that the state density is chosen as $\phi(\mathcal{I}) = \mathcal{I}^\sigma$ with $\sigma > -1$. Then there exists a unique set of equilibrium parameters n, U^μ and γ of f_E satisfying the following identities:*

$$\int_{\mathbb{R}^3}\int_0^\infty Q(f)\phi(\mathcal{I})\,d\mathcal{I}dp = 0, \qquad \int_{\mathbb{R}^3}\int_0^\infty p^\mu(mc^2+\mathcal{I})Q(f)\phi(\mathcal{I})\,d\mathcal{I}dp = 0. \tag{1.5}$$
cancellatio

Indeed, $n = n_f$, $U^\mu = U_f^\mu$, and γ is determined by the nonlinear relation:

$$\frac{\widetilde{M}(\gamma)}{M(\gamma)} = \frac{1}{n_f}\int_{\mathbb{R}^3}\int_0^\infty f\left(1+\frac{\mathcal{I}}{mc^2}\right)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}. \tag{1.6}$$
gamma

where

$$\widetilde{M}(\gamma) = \int_{\mathbb{R}^3}\int_0^\infty e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\frac{\gamma}{mc}p^0}\left(1+\frac{\mathcal{I}}{mc^2}\right)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}.$$

Remark 1.1. *Due to (1.5), the distribution function f verifying (1.2) satisfies the conservation laws of V_f^μ and $T_f^{\mu\nu}$:*

$$\begin{aligned} \partial_\mu V_f^\mu &= m\int_{\mathbb{R}^3}\int_0^\infty \{\partial_t f + \hat{p}\cdot\nabla_x f\}\phi(\mathcal{I})\,d\mathcal{I}\,dp = 0, \\ \partial_\nu T_f^{\mu\nu} &= \frac{1}{mc^2}\int_{\mathbb{R}^3}\int_0^\infty p^\mu\{\partial_t f + \hat{p}\cdot\nabla_x f\}\left(mc^2+\mathcal{I}\right)\phi(\mathcal{I})\,d\mathcal{I}\,dp = 0, \end{aligned}$$

where $\partial_\mu := \partial/\partial x^\mu$. *Furthermore,* (1.5) *implies*

$$\begin{aligned}\int_{\mathbb{R}^3}\int_0^\infty Q(f)\ln f_E\phi(\mathcal{I})\,d\mathcal{I}dp &= \ln\Big\{\frac{n}{M(\gamma)}\Big\}\int_{\mathbb{R}^3}\int_0^\infty Q(f)\phi(\mathcal{I})\,d\mathcal{I}dp - \frac{\gamma}{m^2c^4}U_\mu\int_{\mathbb{R}^3}\int_0^\infty p^\mu(mc^2+\mathcal{I})Q(f)\phi(\mathcal{I})\,d\mathcal{I}dp\\ &= 0,\end{aligned}$$

which gives

$$\begin{aligned}\int_{\mathbb{R}^3}\int_0^\infty Q(f)\ln f\phi(\mathcal{I})\,d\mathcal{I}dp &= \int_{\mathbb{R}^3}\int_0^\infty Q(f)(\ln f-\ln f_E)\phi(\mathcal{I})\,d\mathcal{I}dp\\ &= \frac{cm}{\tau}\int_{\mathbb{R}^3}\int_0^\infty (f_E-f)(\ln f-\ln f_E)\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}\\ &= \frac{cm}{\tau}\int_{\mathbb{R}^3}\int_0^\infty f_E\Big(1-\frac{f}{f_E}\Big)\ln\Big(\frac{f}{f_E}\Big)\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}.\end{aligned}$$

Using the following inequality:

$$(1-x)\ln x \le 0 \quad \textit{for all } x>0,$$

one can obtain the H-theorem:

$$\partial_\mu h^\mu := -k_B\int_{\mathbb{R}^3}\int_0^\infty \{\partial_t + \hat{p}\cdot\nabla_x\} f\ln f\phi(\mathcal{I})\,d\mathcal{I}dp \ge 0.$$

