Dynamic extensions of arrow logic

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Abstract

This paper is devoted to the complete axiomatization of dynamic extensions of arrow logic based on a restriction of propositional dynamic logic with intersection. Our deductive systems contain an unorthodox inference rule: the inference rule of intersection. The proof of the completeness of our deductive systems uses the technique of the canonical model.

Key words: Modal logic, dynamic logic, arrow logic.

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1 Introduction

Within the framework of the research carried out into the subject of arrow logic, arrows may be considered as abstract objects equipped with the ternary relation of composition C, the binary relation of converse R and the unary relation of identity I. Intuitively speaking $C(x, y, z)$ means that arrow x is a composition of arrows y and z, $R(x, y)$ means that arrow x is a converse of arrow y and $I(x)$ means that arrow x is an identity arrow. In this approach arrows have no explicitly stated internal structure, seeing that one uses arrow frames of the form (W, C, R, I) where W is a nonempty set of arrows, C is a ternary relation on W, R is a binary relation on W and I is a unary relation on W. These frames constitute the semantical basis of a propositional modal logic with the binary modality $\bullet$, the unary modality $\otimes$ and the nullary modality id corresponding to the relations of composition, converse and identity between arrows. Venema [20] gives an extensive introduction to this approach.

Another approach is possible and consists in considering arrows as concrete objects with a beginning and an end. In this approach arrows have an explicitly stated internal structure, for the simple reason that one uses arrow frames of the form (Ar, Po, f) where Ar is a nonempty set of arrows, Po is a nonempty set of points and f is a function from $Ar \times \{1, \ldots, n\}$ to Po. Intuitively speaking $f(x, 1)$ defines the beginning point of arrow x, $f(x, 2)$, $\ldots$, $f(x, n-1)$ define the intermediate points of arrow x and $f(x, n)$ defines the end point of arrow x. Using the function f one can define the following binary relations between arrows: for all $i, j \in \{1, \ldots, n\}$, let $xR(i, j)y$ iff $f(x, i) = f(y, j)$. These binary relations explore the different possibilities for two arrows to share points and are used as the semantical basis of a propositional modal logic with the unary modalities $[\pi_{i,j}]$ corresponding to the binary relations $R(i, j)$ between arrows. The first propositional modal logic of this type is introduced by Vakarelov [17,18] who shows that the interest to consider the binary relations $R(i, j)$ between arrows stems from the fact that the first-order conditions which characterize them are modally definable. In other respects it appears that, in the particular case where $n = 2$, the relations of composition, converse and identity have simple definitions in terms of the relations $R(i, j)$: $C(x, y, z)$ iff $xR(1, 1)y$, $xR(2, 2)z$ and $yR(2, 1)z$, $R(x, y)$ iff $xR(1, 2)y$ and $xR(2, 1)y$, $I(x)$ iff $xR(1, 2)x$. Moreover, in the general case where $n \geq 3$, one may consider the extension of arrow logic with the modalities $[\pi_i]$ corresponding to the intersection of the binary relations $R(j, j)$ for all $j \in \{1, \ldots, n\}$ such that $i \neq j$. However neither the modalities $\bullet$, $\otimes$ and id, in the particular case where $n = 2$, nor the modalities $[\pi_i]$, in the general case where $n \geq 3$, are definable in terms of the modalities $[\pi_{i,j}]$. Nevertheless these modalities are definable in a dynamic extension of arrow logic. Our dynamic extension of arrow logic is an iteration-free propositional dynamic logic with intersection, the atomic programs of which correspond to the binary relations $R(i, j)$ between arrows. This paper is devoted to its complete axiomatization. Our deductive systems contain an unorthodox inference rule: the inference rule of intersection. The proof of the completeness of our deductive systems uses the technique of the canonical model. The plan of the paper is as follows. Arrow structures and arrow frames are introduced in section 2. Section 3 presents the basic arrow logic as well as some of its extensions. The iteration-free propositional dynamic logic with intersection is presented in section 4. Our dynamic extension of arrow logic uses the concepts defined in section 3 and section 4. It is introduced and developed in section 5.

2 Arrow structures and arrow frames

Adapted from Vakarelov [18], an *arrow structure* will be any structure of the form $S = (Ar, Po, f)$ where:

- Ar is a nonempty set of arrows;
- Po is a nonempty set of points;
- f is a function with domain $Ar \times \{1, \ldots, n\}$ and range Po.

Arrow structure $S = (Ar, Po, f)$ will be defined to be *normal* if for all $x, y \in Ar$:

- If $f(x, 1) = f(y, 1)$, …, $f(x, n) = f(y, n)$ then $x = y$.

Arrow structures constitute the starting point for the formal examination of the relationships that reflect the ways arrows share points. Given any arrow structure $S = (Ar, Po, f)$, the *arrow frame derived from* S is the structure of the form $F_S = (W_S, \{R_S(i, j)\colon i, j \in \{1, \ldots, n\}\})$ where:

- $W_S = Ar$;
- For all $i, j \in \{1, \ldots, n\}$, $R_S(i, j)$ is the binary relation on W_S defined as follows for all $x, y \in W_S$:
  - $xR_S(i, j)y$ iff $f(x, i) = f(y, j)$.

We leave it to the reader to prove the following result.

