

Dimensions, matroids, and dense pairs of first-order structures Version 2.8

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Abstract

A structure M is pregeometric if the algebraic closure is a pregeometry in all M' elementarily equivalent to M . We define a generalisation: structures with an existential matroid. The main examples are superstable groups of U-rank a power of ω and d-minimal expansion of fields. Ultraproducts of pregeometric structures expanding a field, while not pregeometric in general, do have a unique existential matroid.

Generalising previous results by L. van den Dries, we define dense elementary pairs of structures expanding a field and with an existential matroid, and we show that the corresponding theories have natural completions, whose models also have a unique existential matroid. We also extend the above result to dense tuples of structures.

Key words: Geometric structure; pregeometry; matroid; lovely pair; dense pair.

MSC2000: Primary 03Cxx; Secondary 03C64.

Contents

1	Introduction	2
2	Notations and conventions	4
3	Matroids	4
	3.1 Definable matroids	7
	3.2 Dimension	10
	3.3 Morley sequences	17
	3.4 Local properties of dimension	20
4	Matroids from dimensions	21
5	Expansions	23

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6	Extension to imaginary elements	25
7	Density	28
8	Dense pairs	30
8.1	Small sets	31
8.2	Proof of Theorems 8.3 and 8.5	33
8.3	Additional facts	35
8.4	The small closure	38
8.5	Elimination of imaginaries	42
9	D-minimal topological structures	43
10	Cl-minimal structures	49
11	Connected groups	50
12	Ultraproducts	54
13	Dense tuples of structures	56
13.1	Dense tuples of topological structures	57
14	The (pre)geometric case	58

1 Introduction

A theory T is called pregeometric [HP94, Gagelman05] if, in every model $\mathbb{K}$ of T , acl satisfies the Exchange Principle (and, therefore, acl is a pregeometry on $\mathbb{K}$); if T is complete, it suffices to check that acl satisfies EP in one ω -saturated model of T . T is geometric if it is pregeometric and eliminates the quantifiers $\exists^\infty$. We call a structure $\mathbb{K}$ (pre)geometric if its theory is (pre)geometric (thus, $\mathbb{K}$ is pregeometric iff there exists an ω -saturated elementary extension $\mathbb{K}'$ of $\mathbb{K}$ such that acl satisfies EP in $\mathbb{K}'$).

Note that a pregeometric expansion of a field is geometric ([DMS08, 1.18], see also Lemma 3.46).

In the remainder of this introduction, all theories and all structures expand a field; in the body of the article we will sometimes state definitions and results without this assumption.

Geometric structures are ubiquitous in model theory: if $\mathbb{K}$ is either o-minimal, or strongly minimal, or a p -adic field, or a pseudo-finite field (or more generally a perfect PAC field, see [CDM92] and [HP94, 2.12]), then $\mathbb{K}$ is geometric.

However, ultraproducts of geometric structures (even strongly minimal ones) are not geometric in general. We will show that there is a more general notion, structures with existential matroids, which instead is robust under taking ultraproducts. More in details, we consider structures $\mathbb{K}$ with a matroid cl that satisfies some natural conditions (cl is an “existential matroid”).

Under our hypothesis that $\mathbb{K}$ is a field, then there is at most one existential matroid on $\mathbb{K}$. An (almost) equivalent notion has already been studied by van den Dries [Dries89]: we will show that, if $\mathbb{M}$ is a monster model, an existential matroid on $\mathbb{M}$ induces a (unique) dimension function on the definable subset of $\mathbb{M}^n$, satisfying the axioms in [Dries89], and conversely, any such dimension function, satisfying a slightly stronger version of the axioms, will be induced by a (unique) existential matroid. Moreover, a superstable group $\mathbb{K}$ of U-rank a power of ω is naturally endowed by an existential matroid (van den Dries [Dries89, 2.25] noticed this already in the case when $\mathbb{K}$ is a differential field of characteristic 0).

Given a geometric structure $\mathbb{K}$, there is an abstract notion of dense subsets of $\mathbb{K}$, which specialises to the usual topological notion in the case of o-minimal structures or of formally p -adic fields. More precisely, a subset X of $\mathbb{K}$ is dense in $\mathbb{K}$ if every *infinite* $\mathbb{K}$ -definable subset of $\mathbb{K}$ intersect X [Macintyre75]. If T is a complete geometric theory, then the theory of dense elementary pairs of models of T is complete and consistent (the proof of this fact was already in [Dries98], but the result was stated there only for o-minimal structures).

We consider here the more general case when T is a complete theory and such that a monster model of T has an existential matroid. We show that there is a corresponding abstract notion of density in models of T . Given T as above, consider the theory of pairs $\langle \mathbb{K}, \mathbb{K}' \rangle$, where $\mathbb{K} \prec \mathbb{K}' \models T$ and $\mathbb{K}$ is dense in $\mathbb{K}'$; the theory of such pairs will not be complete in general, but we will show that it will become complete (and consistent) if we add the additional condition that $\mathbb{K}$ is cl-closed in $\mathbb{K}'$ (that is, $\text{cl}(\mathbb{K}) \cap \mathbb{K}' = \mathbb{K}$); we thus obtain the (complete) theory T^d . Moreover T^d also has an existential matroid. This allows us to iterate the above construction, and consider dense cl-closed pairs of models of T^d , which turn out to coincide with nested dense cl-closed triples of models of T ; iterating many times, we can thus study nested dense cl-closed n -tuples of models of T .

Of particular interest are two cases of structures with an existential matroid: the cl-minimal case and the d-minimal one.

A structure $\mathbb{K}$ (with an existential matroid) is cl-minimal if there is only one “generic” 1-type over every subset of $\mathbb{K}$; the prototypes of such structures are given by strongly minimal structures and connected superstable groups of U-rank a power of ω . If T is the theory of $\mathbb{K}$, we show that the condition that $\mathbb{K}$ is dense in $\mathbb{K}'$ is superfluous in the definition of T^d , and that T^d is also cl-minimal.

An first-order topological structure $\mathbb{K}$ (expanding a topological field) is d-minimal if it is Hausdorff, it has an ω -saturated elementary extension $\mathbb{K}'$ such that every definable (unary!) subset of $\mathbb{K}'$ is the union of an open set and finitely many discrete sets, and it satisfies a version of Kuratowski-Ulam’s

theorem for definable subset of $\mathbb{K}^2$ (the “ d ” stands for “discrete”). Examples of d-minimal structures are p -adic fields, o-minimal structures, and d-minimal structures in the sense of Miller. We show that a d-minimal structure has a (unique) existential matroid, and that the notion of density given by the matroid coincides with the topological one. Moreover, if T is the theory of a d-minimal structure, then T^d is the theory of dense elementary pairs of models of T (the condition that $\mathbb{K}$ is a cl-closed subset of $\mathbb{K}'$ is superfluous); hence, in the case when T is o-minimal, we recover van den Dries’ Theorem [Dries98]. However, if T is d-minimal, T^d will not be d-minimal. Moreover, while ultraproducts of o-minimal structures and of formally p -adic fields are d-minimal, ultraproducts of d-minimal structures are not d-minimal in general.

We show that if $\mathbb{K}$ has an existential matroid, then $\mathbb{K}$ is a perfect field: therefore, the theory exposed in this article does not apply to differential fields of finite characteristic, or to separably closed non-perfect fields.

2 Notations and conventions

Let T be a complete theory in some language $\mathcal{L}$, with only infinite models. Let $\kappa > |T|$ be a “big” cardinal. We work inside a κ -saturated and strongly κ -homogeneous model $\mathbb{M}$ of T : we call $\mathbb{M}$ a monster model of T .

A , B , and C , subsets of $\mathbb{M}$ of cardinality less than κ ; by $\bar{a}$, $\bar{b}$, and $\bar{c}$, finite tuples of elements of $\mathbb{M}$; by a , b , and c , elements of $\mathbb{M}$. As usual, we will write, for instance, $\bar{a} \subset A$ to say that $\bar{a}$ is a finite tuple of elements of A , and $A\bar{b}$ to denote the union of A with the set of elements in $\bar{b}$.