Remark 1.2. *In this paper, the state density is chosen as $\phi(\mathcal{I}) = \mathcal{I}^\sigma$ $(\sigma > -1)$, which is used in the proof of Theorem 1.1, see* (2.7)*. However, to obtain* (2.7)*, other types of state densities also seem available if the following conditions are satisfied: (1) $\phi(\mathcal{I})$ does not grow exponentially, (2) $\mathcal{I}\phi(\mathcal{I}) = 0$ when $\mathcal{I} = 0$, and (3) $\phi(\mathcal{I}) + \mathcal{I}\phi'(\mathcal{I})$ is governed by $(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})$ up to constant multiplication.*

2. Proof of theorem 1.1

The proof is divided into two steps. We first show how the equilibrium parameters are determined by using the identities (1.5) as constraints. And then we prove that γ can be uniquely determined through the nonlinear relation (1.6).

• Choice of n, U^μ, γ: By definition of $Q(f)$, (1.5) is written as

$$\int_{\mathbb{R}^3}\int_0^\infty f_E\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} = \int_{\mathbb{R}^3}\int_0^\infty f\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}, \tag{2.1} \quad \boxed{1}$$

and

$$\int_{\mathbb{R}^3}\int_0^\infty p^\mu f_E\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} = \int_{\mathbb{R}^3}\int_0^\infty p^\mu f\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}. \tag{2.2} \quad \boxed{2}$$

To simplify the integral of f_E, we introduce the Lorentz transformation Λ:

$$\Lambda = \begin{bmatrix} c^{-1}U^0 & -c^{-1}U^1 & -c^{-1}U^2 & -c^{-1}U^3\\ -U^1 & 1+(U^0-1)\frac{(U^1)^2}{|U|^2} & (U^0-1)\frac{U^1U^2}{|U|^2} & (U^0-1)\frac{U^1U^3}{|U|^2}\\ -U^2 & (U^0-1)\frac{U^1U^2}{|U|^2} & 1+(U^0-1)\frac{(U^2)^2}{|U|^2} & (U^0-1)\frac{U^2U^3}{|U|^2}\\ -U^3 & (U^0-1)\frac{U^1U^3}{|U|^2} & (U^0-1)\frac{U^2U^3}{|U|^2} & 1+(U^0-1)\frac{(U^3)^2}{|U|^2}\end{bmatrix}$$

which transforms $U^\mu = (\sqrt{c^2+|U|^2}, U)$ into the local rest frame $(c,0,0,0)$. Applying the change of variables $P^\mu := \Lambda p^\mu$, the left-hand side of (2.1) reduces to

$$\begin{aligned}\int_{\mathbb{R}^3}\int_0^\infty f_E\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} &= \frac{n}{M(\gamma)}\int_{\mathbb{R}^3}\int_0^\infty e^{-\frac{\gamma}{mc^2}(1+\frac{\mathcal{I}}{mc^2})U^\mu p_\mu}\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}\\ &= \frac{n}{M(\gamma)}\int_{\mathbb{R}^3}\int_0^\infty e^{-\frac{\gamma}{mc}(1+\frac{\mathcal{I}}{mc^2})P^0}\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dP}{P^0}\end{aligned}$$

where we used the fact that the volume element $\frac{dp}{p^0}$ and the Lorentz inner product $U^\mu p_\mu$ are invariant under the action of Λ. On the other hand, we observe that

$$(mc)^2 = p^\mu p_\mu = (\Lambda p^\mu)(\Lambda p_\mu) = (P^0) - |P|^2, \quad \text{and hence} \quad P^0 = \sqrt{(mc)^2+|P|^2}. \tag{2.3} \quad \boxed{\texttt{P\^{}0}}$$

