# PARTIAL AUTOMORPHISM SEMIGROUPS

JENNIFER CHUBB, VALENTINA HARIZANOV, ANDREI MOROZOV,
SARAH PINGREY, AND ERIC UFFERMAN

Abstract. We study the relationship between algebraic structures and their inverse semigroups of partial automorphisms. We consider a variety of classes of natural structures including equivalence structures, orderings, Boolean algebras, and relatively complemented distributive lattices. For certain subsemigroups of these inverse semigroups, isomorphism (elementary equivalence) of the subsemigroups yields isomorphism (elementary equivalence) of the underlying structures. We also prove that for some classes of computable structures, we can reconstruct a computable structure, up to computable isomorphism, from the isomorphism type of its inverse semigroup of computable partial automorphisms.

## 1. Introduction

A structure with no nontrivial automorphisms may admit nontrivial partial automorphisms. For example, the natural numbers have a unique (trivial) order-preserving automorphism, but there are many order-preserving partial maps. We consider collections of maps of this type and see that they contain a great deal of information about the associated underlying structure.

A *partial automorphism* is a partial injective map on a structure that respects predicates and predicate representations of functions. (Note that we make no substructure requirement on the domain or range of the map.) Some collections of such maps for a structure form semigroups. We will see that unlike the situation that arises when considering the group of automorphisms of a structure, different mutually definable signatures for a structure can yield different semigroups of partial automorphisms.

The reconstruction of algebraic structures from their automorphism groups has been studied for Boolean algebras (see, for example, [11, 12, 13]). For results on the recognition of computable structures from their groups of computable automorphisms, see the survey article [10]. In this article we continue work begun by Lipacheva who established a series of results on the reconstruction of structures from their partial automorphism semigroups [7, 8].

We consider structures for finite languages $L$. When it is convenient, and without loss of generality, we may take $L$ to be a predicate language by interpreting the operations in the signature of a structure as their graphs. When this is the case, constants are understood as unary predicates true on their values only.

The domain $M$ of any countable structure $\mathcal{M}$ can be identified with a subset of $\omega$. A countable structure is *computable* when its atomic diagram is decidable.

We write $\mathcal{M}_0 \equiv \mathcal{M}_1$ for elementarily equivalent structures and use $\mathcal{M}_0 \cong \mathcal{M}_1$ to denote isomorphic structures. Computable structures $\mathcal{M}_0$ and $\mathcal{M}_1$ are *computably isomorphic*, in symbols $\mathcal{M}_0 \cong_c \mathcal{M}_1$, if there is a computable isomorphism from $\mathcal{M}_0$ onto $\mathcal{M}_1$. Details about the existence of computable isomorphisms may be found in [1, 4, 5, 6].

Let $\mathcal{M}_M = (\mathcal{M}, a)_{a \in M}$ be the natural expansion of $\mathcal{M}$ for the language $L_M$, the language of $\mathcal{M}$ expanded by adding a new constant symbol $a$ for every $a \in M$.

For a partial function $p$, $\mathtt{dom}(p)$ and $\mathtt{ran}(p)$ are its domain and range, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures for the same predicate language $L$. We say that a partial function $p$ from $A$ to $B$ is a *partial isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if $p$ is 1–1 and for every atomic formula $\theta = \theta(x_0, \ldots, x_{n-1})$ in $L$, and every $a_0, \ldots, a_{n-1} \in \mathtt{dom}(p)$, we have

$$\mathcal{A}_A \models \theta(a_0, \ldots, a_{n-1}) \Leftrightarrow \mathcal{B}_B \models \theta(p(a_0), \ldots, p(a_{n-1})).$$

A partial function $p$ from $A$ to $B$ is a *finite partial isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if $p$ is a partial isomorphism and finite. If $A$ and $B$ are sets of natural numbers, then a partial function $p$ from $A$ to $B$ is a *partial computable isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if $p$ is a partial isomorphism and a partial computable function.

For a countable structure $\mathcal{M}$, we write $I(\mathcal{M})$, $I_{fin}(\mathcal{M})$, and $I_c(\mathcal{M})$ to denote the set of all partial, finite partial, and partial computable automorphisms of $\mathcal{M}$, respectively. Each of these sets of partial automorphisms forms an inverse semigroup under function composition and the inversion $f \mapsto f^{-1}$. We will consider these as structures for the language of inverse semigroups, $\{\cdot, {}^{-1}\}$, and identify each structure with its universe.

In Section 2, we present techniques and preliminary results that will be of use in all subsequent sections, where we consider a variety of classes of structures.

## 2. Basic interpretations in the semigroups of partial automorphisms

In this section we aim to give a uniform method of interpreting the action of an inverse semigroup $I$ of a structure $\mathcal{M}$ into $I$, where $I_{fin}(\mathcal{M}) \subseteq I \subseteq I(\mathcal{M})$. We begin by interpreting the universe of $\mathcal{M}$ in the semigroup $I$.

First, one can easily check that the set

$$\mathtt{Id}(I) = \{f \in I \mid f^2 = f\}$$

of idempotent elements of $I$ consists of exactly those elements that are the identity on their domain. Let $\mathtt{Id}(x)$ be $x^2 = x$, a corresponding first-order formula in the language of inverse semigroups. We may identify each $f \in \mathtt{Id}(I)$ with $\mathtt{dom}(f)$, a subset of $M$. Next, we define the relation $x \subseteq y$ on elements of $\mathtt{Id}(I)$ in the language of inverse semigroups by the following formula

$$\mathtt{Id}(x) \ \& \ \mathtt{Id}(y) \ \& \ xy = x.$$

The empty function, which we denote here by $\mathbf{\Lambda}$, is defined as the unique element $x \in I$ that satisfies the formula

$$\forall y[\mathtt{Id}(y) \Rightarrow x \subseteq y].$$

We define the set

$$A(\mathcal{M}) = \{\langle a, a\rangle \mid \ a \in M\}$$

as the set of all minimal elements in $I \setminus \{\mathbf{\Lambda}\}$ by the first-order formula

$$x \neq \mathbf{\Lambda} \ \& \ \neg\exists y[\mathbf{\Lambda} \subset y \subset x].$$

Every element $a \in M$ naturally corresponds to a finite partial automorphism $\{\langle a, a\rangle\} \in I$ satisfying this formula. Therefore, the elements of $A(\mathcal{M})$ will be naturally identified with the elements of $M$, and we will usually write $a$ for $\langle a, a\rangle$.

Finally, the natural action of $I$ on $A(\mathcal{M}) \cup \{\mathbf{\Lambda}\}$ is given by the rules

$$\begin{aligned} \mathtt{ap}_{\mathcal{M}}(g, \{\langle a, a\rangle\}) &= \begin{cases} \{\langle g(a), g(a)\rangle\} & \text{if } a \in \mathtt{dom}(g), \\ \mathbf{\Lambda} & \text{otherwise,} \end{cases} \\ \mathtt{ap}_{\mathcal{M}}(g, \mathbf{\Lambda}) &= \mathbf{\Lambda}, \end{aligned}$$

and can be defined by a first-order formula. Indeed, one can easily ascertain that for $f \in A(\mathcal{M}) \cup \{\mathbf{\Lambda}\}$, the following holds

$$\mathtt{ap}_{\mathcal{M}}(g, f) = gfg^{-1}.$$

We will omit the subscript $\mathcal{M}$ in $\mathtt{ap}_{\mathcal{M}}$ when the structure is clear from the context, and usually write $g(x) = y$ for $\mathtt{ap}(g, x) = y$.

The interpretation of the universe $M$ and of the action of elements of $I$ on $M$ into the semigroup as described above suggests a natural two-sorted extension of $I$. We will make use of this extension in subsequent sections and define it as follows.

**Definition 2.1.** Let $\mathcal{M}$ be a structure with universe $M$, and $I$ an inverse subsemigroup of $I(\mathcal{M})$. We define the two-sorted structure $I^*$ naturally extending $I$ as

$$I^* = \left\langle I, M \cup \{\mathbf{\Lambda}\}; \mathtt{ap}, \cdot, {}^{-1} \right\rangle,$$

where the second sort, $M \cup \{\mathbf{\Lambda}\}$, and the function, $\mathtt{ap}$, are the interpretations based on the underlying structure $\mathcal{M}$ as described above.

The following proposition is immediate from the definition of $I^*$.

**Proposition 2.2.** (1) *Assume that $\mathcal{M}_0$ and $\mathcal{M}_1$ are structures, and that $I_i$ are inverse subsemigroups of $I(\mathcal{M}_i)$ for $i = 0, 1$, such that $I_{fin}(\mathcal{M}_i) \subseteq I_i$. Then any isomorphism $\lambda$ from $I_0$ to $I_1$ can be extended to an isomorphism of the two-sorted structures $I_0^*$ and $I_1^*$. That is, there is a bijection $\lambda'$ from $M_0 \cup \{\mathbf{\Lambda}\}$ to $M_1 \cup \{\mathbf{\Lambda}\}$ such that the pair $\langle \lambda, \lambda' \rangle$ is an isomorphism from $I_0^*$ to $I_1^*$.*

(2) *Assume that $\mathcal{M}$ is a structure and that $I$ is an inverse subsemigroup of $I(\mathcal{M})$ such that $I_{fin}(\mathcal{M}) \subseteq I$. Then each first-order formula $\varphi(\bar{x})$ in the language of $I^*$ with all free variables $\bar{x}$ of sort $I$ can be effectively transformed into a formula $\varphi^*(\bar{x})$ in the language of inverse semigroups so that*

$$I^* \models \varphi(\bar{x}) \ \Leftrightarrow \ I \models \varphi^*(\bar{x}).$$

We proceed to study specific classes of structures.