Proposition 1 *Let $S = (Ar, Po, f)$ be an arrow structure and $F_S = (W_S, \{R_S(i, j)\colon i, j \in \{1, \ldots, n\}\})$ be the arrow frame derived from S. For all $i, j, k \in \{1, \ldots, n\}$ and for all $x, y, z \in W_S$:*

T(i) *$xR_S(i, i)x$;*
B(i,j) *If $xR_S(i, j)y$ then $yR_S(j, i)x$;*
4(i,j,k) *If $xR_S(i, j)y$ and $yR_S(j, k)z$ then $xR_S(i, k)z$.*

Moreover, if S is normal then for all $x, y \in W_S$:

($\star$) *If $xR_S(1, 1)y$, …, $xR_S(n, n)y$ then $x = y$.*

Proposition 1 motivates the following definitions. An *arrow frame* is a structure of the form $F = (W, \{R(i, j)\colon i, j \in \{1, \ldots, n\}\})$ where:

- W is a nonempty set of arrows;
- For all $i, j \in \{1, \ldots, n\}$, $R(i, j)$ is a binary relation on W;
- For all $i, j, k \in \{1, \ldots, n\}$ and for all $x, y, z \in W$, the conditions **T(i)**, **B(i,j)** and **4(i,j,k)** of proposition 1 are satisfied.

Arrow frame $F = (W, \{R(i, j)\colon i, j \in \{1, \ldots, n\}\})$ will be defined to be *normal* if for all $x, y \in W$, the condition ($\star$) of proposition 1 is satisfied. An important step in the study of arrow frames is their representability. Generalizing the logical foundations of coincidence relations in arrow structures developed by Vakarelov [17], Vakarelov [18] was the first to prove the following result.

Proposition 2 *(Characterization theorem for arrow frames) Let* $F = (W, \{R(i,j)\colon i,j \in \{1,\ldots,n\}\})$ *be an arrow frame. There is an arrow structure* $S = (Ar, Po, f)$ *such that the arrow frame* $F_S = (W_S, \{R_S(i,j)\colon i,j \in \{1,\ldots,n\}\})$ *derived from* S *is isomorphic to* F*. Moreover, if* F *is normal then* S *is normal.*

Proposition 2 suggests defining propositional modal logics with standard interpretation in arrow frames.

3 Basic arrow logic

The most natural way to define propositional modal logics with standard interpretation in arrow frames is to extend the language of propositional classical logic with the modalities $[\pi_{i,j}]$ corresponding to the binary relations $R(i,j)$. The first propositional modal logics of this type were given by Vakarelov [17] in the particular case where $n = 2$ and Vakarelov [18] in the general case where $n \geq 3$.

3.1 Syntax

The set of all *formulas* is defined as follows:

- $\phi ::= p \mid \bot \mid (\phi \to \psi) \mid [\pi_{i,j}]\phi$;

where p ranges over a countably infinite set of *propositional variables* and i, j range over the set $\{1,\ldots,n\}$. We will use ϕ, ψ, χ, etc, for formulas. It is well worth noting that each formula is a finite string of symbols, these symbols coming from a countable alphabet. It follows that there are countably many formulas. Other connectives are introduced by the usual abbreviations. We shall agree to use the most readable notation for formulas. This permits us to adopt the standard rules for omission of the parentheses.

3.2 Semantics

The standard semantics for this language is a Kripke-style semantics over arrow frames. Let $F = (W, \{R(i,j)\colon i,j \in \{1,\ldots,n\}\})$ be an arrow frame. A function V with domain the set of all propositional variables and range the set of all subsets of W will be defined to be a *valuation on* F. The pair $M = (F, V)$ is called the *model over* F *defined from* V. We define the relation "formula ϕ is true at arrow x in model M", denoted $M, x \models \phi$, as follows:

- $M, x \models p$ iff $x \in V(p)$;
- $M, x \not\models \bot$;
- $M, x \models \phi \to \psi$ iff if $M, x \models \phi$ then $M, x \models \psi$;
- $M, x \models [\pi_{i,j}]\phi$ iff for all $y \in W$, if $xR(i,j)y$ then $M, y \models \phi$.

Formula ϕ is *true in model* M, denoted $M \models \phi$, if for all $x \in W$, $M, x \models \phi$. Formula ϕ is *true in arrow frame* F, denoted $F \models \phi$, if ϕ is true in all models over F. Formula ϕ is *true in a set* Σ *of arrow frames*, denoted $\Sigma \models \phi$, if ϕ is true in all arrow frames of Σ. The set of all formulas true in a set Σ of arrow frames is denoted $L(\Sigma)$. Let us introduce the following sets of arrow frames:

- $ARROW^n$ is the set of all arrow frames;
- $ARROW^n NOR$ is the set of all normal arrow frames;
- $ARROW^n FIN$ is the set of all finite arrow frames;
- $ARROW^n FINNOR$ is the set of all finite normal arrow frames.

3.3 Axiomatization

Let BAL^n be the smallest normal logic that contains the axioms given below:

T(i) $\phi \to \langle\pi_{i,i}\rangle\phi$;
B(i,j) $\phi \to [\pi_{i,j}]\langle\pi_{j,i}\rangle\phi$;
4(i,j,k) $\langle\pi_{i,j}\rangle\langle\pi_{j,k}\rangle\phi \to \langle\pi_{i,k}\rangle\phi$.

A formula ϕ is called *provable in* BAL^n, denoted $\vdash_{BAL^n} \phi$, if ϕ belongs to BAL^n.