Given a set X and $m \leq n \in \mathbb{N}$, denote by Π_m^n the projection from X^n onto the first m coordinates. Given $Y \subseteq X^{n+m}$, $\bar{x} \in X^n$, and $\bar{z} \in X^m$, denote the sections $Y_{\bar{x}} := \{\bar{t} \in X^m : \langle \bar{x}, \bar{t} \rangle \in Y\}$ and $Y^{\bar{z}} := \{\bar{t} \in X^m : \langle \bar{t}, \bar{z} \rangle \in Y\}$.

3 Matroids

sec:matroid

Let cl be a (finitary) closure operator on $\mathbb{M}$: that is, $\text{cl} : \mathcal{P}(\mathbb{M}) \rightarrow \mathcal{P}(\mathbb{M})$ satisfies, for every $X \subseteq \mathbb{M}$:

extension $X \subseteq \text{cl}(X)$;

monotonicity $X \subseteq Y$ implies $\text{cl}(X) \subseteq \text{cl}(Y)$;

idempotency $\text{cl}(\text{cl}(X)) = \text{cl}(X)$;

finitariness $\text{cl}(X) = \bigcup \{\text{cl}(A) : A \subseteq X \text{ \& } A \text{ finite}\}$.

cl is a (finitary) **matroid** (a.k.a. **pregeometry**) if moreover it satisfies the Exchange Principle:

EP $a \in \text{cl}(Xc) \setminus \text{cl}(X)$ implies $c \in \text{cl}(Xa)$.

Proviso. *For the remainder of this section, cl is a finitary matroid on $\mathbb{M}$.*

As is well-known from matroid theory, cl defines notions of rank (which we denote by rk^{cl}), generators, independence, and basis.¹

Definition 3.1. A set A generates C over B if $\text{cl}(AB) = \text{cl}(CB)$. A subset A of $\mathbb{M}$ is independent over B if, for every $a \in A$, $a \notin \text{cl}(Ba' : a' \neq a)$.

lem:Lascar

Lemma 3.2 (Additivity of rank).

$$\text{rk}^{\text{cl}}(\bar{a}\bar{b}/C) = \text{rk}^{\text{cl}}(\bar{a}/\bar{b}C) + \text{rk}^{\text{cl}}(\bar{b}/C).$$

For the axioms of independence relations, we will use the nomenclature in [Adler05].

Definition 3.3. Given an infinite set X , a **pre-independence relation**² on X is a ternary relation $\downarrow$ on $\mathcal{P}(X)$ satisfying the following axioms:

Monotonicity: If $A \downarrow_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \downarrow_C B'$.

Base Monotonicity: If $D \subseteq C \subseteq B$ and $A \downarrow_D B$, then $A \downarrow_C B$.

Transitivity: If $D \subseteq C \subseteq B$, $B \downarrow_C A$, and $C \downarrow_D A$, then $B \downarrow_D A$.

Normality: If $A \downarrow_C B$, then $AC \downarrow_C B$.

Finite Character: If $A_0 \downarrow_C B$ for every finite $A_0 \subseteq A$, then $A \downarrow_C B$.

$\downarrow$ is **symmetric** if moreover it satisfies the following axiom:

Symmetry: $A \downarrow_C B$ iff $B \downarrow_C A$.

¹Sometimes in geometric model theory the “rank” is called “dimension” and/or the “dimension” (defined later) is called “rank”; however, since in many interesting cases (e.g. algebraically closed fields, and o-minimal structures, with the acl matroid) what we call the dimension of a definable set induced by the matroid coincides with the usual notion of dimension given geometrically, our choice of nomenclature is clearly better.

²Pre-independence relations as defined here are slightly different than the ones defined in [Adler05]. However, as we will see later, if cl is definable, then $\downarrow^{\text{cl}}$ is a pre-independence relation in Adler’s sense.

Definition 3.4. The pre-independence relation on $\mathbb{M}$ induced by cl is the ternary relation $\downarrow^{\text{cl}}$ on $\mathcal{P}(\mathbb{M})$ defined by: $X \downarrow_Y^{\text{cl}} Z$ if for every $Z' \subset Z$, if Z' is independent over Y , then Z' remains independent over YX . If $X \downarrow_Y^{\text{cl}} Z$, we say that X and Z are independent over Y (w.r.t. cl).

Remark 3.5. If $X \downarrow_Y^{\text{cl}} Z$, then $\text{cl}(XY) \cap \text{cl}(ZY) = \text{cl}(Y)$.

Lemma 3.6. $\downarrow^{\text{cl}}$ is a symmetric pre-independence relation.

Proof. The same given in [Adler05, Lemma 1.29]. $\square$

Remark 3.7. $\downarrow^{\text{cl}}$ also satisfies the following version of anti-reflexivity:

- $A \downarrow_C^{\text{cl}} B$ iff $\text{cl}(A) \downarrow_{\text{cl}(C)}^{\text{cl}} \text{cl}(B)$;
- $a \downarrow_X^{\text{cl}} a$ iff $a \in \text{cl}(X)$.

Remark 3.8. $X \downarrow_Y^{\text{cl}} Y$.

Lemma 3.9. *T.f.a.e.:*

1. $X \downarrow_Y^{\text{cl}} Z$;
2. $\forall Z'$ such that $Y \subseteq Z' \subseteq \text{cl}(YZ)$, we have $\text{cl}(XZ') \cap \text{cl}(YZ) = \text{cl}(Z')$;
3. there exists $Z' \subseteq Z$ which is a basis of Z/Y , such that Z' remains independent over XY ;
4. for every $Z' \subseteq Z$ which is a basis of Z/Y , Z' remains independent over YX ;
5. if $X' \subseteq X$ is a basis of YX/Y and $Z' \subseteq Z$ is a basis of YZ/Y , then X' and Z' are disjoint, and $X'Z'$ is a basis of XZ over Y ;
6. $\text{rk}^{\text{cl}}(X/YZ) = \text{rk}^{\text{cl}}(X/Y)$.

lem:ind-ternary

Lemma 3.10. Let $\downarrow$ be a symmetric pre-independence relation on some infinite set X . Assume that $\bar{a} \downarrow_C \bar{d}$ and $\bar{a}\bar{d} \downarrow_C \bar{b}$. Then, $\bar{a} \downarrow_C \bar{b}\bar{d}$ and $\bar{d} \downarrow_C \bar{b}\bar{a}$.

Proof. Cf. [Adler05, 1.9]. $\bar{a} \downarrow_C \bar{b}\bar{d}$ implies $\bar{a} \downarrow_{C\bar{b}} \bar{d}$, which implies $\bar{a} \downarrow_{C\bar{b}} \bar{b}\bar{d}$, which, together with $\bar{a} \downarrow_C \bar{d}$, implies $\bar{a} \downarrow_C \bar{b}\bar{d}$. $\square$

lem:ind-sequence

Lemma 3.11. Let $\downarrow$ be a symmetric pre-independence relation on some infinite set X . Let $\langle I, \leq \rangle$ be a linearly ordered set, $(\bar{a}_i : i \in I)$ be a sequence of tuples in X^n , and $C \subset X$. Then, t.f.a.e.:

1. For every $i \in I$, we have $\bar{a}_i \downarrow_C (\bar{a}_j : j < i)$;

2. For every $i \in I$, we have $\bar{a}_i \perp_C (\bar{a}_j : j \neq i)$.

Proof. Assume, for contradiction, that (1) holds, but $a_i \not\perp_C (\bar{a}_j : j \neq i)$, for some $i \in I$. Since $\perp^{\text{cl}}$ satisfies finite character, w.l.o.g. $I = \{1, \dots, m\}$ is finite. Let m' such that $i < m' \leq m$ is minimal with $\bar{a}_i \not\perp_C (\bar{a}_j : j \leq m' \text{ \& } j \neq i)$; w.l.o.g., $m = m'$.

Let $\bar{d} := (a_j : j \neq i \text{ \& } j < m)$. By assumption, $\bar{a}_i \perp_C \bar{d}$ and $\bar{d}\bar{a}_i \perp_C \bar{a}_m$. Then, by Lemma 3.10, we have $\bar{a}_i \perp_C \bar{d}\bar{a}_m$, absurd. $\square$

Definition 3.12. We say that a sequence $(\bar{a}_i : i \in I)$ satisfying one of the above equivalent conditions is an **independent sequence** over C .