Thus one can rewrite P to p, and (2.1) becomes

$$\frac{n}{M(\gamma)}\int_{\mathbb{R}^3}\int_0^\infty e^{-\frac{\gamma}{mc}(1+\frac{\mathcal{I}}{mc^2})p^0}\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} = \int_{\mathbb{R}^3}\int_0^\infty f\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}$$

which gives the relation (1.6). Next, to find n and U^μ, we apply the change of variables $P^\mu := \Lambda p^\mu$ again to (2.2). Then, in a similar way, the left-hand side reduces to

$$\begin{aligned}\int_{\mathbb{R}^3}\int_0^\infty p^\mu f_E \phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} &= \frac{n}{M(\gamma)}\int_{\mathbb{R}^3}\int_0^\infty p^\mu e^{-\frac{\gamma}{mc^2}\left(1+\frac{\mathcal{I}}{mc^2}\right)U^\mu p_\mu}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}\\ &= \frac{n}{M(\gamma)}\int_{\mathbb{R}^3}\int_0^\infty (\Lambda^{-1}P^\mu) e^{-\frac{\gamma}{mc}\left(1+\frac{\mathcal{I}}{mc^2}\right)P^0}\phi(\mathcal{I})\,d\mathcal{I}\frac{dP}{P^0}\\ &= \frac{n}{M(\gamma)}\Lambda^{-1}\int_{\mathbb{R}^3}\int_0^\infty p^\mu e^{-\frac{\gamma}{mc}\left(1+\frac{\mathcal{I}}{mc^2}\right)p^0}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}.\end{aligned}$$

Since $p^0 = \sqrt{(mc)^2+|p|^2}$, it follows from the oddness that

$$\begin{aligned}\int_{\mathbb{R}^3}\int_0^\infty p^\mu e^{-\frac{\gamma}{mc}\left(1+\frac{\mathcal{I}}{mc^2}\right)p^0}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} &= \left(\int_{\mathbb{R}^3}\int_0^\infty e^{-\frac{\gamma}{mc}\left(1+\frac{\mathcal{I}}{mc^2}\right)p^0}\phi(\mathcal{I})\,d\mathcal{I}dp, 0,0,0\right)\\ &= (M(\gamma),0,0,0)\,,\end{aligned}$$

which gives

$$\begin{aligned}\frac{n}{M(\gamma)}\Lambda^{-1}\int_{\mathbb{R}^3}\int_0^\infty p^\mu e^{-\frac{\gamma}{mc}\left(1+\frac{\mathcal{I}}{mc^2}\right)p^0}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} &= \frac{n}{M(\gamma)}\Lambda^{-1}\left(M(\gamma),0,0,0\right)\\ &= \frac{n}{c}\;\Lambda^{-1}(c,0,0,0)\\ &= \frac{1}{c}nU^\mu.\end{aligned}\tag*{(2.4) $\boxed{\texttt{(2)}}$}$$

In the last line, we used the fact that the Lorentz transformation $\Lambda : U^\mu \to (c,0,0,0)$ is invertible and thus $\Lambda^{-1} : (c,0,0,0) \to U^\mu$. On the other hand, the right-hand side of (2.2) can be expressed by (1.3) as

$$\int_{\mathbb{R}^3}\int_0^\infty p^\mu f\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} = \frac{1}{c}n_f U_f^\mu.$$

Therefore, going back to (2.2) with (2.4), we conclude that $n = n_f$ and $U^\mu = U_f^\mu$.

• Unique determination of γ: To prove this part, we will show the existence of a one-to-one correspondence between both sides of (1.6). For this, it suffices to show that (1) $\widetilde{M}/M$ is strictly monotone in $\gamma \in (0,\infty)$, and (2) the ranges of both sides are the same.

(1) Strict monotonicity: Using the change of variables $\frac{p}{mc} \to p$ and the spherical coordinates, one finds

$$\widetilde{M}(\gamma) = 4\pi(mc)^2\int_0^\infty\int_0^\infty \frac{r^2}{\sqrt{1+r^2}}e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\left(1+\frac{\mathcal{I}}{mc^2}\right)^{-1}\phi(\mathcal{I})\,d\mathcal{I}dr =: 4\pi(mc)^2 M_1(\gamma)$$

and

$$M(\gamma) = 4\pi(mc)^3\int_0^\infty\int_0^\infty r^2 e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\,d\mathcal{I}dr =: 4\pi(mc)^3 M_2(\gamma).$$