3.4 Completeness

The proof of the following completeness theorem for BAL^n uses general techniques that can be found in most elementary texts.

Theorem 3 *(Completeness theorem for BAL^n) Let ϕ be a formula. The following conditions are equivalent:*

- $\vdash_{BAL^n} \phi$;
- $\phi \in L(ARROW^n)$;
- $\phi \in L(ARROW^n NOR)$;
- $\phi \in L(ARROW^n FIN)$;
- $\phi \in L(ARROW^n FINNOR)$.

Completeness theorem for BAL^n was first shown by Vakarelov [17] in the particular case where $n = 2$ and Vakarelov [18] in the general case where

$n \geq 3$.

3.5 Extensions of basic arrow logic

The language of our propositional modal logics can be extended in different ways. In the particular case where $n = 2$, we can think of a set of arrows as a device which produces an output for any input. Then, for all arrow frames $F = (W, \{R(i,j) \colon i, j \in \{1,2\}\})$, the following operations on sets of arrows are defined for all subsets σ, τ of W:

Composition: $\sigma \bullet \tau = \{x \colon x \in W$ and there are $y, z \in W$ such that $xR(1,1)y$, $xR(2,2)z$, $yR(2,1)z$, $y \in \sigma$ and $z \in \tau\}$;
Converse: $\otimes\sigma = \{x \colon x \in W$ and there is $y \in W$ such that $xR(1,2)y$, $xR(2,1)y$ and $y \in \sigma\}$;
Identity: $id = \{x \colon x \in W$ and $xR(1,2)x\}$.

Within this context, it is natural to consider the extension of BAL^2 with the modalities $\bullet$, $\otimes$ and id corresponding to the operations of composition, converse and identity on sets of arrows in arrow frames. To be more precise, the set of all *formulas* of the extended language is defined as follows:

- $\phi ::= p \mid \bot \mid (\phi \to \psi) \mid [\pi_{i,j}]\phi \mid (\phi \bullet \psi) \mid \otimes\phi \mid id$;

where the semantics of the new modalities is defined as follows:

- $M, x \models \phi \bullet \psi$ iff there is $y, z \in W$ such that $xR(1,1)y$, $xR(2,2)z$, $yR(2,1)z$, $M, y \models \phi$ and $M, z \models \psi$;
- $M, x \models \otimes\phi$ iff there is $y \in W$ such that $xR(1,2)y$, $xR(2,1)y$ and $M, y \models \phi$;
- $M, x \models id$ iff $xR(1,2)x$.

The extension of BAL^2 with the modalities $\bullet$, $\otimes$ and id has been considered by Arsov [1], Arsov and Marx [2] and Marx [14]. In the general case where $n \geq 3$, Vakarelov [18] has considered the extension of BAL^n with the modalities $[\pi_i]$ corresponding for all arrow frames $F = (W, \{R(i,j) \colon i, j \in \{1,2\}\})$ to the intersection of the binary relations $R(j,j)$ on W for all $j \in \{1, \ldots, n\}$ such that $i \neq j$. Formally, the set of all *formulas* of the extended language is defined as follows:

- $\phi ::= p \mid \bot \mid (\phi \to \psi) \mid [\pi_{i,j}]\phi \mid [\pi_i]\phi$;

where the semantics of the new modalities is defined as follows:

- $M, x \models [\pi_i]\phi$ iff for all $y \in W$, if $xR(j,j)y$ for all $j \in \{1, \ldots, n\}$ such that $i \neq j$, then $M, y \models \phi$.

Let us be clear that neither the modalities $\bullet$, $\otimes$ and id, in the particular case where $n = 2$, nor the modalities $[\pi_i]$, in the general case where $n \geq 3$, are modally definable in the basic language of arrow logic. However, these modalities will become definable in our dynamic extension of arrow logic, an iteration-free PDL with intersection the atomic programs of which correspond for all arrow frames $F = (W, \{R(i, j)\colon i, j \in \{1, 2\}\})$ to the binary relations $R(i, j)$.

4 Iteration-free PDL with intersection

Propositional dynamic logic, PDL, is a polymodal logic with the following operations in the set of modalities:

- Composition $\alpha; \beta$: sequential execution of programs α and β corresponding to the composition of the accessibility relations $R(\alpha)$ and $R(\beta)$;
- Disjunction $\alpha \vee \beta$: nondeterministic choice of programs α and β corresponding to the union of $R(\alpha)$ and $R(\beta)$;
- Iteration $\alpha^\star$: nondeterministic iteration of program α corresponding to the transitive closure of $R(\alpha)$;
- Test ϕ?: an operation transforming the formula ϕ into a program ϕ? corresponding to the partial identity relation in the states of the PDL-models where the formula ϕ is true.

Balbiani and Vakarelov [6] were the first to propose a complete axiomatization of iteration-free PDL with intersection, an extension of the iteration-free fragment of PDL with the following operation in the set of modalities:

- Intersection $\alpha \wedge \beta$: conjunction of programs α and β corresponding to the intersection of $R(\alpha)$ and $R(\beta)$.

The question of the complete axiomatization of iteration-free PDL with intersection lies outside the scope of this paper. However we present the line of reasoning suggested by Balbiani and Vakarelov [6], because we will follow the same line of reasoning with respect to the complete axiomatization of our dynamic extension of arrow logic.