Remark 3.13. Let $(a_i : i \in I)$ be a sequence of elements of $\mathbb{M}$. There is a clash with the previous definition of independence; more precisely, let $J := \{i \in I : a_i \notin \text{cl}(C)\}$; then, $(a_i : i \in I)$ is an independent sequence over C according to $\perp^{\text{cl}}$ iff all the a_j are pairwise distinct for $j \in J$, and the set $\{a_j : j \in J\}$ is independent over C according to cl . Hopefully, this will not cause confusion.

3.1 Definable matroids

def:definable

Definition 3.14. Let $\phi(x, \bar{y})$ be an $\mathcal{L}$ -formula. We say that ϕ is **x -narrow** if, for every $\bar{b}$ and every a , if $\mathbb{M} \models \phi(a, \bar{b})$, then $a \in \text{cl}(\bar{b})$. We say that cl is **definable** if, for every A ,

$$\text{cl}(A) = \bigcup \{ \phi(\mathbb{M}, \bar{a}) : \phi(x, \bar{y}) \text{ is } x\text{-narrow}, \bar{a} \in A^n, n \in \mathbb{N} \}.$$

Proviso. For the rest of the section, cl is a definable matroid.

Remark 3.15. For every A and every $\sigma \in \text{Aut}(\mathbb{M})$, $\sigma(\text{cl}(A)) = \text{cl}(\sigma(A))$.

lem:cl-definable

Lemma 3.16. 1. $\perp^{\text{cl}}$ satisfies the Invariance axiom: if $A \perp_B^{\text{cl}} C$ and $\langle A', B', C' \rangle \equiv \langle A, B, C \rangle$, then $A' \perp_{B'}^{\text{cl}} C'$.

2. $\perp^{\text{cl}}$ satisfies the Strong Finite Character axiom: if $A \not\perp_C^{\text{cl}} B$, then there exist finite tuples $\bar{a} \subset A$, $\bar{b} \subset B$, and $\bar{c} \subset C$, and a formula $\phi(\bar{x}, \bar{y}, \bar{z})$ without parameters, such that

- $\mathbb{M} \models \phi(\bar{a}, \bar{b}, \bar{c})$;
- $\bar{a}' \not\perp_C^{\text{cl}} B$ for all $\bar{a}'$ satisfying $\mathbb{M} \models \phi(\bar{a}', \bar{b}, \bar{c})$.

3. For every $\bar{a}$, B , and C , if $\text{tp}(\bar{a}/BC)$ is finitely satisfied in B , then $\bar{a} \perp_B^{\text{cl}} C$.

4. $\downarrow^{\text{cl}}$ satisfies the Local Character axiom: for every A, B there exists a subset C of B such that $|C| \leq |T| + |A|$ and $A \downarrow_C^{\text{cl}} B$.

Proof. (1) is obvious.

(2) Assume that $A \not\downarrow_C^{\text{cl}} B$. Hence, there exists $\bar{b} \in B^n$ independent over C , such that $\bar{b}$ is not independent over AC . Hence, there exists $\bar{a} \subset A$ and $\bar{c} \subset C$ finite tuples, such that, w.l.o.g., $b_1 \in \text{cl}(\bar{c}\bar{a}\bar{b})$, where $\bar{b} := \langle b_2, \dots, b_n \rangle$. Let $\alpha(x, \tilde{x}, \tilde{y}, \tilde{z})$ be an x -narrow formula, such that $\mathbb{M} \models \alpha(b_1, \tilde{b}, \bar{c}, \bar{a})$. If $\bar{a}' \subset \mathbb{M}$ satisfies $\alpha(\bar{b}, \bar{c}, \bar{a}')$, then $\bar{a}' \not\downarrow_C^{\text{cl}} B$.

(3) and (4) follow as in [Adler05, 2.3–4]. Here is a direct proof of the Local Character axiom: let A and B be given. Let $B' \subseteq B$ be a basis of AB over A , $A' \subseteq A$ be a basis of A , and $C \subseteq B$ be a basis of B over B' . Notice that CB' is a basis of AB and $A'B'$ is a set of generators of AB ; hence, by the Exchange Principle, $|C| \leq |A'| = \text{rk}^{\text{cl}}(A) \leq |A|$. Moreover, $A \downarrow_C^{\text{cl}} B$. $\square$

Definition 3.17. Let $\downarrow$ be a pre-independence relation on $\mathbb{M}$. We say that $\downarrow$ is an independence relation on $\mathbb{M}$ if it moreover satisfies Invariance, Local Character, and

Extension: If $A \downarrow_C B$ and $D \supseteq B$, then there exists $A' \equiv_{BC} A$ such that $A' \downarrow_C D$.

We also define the following axiom:

Existence: For any A, B , and C , there exists $A' \equiv_C A$ such that $A' \downarrow_C B$.

The following result follows from [Adler05].

Corollary 3.18. *If $\downarrow^{\text{cl}}$ satisfies either the Extension or the Existence axiom, then it is an independence relation (and satisfies the Existence axiom).*

Proof. See [Adler05, Thm. 2.5]. $\square$

def:existence

Definition 3.19. cl satisfies Existence if:

For every a, B , and C , if $a \notin \text{cl } B$, then there exists $a' \equiv_B a$ such that $a' \notin \text{cl}(BC)$.

Denote by $\text{Aut}(\mathbb{M}/B)$ the set of automorphisms of $\mathbb{M}$ which fix B pointwise. Denote by $\Xi(a/B)$ the set of conjugates of a over B : $\Xi(a/C) := \{a^\sigma : \sigma \in \text{Aut}(\mathbb{M}/B)\}$.

lem:cl-E

Lemma 3.20. *T.f.a.e.:*

1. cl satisfies Existence.

2. For every a , B , and C , if $\Xi(a/B) \subseteq \text{cl}(BC)$, then $a \in \text{cl}(B)$.
3. For every a , $\bar{b}$, and $\bar{c}$, if $a \notin \text{cl}(\bar{b})$, then there exists $a' \equiv_{\bar{b}} a$ such that $a' \notin \text{cl}(\bar{b}\bar{c})$.
4. For every a , $\bar{b}$, and $\bar{c}$, and every x -narrow formula $\phi(x, \bar{y}, \bar{z})$, if $\mathbb{M} \models \phi(a', \bar{b}, \bar{c})$ for every $a' \equiv_{\bar{b}} a$, then $a \in \text{cl}(\bar{b})$.
5. For every formula (without parameters) $\phi(x, \bar{y})$ and every x -narrow formula $\psi(x, \bar{y}, \bar{z})$, if $\mathbb{M} \models \forall \bar{y} \exists \bar{z} \forall x (\phi(x, \bar{y}) \rightarrow \psi(x, \bar{y}, \bar{z}))$, then ϕ is x -narrow.
6. For every a and B , if $\text{rk}^{\text{cl}}(\Xi(a/B))$ is finite, then $a \in \text{cl}(B)$.
7. For every a and B , if $\text{rk}^{\text{cl}}(\Xi(a/B)) < \kappa$, then $a \in \text{cl}(B)$.
8. $\perp^{\text{cl}}$ is an independence relation.

Remark 3.21. If cl satisfies Existence, then $\text{acl } A \subseteq \text{cl } A$.

-Skolem-existence

Lemma 3.22. Assume that $\text{cl}(A)$ is an elementary substructure of $\mathbb{M}$, for every $A \subset \mathbb{M}$. Then, cl satisfies Existence, and therefore $\perp^{\text{cl}}$ is an independence relation. Hence, if T has definable Skolem functions and cl extends acl , then cl satisfies Existence.