## 3. Equivalence structures

Here we focus on *equivalence structures*, $\mathcal{M} = \langle M, E \rangle$, where $E$ is an equivalence relation on $M$. We call an equivalence relation $E$ on a set $M$ (and the corresponding equivalence structure) *nontrivial* if $E$ differs from the diagonal relation $\{\langle a, a \rangle \mid a \in M\}$ and from the set $M \times M$. The $\mathcal{M}$-equivalence class of $a \in M$ is

$$[a]_E = \{x \in M : xEa\}.$$

The following theorem demonstrates that the isomorphism class or elementary type of a nontrivial equivalence structure can be determined by the corresponding classification of the semigroup of its partial automorphisms. In particular, countable structures can be recovered up to isomorphism from the elementary type of their semigroups of finite partial automorphisms.

**Theorem 3.1.** *Let $\mathcal{M}_0 = \langle M_0, E_0\rangle$ and $\mathcal{M}_1 = \langle M_1, E_1\rangle$ be nontrivial equivalence structures. Let $I_0$ and $I_1$ be inverse semigroups such that $I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i)$ for $i = 0, 1$. Then*

(1) $I_0 \cong I_1 \Rightarrow \mathcal{M}_0 \cong \mathcal{M}_1$,
(2) $I_0 \equiv I_1 \Rightarrow \mathcal{M}_0 \equiv \mathcal{M}_1$.
(3) *If both structures $\mathcal{M}_0$ and $\mathcal{M}_1$ are countable, then*

$$I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \;\Leftrightarrow\; \mathcal{M}_0 \cong \mathcal{M}_1.$$

*Proof.* First assume that $\mathcal{M} = \langle M, E\rangle$ is a nontrivial equivalence structure, and that $I$ is an inverse subsemigroup of $I(\mathcal{M})$ such that $I_{fin}(\mathcal{M}) \subseteq I \subseteq I(\mathcal{M})$. We interpret $E$ in $I$ in the following way. Let

$$p, q \sim r, s$$

be an abbreviation for the formula

$$\exists f\, [f(p) = r \;\&\; f(q) = s]$$

in the language of $I^*$. Let

$$\widetilde{E}(a, b) =_{\texttt{def}} \forall x\, \forall y\, \forall z\, [(a, b \sim x, y \;\&\; a, b \sim y, z) \Rightarrow (x = z \vee a, b \sim x, z)].$$

Thus, for all $a, b \in M$,

$$\mathcal{M} \models E(a, b) \Leftrightarrow I^* \models \widetilde{E}(a, b).$$

Indeed, by transitivity of $E$, we have that $\mathcal{M} \models E(a, b)$ implies $\mathcal{M}^* \models \widetilde{E}(a, b)$. To prove the converse, assume that $\mathcal{M}^* \models \widetilde{E}(a, b)$, but $\mathcal{M} \models \neg E(a, b)$. We have that $a \neq b$. The relation $E$ is nontrivial, so choose pairwise distinct elements $x, y, z \in M$ so that $E(x, z)$ and $\neg E(x, y)$. Then we have $a, b \sim x, y$ and $a, b \sim y, z$. However, $a, b \nsim x, z$, which contradicts $\widetilde{E}(a, b)$.

Now, suppose that $I_0 \cong I_1$. By Proposition 2.2 and the properties of the formula $\widetilde{E}$ above, we have $\mathcal{M}_0 \cong \mathcal{M}_1$. The implication $\mathcal{M}_0 \cong \mathcal{M}_1 \Rightarrow I_{fin}(\mathcal{M}_0) \cong I_{fin}(\mathcal{M}_1)$ is trivial.

Similarly, (2) follows from Proposition 2.2 and the properties of the formula $\widetilde{E}$.

For (3), the direction ($\Leftarrow$) is trivial.

For the other direction, take nontrivial equivalence structure $\mathcal{M}$, and for each $m \in \omega$ and $n \in \omega \cup \{\infty\}$ define the property $\varphi_{m,n}$ as follows:

$$\varphi_{m,n} =_{\texttt{def}} \text{“}E \text{ has at least } m\ n\text{-element equivalence classes.”}$$

It is clear that any two countable equivalence structures are isomorphic if and only if they satisfy the same $\varphi_{m,n}$ for all $m \in \omega$, $n \in \omega \cup \{\infty\}$.

If we prove that all of these properties $\varphi_{m,n}$ are expressible by first-order sentences in the language of $(I_{fin}(\mathcal{M}))^*$, then the result will follow from Proposition 2.2(2). Using the properties of formula $\widetilde{E}(x,y)$, we can easily express $\varphi_{m,n}$ for finite $m$ and $n$ by a sentence in the language of $(I_{fin}(\mathcal{M}))^*$. To express this property for $n=\infty$, it suffices to express "$a$ is a member of an infinite equivalence class" by a first-order formula in the language of $(I_{fin}(\mathcal{M}))^*$. To do so, first note that $a \in M$ is a member of an infinite equivalence class if and only if

$$\neg\exists f \ [\forall x \ (\widetilde{E}(a,x) \Rightarrow x \in \mathtt{dom}(f))],$$

since each $f$ has a finite domain. It remains to note that the condition $y \in \mathtt{dom}(f)$ is equivalent to $f(y) \neq \mathbf{\Lambda}$. $\square$

**Remark 3.2.** The converse of (2) fails. Indeed, consider an equivalence relation $E_0$ on an infinite set $M_0$ such that all of its classes are finite and $E_0$ has infinitely many $n$-element classes for every $n \in \omega$. Let $E_1$ be an equivalence structure on a set $M_1$ with infinitely many equivalence classes of each cardinality in $\omega \cup \{\omega\}$, so that $\langle M_0; E_0\rangle \equiv \langle M_1; E_1\rangle$, but $\langle M_0; E_0\rangle \not\cong \langle M_1; E_1\rangle$. The elementary equivalence $I_{fin}(\langle M_0; E_0\rangle) \equiv I_{fin}(\langle M_1; E_1\rangle)$ must fail, because

$$\forall a \exists f \ [\forall x \ (\widetilde{E}(a,x) \Rightarrow x \in \mathtt{dom}(f))]$$

holds in $\langle M_0; E_0\rangle$ but not in $\langle M_1; E_1\rangle$.

**Remark 3.3.** We cannot omit the cardinality condition in (3). Consider two equivalence structures $\mathcal{M}_0$ and $\mathcal{M}_1$ of distinct infinite cardinalities such that all of their equivalence classes are infinite and each of them has infinitely many equivalence classes. Then we have $\mathcal{M}_0 \not\cong \mathcal{M}_1$, but $I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1)$.

To see that these structures are elementarily equivalent, recall that a family $S$ of partial isomorphisms from a structure $\mathcal{A}$ into a structure $\mathcal{B}$ satisfies the *back-and-forth property* from $\mathcal{A}$ to $\mathcal{B}$ if for every $f \in S$ :

(1) for all $a \in A$, there exists $f' \in S$ such that $f \subseteq f'$ and $a \in \mathtt{dom}(f')$,
(2) for all $b \in B$, there exists $f' \in S$ such that $f \subseteq f'$ and $b \in \mathtt{ran}(f')$.

The structures $\mathcal{A}$ and $\mathcal{B}$ are called *partially isomorphic* if there exists a back-and-forth property from $\mathcal{A}$ to $\mathcal{B}$. It follows from Karp's Theorem (see [2, Theorem VII, 5.3]) that if two structures are partially isomorphic, then they are elementary equivalent. To prove that the structures $I_{fin}(\mathcal{M}_0)$ and $I_{fin}(\mathcal{M}_1)$ are partially isomorphic, it suffices to prove that a family $S$ defined by

$$S = \{G_f \mid f \text{ is a finite partial isomorphism from } \mathcal{M}_0 \text{ to } \mathcal{M}_1\},$$

where

$$G_f = \{\langle h, fhf^{-1}\rangle \mid \mathtt{dom}(h) \cup \mathtt{ran}(h) \subseteq \mathtt{dom}(f)\}$$

satisfies the back-and-forth property from $I_{fin}(\mathcal{M}_0)$ to $I_{fin}(\mathcal{M}_1)$. The details can be checked routinely.

The following theorem shows that we can restore nontrivial equivalence structures up to computable isomorphism from the first-order theory of their inverse semigroups of partial computable automorphisms. Moreover, the isomorphism type of these structures can be characterized by a single formula in the language of inverse semigroups.

**Theorem 3.4.** *Let $\mathcal{M}_0$ be a nontrivial computable equivalence structure. Then there exists a first-order sentence $\varphi$ in the language of inverse semigroups such that for any nontrivial computable equivalence structure $\mathcal{M}_1$, the condition $I_c(\mathcal{M}_1) \models \varphi$ implies $\mathcal{M}_1 \cong_c \mathcal{M}_0$.*

*Proof.* First, we establish two lemmas.

**Lemma 3.4.1.** *Let $\mathcal{M} = \langle M; E\rangle$ be a computable equivalence structure. Then the following first-order formula of the language $I_c^*(\mathcal{M})$ is satisfied by exactly those $f \in I_c(\mathcal{M})$ with infinite $\mathtt{dom}(f)$:*

(1) $$\exists g[\mathtt{dom}(g) \subseteq \mathtt{dom}(f) \ \&\ \mathtt{ran}(g) \subset \mathtt{dom}(g)].$$

*Proof.* If (1) is true, then $\mathtt{dom}(g)$ is in 1–1 correspondence with some proper subset of itself, and, since $\mathtt{dom}(g) \subseteq \mathtt{dom}(f)$, we conclude that $\mathtt{dom}(f)$ is infinite.