4.1 Syntax

We now give a formal definition of the syntax of iteration-free PDL with intersection, $PDL_0^\cap$. The set of all *formulas* and the set of all *programs* of the language of $PDL_0^\cap$ are defined as follows:

- $\phi ::= p \mid \bot \mid (\phi \to \psi) \mid [\alpha]\phi$;
- $\alpha ::= \pi \mid (\alpha; \beta) \mid (\alpha \vee \beta) \mid (\alpha \wedge \beta) \mid \phi?$;

where p ranges over a countably infinite set of *propositional variables* and π ranges over a countable set of *atomic programs*. We will use α, β, γ, etc, for programs. The method developed by Balbiani and Vakarelov [6] uses a special inference rule, the inference rule of intersection. For its definition, the concept of *admissible form* will be of use to us. Each admissible form has a positive integer as a *rank* and the definition of admissible forms is by induction on the rank. Let the syntax be extended with a new propositional variable $\sharp$. If $\alpha(\sharp?)$ is a program with a unique occurrence of the test $\sharp?$ as a part of it then for all formulas ϕ, $\alpha(\phi?)$ will denote the program obtained as the result of the replacement of the propositional variable $\sharp$ in its place in $\alpha(\sharp?)$ with the formula ϕ. The admissible forms are defined as follows:

- For all programs $\alpha(\sharp?)$ with a unique occurrence of the test $\sharp?$ as a subprogram, $\alpha(\sharp?)$ is an admissible form of rank 0;
- For all positive integers a, for all programs $\alpha(\sharp?)$ with a unique occurrence of the test $\sharp?$ as a subprogram, for all admissible forms $\beta(\sharp?)$ of rank a and for all formulas ϕ with no occurrence of the propositional variable $\sharp$ as a part of it, $\alpha(\neg[\beta(\sharp?)]\phi?)$ is an admissible form of rank $a + 1$.

Note that each admissible form $\alpha(\sharp?)$ contains a unique occurrence of the test $\sharp?$ as a part of it. What is more, test $\sharp?$ occurs as a subprogram of admissible form $\alpha(\sharp?)$ only if $\alpha(\sharp?)$ is of rank 0.

4.2 Semantics

The standard semantics for the language of $PDL_0^{\cap}$ uses the concept of PDL-*frame*, i.e., structures of the form $F = (W, R)$ where W is a nonempty set of states and R is a function with domain the set of all atomic programs and range the set of all binary relations on W. A function V with domain the set of all propositional variables and range the set of all subsets of W will be called *valuation on* F. We shall say that the pair $M = (F, V)$ is the PDL-*model over* F *defined from* V. The relation "formula ϕ is true at state x in PDL-model M", denoted $M, x \models \phi$, is inductively defined as follows:

- $M, x \models p$ iff $x \in V(p)$;
- $M, x \not\models \bot$;
- $M, x \models \phi \to \psi$ iff if $M, x \models \phi$ then $M, x \models \psi$;
- $M, x \models [\alpha]\phi$ iff for all $y \in W$, if $x\overline{R}(\alpha)y$ then $M, y \models \phi$;

where the binary relations $\overline{R}(\alpha)$ on W corresponding to the modalities $[\alpha]$ reflect the intended meanings of programs α:

- $\overline{R}(\pi) = R(\pi)$;
- $\overline{R}(\alpha;\beta) = \overline{R}(\alpha) \circ \overline{R}(\beta)$;
- $\overline{R}(\alpha \vee \beta) = \overline{R}(\alpha) \cup \overline{R}(\beta)$;
- $\overline{R}(\alpha \wedge \beta) = \overline{R}(\alpha) \cap \overline{R}(\beta)$;
- $\overline{R}(\phi?) = \{(x,x)\colon M,x \models \phi\}$.

Formula ϕ is *true in PDL-model M*, denoted $M \models \phi$, if for all $x \in W$, $M,x \models \phi$. Formula ϕ is *true in PDL-frame F*, denoted $F \models \phi$, if ϕ is true in all PDL-models over F. Formula ϕ is *true in a set Σ of PDL-frames*, denoted $\Sigma \models \phi$, if ϕ is true in all PDL-frames of Σ. The set of all formulas true in a set Σ of PDL-frames is denoted $L(\Sigma)$. Let PDL be the set of all PDL-frames.

4.3 Axiomatization

The axiomatic system for $PDL_0^{\cap}$ developed by Balbiani and Vakarelov [6] is the smallest normal logic in the language of $PDL_0^{\cap}$ that contains all the instances of the following axiom schemata:

- $\langle\alpha;\beta\rangle\phi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\phi$;
- $\langle\alpha \vee \beta\rangle\phi \leftrightarrow \langle\alpha\rangle\phi \vee \langle\beta\rangle\phi$;
- $\langle\alpha \wedge \beta\rangle\phi \rightarrow \langle\alpha\rangle\phi \wedge \langle\beta\rangle\phi$;
- $\langle\alpha \wedge (\beta \vee \gamma)\rangle\phi \leftrightarrow \langle\alpha \wedge \beta\rangle\phi \vee \langle\alpha \wedge \gamma\rangle\phi$;
- $\langle\phi?\rangle\psi \leftrightarrow \phi \wedge \psi$;

and is closed under the following inference rule:

- If for all propositional variables p, $\vdash_{PDL_0^{\cap}} [\alpha(\neg(\langle\beta\rangle(\phi \wedge p) \vee \langle\gamma\rangle(\phi \wedge \neg p))?)]\psi$ then $\vdash_{PDL_0^{\cap}} [\alpha(\neg\langle\beta \wedge \gamma\rangle\phi?)]\psi$;

where $\alpha(\sharp?)$ is an admissible form. Following standard usage, $\vdash_{PDL_0^{\cap}} \phi$ means that ϕ is a theorem of $PDL_0^{\cap}$. The inference rule of intersection is similar in some sense to the inference rules considered by Goranko [11] and Venema [19]. It is based on the following idea. Although intersection of programs is not definable in ordinary quantifier-free polymodal logics, intersection of programs becomes definable in polymodal logics with quantification over propositional variables. In the language of $PDL_0^{\cap}$, the inference rule of intersection simulates this definition of the operation of intersection. We conclude this section by the soundness theorem of $PDL_0^{\cap}$.

Theorem 4 *(Soundness theorem of $PDL_0^{\cap}$) Let ϕ be a formula. If $\vdash_{PDL_0^{\cap}} \phi$ then $\phi \in L(PDL)$.*

PROOF. See Balbiani and Vakarelov [6].

4.4 Completeness

We now want to see that every formula true in the set of all PDL-frames is provable in $PDL_0^{\cap}$. The general technique of the canonical model has to be modified in many details for our situation, for the simple reason that $PDL_0^{\cap}$ is closed for the inference rule of intersection, an infinitary rule of inference. In this respect, Balbiani and Vakarelov [6] introduced the concept of *theory*. Following the line of reasoning suggested by Goldblatt [9,10] within his framework for infinitary modal logic, a set S of formulas is said to be a theory if it satisfies the following conditions:

- If $\vdash_{PDL_0^{\cap}} \phi$ then $\phi \in S$;
- If $\phi \in S$ and $\phi \to \psi \in S$ then $\psi \in S$;
- If for all propositional variables p, $[\alpha(\neg(\langle\beta\rangle(\phi \wedge p) \vee \langle\gamma\rangle(\phi \wedge \neg p))?)]\psi \in S$ then $[\alpha(\neg\langle\beta \wedge \gamma\rangle\phi?)]\psi \in S$;

where $\alpha(\sharp?)$ is an admissible form, i.e., a theory is any set of formulas that contains every formula provable in $PDL_0^{\cap}$ and is closed under the inference rule of modus ponens and the inference rule of intersection. We will use S, T, U, etc, for theories. A theory S is called *consistent* if $\perp \notin S$. By a *maximal theory* we mean a consistent theory S such that for all formulas ϕ, $\phi \in S$ or $\neg\phi \in S$. The method of the canonical model uses the following important lemma.

Lemma 5 *(Lindenbaum's lemma for $PDL_0^{\cap}$) Let S be a consistent theory. There is a maximal theory T such that $S \subseteq T$.*

PROOF. See Balbiani and Vakarelov [6].

The *canonical frame for $PDL_0^{\cap}$* is the PDL-frame $F_c = (W_c, R_c)$ where W_c is the set of all maximal theories and R_c is the function with domain the set of all atomic programs and range the set of all binary relations on W_c such that for all atomic programs π and for all maximal theories S, T, $SR_c(\pi)T$ iff $[\pi]S = \{\phi\colon [\pi]\phi \in S\} \subseteq T$. The *canonical model for $PDL_0^{\cap}$* is the PDL-model over F_c defined from the valuation V_c on F_c such that for all propositional variables p, $V_c(p) = \{S\colon p \in S\}$. The next lemma is the fundamental lemma for canonical models.

Lemma 6 *(Truth lemma for $PDL_0^{\cap}$) Let ϕ be a formula. For all maximal theories S, the following conditions are equivalent:*

- *$M_c, S \models \phi$;*
- *$\phi \in S$.*

PROOF. For normal modal logics such as K, T, $S4$, etc, the proof of the truth lemma can be done by induction on the complexity of ϕ. To prove the truth lemma for $PDL_0^\cap$, we have to use in parallel an additional induction. This additional induction involves the new concept of maximal program. Like maximal theories, which are special sets of formulas, maximal programs are special sets of programs, with a precise definition that lies outside the scope of this paper. The interested reader is invited to consult the paper of Balbiani and Vakarelov [6] for details.

We are now ready for the completeness theorem of $PDL_0^\cap$.

Theorem 7 *(Completeness theorem of $PDL_0^\cap$) Let ϕ be a formula. If $\phi \in L(PDL)$ then $\vdash_{PDL_0^\cap} \phi$.*

PROOF. By lemmas 5 and 6.