Proof. Let $\Xi(a/B) \subseteq \text{cl}(BC)$. We want to prove that $a \in \text{cl}(B)$. Let B' and C' be elementary substructures of $\mathbb{M}$, such that $B \subseteq B' \subset \text{cl}(B)$, $B'C \subseteq C' \subset \text{cl}(BC)$, $|B'| < \kappa$, and $|C'| < \kappa$ (B' and C' exist by hypothesis on cl). By substituting B with B' and C with C' , w.l.o.g. we can assume that $B \preceq C \prec \mathbb{M}$. By saturation, there exist an x -narrow formula $\phi(x, \bar{y}, \bar{z})$, $\bar{b} \subset B$, and $\bar{c} \subset C$, such that $\Xi(a/B) \subseteq \phi(\mathbb{M}, \bar{b}, \bar{c})$. Let $p := \text{tp}(a/B)$, let $q \in S_1(C)$ be a heir of p , and a' be a realisation of q . Since $\phi(x, \bar{b}, \bar{c}) \in p$, there exists $\bar{b}' \in B$ such that $\phi(x, \bar{b}, \bar{b}') \in q$. Hence, $a' \in \text{cl}(B)$; since $a' \equiv_B a$, $a \in \text{cl}(B)$. $\square$

Definition 3.23. The **trivial matroid** cl^0 is given by $\text{cl}^0(X) = X$ for every $X \subseteq \mathbb{M}$. cl^0 is a definable matroid and satisfies Existence. It induces the trivial pre-independence relation $\perp^0$, such that $A \perp_B^0 C$ for every A , B , and C . Notice that $\perp^0$ is an independence relation.

def:existential

Definition 3.24. We say that cl is an **existential matroid** if cl is a definable matroid, satisfies Existence, and is non-trivial (i.e., different from cl^0).

Examples 3.25. 1. Given $n \in \mathbb{N}$, the uniform matroid of rank n is defined as: $\text{cl}^n(X) := X$, if $|X| < n$, or $\mathbb{M}$ if $|X| \geq n$. cl^n is a definable matroid, but does not satisfy Existence in general (unless $n = 0$).

2. Define $\text{id}(X) := X$. id is a definable matroid, but does not satisfy Existence in general. The pre-independence relation induced by id is given by $A \downarrow_B^{\text{id}} C$ iff $A \cap C \subseteq B$.

Remark 3.26. Let $\mathbb{M}'$ be another monster model of T . We can define an operator cl' on $\mathbb{M}'$ in the following way:

$$\text{cl}(X') := \bigcup \{ \phi(\mathbb{M}', \bar{a}) : \phi(x, \bar{y}) \text{ } x\text{-narrow} \ \& \ \bar{a}' \subset X' \}.$$

Then, cl' is a definable matroid. If cl satisfies existence, then cl' also satisfies existence.

general-closure

Remark 3.27. Notice that the definitions of “definable” (3.14) and “existential” (3.24 and 3.19) make sense also for finitary closure operators (and not only for matroids).

However, we will not need such more general definitions.

Proviso. *For the remainder of this section, cl is an existential matroid.*

Summarising, we have:

If cl is an existential matroid, then $\downarrow^{\text{cl}}$ is an independence relation, satisfying the strong finite character axiom. In particular, if $\mathbb{M}$ is a pregeometric structure, then $\downarrow^{\text{acl}}$ is an independence relation.

3.2 Dimension

def:dimension

Definition 3.28. Given a set $V \subseteq \mathbb{M}^n$, definable with parameters A , the **dimension** of V (w.r.t. to the matroid cl) is given by

$$\dim^{\text{cl}}(V) := \max \{ \text{rk}^{\text{cl}}(\bar{b}/A) : \bar{b} \in V \},$$

with $\dim^{\text{cl}}(V) := -\infty$ iff $V = \emptyset$. More generally, the dimension of a partial type p with parameters A is given by

$$\dim^{\text{cl}}(p) := \max \{ \text{rk}^{\text{cl}}(\bar{b}/A) : \bar{b} \models p \}.$$

The following lemma shows that the above notion is well-posed: in its proof, it is important that cl satisfies existence.

Lemma 3.29. *Let V be a type-definable subset of $\mathbb{M}^n$. Then, $\dim^{\text{cl}}(V) \leq n$, and $\dim^{\text{cl}}(V)$ does not depend on the choice of the parameters.*

Remark 3.30. For every $d \leq n \in \mathbb{N}$, the set of complete types in $S_n(A)$ of $\dim^{\text{cl}}$ equal to d is closed (in the Stone topology). That is, $\dim^{\text{cl}}$ is continuous in the sense of [Poizat85, §17.b].

Lemma 3.31. *Let p be a partial type over A . Then,*

$$\dim^{\text{cl}}(p) := \min\{\dim^{\text{cl}}(V) : V \text{ is } A\text{-definable} \ \& \ V \in p\}.$$

Moreover, if p is a complete type, then, for every $\bar{b} \models p$, $\text{rk}^{\text{cl}}(\bar{b}/A) = \dim^{\text{cl}}(p)$.

Proof. Let $d := \dim^{\text{cl}}(p)$, $e := \min\{\dim^{\text{cl}}(V) : V \text{ is } A\text{-definable} \ \& \ V \in p\}$, and $\bar{b} \models p$, such that $d = \text{rk}^{\text{cl}}(\bar{b}/A)$. If $V \in p$, then $\bar{b} \in V$, and therefore

$$e \geq \dim^{\text{cl}}(V) \geq \text{rk}^{\text{cl}}(\bar{b}/A) = d.$$

For the opposite inequality, first assume that p is a complete type. W.l.o.g. $\tilde{b} := \langle b_1, \dots, b_d \rangle$ are cl-independent over A , and therefore $b_i \in \text{cl}(A\tilde{b})$ for every $i = d+1, \dots, n$. For every $i \leq n$, $\phi_i(x, \bar{y}, \bar{z})$ be an x -narrow formula such that $\mathbb{M} \models \phi(b_i, \tilde{b}, \bar{a})$ (where $\bar{a} \subset A$), $\phi(\bar{x}, \bar{y}, \bar{z}) := \bigcap_{i=1}^n \phi_i(x_i, x_1, \dots, x_d, \bar{z})$, and $V := \phi(\mathbb{M}^n, \mathbb{M}^d, \bar{a})$. Then, for every $\bar{b}' \in V$, $\text{rk}^{\text{cl}}(\bar{b}'/A) \leq d$, and therefore $\dim^{\text{cl}}(V) \leq d$. Moreover, $\tilde{b} \in V$, hence $V \in p$, and therefore $e \leq d$.

The general case when p is a partial type follows from the complete case, the fact that the set of complete types extending p is a closed (and hence compact) subset of $S_n(A)$, and the previous remark. $\square$

Remark 3.32. $\dim^{\text{cl}}(\mathbb{M}^n) = n$. Moreover, $\dim^{\text{cl}}$ is monotone: if $U \subseteq V \subseteq \mathbb{M}^n$, then $\dim^{\text{cl}}(U) \leq \dim^{\text{cl}}(V)$.

Definition 3.33. Given $p \in S_n(B)$, $q \in S_n(C)$, with $B \subseteq C$, we say that q is a non-forking extension of p (w.r.t. cl), and write $p \sqsubseteq q$, if q extends p and $\dim^{\text{cl}}(q) = \dim^{\text{cl}}(p)$. We write $q \perp_B^{\text{cl}} C$ if $q \upharpoonright_B \sqsubseteq q$.

Remark 3.34. Let $B \subseteq C$ and $q \in S_n(C)$. Then, $q \perp_B^{\text{cl}} C$ iff, for some (for all) $\bar{a}$ realising q , $\bar{a} \perp_B^{\text{cl}} C$.

Remark 3.35. Let $p \in S_n(B)$ and $B \subseteq C$. Then, for every $q \in S_n(C)$ extending p , $\dim^{\text{cl}}(q) \leq \dim^{\text{cl}}(p)$. Moreover, there exists $q \in S_n(C)$ which is a non-forking extension of p .

lem:ind-heir

Lemma 3.36. *Let $\perp^{\text{f}}$ be Shelah's forking relation on $\mathbb{M}$. Then, for every A , B , and C subsets of $\mathbb{M}$, if $A \perp_B^{\text{f}} C$, then $A \perp_B^{\text{cl}} C$. In particular, if $\mathbb{K} \prec \mathbb{M}$, $\mathbb{K} \subseteq C$, and $q \in S_n(C)$, and q is either a heir or a coheir of $q \upharpoonright_{\mathbb{K}}$, then $q \perp_{\mathbb{K}}^{\text{cl}} C$.*

Proof. The fact that $\perp^{\text{f}}$ implies $\perp^{\text{cl}}$ is a particular case of [Adler05, Remark 1.27]. For the case when q is a heir of $p := q \upharpoonright_{\mathbb{K}}$, see also [Adler05, Remark 2.3]. $\square$

Corollary 3.37. *Assume that T is super-simple and $p \in S_n(A)$ for some $A \subseteq \mathbb{M}$. Then, $SU(p) \geq \dim^{\text{cl}}(p)$, where SU is the SU -rank (see [Wagner00]).*

forking-closed

Remark 3.38. Given $B \supseteq A$, let $N_n(B/A)$ be the set of all n -types over B that do not fork over A . $N_n(B, A)$ is closed in $S_n(B)$. The same is true for any independence relation $\perp$, instead of $\perp^{\text{cl}}$.