Assume now that $\mathtt{dom}(f)$ is infinite. Consider the following two cases.
*Case 1.* There exists $a \in \mathtt{dom}(f)$ such that $\mathtt{dom}(f) \cap [a]_E$ is infinite.

In this case, the set $\mathtt{dom}(f) \cap [a]_E$ is an infinite c.e. set. Hence we may take $g \in I_c(\mathcal{M})$ to be any partial computable 1–1 function such that $\mathtt{dom}(g) = \mathtt{dom}(f) \cap [a]_E$ and $\mathtt{ran}(g) \subset \mathtt{dom}(g)$, so (1) is satisfied.
*Case 2.* The set $\mathtt{dom}(f)$ has a nonempty intersection with infinitely many equivalence classes.

In this case, there exists an infinite c.e. set $S \subseteq \mathtt{dom}(f)$ so that any two distinct members are in different equivalence classes. The result follows analogously to Case 1. □

From Lemma 3.4.1 we see that the structure $I^{\diamondsuit}(\mathcal{M}) = \langle I_c^*(\mathcal{M}); \mathtt{Fin}, \in\rangle$, where $\mathtt{Fin}$ is the set of all finite subsets of $M$ and $\in$ is the membership relation on $\mathcal{M} \times \mathtt{Fin}$, is elementary definable without parameters in $I_c^*(\mathcal{M})$. This means that each formula in the language of $I^{\diamondsuit}(\mathcal{M})$ expressing a first-order property of elements of $I_c^*(\mathcal{M})$ can be transformed into a first-order formula expressing the same property in the language of $I_c^*(\mathcal{M})$. Indeed, we can identify finite sets with elements $f \in I_c(\mathcal{M})$ not satisfying the formula (1). Any such $f_0$ and $f_1$ are identified with the same finite set if and only if

$$\forall x\, [x \in \mathtt{dom}(f_0) \Leftrightarrow x \in \mathtt{dom}(f_1)].$$

Finally, $a$ belongs to the set identified with $f$ if and only if $a \in \mathtt{dom}(f)$.

**Lemma 3.4.2.** *There exists a first-order formula* $\mathtt{nat}(v)$ *in the language of* $I_c^*(\mathcal{M})$ *that distinguishes* $p \in I_c(\mathcal{M})$ *for which there is a computable* $1-1$ *function* $f : \omega \to M$ *with the following properties:*

(1) $\mathtt{dom}(p) = \{f(i) \mid i \in \omega\}$,
(2) $\forall i\, [p(f(i)) = f(i+1)]$,
(3) $\forall i\, \forall j\, [i \neq j \Rightarrow \langle f(i), f(j)\rangle \notin E]$,
(4) $\forall x \exists i [\langle f(i), x\rangle \in E]$.

*Proof.* We will show that the required formula $\mathtt{nat}(v)$ can be taken as the conjunction of three first-order formulas, each expressing one of the following conditions:

(a) The set $\mathtt{dom}(v) \setminus \mathtt{ran}(v)$ contains only one element.
(b) Let $a_0$ be the unique element in the set $\mathtt{dom}(v) \setminus \mathtt{ran}(v)$. Then for all $x$, there exists $x_1 \in \mathtt{dom}(v)$ such that $\langle x, x_1\rangle \in E$, and there exists a finite set $F \subseteq \mathtt{dom}(v)$ such that $a_0 \in F$ and for all $t \in F \setminus \{x_1\}$, we have $v(t) \in F$.
(c) Distinct elements of $\mathtt{dom}(v)$ are not $E$-equivalent.

It follows immediately that if $p$ and $f$ satisfy (1)–(4), then $p$ satisfies $\mathtt{nat}$.

On the other hand, if $p \in I_c(\mathcal{M})$ satisfies (a)–(c), define $f : \omega \to M$ by $f(n) =_{def} p^n(a_0)$ so that (2) holds. Take an arbitrary $x \in \mathtt{dom}(p)$ and fix a finite set $F$ satisfying (b). Consider the sequence $a_0, p(a_0), p^2(a_0), p^3(a_0), \ldots$. If there is no $m \in \omega$ such that $p^m(a_0) = x$, then by (b), all elements in this sequence belong to $F$. The elements in the sequence are pairwise distinct, for if $p^k(a_0) = p^l(a_0)$ for some $k < l$, then $a_0 = p^{l-k}(a_0)$ and thus $a_0 \in \mathtt{ran}(p)$, which is a contradiction. It follows that $F$ contains an infinite subset, which is impossible. So (1) must hold. Clearly (3) and (4) hold. □

We split the proof of the theorem into two cases.

*Case 1.* There are finitely many $E$-equivalence classes.

This case is distinguished by the sentence saying that there is no $p$ satisfying $\mathtt{nat}(p)$.

It follows from the interpretability of both $E$ and of the finiteness of subsets that the property "the equivalence class of $a$ is finite" can be expressed by a first-order formula in the language of $I_c^*(\mathcal{M})$ saying

$$(\exists F \in \mathtt{Fin})(\forall x)[\langle a, x\rangle \in E \Rightarrow x \in F].$$

Note that for each $n \in \omega$, $n \neq 0$, there exists a first-order formula in the language of $I_c^*(\mathcal{M})$ asserting that "the equivalence class of $x$ consists of $n$ elements."

Now assume that $E$ contains $m$ equivalence classes, and that they have cardinalities $k_0, \dots, k_{m-1}$, where $k_i \in \omega \cup \{\infty\}$. Such a property can be expressed in the language of $I_c^*(\mathcal{M})$ by a sentence saying

$$\exists x_0, \dots, x_{m-1}[\, \bigwedge_{i<j<m} \langle x_i, x_j\rangle \notin E \;\&\; \forall x\,[\, \bigvee_{i<m} \langle x, x_i\rangle \in E] \;\&\; \bigwedge_{i<m}([x_i]_E \text{ contains } k_i \text{ elements})].$$

The conjunction of this sentence with the sentence $\neg\exists p\ \mathtt{nat}(p)$ is satisfied in $I_c^*(\mathcal{M})$ if and only if $\mathcal{M}$ has $m$ equivalence classes with cardinalities $k_0, \dots, k_{m-1}$. Since this sentence is equivalent to a sentence in the language of inverse semigroups, and any two computable equivalence structures with the same finite number of equivalence classes of cardinalities $k_0, \dots, k_{m-1}$ are computably isomorphic, this completes the proof for this case.

*Case 2.* There are infinitely many $E$-equivalence classes.

First we define a standard model of arithmetic in $I_c(\mathcal{M})$. We will need an arbitrary parameter $p$ satisfying the formula $\mathtt{nat}$, and will use the property that the action of $p$ looks much like the successor function on the natural numbers.

Define the zero element $0_p$ as the unique element in the set $\mathtt{dom}(p) \setminus \mathtt{ran}(p)$. Define the successor function $s_p$ on $\mathtt{dom}(p)$ as $s_p(a) = p(a)$.

Next, define the ordering $<_p$ on $\mathtt{dom}(p)$, which corresponds to the usual ordering on the natural numbers,

$$p^m(0_p) <_p p^n(0_p) \ \Leftrightarrow_{\mathtt{def}}\ m < n,$$

as follows:

$$\begin{aligned} a <_p b \quad \Leftrightarrow_{\mathtt{def}} \quad & a \neq b \;\&\; (\exists S \in \mathtt{Fin})[\, 0_p \in S \;\&\; a \in S \;\&\; b \notin S \;\&\; \\ & (\forall t \in S \setminus \{a\})[p(t) \in S]\, ]. \end{aligned}$$

We define the operations $+_p$ and $\times_p$ that correspond to the usual addition and multiplication as follows:

$$\begin{aligned} a +_p b = c \quad \Leftrightarrow_{\mathtt{def}} \quad & \exists f\,[\, \mathtt{dom}(f) \supseteq \{t \mid\ 0_p \leqslant_p t \leqslant_p b\} \;\&\; f(0_p) = a \;\&\; \\ & (\forall t <_p b)[s_p(f(t)) = f(s_p(t))] \;\&\; f(b) = c\, ], \\ a \times_p b = c \quad \Leftrightarrow_{\mathtt{def}} \quad & \exists f\,[\, \mathtt{dom}(f) \supseteq \{t \mid\ 0_p \leqslant_p t \leqslant_p b\} \;\&\; \\ & f(0_p) = 0_p \;\&\; (\forall t <_p b)[(f(s_p(t))) = f(t) +_p a] \;\&\; \\ & f(b) = c\, ]. \end{aligned}$$

Now we consider the following two subcases.

*Subcase 1.* The set of cardinalities of $E$-equivalence classes is finite.

The sentence $C_1$, saying there exists a finite set $F$ such that for any $x$, there exists $y \in F$ and $f \in I_c(\mathcal{M})$ that is a bijection from $[x]_E$ onto $[y]_E$, distinguishes this case. Let

$$K = \{k_0 < k_1 < \ldots < k_{m-1}\}$$

be the set of all possible cardinalities of classes of $E$. We do not exclude the possibility that $k_{m-1} = \infty$. Let $\psi_i(v)$ be a formula (in the language of inverse semigroups) requiring that the cardinality of $[v]_E$ be $k_i$. Since $E$ is computable and $p$ is a partial computable function, the set

$$\{n \mid \text{ the cardinality of } [p^n(0_p)]_E \text{ equals } k_i\}$$

is arithmetical for any $i < m$; that is, it can be expressed by a formula $\varphi_i(n)$ in the language of arithmetic. We can now describe all the cardinalities along $\mathtt{dom}(p)$ by the following statement:

$$\forall t \bigvee_{i<m} (\varphi_i(t) \ \& \ \psi_i(t)).$$

Thus, the equivalence structure can be characterized, up to computable isomorphism, by the following statement, which can be translated into an equivalent statement in the language of inverse semigroups:

$$C_1 \ \& \ \exists p[\mathtt{nat}(p) \ \& \ \forall t \bigvee_{i<m} (\varphi_i(t) \ \& \ \psi_i(t))].$$

*Subcase 2.* The set of cardinalities of $E$-equivalence classes is infinite.