We have no idea whether the resort to the inference rule of intersection is necessary. In other words, we do not know whether the set of all formulas true in the set of all PDL-frames is finitely axiomatizable by an orthodox derivation system or not. Orthodox completeness results for a syntactically restricted version of $PDL_0^\cap$ in which programs are built up from atomic programs by means of the operations of composition and intersection are given in [3,5]. The completeness proof treated in Balbiani [3] draws from the subordination method of Hughes and Cresswell [13] whereas Balbiani and Fariñas del Cerro [5] base their line of reasoning on a suitable modification of the mosaic method of Marx and Venema [15]. Both arguments consist in a step-by-step method for constructing irreflexive models. This brings us to the question of whether the proofs in [3,5] can be extended in the presence of tests, a question that remains unsolved, although [4] brings new ideas that may lead to a positive answer. Additional topics related to $PDL_0^\cap$, which space does not permit us to discuss in depth, include the decidability/complexity issue of the satisfiability problem. Decidability of the satisfiability problem for PDL with intersection — $PDL^\cap$ — is proved in Danecki [8], but it is not known at present whether the upper bound of deterministic exponential-time obtained for PDL in [16] carries over to $PDL^\cap$. Hence, the satisfiability problem for $PDL_0^\cap$ is decidable. However, there is no known results concerning its inherent complexity. The intersection operator is also investigated in Harel [12] which gives the proof that the satisfiability problem for $PDL^\cap$ is undecidable if the semantics is modified so as to refer only to deterministic programs variables.

5 Dynamic extensions of arrow logic

In this section, we shall suppose that the set of atomic programs is equal to the set $\{\pi_{i,j}\colon i,j \in \{1,\ldots,n\}\}$, these atomic programs corresponding for all arrow frames $F = (W, \{R(i,j)\colon i,j \in \{1,2\}\})$, to the binary relations $R(i,j)$. A number of constructs can be defined from them such as the constructs $\bullet$, $\otimes$ and *id* considered in [1,2,14] or the constructs π_i considered in [18].

5.1 Syntax

The language of $BAL^nPDL_0^\cap$ is obtained from the language of $PDL_0^\cap$ by supposing that the set of *atomic programs* is equal to the set $\{\pi_{i,j}\colon i,j \in \{1,\ldots,n\}\}$. Hence, the set of all *formulas* and the set of all *programs* of the language of $BAL^nPDL_0^\cap$ are defined as follows:

- $\phi ::= p \mid \bot \mid (\phi \to \psi) \mid [\alpha]\phi$;
- $\alpha ::= \pi_{i,j} \mid (\alpha;\beta) \mid (\alpha \vee \beta) \mid (\alpha \wedge \beta) \mid \phi?$;

where p ranges over a countably infinite set of *propositional variables* and i,j range over the set $\{1,\ldots,n\}$. What gives our language its interest is the possibility of defining modalities that are not definable in the language of basic arrow logic. To illustrate the truth of this, one can consider formulas like $\langle(\pi_{1,1};\phi?;\pi_{2,1}) \wedge \pi_{2,2}\rangle\psi$, $\langle\pi_{1,2} \wedge \pi_{2,1}\rangle\phi$ and $\langle\pi_{1,2} \wedge \top?\rangle\top$, in the particular case where $n = 2$, or formulas like $[\pi_{1,1} \wedge \ldots \pi_{i-1,i-1} \wedge \pi_{i+1,i+1} \wedge \ldots \wedge \pi_{n,n}]\phi$, in the general case where $n \geq 3$. The former formulas correspond to the formulas $\phi \bullet \psi$, $\otimes\phi$ and *id* considered by Arsov [1], Arsov and Marx [2] and Marx [14] whereas the latter formulas correspond to the formulas $[\pi_i]\phi$ considered by Vakarelov [18].

5.2 Semantics

The standard semantics for this language is a Kripke-style semantics over arrow frames. Let $F = (W, \{R(i,j)\colon i,j \in \{1,\ldots,n\}\})$ be an arrow frame, V be a valuation on F and $M = (F,V)$ be the model over F defined from V. We define the relation "formula ϕ is true at arrow x in model M", denoted $M,x \models \phi$, as follows:

- $M,x \models p$ iff $x \in V(p)$;
- $M,x \not\models \bot$;
- $M,x \models \phi \to \psi$ iff if $M,x \models \phi$ then $M,x \models \psi$;
- $M,x \models [\alpha]\phi$ iff for all $y \in W$, if $x\overline{R}(\alpha)y$ then $M,y \models \phi$;

where the binary relations $\overline{R}(\alpha)$ on W corresponding to the modalities $[\alpha]$ reflect the intended meanings of programs α:

- $\overline{R}(\pi_{i,j}) = R(i,j)$;
- $\overline{R}(\alpha;\beta) = \overline{R}(\alpha) \circ \overline{R}(\beta)$;
- $\overline{R}(\alpha \vee \beta) = \overline{R}(\alpha) \cup \overline{R}(\beta)$;
- $\overline{R}(\alpha \wedge \beta) = \overline{R}(\alpha) \cap \overline{R}(\beta)$;
- $\overline{R}(\phi?) = \{(x,x)\colon M, x \models \phi\}$.

The notions of *truth in a model*, *truth in an arrow frame* and *truth in a set of arrow frames* are those defined in the section devoted to basic arrow logic. As a result, $L(ARROW^n)$ and $L(ARROW^n NOR)$ will respectively denote the set of all formulas true in the set of all arrow frames and the set of all formulas true in the set of all normal arrow frames.