Lemma 3.39. *For every complete type p , $\dim^{\text{cl}}(p)$ is the maximum of the cardinalities n of chains of complete types $p = q_0 \subset q_1 \subset \dots \subset q_n$, such that each q_{i+1} is a forking extension of q_i .*

Proof. Let A be the set of parameters of p , and $\bar{b} \models p$. Let $d := \dim^{\text{cl}}(p)$; w.l.o.g., $\bar{b} := \langle b_1, \dots, b_d \rangle$ are independent over A . For every $i \leq n$ let $A_i := Ab_1 \dots b_i$, and $q_i := \text{tp}(\bar{b}/A_i)$. Then, $p = q_0 \subset \dots \subset q_d$, and each q_{i+1} is a forking extension of q_i .

Conversely, assume that $p = q_0 \subset \dots \subset q_n$, and each q_{i+1} is a forking extension of q_i , and A_i be the set of parameters of q_i .

Claim 1. For every $i \leq n$, $\dim^{\text{cl}}(q_{n-1}) \geq i$; in particular, $\dim^{\text{cl}}(p) \geq n$.

By induction on i . The case $i = 0$ is clear. Assume that we have proved the claim for i , we want to show that it holds for $i + 1$. Since q_i is a forking extension of q_{i+1} , $\dim^{\text{cl}}(q_i) > \dim^{\text{cl}}(q_{i+1})$, and we are done. $\square$

lem:dim-rk

Lemma 3.40. *Let $V \subseteq \mathbb{M}^n$ be non-empty and definable with parameters $\bar{a}$. Then, either $\dim^{\text{cl}}(V) = 0 = \text{rk}^{\text{cl}}(V/\bar{a})$, or $\dim^{\text{cl}}(V) > 0$ and $\text{rk}^{\text{cl}}(V) \geq \kappa$.*

Lemma 3.41. *A formula $\phi(x, \bar{y})$ is x -narrow iff, for every $\bar{b}$, $\dim^{\text{cl}}(\phi(\mathbb{M}, \bar{b})) = 0$.*

lem:type-def

Lemma 3.42. *Let $\phi(x, \bar{y})$ be a formula without parameters, and $\bar{a} \in \mathbb{M}^n$. Then, $\dim^{\text{cl}}(\phi(\mathbb{M}, \bar{a})) = 0$ iff there exists an x -narrow formula $\psi(x, \bar{y})$ such that $\forall x (\phi(x, \bar{a}) \rightarrow \psi(x, \bar{a}))$. Therefore, define*

$$\Gamma_\phi(\bar{y}) := \{\neg\theta(\bar{y}) : \theta(\bar{y}) \text{ formula without parameters s.t.}$$

$$\forall \bar{a} (\theta(\bar{a}) \rightarrow \dim^{\text{cl}}(\phi(\mathbb{M}, \bar{a})) = 0)\},$$

$$U_\phi^1 := \{\bar{a} \in \mathbb{M}^n : \dim^{\text{cl}}(\phi(\mathbb{M}, \bar{a})) = 1\}.$$

Then, $U_\phi^1 = \{\bar{a} \in \mathbb{M}^n : \mathbb{M} \models \Gamma_\phi(\bar{a})\}$, and in particular U_ϕ^1 is type-definable (over the empty set).

More generally, let $k \leq n$, $\bar{x} := \langle x_1, \dots, x_n \rangle$, and $\phi(\bar{x}, \bar{y})$ be a formula without parameters. Define

$$U_\phi^{\geq k} := \{\bar{a} \in \mathbb{M}^n : \dim^{\text{cl}}(\phi(\mathbb{M}^n, \bar{a})) \geq k\}.$$

Then, $U_\phi^{\geq k}$ is type-definable over the empty set.

lem:cl-function

Lemma 3.43 (Fibre-wise dimension inequalities). $U \subseteq \mathbb{M}^{m_1}$, $V \subseteq \mathbb{M}^{m_2}$, and $F : U \rightarrow V$ be definable, with parameters C . Let $X \subseteq U$ and $Y \subseteq V$ be type-definable, such that $F(X) \subseteq Y$. Define $f := F \upharpoonright X : X \rightarrow Y$. For every $\bar{b} \in Y$, let $X_{\bar{b}} := f^{-1}(\bar{b}) \subseteq X$, and $m := \dim^{\text{cl}}(Y)$.

1. If, for every $\bar{b} \in Y$, $\dim^{\text{cl}}(X_{\bar{b}}) \leq n$, then $\dim^{\text{cl}}(X) \leq m + n$.
2. If f is surjective and, for every $\bar{b} \in Y$, $\dim^{\text{cl}}(X_{\bar{b}}) \geq n$, then $\dim^{\text{cl}}(X) \geq m + n$.
3. If f is surjective, then $\dim^{\text{cl}}(X) \geq m$.
4. If f is injective, then $\dim^{\text{cl}}(X) \leq m$.
5. If f is bijective, then $\dim^{\text{cl}}(X) = m$.

Proof. 1) Assume, for contradiction, that $\dim^{\text{cl}}(X) > m + n$. Let $\bar{a} \in X$ such that $\text{rk}^{\text{cl}}(\bar{a}/C) > m + n$, and $\bar{b} := F(\bar{a})$. Since $\bar{a} \in X_{\bar{b}}$, and $X_{\bar{b}}$ is type-definable with parameters $C\bar{b}$, $\text{rk}^{\text{cl}}(\bar{a}/\bar{b}C) \leq n$. Hence, by Lemma 3.2, $\text{rk}^{\text{cl}}(\bar{a}/C) \leq \text{rk}^{\text{cl}}(\bar{a}\bar{b}/C) \leq m + n$, absurd.

2) Let $\bar{b} \in Y$ such that $\dim^{\text{cl}}(\bar{b}/C) = m$. Let $\bar{a} \in X_{\bar{b}}$ such that $\dim^{\text{cl}}(\bar{a}/\bar{b}C) \geq n$. Then, by Lemma 3.2, $\text{rk}^{\text{cl}}(\bar{a}\bar{b}/C) \geq m + n$. However, since $\bar{a} = F(\bar{b})$, $\bar{a} \in \text{cl}(\bar{b}C)$, and therefore $\text{rk}^{\text{cl}}(\bar{b}/C) = \text{rk}^{\text{cl}}(\bar{a}\bar{b}/C) \geq m + n$.

(3) follows from (2) applied to $n = 0$. The other assertions are clear. $\square$

lem:cl-equality

Lemma 3.44. Let cl' be another existential matroid on $\mathbb{M}$. T.f.a.e.:

1. $\text{cl} \subseteq \text{cl}'$;
2. $\text{rk}^{\text{cl}} \geq \text{rk}^{\text{cl}'}$;
3. $\dim^{\text{cl}} \geq \dim^{\text{cl}'}$ on definable sets;
4. $\dim^{\text{cl}} \geq \dim^{\text{cl}'}$ on complete types;
5. for every definable set $X \subseteq \mathbb{M}$, if $\dim^{\text{cl}'}(X) = 0$, then $\dim^{\text{cl}}(X) = 0$.

T.f.a.e.:

1. $\text{cl} = \text{cl}'$;
2. $\text{rk}^{\text{cl}} = \text{rk}^{\text{cl}'}$;
3. $\dim^{\text{cl}} = \dim^{\text{cl}'}$ on definable sets;
4. $\dim^{\text{cl}} = \dim^{\text{cl}'}$ on complete types;

5. for every definable set $X \subseteq \mathbb{M}$, $\dim^{\text{cl}}(X) = 0$ iff $\dim^{\text{cl}'}(X) = 0$.