Suppose that we have already expressed the following property of $a$ and $b$ by a first-order formula $\mathtt{Card}(a, b, p)$ in the language of $I^{\diamondsuit}(\mathcal{M})$:

*"$a, b \in \mathtt{dom}(p)$ and there exists $n \in \omega$ such that $b = p^n(0_p)$ and the cardinality of $[a]_E$ equals $n$."*

Note that $\mathtt{nat}(p)$ implies that the relation

$$\begin{aligned} C(n,m) =_{\mathtt{def}} \{\langle n,m\rangle \mid \ & (n = 0 \ \& \ [p^m(0_p)]_E \text{ is infinite}) \vee \\ & (n \neq 0 \ \& \ [p^m(0_p)]_E \text{ contains exactly } n \text{ elements})\} \end{aligned}$$

is arithmetical.

We can express the relation describing all of the cardinalities of classes along the sequence $0_p, p(0_p), p^2(0_p), \ldots$ by a formula saying:

$$\begin{aligned} &\exists p\, [\ (\forall m \in \mathtt{dom}(p))\, (\exists n \in \mathtt{dom}(p))\, [\ C'(n,m)\ \& \\ &[(n = \ 0_p \ \& \ ([m]_E \text{ is infinite})) \vee (n \neq 0_p \ \& \ (\mathtt{Card}(m,n,p)))]\ ]\ ], \end{aligned} \tag{2}$$

where $C'$ is obtained from $C$ by replacing all the occurrences of $0$, $s$, $<$, $+$, $\times$ with $0_p$, $s_p$, $<_p$, $+_p$, $\times_p$, respectively. It can be shown that the

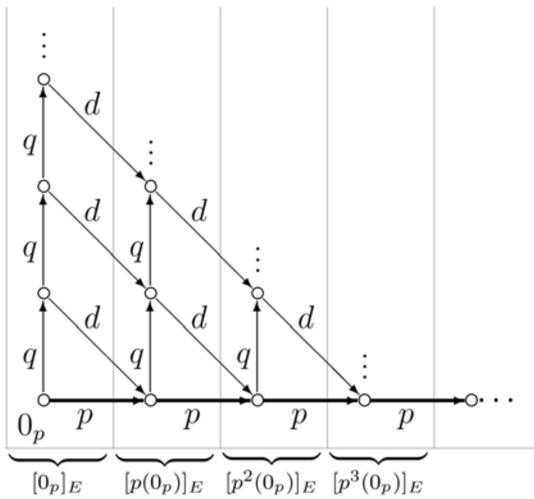


FIGURE 1.

conjunction of the formula (2) with $\exists p\, \mathtt{nat}(p)\ \&\ \neg C_1$ characterizes the equivalence structure up to computable isomorphism. Indeed, if this conjunction is satisfied by two computable equivalence structures $\mathcal{M}_0$ and $\mathcal{M}_1$ of type $I_c^{\diamondsuit}(\mathcal{M})$, then there exist computable sequences

$$a_0, a_1, \ldots, \quad \text{and} \quad b_0, b_1, \ldots$$

of representatives of the equivalence classes such that the cardinalities of $[a_i]_{E^{\mathcal{M}_0}}$ and $[b_i]_{E^{\mathcal{M}_1}}$ are the same for all $i \in \omega$. Using this fact, we can easily construct a computable isomorphism between these equivalence structures using a back-and-forth argument.

It only remains to prove the existence of the formula $\mathtt{Card}(m, n, p)$. Note that it suffices to express the fact that $[m]_E$ has at least $k$ elements, where $n = p^k(0_p)$, by a first-order formula.

Observe that there is a $1-1$ correspondence between the sets $[m]_E$ and $A_n =_{\mathtt{def}} \{a \mid a <_p n\}$. This correspondence cannot be established by a partial computable automorphism because all elements in $[m]_E$ are $E$-related, while the elements of $A_n$ are not. We will need additional partial computable automorphisms $q$ and $d$, satisfying special conditions, to express this correspondence. The idea is shown in Figure 1.

In Figure 1, $p$ satisfies the property $\mathtt{nat}(p)$. All the arrows in this figure act like partial automorphisms. Partial automorphism $q$ has the property that for each $x \in \mathtt{dom}(p)$, it generates the whole class $[x]_E$ as the set

$$\{y \mid \ (\exists i \in \omega)\ [q^i(x)\downarrow \ \&\ y = q^i(x)]\}.$$

The action of $d$ serves to establish a direct $1-1$ correspondence between a subset of $[0_p]_E$ and a subset $\{t \mid 0_p \leqslant_p t <_p u\}$ by means of diagonal arrows of different lengths: $d$, $d^2$, $d^3$, etc. Its behavior outside the region drawn is of no importance. In the concrete situation displayed in Figure 1, we can assert that the number of elements in $[0_p]_E$ is greater than or equal to 4. This idea will help us to express the property of the number of elements in $[0_p]_E$ only.

We must express this property for an arbitrary element $a \in \mathtt{dom}(p)$. Fix $p$ such that $\mathtt{nat}(p)$ and use the assumption of Subcase 2 to find appropriate $p'$ and $q'$ to create a configuration as in this figure, saying that the number of elements in $[a]_E$ is greater than or equal to the number of the position of some $y'$ in the sequence $0_{p'}, p'(0_{p'}), p'^2(0_{p'}), \ldots$ Then we identify $y'$ with an appropriate element $y$ in the standard sequence $0_p, p(0_p), p^2(0_p), \ldots$

We can now describe these figures by first-order formulas. The action of $p$ and $q$ is specified by the conjunction of formulas expressing the following conditions:

(a) $\mathtt{nat}(p)$,
(b) $\forall x\,[\langle q(x), x\rangle \in E]$,
(c) $\forall x\,[q(x)\downarrow \Rightarrow q(x) \neq x]$,
(d) $(\forall a \in \mathtt{dom}(p))\,(\forall y \in [a]_E)\ [\ y \neq a \Rightarrow (\exists F \in \mathtt{Fin})\ [a, y \in F\ \&\ (\forall t \in F \setminus \{y\})[q(y)\downarrow \in F]]\ ]$.

An immediate check shows that if $p$ and $q$ satisfy the conditions above, then $p$ defines the complete system of representatives $\{p^i(0_p) \mid i \in \omega\}$ for $E$, and every equivalence class of $E$ has the form

$$\{q^k(p^i(0_p)) \mid \ k \in \omega\ \&\ q^k(p^i(0_p))\downarrow\}$$

for an appropriate $i \in \omega$.

The behavior of $d$, whose action draws diagonals on Figure 1 up to the diagonal with the lower right end being an element $y \in \mathtt{dom}(p)$, is specified by the following conditions:

(e) For every $x \in \mathtt{dom}(p)$, the property $0_p \leqslant_p x <_p y$ implies $q(x)\downarrow$, $d(q(x))\downarrow$, and $d(q(x)) = p(x)$;
(f) For all $t$, if $t$ is equivalent to some $u$ such that $0_p \leqslant_p u <_p y$, $q(t)\downarrow$, $d(t)\downarrow$, and $d(q(t))\downarrow$, then we have $d(q(t)) = q(d(t))$.

Let $\Theta(p, q, d, y)$ be the conjunction of (a)–(f). If $\Theta$ is satisfied on the sequence $p, q, d, y$, then the number of elements in the class $0_p$ is greater than or equal to the natural number corresponding to $y$. On the other hand, if the cardinality of the class $[a]_E$ is greater than or equal to $n+1$, then there exist $p, q, d, y$ such that $a = 0_p$, $\Theta(p, q, d, y)$, and $y = p^n(0_p)$.

Finally, we can formulate the property $\texttt{Card}(x, y, p)$ as follows:

There exist partial computable automorphisms $\gamma, p', q, d$ and $y' \in \texttt{dom}(p')$ such that $x = 0_{p'}$, $\Theta(p', q, d, y')$, $\texttt{dom}(p') \subseteq \texttt{dom}(\gamma)$, $\texttt{dom}(p) \subseteq \texttt{dom}(\gamma^{-1})$, $\gamma p \gamma^{-1} = p'$, $\gamma^{-1} p' \gamma = p$, and $\gamma(y') = y$.
Since this assertion can be expressed in the language of semigroups, we have established the existence of the formula $\texttt{Card}$. This completes the proof of Subcase 2. □

## 4. Partial Orderings

We now consider partial orderings and show that the elementary equivalence of the semigroups of finite partial automorphisms of partial orderings yields elementary equivalence of the orderings themselves, up to a possible inversion of the ordering. For convenience, we will assume the ordering is strict.