5.3 Axiomatization

What we have in mind is to propose complete axiomatic systems for $L(ARROW^n)$ and $L(ARROW^n NOR)$. Concerning the axiomatization of $L(ARROW^n)$, let $BAL^n PDL_0^{\cap}$ be the smallest normal logic in our extended language that contains all the instances of the following axiom schemata:

- $\phi \to \langle \pi_{i,i} \rangle \phi$;
- $\phi \to [\pi_{i,j}]\langle \pi_{j,i} \rangle \phi$;
- $\langle \pi_{i,j} \rangle \langle \pi_{j,k} \rangle \phi \to \langle \pi_{i,k} \rangle \phi$;
- $\langle \alpha;\beta \rangle \phi \leftrightarrow \langle \alpha \rangle \langle \beta \rangle \phi$;
- $\langle \alpha \vee \beta \rangle \phi \leftrightarrow \langle \alpha \rangle \phi \vee \langle \beta \rangle \phi$;
- $\langle \alpha \wedge \beta \rangle \phi \to \langle \alpha \rangle \phi \wedge \langle \beta \rangle \phi$;
- $\langle \alpha \wedge (\beta \vee \gamma) \rangle \phi \leftrightarrow \langle \alpha \wedge \beta \rangle \phi \vee \langle \alpha \wedge \gamma \rangle \phi$;
- $\langle \phi? \rangle \psi \leftrightarrow \phi \wedge \psi$;

and is closed under the following inference rule:

- If for all propositional variables p, $\vdash_{BAL^n PDL_0^{\cap}} [\alpha(\neg(\langle \beta \rangle (\phi \wedge p) \vee \langle \gamma \rangle (\phi \wedge \neg p))?)]\psi$ then $\vdash_{BAL^n PDL_0^{\cap}} [\alpha(\neg \langle \beta \wedge \gamma \rangle \phi?)]\psi$;

where $\alpha(\sharp?)$ is an admissible form. Concerning the axiomatization of $L(ARROW^n NOR)$, let $BAL^n PDL_0^{\cap} NOR$ be the smallest normal logic in our extended language that contains $BAL^n PDL_0^{\cap}$ together with all the instances of the following axiom schema:

- $\phi \to [\pi_{1,1} \wedge \ldots \wedge \pi_{n,n}]\phi$.

Seeing that the axiomatic system $BAL^n PDL_0^{\cap}$ is obtained by adding the axiomatization of $PDL_0^\cap$ to the axiomatization of BAL^n, soundness of $BAL^nPDL_0^\cap$ is easy to check. Seeing that the schema $\phi \to [\pi_{1,1} \wedge \ldots \wedge \pi_{n,n}]\phi$ modally defines $ARROW^nNOR$ within $ARROW^n$, soundness of $BAL^nPDL_0^\cap NOR$ follows immediately.

Theorem 8 *(Soundness theorem of $BAL^nPDL_0^\cap$) Let ϕ be a formula. If $\vdash_{BAL^nPDL_0^\cap} \phi$ then $\phi \in L(ARROW^n)$.*

Theorem 9 *(Soundness theorem of $BAL^nPDL_0^\cap NOR$) Let ϕ be a formula. If $\vdash_{BAL^nPDL_0^\cap NOR} \phi$ then $\phi \in L(ARROW^nNOR)$.*

5.4 Completeness

Following the line of reasoning suggested in the proof of the completeness theorem of $PDL_0^\cap$, the method of the canonical model can be used to demonstrate that every formula true in the set of all arrow frames is provable in $BAL^nPDL_0^\cap$ and every formula true in the set of all normal arrow frames is provable in $BAL^nPDL_0^\cap NOR$. Let L be either $BAL^nPDL_0^\cap$ or $BAL^nPDL_0^\cap NOR$. The proof of the Lindenbaum's lemma for L is similar to the proof of the Lindenbaum's lemma for $PDL_0^\cap$.

Lemma 10 *(Lindenbaum's lemma for L) Let S be a consistent theory. There is a maximal theory T such that $S \subseteq T$.*

The *canonical frame for L* is the structure of the form $F_c = (W_c, \{R_c(i,j)\colon i,j \in \{1,\ldots,n\}\})$ where W_c is the set of all maximal theories and for all $i,j \in \{1,\ldots,n\}$, $R_c(i,j)$ is the binary relation on W_c such that for all maximal theories S,T, $SR_c(i,j)T$ iff $[\pi_{i,j}]S = \{\phi\colon [\pi_{i,j}]\phi \in S\} \subseteq T$. The *canonical model for L* is the model over F_c defined from the valuation V_c on F_c such that for all propositional variables p, $V_c(p) = \{S\colon p \in S\}$. The proof of the truth lemma for L is similar to the proof of the truth lemma for $PDL_0^\cap$.

Lemma 11 *(Truth lemma for L) Let ϕ be a formula. For all maximal theories S, the following conditions are equivalent:*

- $M_c, S \models \phi$;
- $\phi \in S$.

Using the schemata $\phi \to \langle\pi_{i,i}\rangle\phi$, $\phi \to [\pi_{i,j}]\langle\pi_{j,i}\rangle\phi$ and $\langle\pi_{i,j}\rangle\langle\pi_{j,k}\rangle\phi \to \langle\pi_{i,k}\rangle\phi$, the reader may easily verify that for all maximal theories S,T,U:

- $[\pi_{i,i}]S \subseteq S$;
- If $[\pi_{i,j}]S \subseteq T$ then $[\pi_{j,i}]T \subseteq S$;
- If $[\pi_{i,j}]S \subseteq T$ and $[\pi_{j,k}]T \subseteq U$ then $[\pi_{i,k}]S \subseteq U$.

Hence, the canonical frame for L is an arrow frame. To see that the canonical frame for $BAL^nPDL_0^{\cap}NOR$ is normal, we first need a useful lemma concerning the inference rule of intersection.