We will show that, for many interesting theories, there is at most one existential matroid.

Define $T_{R \setminus 0}$ to be the theory of rings without zero divisors, in the language of rings $\mathcal{L}_R := (0, 1, +, \cdot)$.

Definition 3.45. If $\mathbb{K}$ expands a ring without zero divisors, define $F : \mathbb{K}^4 \rightarrow \mathbb{K}$ the function, definable without parameters in the language $\mathcal{L}_R$,

$$\langle x_1, x_2, y_1, y_2 \rangle \mapsto \begin{cases} t & \text{if } y_1 \neq y_2 \text{ \& } t \cdot (y_1 - y_2) = x_1 - x_2; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that F is well-defined, because in a ring without zero divisors, if $y_1 \neq y_2$, then, for every x , there exists at most one t such that $t \cdot (y_1 - y_2) = x$.

l-ring-function

Lemma 3.46 ([DMS08, 1.18]). *Assume that T expands $T_{R \setminus 0}$. Let $A \subseteq \mathbb{M}$ be definable. Then, $\dim^{\text{cl}}(A) = 1$ iff $\mathbb{M} = F(A^4)$.*

Proof. Same as [DMS08, 1.18]. Assume for contradiction that $\dim^{\text{cl}}(A) = 1$, but there exists $c \in \mathbb{M} \setminus F(A^4)$. Since $c \notin F(A^4)$, the function $\langle x_1, x_2 \rangle \mapsto c \cdot x_1 + x_2 : A^2 \rightarrow \mathbb{M}$ is injective. Hence, by Lemma 3.43, $\dim^{\text{cl}}(\mathbb{M}) \geq \dim^{\text{cl}}(A^2) = 2$, absurd.

Conversely, by Lemma 3.43 again, if $f(A^4) = \mathbb{M}$, then $\dim(A) = 1$. $\square$

thm:cl-unique

Theorem 3.47. *If T expands $T_{R \setminus 0}$, then cl is the only existential matroid on $\mathbb{M}$. If S is a definable subfield of $\mathbb{M}$ of dimension 1, then $S = \mathbb{M}$.*

Proof. Let $A \subseteq \mathbb{M}$ be definable. By the previous lemma, $\dim(A) = 1$ iff $F(A^4) = \mathbb{M}$. Since the same holds for any existential matroid cl' on $\mathbb{M}$, we conclude that, for every definable set $A \subseteq \mathbb{M}$, $\dim^{\text{cl}}(A) = 0$ iff $\dim^{\text{cl}'}(A) = 0$, and hence $\dim^{\text{cl}} = \dim^{\text{cl}'}$.

Given S a subfield of $\mathbb{M}$, $F(S^4) = S$. Hence, if $\dim^{\text{cl}}(S) = 1$, then $S = \mathbb{M}$. $\square$

Example 3.48. In the above theorem, we cannot drop the hypothesis that T expands $T_{R \setminus 0}$. In fact, let $\mathbb{M}_0$ be an infinite connected graph, such that $\mathbb{M}_0$ is a monster model, and acl is a matroid in $\mathbb{M}_0$ (e.g., $\mathbb{M}_0$ equal to a monster model of the theory of random graphs). Let $\mathbb{M}$ be the disjoint union of κ copies of $\mathbb{M}_0$: notice that $\mathbb{M}$ is a monster model. For every $a \in \mathbb{M}$, let $\text{cl}(a)$ be the connected component of $\mathbb{M}$ containing a (it is a copy of $\mathbb{M}_0$), and $\text{cl}(A) := \bigcup_{a \in A} \text{cl}(a)$. Then, acl and cl are two different existential matroids on $\mathbb{M}$.

Example 3.49. In Lemma 3.46 and Theorem 3.47 we cannot even relax the hypothesis to “ T expands the theory of a vector space”. In fact, let $\mathbb{F}$ be an ordered field, considered as a vector space over itself, in the language $\langle 0, 1, +, <, \lambda_c \rangle_{c \in \mathbb{F}}$, and let T be its theory. Let T^d be the theory of dense pairs of models of T . [DMS08, 5.8] show that T^d has elimination of quantifiers, and acl is a matroid on T^d . However, as the reader can verify, the small closure Scl is another existential matroid on T^d (cf. §8.4), and it is different from acl .

Corollary 3.50. *If $\mathbb{M}$ expands a field, then $\mathbb{M}$ must be a perfect field. In particular, the theory of separably closed and non-algebraically closed fields, and the theory of differentially closed fields of finite characteristic do not admit an existential matroid.*

Proof. Cf. [Dries89, 1.6]. If $\mathbb{M}$ is not perfect, then $\mathbb{M}^p$ is a proper definable subfield of $\mathbb{M}$, where $p := \text{char}(\mathbb{M})$, and therefore $\dim^{\text{cl}}(\mathbb{M}^p) = 0$. However, the map $x \mapsto x^p$ is a bijection from $\mathbb{M}$ to $\mathbb{M}^p$; therefore, $\dim^{\text{cl}}(\mathbb{M}) = 0$, absurd. $\square$

Corollary 3.51. *Let cl' be a non-trivial definable matroid on some monster model $\mathbb{M}'$. Assume that $\mathbb{M}'$ expands a model of $T_{R|0}$. Then, t.f.a.e.:*

1. cl' is an existential matroid;
2. for every formula (without quantifiers) $\phi(x, \bar{y})$, ϕ is x -narrow iff, for y every $\bar{b}$, $F^4(\phi(\mathbb{M}', \bar{b})) \neq \mathbb{M}'$.

Proof. $(1 \Rightarrow 2)$ is clear.

$(2 \Rightarrow 1)$ follows from Lemma 3.20-5. $\square$

Lemma 3.52. *Let $\mathbb{K}$ be a ring without zero divisors definable in $\mathbb{M}$, of dimension $n \geq 1$. Let $\mathbb{F} \subseteq \mathbb{K}$ be a definable subring such that $\mathbb{F}$ is a skew field. If $\dim^{\text{cl}}(\mathbb{K}) = n$, then $\mathbb{K} = \mathbb{F}$.*

Proof. Assume, for contradiction, that there exists $c \in \mathbb{K} \setminus \mathbb{F}$. Define $h : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{K}$, $h(x, y) := x + cy$. Since $c \notin \mathbb{F}$ and $\mathbb{F}$ is a skew field, h is injective. Thus, $2n = \dim(\mathbb{F}^2) \leq \dim(\mathbb{K}) = n$, a contradiction. $\square$

Corollary 3.53. *Let $\mathbb{K} \subseteq \mathbb{M}^n$ be a definable field, such that $\dim^{\text{cl}}(\mathbb{K}) \geq 1$. Then, $\mathbb{K}$ is perfect.*

The assumption that $\dim^{\text{cl}}(\mathbb{K}) \geq 1$ is necessary: non-perfect definable fields of dimension 0 can exist.

Proof. Let $p := \text{char } \mathbb{K}$, and $\phi : \mathbb{K} \rightarrow \mathbb{K}$ be the Frobenius automorphism $\phi(x) = x^p$. Since ϕ is injective, $\dim^{\text{cl}}(\mathbb{K}^p) = \dim^{\text{cl}}(\mathbb{K})$, and therefore $\mathbb{K}^p = \mathbb{K}$. $\square$

ex:group

Example 3.54. Let λ be an ordinal, which is an ordinal power of ω (e.g., $\lambda = 1$, $\lambda = \omega, \dots$). Let $\mathbb{G}$ be a monster model of a superstable group, such that $U(\mathbb{G}) = \lambda$, where U is Lascar's rank. For every a and B , define $a \in \text{cl}(B)$ iff $U(a/B) < \lambda$. Then, cl is an existential matroid. If X is a definable subset of $\mathbb{G}$, then $\dim^{\text{cl}}(X) = 1$ iff X is generic, that is finitely many left translates of X cover $\mathbb{G}$.