If $\mathcal{M} = \langle M, < \rangle$ is an ordering, we denote its reverse ordering by

$$\mathcal{M}^{\texttt{rev}} = \langle M, <^{\texttt{rev}} \rangle .$$

**Theorem 4.1.** *Let $\mathcal{M}_0 = \langle M_0, <_0 \rangle$ and $\mathcal{M}_1 = \langle M_1, <_1 \rangle$ be strict partial orderings, and let $I_i$ be inverse subsemigroups of $I(\mathcal{M}_i)$ for $i = 0, 1$, such that $I_{fin}(\mathcal{M}_i) \subseteq I_i$. Then*

$$I_0 \equiv I_1 \;\Rightarrow\; [\mathcal{M}_0 \equiv \mathcal{M}_1 \vee \mathcal{M}_0 \equiv \mathcal{M}_1^{\texttt{rev}}].$$

*Proof.* Note that for every pair $a < b$ of elements of an ordering $\mathcal{M}$, we can define the structure's ordering $x < y$ in the semigroup $I$ by saying "$x$ and $y$ are in the same relative ordering as $a$ and $b$," i.e., with the formula

$$\exists p\, [p(a) = x \;\&\; p(b) = y].$$

In nontrivial situations, we can distinguish the pairs of comparable elements $a$ and $b$ and use them as parameters to define the relation "$x$ and $y$ are in the same relation as $a$ and $b$," which means $x < y$ or $x <^{\texttt{rev}} y$. In general, however, we will not be able to determine which of $x < y$ or $x <^{\texttt{rev}} y$ holds as we will know only that $a$ and $b$ are comparable in the ordering, and not how they are ordered.

More formally, assume that $\mathcal{M} = \langle M, < \rangle$ is an arbitrary strict partial ordering and that $I$ is an arbitrary inverse subsemigroup of $I(\mathcal{M})$ containing $I_{fin}(\mathcal{M})$.

Define a binary relation $<_{a,b}$ on the elements of the sort $M$ of the structure $I^*$ as

$$x <_{a,b} y \Leftrightarrow \exists p\, [p(a) = x \;\&\; p(b) = y]. \tag{3}$$

Let

$$\texttt{Comp}(a,b) =_{\texttt{def}} (a \neq b) \;\&\; \neg\exists p\,[p(a) = b \;\&\; p(b) = a]. \tag{4}$$

Note that for any $a, b \in M$, the condition $\texttt{Comp}(a,b)$ is satisfied in $I^*$ if and only if $a$ and $b$ are distinct and comparable, that is, $a < b$ or $b < a$.

We must transform an arbitrary formula $\varphi$ in the language of strict partial orderings to an equivalent formula $\varphi^s$ in the language of inverse semigroups. Let $u$ and $v$ be variables not occurring in $\varphi$, and set

$$\varphi^s =_{\texttt{def}} (\exists u \exists v[\texttt{Comp}(u,v) \;\&\; \widetilde{\varphi}] \vee \neg\exists u\exists v[\texttt{Comp}(u,v) \;\&\; \varphi'])^*,$$

where $^*$ is the transformation of formulas from Proposition 2.2(2), $\widetilde{\varphi}$ is obtained from $\varphi$ by replacing each of its atomic subformulas of the form $x < y$ with $x <_{u,v} y$, and $\varphi'$ is obtained from $\varphi$ by replacing all of its subformulas $x < y$ with the false sentence $\neg\forall x\,(x = x)$.

**Lemma 4.1.1.** *$\mathcal{M} \models \varphi \Rightarrow I \models \varphi^s$.*

*Proof.* If there are comparable elements in $M$, then, since $\mathcal{M} \models \varphi$, we must have

$$I^* \models \exists u\exists v[\texttt{Comp}(u,v) \;\&\; \widetilde{\varphi}].$$

If there are no elements $a, b \in M$ such that $a < b$, then all atomic subformulas of $\varphi$ containing $<$ are false, so $\varphi$ is equivalent to $\varphi'$. Furthermore, $I^* \models \neg\exists u\exists v\texttt{Comp}(u,v)$. It follows that

$$I^* \models \exists u\exists v[\texttt{Comp}(u,v) \;\&\; \widetilde{\varphi}] \vee \neg\exists u\exists v[\texttt{Comp}(u,v) \;\&\; \varphi'].$$

By Proposition 2.2, we have that $I \models \varphi^s$. □

**Lemma 4.1.2.** *$I \models \varphi^s \Rightarrow [\mathcal{M} \models \varphi \vee \mathcal{M}^{\texttt{rev}} \models \varphi]$.*

*Proof.* Assume that $I \models \varphi^s$. By Proposition 2.2, we have that

$$I^* \models \exists u\exists v[\texttt{Comp}(u,v) \;\&\; \widetilde{\varphi}] \vee \neg\exists u\exists v[\texttt{Comp}(u,v) \;\&\; \varphi']. \tag{5}$$

We will consider two cases.

*Case 1.* There are no comparable elements in $\mathcal{M}$.

By (5), we have that

$$I^* \models \neg\exists u\exists v[\texttt{Comp}(u,v) \;\&\; \varphi'],$$

and, in particular, $I^* \models \varphi'$. Since all subformulas of $\varphi$ containing $<$ are false, we have that $\mathcal{M} \models \varphi$.

*Case 2.* There exists a pair of comparable elements in $\mathcal{M}$.

By (5), we have that

$$I^* \models \exists u\exists v[\texttt{Comp}(u,v) \;\&\; \widetilde{\varphi}].$$

Let $a$ and $b$ from $M$ witness that $\texttt{Comp}(u,v) \;\&\; \widetilde{\varphi}$ holds. If $a <_{\mathcal{M}} b$, then $<_{a,b}$ is an exact interpretation of $<_{\mathcal{M}}$. That is, for $x$ and $y$ in $M$, we have $x <_{\mathcal{M}} y$ if and only if $x <_{a,b} y$, and thus, $\mathcal{M} \models \varphi$. If $b <_{\mathcal{M}} a$, then we have for $x$ and $y$ in $M$, $x <_{\mathcal{M}} y$ if and only if $y <_{a,b} x$ in $I$, so $\mathcal{M}^{\mathtt{rev}} \models \varphi$. □

Now, suppose that $\mathcal{M}_0$ and $\mathcal{M}_1$ satisfy the assumption of the theorem, and that neither $\mathcal{M}_0 \equiv \mathcal{M}_1$ nor $\mathcal{M}_0 \equiv \mathcal{M}_1^{\mathtt{rev}}$. Let $\varphi$ be a sentence true in $\mathcal{M}_0$ but not in $\mathcal{M}_1$, and let $\psi$ be a sentence true in $\mathcal{M}_0$ but not in $\mathcal{M}_1^{\mathtt{rev}}$. Then $\mathcal{M}_0 \models \varphi \,\&\, \psi$, so, by Lemma 4.1.1, we have $I_0 \models (\varphi \,\&\, \psi)^s$. By hypothesis, $I_1 \models (\varphi \,\&\, \psi)^s$ as well, but $\varphi \,\&\, \psi$ is true neither in $\mathcal{M}_1$ nor in $\mathcal{M}_1^{\mathtt{rev}}$. This contradicts Lemma 4.1.2. □

The following theorem is a generalization of a result by Lipacheva from [7, 8].

**Theorem 4.2.** *If $\mathcal{M}_0$ and $\mathcal{M}_1$ are strict partial orderings and $I_i$ are inverse semigroups such that*

$$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i) \text{ for } i = 0, 1,$$

*then*

$$I_0 \cong I_1 \Rightarrow (\mathcal{M}_0 \cong \mathcal{M}_1 \vee \mathcal{M}_0 \cong \mathcal{M}_1^{\mathtt{rev}}).$$

*Proof.* Using Proposition 2.2, extend the initial isomorphism between the inverse semigroups $I_0$ and $I_1$ to an isomorphism between $I_0^*$ and $I_1^*$. Without loss of generality, we may consider the sorts of the structures $I_i^*$ to be disjoint sets, so that we may denote the union of both components of this isomorphism by $\lambda$. Fix a pair $a, b \in M_0$ such that $a < b$. It follows from the considerations above that the elements $\lambda(a)$ and $\lambda(b)$ are comparable.

If $\lambda(a) < \lambda(b)$, then the relation $x <_{\lambda(a),\lambda(b)} y$ defines the ordering $<$ on $M_1$. Thus, we have $\mathcal{M}_0 \cong \mathcal{M}_1$. If $\lambda(b) < \lambda(a)$, then the same formula $x <_{\lambda(a),\lambda(b)} y$ defines the ordering $<^{\mathtt{rev}}$. Thus, we have $\mathcal{M}_0 \cong \mathcal{M}_1^{\mathtt{rev}}$. □

## 5. Relatively complemented distributive lattices

A strict partial ordering $\mathcal{B} = \langle B, < \rangle$ with smallest element 0 is called a *relatively complemented distributive lattice* (RCDL) if it is a distributive lattice and for all $a, b$ with $a \leqslant b$ in $B$, there exists the unique relative complement of $a$ in $b$ (that is, an element $a'$ such that $\sup\{a, a'\} = b$ and $\inf\{a, a'\} = 0$). It can be proven that for each element $a \in B$, the structure $\langle \widehat{a}, < \rangle$, where $\widehat{a} = \{x \in B \mid x \leqslant a\}$, is a Boolean algebra in which $a$ is the maximal element. Relatively complemented distributive lattices can also be considered in the language $\{\cap, \cup, \setminus, 0\}$, where 0 is the smallest element of $\mathcal{B}$, $x \cap y = \inf\{x, y\}$, $x \cup y = \sup\{x, y\}$, and

$x \setminus y = z$ if $z$ is the relative complement of $x \cap y$ in $x$. For more details on RCDLs, see [3].

If we change the language of a RCDL from $\{<\}$ to $\{\cap, \cup, \setminus, 0\}$, the semigroup $I_{fin}$ changes as well. To see this, let $\mathcal{B}$ be a RCDL, let $a$, $b$, $c$ be pairwise distinct elements such that $a \cap b = c$, and let $a'$, $b'$, $c'$ be pairwise distinct elements such that $a' \cap b' \neq c'$ but $c' < a', b'$, and $a'$, $b'$ are incomparable. Then the mapping $p$ with $\mathtt{dom}(p) = \{a, b, c\}$ taking $a$ to $a'$, $b$ to $b'$, and $c$ to $c'$ is a member of $I_{fin}(\mathcal{B})$ with respect to the language $\{<\}$ but not with respect to the language $\{\cap, \cup, \setminus, 0\}$. Thus, different languages must be handled separately.