Also, it is straightforward that

$$\frac{d}{d\gamma}\{\widetilde{M}(\gamma)\} = -4\pi(mc)^2\int_0^\infty\int_0^\infty r^2 e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\,d\mathcal{I}dr = -4\pi(mc)^2 M_2(\gamma)$$

and

$$\frac{d}{d\gamma}\{M(\gamma)\} = -4\pi(mc)^3\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\left(1+\frac{\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})\,d\mathcal{I}dr =: -4\pi(mc)^3 M_3(\gamma).$$

From these observations, we have

$$\frac{d}{d\gamma}\left\{\frac{\widetilde{M}(\gamma)}{M(\gamma)}\right\} = \frac{\widetilde{M}'(\gamma)M(\gamma) - \widetilde{M}(\gamma)M'(\gamma)}{M^2(\gamma)} = \frac{1}{mc}\frac{M_1(\gamma)M_3(\gamma) - \{M_2(\gamma)\}^2}{\{M_2(\gamma)\}^2}\tag*{(2.5) $\boxed{\texttt{M/M}}$}$$

where

$$\begin{aligned}&M_1(\gamma)M_3(\gamma) - \{M_2(\gamma)\}^2\\ &= \int_0^\infty\int_0^\infty \frac{r^2}{\sqrt{1+r^2}}e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\frac{1}{1+\frac{\mathcal{I}}{mc^2}}\phi(\mathcal{I})\,d\mathcal{I}dr\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\left(1+\frac{\mathcal{I}}{mc^2}\right)d\mathcal{I}dr\\ &- \left(\int_0^\infty\int_0^\infty r^2 e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\,d\mathcal{I}dr\right)^2.\end{aligned}$$

Therefore, since the integrands above are linearly independent, by Hölder's inequality we conclude that $\widetilde{M}/M$ is strictly increasing in $\gamma \in (0,\infty)$.
(2) Ranges: It follows from (1.4) and the change of variables $P^\beta := \Lambda p^\beta$ that

$$
\begin{aligned}
\frac{1}{n_f}\int_{\mathbb{R}^3}\int_0^\infty f\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0} &= c\frac{\int_{\mathbb{R}^3}\int_0^\infty f\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dp}{p^0}}{\int_{\mathbb{R}^3}\int_0^\infty U_{f\mu}p^\mu f\phi(\mathcal{I})\,d\mathcal{I}\,\frac{dp}{p^0}}\\
&= \frac{\int_{\mathbb{R}^3}\int_0^\infty f_\Lambda\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\frac{dP}{P^0}}{\int_{\mathbb{R}^3}\int_0^\infty f_\Lambda\phi(\mathcal{I})\,d\mathcal{I}\,dP}\\
&= \frac{1}{mc}\frac{\int_{\mathbb{R}^3}\int_0^\infty \frac{1}{\sqrt{1+|\frac{P}{mc}|^2}}f_\Lambda\Big(1+\frac{\mathcal{I}}{mc^2}\Big)^{-1}\phi(\mathcal{I})\,d\mathcal{I}\,dP}{\int_{\mathbb{R}^3}\int_0^\infty f_\Lambda\phi(\mathcal{I})\,d\mathcal{I}\,dP}
\end{aligned}
$$

where $f_\Lambda := f(x^\alpha,\Lambda^{-1}P^\beta)$ and we used (2.3) in the last line. Since the above is strictly less than $\frac{1}{mc}$, it only remains to show that Range$(\widetilde{M}/M)=(0,\frac{1}{mc})$. For this, we employ the integration by parts with respect to r to see

$$
\begin{aligned}
\int_0^\infty\int_0^\infty r^2e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\,d\mathcal{I}dr &= \frac{1}{\gamma}\int_0^\infty\int_0^\infty\left\{\sqrt{1+r^2}+\frac{r^2}{\sqrt{1+r^2}}\right\}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\frac{1}{1+\frac{\mathcal{I}}{mc^2}}\phi(\mathcal{I})\,d\mathcal{I}dr\\
&\geq \frac{1}{\gamma}\int_0^\infty\int_0^\infty\frac{r^2}{\sqrt{1+r^2}}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\frac{1}{1+\frac{\mathcal{I}}{mc^2}}\phi(\mathcal{I})\,d\mathcal{I}dr,
\end{aligned}
$$

which combined with (2.5) gives

$$
\frac{\widetilde{M}(\gamma)}{M(\gamma)}\leq\frac{\gamma}{mc},\quad \text{and hence}\quad \frac{\widetilde{M}(\gamma)}{M(\gamma)}\to 0\ \text{ as }\ \gamma\to 0.
$$