Proof. See [Poizat87]. $\square$

Example 3.55. Let $\mathbb{K}$ be a monster differentially closed field, and $p \geq 0$ be its characteristic. If $p = 0$, then $\mathbb{K}$ is superstable, and $U(\mathbb{K}) = \omega$; hence, by the previous example, there exists a (unique) existential matroid cl on $\mathbb{K}$. It is easy to see that, if A is a differential subfield of $\mathbb{K}$ and $b \in \mathbb{K}$, then $b \in \text{cl}(A)$ iff b is differential-algebraic over A (that is, iff $b, db, d^2b, \dots$ are algebraically dependent over A); see [Wood76] and [Dries89, 2.25]. On the other hand, if $p > 0$, then there is no existential matroid on $\mathbb{K}$, because $\mathbb{K}$ is not perfect.

Definition 3.56. Let $X \subseteq \mathbb{K}^n$ any $Y \subseteq \mathbb{K}^m$ be definable. Let $f : X \rightsquigarrow Y$ be a definable application (i.e., a multi-valued partial function), with graph F . For every $x \in X$, let $f(x) := \{y \in Y : \langle x, y \rangle \in F\} \subseteq Y$. Such an application f is a **Z-application** if, for every $x \in X$, $\dim^{\text{cl}}(f(x)) \leq 0$.

em:Zapplication

Remark 3.57. Let $A \subseteq \mathbb{K}$, and $b \in \mathbb{K}$. Then, $b \in \text{cl}(A)$ iff there exists an $\emptyset$ -definable Z-application $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ and $\bar{a} \in A$, such that $b \in f(\bar{a})$. Moreover, if $\bar{c} \in \mathbb{K}^n$, then $b \in \text{cl}(A\bar{c})$ iff there exists an A -definable Z-application $f : \mathbb{K}^n \rightarrow \mathbb{K}$, such that $b \in f(\bar{c})$.

Definition 3.58. We say that $\dim^{\text{cl}}$ is **definable** if, for every X definable subset of $\mathbb{M}^m \times \mathbb{M}^n$, the set $\{\bar{a} \in \mathbb{M}^m : \dim^{\text{cl}}(X_{\bar{a}}) = d\}$ is definable.

Lemma 3.59. *T.f.a.e.:*

1. $\dim^{\text{cl}}$ is definable;
2. for every X definable subset of $\mathbb{M}^m \times \mathbb{M}$, the set $X^{1,1} := \{\bar{a} \in \mathbb{M}^m : \dim^{\text{cl}}(X_{\bar{a}}) = 1\}$ is also definable;
3. for every $k \leq n$, every m , and every X definable subset of $\mathbb{M}^m \times \mathbb{M}^n$, the set $X^{n,k} := \{\bar{a} \in \mathbb{M}^m : \dim^{\text{cl}}(X_{\bar{a}}) = k\}$ is also definable, with the same parameters as X .

Proof. $(3 \Rightarrow 1 \Rightarrow 2)$ is obvious.

$(2 \Rightarrow 1)$ is obvious. We will prove by induction on n that, for every Y definable subset of $\mathbb{K}^n \times \mathbb{K}^m$, the set $Y^{n, \geq k} := \{\bar{a} \in \mathbb{M}^m : \dim^{\text{cl}}(X_{\bar{a}}) \geq k\}$ is definable. The case $k = 0$ is clear. The case $k = 1$ follows from the assumption and the observation that, for every Z definable subset of $\mathbb{K}^n$, $\dim^{\text{cl}}(Z) \geq 1$ iff $\dim^{\text{cl}}(\theta(Z)) \geq 1$ for some θ projection from $\mathbb{K}^n$ to a coordinate axis. The inductive step follows from the fact that

$$X^{n, \geq k} = (\Pi_{n+m-1}^{n+m}(X))^{n-1, \geq k} \cup (X^{n+m-1, \geq 1})^{n-1, \geq k-1}. \quad \square$$

$(1 \Rightarrow 3)$ Let $X \subseteq \mathbb{K}^{n+m}$ be definable with parameters A . Then, $X^{n,k}$ is $\mathbb{M}$ -definable, by assumption. Moreover, by Lemma 3.42, $X^{n,k}$ is type-definable over A , and therefore invariant under automorphisms that fix A pointwise. Hence, by Beth's definability theorem, $X^{n,k}$ is definable over A .

Remark 3.60. If T expands $T_{R \setminus \emptyset}$, then $\dim^{\text{cl}}$ is definable.

3.3 Morley sequences

Most of the results of this subsection remain true for an arbitrary independence relation $\perp$ instead of $\perp^{\text{cl}}$.

Definition 3.61. Let $C \subseteq B$, $p(\bar{x}) \in S_n(B)$, and $\langle I, \leq \rangle$ be a linear order. A **Morley sequence** over C indexed by I in p is a sequence $(\bar{a}_i : i \in I)$ of tuples in $\mathbb{M}^n$, such that $(\bar{a}_i : i \in I)$ is order-indiscernibles over B and independent over C , and every $\bar{a}_i$ realises $p(\bar{x})$.

A Morley sequence over C is a Morley sequence over C in some $p \in S_n(C)$.

A Morley sequence in p is a Morley sequence over B in p .

lem:Morley-seq

Lemma 3.62. Let $\langle I, \leq \rangle$ be a linear order, with $|I| < \kappa$. Let $p(\bar{x}) \in S_n(C)$. Then, there exists a Morley sequence over C indexed by I in $p(\bar{x})$. If moreover $\bar{b} \perp_C^{\text{cl}} \bar{d}$, then there exists a Morley sequence $(\bar{a}_i : i \in I)$ over C indexed by I in $p(\bar{x})$, such that $(\bar{b}\bar{a}_i : i \in I)$ is order-indiscernibles over $C\bar{d}$ and, for every $i \in I$, $\bar{b}\bar{a}_i \perp_C^{\text{cl}} \bar{d}(\bar{a}_j : j \neq i)$.

Proof. Let $(\bar{x}_i : i \in I)$ be a sequence of n -tuples of variables. Consider the following set of C -formulae:

$$\Gamma_1(\bar{x}_i : i \in I) := \bigwedge_{i \in I} p(\bar{x}_i) \ \& \ \bigwedge_{i \in I} \bar{x}_i \perp_C^{\text{cl}} (\bar{x}_j : j < i).$$

First, notice that, by Remark 3.38 Γ_1 is a set of formulae.

Consider the following set of C -formulae:

$$\Gamma_2(\bar{x}_i : i \in I) := \Gamma_1(\bar{x}_i : i \in I) \ \& \ (\bar{x}_i : i \in I) \text{ is an order-indiscernible sequence over } C.$$

By [Adler05, 1.12], Γ_2 is consistent.

We give an alternative proof of the above fact, which does not use Erdős-Rado.

Claim 2. Γ_1 is consistent.

It is enough to prove that Γ_1 is finitely satisfiable; hence, w.l.o.g. $I = \{0, \dots, m\}$ is finite. Let $\bar{a}_0$ be any realisation of $p(\bar{x})$. Let $\bar{a}_1 \equiv_C \bar{a}_0$ such that $\bar{a}_i \downarrow_C^{\text{cl}} \bar{a}_0, \dots$, let $\bar{a}_m \equiv_C \bar{a}_0$ such that $\bar{a}_m \downarrow_C^{\text{cl}} \bar{a}_0 \dots \bar{a}_{m-1}$.

By Ramsey's Theorem, Γ_2 is also consistent.

Since $|I| < \kappa$, there exists a realisation $(\bar{a}_i : i \in I)$ of Γ_2 . Then, by Lemma 3.11 $(\bar{a}_i : i \in I)$ is a Morley sequence in $p(\bar{x})$ over C .

If moreover $\bar{b}$ and $\bar{d}$ satisfy $\bar{b} \downarrow_C^{\text{cl}} \bar{d}$, let $q(\bar{x}, \bar{y}, \bar{z})$ be the extension of $p(\bar{x})$ to $S^*(C\bar{b}\bar{d})$ satisfying $\bar{y} = \bar{b}$ and $\bar{z} = \bar{d}$. Let $(\bar{a}_i \bar{b} \bar{d} : i \in I)$ be a Morley sequence in $q(\bar{x}, \bar{y}, \bar{z})$. By Lemma 3.10, for every $i \in I$ we have $\bar{b} \bar{a}_i \downarrow_C^{\text{cl}} \bar{d}(\bar{a}_j : i \neq j \in I)$. $\square$

Lemma 3.63. *A type $p \in S_n(A)$ is **stationary** if, for every $B \supseteq A$, there exists a unique $q \in S_n(B)$ such that $p \sqsubseteq q$.*

Remark 3.64. Let $p \in S_n(A)$. If $\dim^{\text{cl}}(p) = 0$, then p is stationary iff p is realised in $\text{dcl}(A)$.