First, we consider RCDLs in the language $\{<\}$. From Theorems 4.1 and 4.2, we immediately obtain the following result.

**Corollary 5.1.** *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be RCDLs considered in the language $\{<\}$. Let $I_i$ be inverse semigroups such that $I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i)$ for $i = 0, 1$. Then*

(1) $I_0 \equiv I_1 \Rightarrow \mathcal{B}_0 \equiv \mathcal{B}_1$, *and*
(2) $I_0 \cong I_1 \Rightarrow \mathcal{B}_0 \cong \mathcal{B}_1$.

*Proof.* If $I_0 \equiv I_1$ (or $I_0 \cong I_1$), then Theorem 4.1 implies that $\mathcal{B}_0 \equiv \mathcal{B}_1$ or $\mathcal{B}_0 \equiv \mathcal{B}_1^{\mathtt{rev}}$ (and, by Theorem 4.2, the respective result holds in the case when $I_0 \cong I_1$). If $\mathcal{B}_0 \equiv \mathcal{B}_1$, then the results follow immediately. If $\mathcal{B}_0 \equiv \mathcal{B}_1^{\mathtt{rev}}$, then we have that both $\mathcal{B}_0$ and $\mathcal{B}_1$ have greatest elements, so they are Boolean algebras. However, for any Boolean algebra $\mathcal{B}$, the mapping taking each element $b$ to its complement $\overline{b}$ in $\mathcal{B}$ is an isomorphism from the ordering $\mathcal{B}$ onto the ordering $\mathcal{B}^{\mathtt{rev}}$, since it is $1-1$ and $x < y$ is equivalent to $\overline{y} < \overline{x}$. Thus, $\mathcal{B}_1 \cong \mathcal{B}_1^{\mathtt{rev}}$, which implies $\mathcal{B}_1 \equiv \mathcal{B}_1^{\mathtt{rev}}$, and the results follow immediately. □

Similar results can be obtained for RCDLs when the language is $\{\cap, \cup, \setminus, 0\}$.

**Theorem 5.2.** *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be RCDLs considered in the predicate language $\{\cap, \cup, \setminus, 0\}$ (we mean that all the mentioned operations are replaced with their graphs). Let $I_i$ be an inverse semigroup such that $I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i)$ for $i = 0, 1$. Then*

(1) $I_0 \equiv I_1 \Rightarrow \mathcal{B}_0 \equiv \mathcal{B}_1$, *and*
(2) $I_0 \cong I_1 \Rightarrow \mathcal{B}_0 \cong \mathcal{B}_1$.

*Proof.* If $\mathcal{B}$ is a RCDL and $I_{fin}(\mathcal{B}) \subseteq I \subseteq I(\mathcal{B})$, we can distinguish the element 0 of $B$ in the structure $I^*$ as it is the unique element of the universe satisfying the first-order formula:

$$(x \neq \mathbf{\Lambda}) \ \& \ \forall p \ [\ p(x) = x \vee p(x) = \mathbf{\Lambda}\ ].$$

Using this formula, we can distinguish the set of pairs of distinct comparable elements of the set $B\setminus\{0\}$ in the structure $I^*$ with a first-order formula $\mathtt{Comp}_1(a,b)$:

$$(a \neq b \ \&\ a, b \notin \{0, \mathbf{\Lambda}\}) \ \&\ \neg\exists p\ [\ p(a) = b \ \&\ p(b) = a\ ].$$

Indeed, if $a, b \neq 0$ are comparable (for instance, if $a < b$, so $a \cap b = a$), and a witness $p$ for the subformula following the negation does exist, then we would have that $p(a) \cap p(b) = p(a)$, which implies the false statement $a = b \cap a = b$. On the other hand, if $\mathtt{Comp}_1(a,b)$ holds, and $a$ and $b$ are incomparable, then let $p = \{\langle a, b\rangle, \langle b, a\rangle\}$. If $p \in I_{fin}(\mathcal{B})$, we will have obtained a contradiction. The preservation (under $p$) of the predicates $\cap$ and $\cup$ is obvious. As for the predicate $\setminus$, the cases $a \setminus a \in \{a, b\}$, $b \setminus b \in \{a, b\}$, $a \setminus b = b$, and $b \setminus a = a$ are impossible, since $a$ and $b$ are nonzero. Now, assume that $a \setminus b = a$. This is equivalent to the condition $b \setminus a = b$, so, by definition of $p$, $p(a) \setminus p(b) = p(a)$ and $p$ preserves $\setminus$. We conclude that $p$ is a finite partial automorphism.

Now we can distinguish the pairs $\langle a, b\rangle$ such that $\mathtt{Comp}_1(a,b)$ and $a < b$. One can easily check that this property is equivalent to the following condition, which can be easily expressed by a first-order formula:

$$\begin{aligned}&\mathtt{Comp}_1(a,b)\ \&\\ &\exists x \neq 0\,[x <_{a,b} b \ \&\ (a \text{ and } x \text{ have no } <_{a,b}\text{-lower bound in } B\setminus\{0\})].\end{aligned}$$

Using this property and the fact that for such $a$ and $b$, the ordering $<_{a,b}$ coincides with the usual ordering given by the formula

$$x < y \Leftrightarrow_{\mathtt{def}} x \cap y = x,$$

we can define the ordering $<$ on the whole $B$ as

$$<_{a,b} \cup \{\langle 0, b\rangle \mid b \in B \ \&\ b \neq 0\}.$$

Then we can define the operations $\cap$, $\cup$, and $\setminus$ by first-order formulas using $<$ as we have just interpreted it.

The considerations above together with Proposition 2.2 imply the existence of a function $^\sharp$ from the language $\sigma = \{\cap, \cup, \setminus, 0\}$ into the language of inverse semigroups such that for every RCDL $\mathcal{B}$ and every inverse semigroup $I$ with $I_{fin}(\mathcal{B}) \subseteq I \subseteq I(\mathcal{B})$, we have that for every sentence $\varphi$ of the language $\sigma$,

$$\mathcal{B} \models \varphi \ \Leftrightarrow\ I \models \varphi^\sharp$$

holds, which establishes (1). Moreover, it follows that such $I$ uniquely defines the isomorphism type of $\mathcal{B}$, which gives (2). $\square$

To formulate and establish our further results, we need some results on presentations of countable RCDLs. Similar results for Boolean algebras are well-known and may be found, for instance, in [5]. The proofs of these results for RCDLs are based on similar ideas.

As usual, a nonzero element $a$ of a RCDL is called an *atom* if there is no $x$ with the property $0 < x < a$. An element $b$ is called *atomless* if there is no atom $a \leqslant b$. An RCDL is called *atomless* if it contains no atoms.

In the following proposition, we consider RCDLs in the language $\{\cap, \cup, \setminus, 0\}$.

**Proposition 5.3.** (1) *There exists a unique, up to isomorphism, countable RCDL with no atoms and no greatest element.*
(2) *There exists a unique, up to computable isomorphism, countable computable RCDL with no atoms and no greatest element.*

*Proof.* Let $\mathcal{B}$ be an arbitrary RCDL. Note that a RCDL generated by a finite family $\{a_0, \ldots, a_{n-1}\} \subseteq B$ is finite and consists of all possible unions of elements of the form

$$a_0^{\varepsilon_0} \cap a_1^{\varepsilon_1} \cap \ldots \cap a_{n-1}^{\varepsilon_{n-1}}, \tag{6}$$

where $\varepsilon_i \in \{0, 1\}$ for $i < n$ and

$$a_i^{\varepsilon_i} = \begin{cases} a_i & \text{if } \varepsilon_i = 1, \\ \left[\bigcup_{j<n} a_j\right] \setminus a_i & \text{if } \varepsilon_i = 0. \end{cases}$$

The nonzero elements of the form as in (6) are atoms of this algebra. It follows that each element is the union of a finite family of these atoms.

To see that there is a countable atomless RCDL with no greatest element, consider the algebra of all subsets of the ordered set of rationals $\mathbb{Q}$ with the usual set-theoretic operations $\cap, \cup, \setminus, \emptyset$. Its subalgebra generated by all elements of the form $[a, b) = \{x \in \mathbb{Q} \mid a \leqslant x < b\}$, where $a, b \in \mathbb{Q}$, is an atomless countable RCDL. Moreover, using an appropriate coding of the rational numbers, we can easily prove that this algebra has a computable isomorphic copy.

Now, consider two countable atomless RCDLs $\mathcal{A}$ and $\mathcal{B}$ with no greatest elements, and fix enumerations of their respective universes:

$$A = \{a_0, a_1, \ldots\},$$

$$B = \{b_0, b_1, \ldots\}.$$

Assume that we have already established an isomorphism $f$ between two finitely generated subalgebras $\mathcal{A}' \leqslant \mathcal{A}$ and $\mathcal{B}' \leqslant \mathcal{B}$.

Take an arbitrary element $a \in A$. We extend this isomorphism to an isomorphism $f' \supseteq f$ between the finitely generated extensions of $\mathcal{A}'$ and $\mathcal{B}'$ so that $a \in \mathtt{dom}(f')$.