Lemma 12 *Let α, β be programs. For all maximal theories S, T, the following conditions are equivalent:*

- $[\alpha \wedge \beta]S \subseteq T$;
- $[\alpha]S \subseteq T$ *and* $[\beta]S \subseteq T$.

PROOF. Suppose $[\alpha \wedge \beta]S \subseteq T$, we demonstrate $[\alpha]S \subseteq T$ and $[\beta]S \subseteq T$. If $[\alpha]S \not\subseteq T$ then there is a formula ϕ such that $\phi \in [\alpha]S$ and $\phi \notin T$. Consequently, $[\alpha]\phi \in S$. Hence, $[\alpha \wedge \beta]\phi \in S$. It follows that $\phi \in [\alpha \wedge \beta]S$. Since $[\alpha \wedge \beta]S \subseteq T$, then $\phi \in T$, a contradiction. As a conclusion, $[\alpha]S \subseteq T$. The proof that $[\beta]S \subseteq T$ is similar.
Suppose $[\alpha]S \subseteq T$ and $[\beta]S \subseteq T$, we demonstrate $[\alpha \wedge \beta]S \subseteq T$. If $[\alpha \wedge \beta]S \not\subseteq T$ then there is a formula ϕ such that $\phi \in [\alpha \wedge \beta]S$ and $\phi \notin T$. It follows that $\neg\phi \in T$. In order to show that $[\neg\langle\alpha \wedge \beta\rangle\neg\phi?]\bot \in S$, we take an arbitrary propositional variable p. Obviously, $\neg\phi \wedge p \in T$ or $\neg\phi \wedge \neg p \in T$. Since $[\alpha]S \subseteq T$ and $[\beta]S \subseteq T$, then $\langle\alpha\rangle(\neg\phi \wedge p) \in S$ or $\langle\beta\rangle(\neg\phi \wedge \neg p) \in S$. Thus, $\langle\alpha\rangle(\neg\phi \wedge p) \vee \langle\beta\rangle(\neg\phi \wedge \neg p) \in S$ and $[\neg(\langle\alpha\rangle(\neg\phi \wedge p) \vee \langle\beta\rangle(\neg\phi \wedge \neg p))?]\bot \in S$. Seeing that S is closed under the inference rule of intersection, therefore $[\neg\langle\alpha \wedge \beta\rangle\neg\phi?]\bot \in S$ and $\neg[\alpha \wedge \beta]\phi \in S$. Consequently, $[\alpha \wedge \beta]\phi \notin S$ and $\phi \notin [\alpha \wedge \beta]S$, a contradiction. As a conclusion, $[\alpha \wedge \beta]S \subseteq T$.

To infer that the canonical frame for $BAL^nPDL_0^{\cap}NOR$ is normal, let S, T be maximal theories such that $[\pi_{1,1}]S \subseteq T$, ..., $[\pi_{n,n}]S \subseteq T$, we demonstrate $S = T$. Since $[\pi_{1,1}]S \subseteq T$, ..., $[\pi_{n,n}]S \subseteq T$, then, by the lemma above, $[\pi_{1,1} \wedge \ldots \wedge \pi_{n,n}]S \subseteq T$. Seeing that the schema $\phi \leftrightarrow [\pi_{1,1} \wedge \ldots \wedge \pi_{n,n}]\phi$ is provable in $BAL^nPDL_0^{\cap}NOR$, therefore $[\pi_{1,1} \wedge \ldots \wedge \pi_{n,n}]S = S$. Since $[\pi_{1,1} \wedge \ldots \wedge \pi_{n,n}]S \subseteq T$, then $S = T$. From all this, the completeness theorem of $BAL^nPDL_0^{\cap}$ and the completeness theorem of $BAL^nPDL_0^{\cap}NOR$ easily follow.

Theorem 13 *(Completeness theorem of $BAL^nPDL_0^{\cap}$) Let ϕ be a formula. If $\phi \in L(ARROW^n)$ then $\vdash_{BAL^nPDL_0^{\cap}} \phi$.*

Theorem 14 *(Completeness theorem of $BAL^nPDL_0^{\cap}NOR$) Let ϕ be a formula. If $\phi \in L(ARROW^nNOR)$ then $\vdash_{BAL^nPDL_0^{\cap}NOR} \phi$.*

Again, it is not known whether there exists a sound and complete orthodox proof system capable of dealing with either $BAL^nPDL_0^{\cap}$ or $BAL^nPDL_0^{\cap}NOR$. Similarly, the decidability of the satisfiability problem for either $BAL^nPDL_0^{\cap}$ or $BAL^nPDL_0^{\cap}NOR$ is still open.

6 Conclusion

In this paper we have presented deductive systems for $BAL^nPDL_0^\cap$ and $BAL^nPDL_0^\cap NOR$ which explore the following idea: although intersection of binary relations is not definable in the ordinary language of modal logic it becomes definable in a modal language strengthened by the introduction of propositional quantifiers. The one drawback is that our deductive systems use an unorthodox inference rule: the inference rule of intersection. The interesting question of course is to know whether the use of this unorthodox inference rule is essential or not. In other words it is of the utmost importance to determine whether the unorthodox inference rule of intersection can be replaced with a recursively enumerable set of axioms, a question that remains unsolved.

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