On the other hand, it follows from the Hölder inequality that

$$
\begin{aligned}
\frac{\widetilde{M}(\gamma)}{M(\gamma)} &= \frac{1}{mc}\frac{\int_0^\infty\int_0^\infty\frac{r^2}{\sqrt{1+r^2}}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\frac{1}{1+\frac{\mathcal{I}}{mc^2}}\phi(\mathcal{I})\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\,d\mathcal{I}dr}\\
&\geq \frac{1}{mc}\left(\frac{\int_0^\infty\int_0^\infty\frac{r^2}{\sqrt{1+r^2}}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\frac{1}{1+\frac{\mathcal{I}}{mc^2}}\phi(\mathcal{I})\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr}\right)^{1/2}
\end{aligned}
$$

Since $\frac{1}{1+x}\geq(1+x)-2x$ for all $x\geq 0$, we obtain

$$
\begin{aligned}
\frac{\widetilde{M}(\gamma)}{M(\gamma)} \geq \frac{1}{mc}\Bigg(&\frac{\int_0^\infty\int_0^\infty\frac{r^2}{\sqrt{1+r^2}}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr}\\
&-\frac{2}{mc^2}\frac{\int_0^\infty\int_0^\infty\frac{r^2}{\sqrt{1+r^2}}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\mathcal{I}\phi(\mathcal{I})\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr}\Bigg)^{1/2}\\
=:&\ \frac{1}{mc}(I-II)^{\frac12}.
\end{aligned}
\tag{2.6}
$$

I+II

For I, we use the identity

$$
\frac{1}{\sqrt{1+r^2}}=\sqrt{1+r^2}-\frac{r^2}{\sqrt{1+r^2}}
$$

to obtain

$$
I = 1-\frac{\int_0^\infty\int_0^\infty\frac{r^4}{\sqrt{1+r^2}}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr}.
$$

Applying the integration by parts with respect to r, we get

$$
\begin{aligned}
I &= 1-\frac{3}{\gamma}\frac{\int_0^\infty\int_0^\infty r^2e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-(1+\frac{\mathcal{I}}{mc^2})\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr}\\
&\geq 1-\frac{3}{\gamma}.
\end{aligned}
$$

For II, since we have set $\phi(\mathcal{I}) = \mathcal{I}^\sigma$ $(\sigma > -1)$, applying the integration by parts with respect to $\mathcal{I}$ gives

$$\begin{aligned} II &= \frac{2}{mc^2}\frac{\int_0^\infty\int_0^\infty \frac{r^2}{\sqrt{1+r^2}} e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\mathcal{I}\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr} \\ &= \frac{2(\sigma+1)}{\gamma}\frac{\int_0^\infty\int_0^\infty \frac{r^2}{1+r^2} e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}\phi(\mathcal{I})\,d\mathcal{I}dr}{\int_0^\infty\int_0^\infty r^2\sqrt{1+r^2}e^{-\left(1+\frac{\mathcal{I}}{mc^2}\right)\gamma\sqrt{1+r^2}}(1+\frac{\mathcal{I}}{mc^2})\phi(\mathcal{I})\,d\mathcal{I}dr} \\ &\le \frac{2(\sigma+1)}{\gamma}. \end{aligned} \tag{2.7}$$

phi lower

Going back to (2.6) with estimates of I and II, we conclude that

$$\frac{\widetilde{M}(\gamma)}{M(\gamma)} \ge \frac{1}{mc}\left(1-\frac{3}{\gamma}-\frac{2(\sigma+1)}{\gamma}\right)^{\frac{1}{2}}, \qquad \text{and hence} \qquad \frac{\widetilde{M}(\gamma)}{M(\gamma)} \to \frac{1}{mc} \quad \text{as} \quad \gamma\to\infty.$$

This completes the proof.

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