For $x \in \mathtt{dom}(f)$, let $f'(x) = f(x)$. Let $c_0, \ldots, c_{k-1}$ be a list of all atoms of $\mathcal{A}'$. Then the atoms of the RCDL generated by the set $A' \cup \{a\}$ are exactly all nonzero elements among those of the form $c_i \cap a$ or $c_i \setminus a$, for $i < k$, together with $a \setminus \big(\bigcup_{i<k} c_i\big)$. If $a \setminus \big(\bigcup_{i<k} c_i\big) \neq 0$, find an element $b \in B$ greater than $\bigcup_{i<k} f(c_i)$, and let

$$f'(a \setminus \left(\bigcup_{i<k} c_i\right)) = b \setminus \left(\bigcup_{i<k} f(c_i)\right).$$

Then for each $i < k$, if $c_i \cap a \neq 0$ and $c_i \setminus a \neq 0$, find an element $b$ strictly between 0 and $f(c_i)$, and let $f'(c_i \cap a) = b$ and $f'(c_i \setminus a) = f(c_i) \setminus b$. Thus, $f'$ is defined on all atoms of the RCDL generated by the set $\mathcal{A}' \cup \{a\}$. Extend the mapping defined so far to an isomorphism $f'$ from this algebra into $\mathcal{B}$. One can easily see that $f \subseteq f'$.

Similarly, we can show that for an arbitrary element $b \in B$, we can extend $f$ to an isomorphism $f' \supseteq f$ between the finitely generated extensions of $\mathcal{A}'$ and $\mathcal{B}'$ so that $b \in \mathtt{ran}(f')$.

Using this construction, we can start from the empty funciton $f_0 = \emptyset$ and extend it by adding new elements to its domain and range so that each $a_i \in A$ and $b_i \in B$ eventually appear in the domain and range of the isomorphism from $\mathcal{A}$ to $\mathcal{B}$ we are constructing.

If $\mathcal{A}$ and $\mathcal{B}$ are computable algebras, we can assume that $a_i = i$ and $b_i = i$ for $i = 0, 1, \ldots$ Then we can execute the above procedure algorithmically by selecting the least element available at each stage. The resulting isomorphism between $\mathcal{A}$ and $\mathcal{B}$ will be computable. $\square$

The presentations of countable Boolean algebras by subtrees of $2^{<\omega}$ are well-known (see [5]). For presentations of RCDLs, we modify this approach since the "trees" appear to be not well-founded, since a RCDL may have no greatest element. Specifically, let $2^{<\mathbb{Z}}$ be the countable set consisting of all functions $f$ from sets of the form $\mathbb{Z} \upharpoonright m = \{x \in \mathbb{Z} \mid x < m\}$ into the set $2 = \{0, 1\}$ such that the set $\{x \mid f(x) \neq 0\}$ is finite. The set $2^{<\mathbb{Z}}$ admits a usual coding by natural numbers such that given the number of any $f \in 2^{<\mathbb{Z}}$, we can effectively compute the maximal element of $\mathtt{dom}(f)$ and the index for the finite set $\{x \mid f(x) \neq 0\}$. Fix such a coding and identify elements of $2^{<\mathbb{Z}}$ with their codes.

For a function $f \in 2^{<\mathbb{Z}}$, if $k = \sup(\mathtt{dom}(f))$, we let

$$\begin{aligned} f^- &= f \setminus \{\langle k, f(k)\rangle\}, \\ f^\frown 0 &= f \cup \{\langle k+1, 0\rangle\}, \\ f^\frown 1 &= f \cup \{\langle k+1, 1\rangle\}. \end{aligned}$$

The element $s^-$ is called the *predecessor* of $f$. The elements $f^\frown 0$ and $f^\frown 1$ are called the *successors* of $f$. The elements of the set $2^{<\mathbb{Z}}$ could be thought of as elements of an infinite $\{0,1\}$-branching tree with no root, so each $f \in 2^{<\mathbb{Z}}$ splits into $f^\frown 0$ and $f^\frown 1$, and $f^-$ precedes $f$, and so on. The set $2^{<\mathbb{Z}}$ is naturally ordered by inclusion.

Denote by $\mathcal{F}$ the computable nontrivial atomless RCDL with no greatest element.

**Proposition 5.4.** *There exists a computable $1-1$ function $\theta$ from the set $2^{<\mathbb{Z}}$ into $\mathcal{F}$ so that the following conditions are satisfied.*

(1) *The set $\mathtt{ran}(\theta)$ generates $\mathcal{F}$. Moreover, each element of $\mathcal{F}$ is the union of a finite subset of $\mathtt{ran}(\theta)$.*
(2) *For all $f, g \in 2^{<\mathbb{Z}}$,*
$$f \subseteq g \Leftrightarrow \theta(f) \geqslant \theta(g).$$
(3) *For all $f \in 2^{<\mathbb{Z}}$,*
$$\theta(f^\frown 0) \cup \theta(f^\frown 1) = \theta(f) \text{ and } \theta(f^\frown 0) \cap \theta(f^\frown 1) = 0.$$
(4) *There exists an element $a$ in $\mathcal{F}$ and computable automorphisms $\varphi, \psi$ of $\mathcal{F}$ such that for every $b \in \mathtt{ran}(\theta)$, there exist $k, m \in \mathbb{Z}$ and $l \in \omega$ such that $b = \varphi^k \psi^l \varphi^m(a)$.*

*Proof.* We assume that the universe of $\mathcal{F}$ is the set of natural numbers, all the operations $\cap, \cup, \setminus, 0$ are computable, and that the natural number $0$ is the value for the constant $0_{\mathcal{F}}$ denoting the least element. We will write $a_i$ instead of $i$ when we refer to elements of $\mathcal{F}$. Define the function $\theta$ as follows.

First, define $f_0 \in 2^{<\mathbb{Z}}$ by

$$f_0 = \{\langle z, 0\rangle \mid z \in \mathbb{Z} \ \&\ z \leqslant 0\}.$$

Let $\theta_0(f_0) = a_1$.

Assume that a finite function $\theta_i$, for $i \in \omega$, is already defined and that the following conditions are satisfied:

1. $\mathtt{dom}(\theta_i)$ contains a least element $f^*$ under inclusion such that $\mathtt{ran}(f^*) = \{0\}$,
2. for all $f$, if $f \in \mathtt{dom}(\theta_i) \setminus \{f^*\}$, then $f^- \in \mathtt{dom}(\theta_i)$,
3. for all $f \in \mathtt{dom}(\theta_i)$, $f^\frown 0 \in \mathtt{dom}(\theta_i) \Leftrightarrow f^\frown 1 \in \mathtt{dom}(\theta_i)$,
4. $0 \notin \mathtt{ran}(\theta_i)$,

5. for all $f$, if $f^\frown 0, f^\frown 1 \in \mathtt{dom}(\theta_i)$, then $\theta_i(f^\frown 0) \cup \theta_i(f^\frown 1) = \theta_i(f)$ and $\theta_i(f^\frown 0) \cap \theta_i(f^\frown 1) = 0$.

It follows that if $f \in \mathtt{dom}(\theta_i)$ has no successors in $\mathtt{dom}(\theta_i)$, then $\theta_i(f)$ is an atom of the algebra generated by $\mathtt{ran}(\theta_i)$. This algebra is generated by such atoms, and each of its elements is the union of a finite set of such atoms. Furthermore, all atoms of this algebra have this form.

For each pair of elements $a_{i+2} \cap \theta_i(f)$ and $a_{i+2} \setminus \theta_i(f)$ such that both of these elements differ from 0, and $f$ has no successors in $\mathtt{dom}(\theta_i)$, let $\theta_{i+1}(f^\frown 0) = a_{i+2} \cap \theta_i(f)$ and $\theta_{i+1}(f^\frown 1) = a_{i+2} \setminus \theta_i(f)$. Furthermore, if $a_{i+2} \cup \theta_i(f^*) > \theta_i(f^*)$, we let $\theta_{i+1}((f^*)^-) = \theta_i(f^*) \cup a_{i+2}$ and $\theta_{i+1}((f^*)^{-\frown} 1) = a_{i+2} \setminus \theta_i(f^*)$. We also let $\theta_{i+1}(f) = \theta_i(f)$ for all $f \in \mathtt{dom}(\theta_i)$.

Let $\theta = \bigcup_{i\in\omega} \theta_i$. Since $\mathcal{F}$ is atomless and has no greatest element, $\mathtt{dom}(\theta) = 2^{<\mathbb{Z}}$. Parts (1)–(3) of the proposition follow immediately from the construction.

To establish (4), we define the automorphisms $\varphi$ and $\psi$. The automorphism $\varphi$ shifts the generators of $\mathcal{F}$. It is defined on generators $\theta(f)$ as follows:

$$\varphi(\theta(f)) = \theta(f^+),$$

where

$$f^+(i+1) = \begin{cases} f(i) & \text{if } i \in \mathtt{dom}(f), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We wish to define the automorphism $\psi$ so that it will permute generators smaller than $\theta(f_0)$ in such a way that each $\theta(f_0{}^\frown\varepsilon_1{}^\frown \ldots {}^\frown\varepsilon_k)$ for $\varepsilon_1, \ldots, \varepsilon_k \in \{0,1\}$ can be expressed as $\psi^j(\theta(f_0{}^\frown \underbrace{0^\frown \cdots {}^\frown 0}_{k \text{ times}}))$ for some $j \in \omega$. First, we define an automorphism $\tau$ on the structure $\langle 2^{<\mathbb{Z}}; \subseteq \rangle$, and then we let $\psi(\theta(f)) = \theta(\tau(f))$. Let $\tau(f) = f$ for all $f \in 2^{<\mathbb{Z}}$ such that $f \subseteq f_0$. In particular, $\tau(f_0) = f_0$. Assume that we have already defined all values of $\tau$ on the set

$$F_k = \{ f_0{}^\frown\varepsilon_1{}^\frown \ldots {}^\frown\varepsilon_k \mid \varepsilon_1, \ldots, \varepsilon_k \in \{0,1\}\}$$

for some $k \in \omega$, so that $\tau$ forms a single cycle on the set $F_k$. That is, $\tau$ acts on $F_k$ as follows:

$$g_0 \xrightarrow{\tau} g_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} g_{2^k} \xrightarrow{\tau} g_0.$$

Then $\tau$ will act on $F_{k+1}$ as follows:

$$g_0{}^\frown 0 \xrightarrow{\tau} g_1{}^\frown 0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} g_{2^k}{}^\frown 0 \xrightarrow{\tau} g_0{}^\frown 1 \xrightarrow{\tau}$$
$$\xrightarrow{\tau} g_1{}^\frown 1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} g_{2^k}{}^\frown 1 \xrightarrow{\tau} g_0{}^\frown 0.$$

We now let $\psi(\theta(f)) = \theta(\tau(f))$.

The automorphism $\varphi^k$ moves $\theta(f_0)$ along some branch of the tree $2^{<\mathbb{Z}}$, consisting of all elements of the form $\theta(f)$ with $\mathtt{ran}\,(f) = \{0\}$, and shifts all of the elements below them. The automorphism $\psi^k$ identifies all the elements on the same level below $\theta(f_0)$. It is immediate from the construction that $\varphi$ and $\psi$ are computable. Thus, we have established (4). □

**Proposition 5.5.** *For each computable RCDL $\mathcal{B}$, there exists a computable isomorphic embedding from $\mathcal{B}$ into $\mathcal{F}$.*

*Proof.* We construct this embedding as the union of an increasing chain of embeddings of finitely generated subalgebras. Let $B = \{b_0, b_1, \ldots\}$, and let all the basic operations be computable on the indices of the elements $b_i$.

*Stage* 0. Let $f_0 = \{\langle 0_{\mathcal{B}}, 0_{\mathcal{F}}\rangle\}$, where $0_{\mathcal{B}}$ and $0_{\mathcal{F}}$ are the least elements in the corresponding structures.

*Stage* $t+1$. Assume that $f_t$ is already defined and its domain is a finite subalgebra of $\mathcal{B}$. For all $x \in \mathtt{dom}(f_t)$, we set $f_{t+1}(x) = f_t(x)$. For each atom $\alpha$ in $\mathcal{B}$, we execute the following. If both elements $\alpha \cap b_t$ and $\alpha \setminus b_t$ are not equal to 0, we let $f_{t+1}(\alpha \cap b_t)$ be equal to the element $c \in \mathcal{F}$ with minimal index so that $0 < c < f_t(\alpha)$, and we let $f_{t+1}(\alpha \setminus b_t)$ be $f_t(\alpha) \setminus c$. Now extend $f_{t+1}$ to an isomorphism from the RCDL generated by its domain into $\mathcal{F}$.

It is clear that $f_0 \subseteq f_1 \subseteq \ldots \subseteq f_k \subseteq \ldots$ and that $\bigcup_{i\in\omega} f_i$ is computable. Since $b_t \in \mathtt{dom}(f_{t+1})$, we have $\mathtt{dom}(f) = B$, and $f$ is the required embedding. □

The following theorem shows that in many cases we can restore a computable RCDL, up to computable isomorphism, from its inverse semigroup of partial computable automorphisms.

**Theorem 5.6.** *Assume that $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable RCDLs in the language $\{\cap, \cup, \setminus, 0\}$. Suppose that there exists a computable isomorphic embedding of $\mathcal{F}$ into $\mathcal{B}_0$. Then*

$$I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1) \Rightarrow \mathcal{B}_0 \cong_c \mathcal{B}_1.$$

*Proof.* Assume that $\mathcal{B}$ is a computable RCDL. It was established in the proof of Theorem 5.2 that the ordering $<$, the operations $\cap$, $\cup$, and $\setminus$, and the constant 0 are first-order definable in $I^*$, for any inverse subsemigroup $I$ such that $I_{fin}(\mathcal{B}) \subseteq I \subseteq I(\mathcal{B})$. Furthermore, the defining formulas do not depend on $I$. Thus, if $I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1)$, then

$$\langle I_c(\mathcal{B}_0)^*; <, \cap, \cup, \setminus, 0\rangle \cong \langle I_c(\mathcal{B}_1)^*; <, \cap, \cup, \setminus, 0\rangle\,,$$

or, more specifically,

$$\langle I_c(\mathcal{B}_0), B_0; \mathtt{ap}\,, \cdot, {}^{-1}, <, \cap, \cup, \setminus, 0\rangle \cong \langle I_c(\mathcal{B}_1), B_1; \mathtt{ap}\,, \cdot, {}^{-1}, <, \cap, \cup, \setminus, 0\rangle\,. \tag{7}$$

Let $\beta$ be a computable isomorphic embedding of $\mathcal{F}$ into $\mathcal{B}_0$. Fix a computable isomorphic embedding $\gamma$ of $\mathcal{B}_0$ into $\mathcal{F}$, which exists by Proposition 5.5. Their composition $\xi = \beta \circ \gamma$ is a computable isomorphic embedding from $\mathcal{B}_0$ to $\mathcal{B}_0$. Fix partial computable automorphisms $\varphi, \psi \in I_c(\mathcal{F})$ together with an element $a$, as in Proposition 5.4.

Take an arbitrary $b \in B_0$. By Proposition 5.4, the element $\gamma(b)$ can be represented as the union

$$\gamma(b) = \bigcup_{i=1}^{n-1} \varphi^{k_i}\psi^{l_i}\varphi^{m_i}(a)$$

for appropriate $n \in \omega$, and $k_i$, $l_i$, $m_i$ for $i = 1, \dots, n$. It follows that the element $\xi(b) = \beta\gamma(b)$ can be represented as

$$\xi(b) = \bigcup_{i=1}^{n-1} (\beta\varphi\beta^{-1})^{k_i}(\beta\psi\beta^{-1})^{l_i}(\beta\varphi\beta^{-1})^{m_i}\beta(a).$$

Denote $\Phi = \beta\varphi\beta^{-1}$, $\Psi = \beta\psi\beta^{-1}$, and $a' = \beta(a)$. Note that $\xi, \Phi, \Psi \in I_c(\mathcal{B}_0)$ and that the following condition is satisfied:

*For all $b \in B_0$, there exist $n \in \omega$ and integers $k_i, l_i, m_i$,* for $i < n$, *such that*

$$\xi(b) = \bigcup_{i=1}^{n-1} \Phi^{k_i}\Psi^{l_i}\Phi^{m_i}(a'). \tag{8}$$

Denote the isomorphic images of $\xi$, $\Phi$, $\Psi$, $b$, $a'$ with respect to the isomorphism (7) by $\xi_1$, $\Phi_1$, $\Psi_1$, $b_1$, $a_1$, respectively. Then we have

$$\xi_1(b_1) = \bigcup_{i=1}^{n-1} \Phi_1^{k_i}\Psi_1^{l_i}\Phi_1^{m_i}(a_1). \tag{9}$$

This gives us the following algorithm to compute the isomorphism between $\mathcal{B}_0$ and $\mathcal{B}_1$.

> Given $b \in B_0$, use exhaustive search over all $n, k_i, l_i, m_i$, for $i < n$, to find a decomposition for $\xi(b)$ of the form in (8). Then define the isomorphic image $b_1$ of $b$ as the unique element of $B_1$ satisfying (9).

$\square$

Let $a$ be an element of a Boolean algebra $\mathcal{B}$. We let

$$\widehat{a} = \{x \in B \mid \ x \leqslant a\}.$$

Denote by $\mathcal{F}_1$ a fixed computable atomless Boolean algebra. It is unique, up to computable isomorphism (for example, see [5]).

Lipacheva [7, 8] proved that if $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable Boolean algebras such that $I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1)$, and $\mathcal{B}_0$ contains a nontrivial atomless element, then $\mathcal{B}_0$ and $\mathcal{B}_1$ are computably isomorphic. The proof of her result gives us the analogue of Theorem 5.6 for Boolean algebras.

**Theorem 5.7.** *Assume that $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable Boolean algebras in the language $\{\cap, \cup, \overline{\phantom{x}}, 0, 1\}$ and that there exists a computable isomorphic embedding of $\mathcal{F}_1$ into $\mathcal{B}_0$. Then*

$$I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1) \Rightarrow \mathcal{B}_0 \cong_c \mathcal{B}_1.$$

Note that if $\mathcal{B}_0$ contains a nontrivial atomless element, then it satisfies the condition of the theorem. However, there exist Boolean algebras without nontrivial atomless elements satisfying this condition. Thus, it is natural to look for counterexamples among atomic Boolean algebras.

Morozov [9] showed that if $\mathcal{B}_0$ is a nontrivial atomic computable Boolean algebra with a computable set of atoms and $I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1)$ for a Boolean algebra $\mathcal{B}_1$, then $\mathcal{B}_0$ and $\mathcal{B}_1$ are computably isomorphic. It follows easily from our analysis above that the group of computable automorphisms of a computable Boolean algebra is definable in its inverse semigroup of partial computable automorphisms. Therefore, the implication

$$I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1) \Rightarrow \mathcal{B}_0 \cong_c \mathcal{B}_1$$

remains true when $\mathcal{B}_0$ is a nontrivial atomic computable Boolean algebra with a computable set of atoms.

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Department of Mathematics, George Washington University, Washington, DC 20052, jchubb@gwu.edu

Department of Mathematics, George Washington University, Washington, DC 20052, harizanv@gwu.edu

Sobolev Institute of Mathematics, 630090 Novosibirsk, Russia, morozov@math.nsc.edu

Department of Mathematics, George Washington University, Washington, DC 20052, spingrey@gwu.edu

Department of Mathematics, George Washington University, Washington, DC 20052, ufferman@gmail.com