Hence, unlike the stable case, if $\text{cl} \neq \text{acl}$, then there are types over models which are not stationary.

ident-morley-seq

Lemma 3.65. *Let $C \supseteq B$, and $q \in S_n(C)$ such that $q \downarrow_B^{\text{cl}} C$. Let $(\bar{a}_i : i \in I)$ be a sequence of realisations of q independent over C . Then, $(\bar{a}_i : i \in I)$ is also independent over B . If moreover q is stationary, then*

1. $(\bar{a}_i : i \in I)$ is a totally indiscernible set over C , and in particular it is a Morley sequence for q over B .
2. If $(\bar{a}' : i \in I)$ is another sequence of realisations of q independent over C , then $(\bar{a}_i : i \in I) \equiv_C (\bar{a}' : i \in I)$.

Proof. Standard proof. More precisely, for every $i \in I$, let $\bar{d}_i := (a_j : i \neq j \in I)$. By assumption, $\bar{a}_i \downarrow_C^{\text{cl}} \bar{d}_i$, and, since $q \downarrow_B^{\text{cl}} C$, $\bar{a}_i \downarrow_B^{\text{cl}} C$, and therefore $\bar{a}_i \downarrow_B^{\text{cl}} \bar{d}_i$, proving that $(\bar{a}_i : i \in I)$ is independent over B .

Let us prove Statement (2). By compactness, w.l.o.g. $I = \{1, \dots, m\}$ is finite. Assume, for contradiction, that $(\bar{a} : i \leq m) \not\equiv_C (\bar{a}' : i \leq m)$; by induction on m , we can assume that $(\bar{a}_i : i \leq m-1) \equiv_C (\bar{a}'_i : i \leq m-1)$, and therefore, w.l.o.g., that $\bar{a}_i = \bar{a}'_i$ for $i = 1, \dots, m-1$. However, since q is stationary, $\bar{a}_m \equiv_C \bar{a}'_m$, $\bar{a}_m \downarrow_C^{\text{cl}} (\bar{a}_i : i \leq m-1)$, and $\bar{a}'_m \downarrow_C^{\text{cl}} (\bar{a}_i : i \leq m-1)$, we have that $\bar{a}_m \equiv_{C(\bar{a}_i : i \leq m-1)} \bar{a}'_m$, absurd.

Finally, it remains to prove that the set $(\bar{a}_i : i \in I)$ is totally indiscernible over C . If σ is any permutation of I , then $(\bar{a}_{\sigma(i)} : i \in I)$ is also a sequence of realisations of q independent over C , and therefore, by Statement (2), $(\bar{a}_{\sigma(i)} : i \in I) \equiv_C (\bar{a}_i : i \in I)$. $\square$

Corollary 3.66. *Assume that there is a definable linear ordering on $\mathbb{M}$. Then, $p \in S_n(A)$ is stationary iff p is realised in $\text{dcl}(A)$. Hence, if $\dim^{\text{cl}}(p) > 0$, every non-forking extension of p is not stationary.*

Contrast the above situation to the case of stable theories, where instead every type has at least one stationary non-forking extension.

Proof. Assume that p is stationary, but, for contradiction, that $\dim^{\text{cl}}(p) > 0$. Then, there is a Morley sequence in p with at least two elements $\bar{a}_0$ and $\bar{a}_1$. Since $\dim^{\text{cl}}(p) > 0$, $\bar{a}_0 \neq \bar{a}_1$. By Lemma 3.65, $\bar{a}_0$ and $\bar{a}_1$ are indiscernibles over A , absurd. $\square$

cor:ind-morley

Corollary 3.67. *Let $B \subseteq C$ and $q \in S_n(C)$. Then, t.f.a.e.:*

1. $q \downarrow_B^{\text{cl}} C$;
2. *there exists an infinite sequence of realisations of q that are independent over B ;*
3. *every sequence $(\bar{a}_i : i \in I)$ of realisations of q that are independent over C are independent also over B ;*
4. *there exists an infinite Morley sequence in q over B .*

Proof. Cf. [Adler05, 1.12–13].

(1 $\Rightarrow$ 3) Let $(\bar{a}_i : i \in I)$ be a sequence of realisations of q independent over C . For every $i \in I$, let $\bar{d}_i := (\bar{a}_j : i \neq j \in I)$. Since $\bar{a}_i \downarrow_C \bar{d}_i$ and $\bar{a}_i \downarrow_B C$, we have $\bar{a}_i \downarrow_B \bar{d}_i$.

(3 $\Rightarrow$ 4) Let $(\bar{a}_i : i \in I)$ be an infinite Morley sequence in q over C : such sequence exists by Lemma 3.62 (or by [Adler05, 1.12]). Then, $(\bar{a}_i : i \in I)$ is independent also over B , and hence a Morley sequence for q over B .

(4 $\Rightarrow$ 2) is obvious.

(2 $\Rightarrow$ 1) Choose $\lambda < \kappa$ a regular cardinal large enough. Let $(\bar{a}'_i : i < \omega)$ be a sequence of realisations of q independent over C . By saturation, there exists $(\bar{a}_i : i < \lambda)$ a sequence of realisations of q independent over C . By Local Character, and since λ is regular, there exists $\alpha < \lambda$ such that $\bar{a}_\alpha \downarrow_{B\bar{d}}^{\text{cl}} C$, where $\bar{d} := (\bar{a}_i : i < \alpha)$. Since moreover $\bar{a}_\alpha \downarrow_B^{\text{cl}} \bar{d}$, we have $\bar{a}_\alpha \downarrow_B^{\text{cl}} C$, and therefore $q \downarrow_B^{\text{cl}} C$. $\square$

3.4 Local properties of dimension

In this subsection, we will show that the dimension of a set can be checked locally: what this means precisely will be clear in §9, where the results given here will be applied to a “concrete” situation.

Definition 3.68. A quasi-ordered set $\langle I, \leq \rangle$ is a **directed set** if every pairs of elements of I has an upper bound.

lem:order-ind

Lemma 3.69. *Let $\langle I, \leq \rangle$ be a directed set, definable in $\mathbb{M}$ with parameters $\bar{c}$. Then, for every $\bar{a} \in I$ and $\bar{d} \subset \mathbb{M}$ there exists $\bar{b} \in I$ such that $\bar{b} \geq \bar{a}$ and $\bar{d}\bar{a} \downarrow_{\bar{c}}^{\text{cl}} \bar{b}$.*

Proof. Fix $\bar{a} \in I$ and $\bar{d} \subset \mathbb{M}$, and assume, for contradiction, that every $\bar{b} \geq \bar{a}$ satisfies $\bar{d}\bar{a} \not\downarrow_{\bar{c}}^{\text{cl}} \bar{b}$.

W.l.o.g., $\bar{c} = \emptyset$. Let λ be a large enough cardinal; at the price of increasing κ if necessary, we may assume that $\lambda < \kappa$. By Lemma 3.62, there exists a Morley sequence $(\bar{d}'\bar{a}'_i : i < \lambda)$ in $\text{tp}(\bar{d}\bar{a}/\emptyset)$ over $\emptyset$. Consider the following set of formulae over $\{\bar{a}'_i : i < \lambda\}$:

$$\Lambda(\bar{x}) := \{\bar{x} \in I, \bar{x} \geq \bar{a}'_i : i < \lambda\}.$$

Since $\langle I, \leq \rangle$ is a directed set, Λ is consistent: let $\bar{b} \in I$ be a realisation of Λ . By Erdős-Rado’s Theorem, there exists a Morley sequence $(\bar{d}_i\bar{a}_i : i < \omega)$ in $\text{tp}(\bar{d}\bar{a}/\emptyset)$ over $\emptyset$, such that all the $\bar{d}_i\bar{a}_i$ satisfy the same type $q(\bar{x}, \bar{y})$ over $\bar{b}$. Therefore, by Corollary 3.67, $q \downarrow_{\bar{b}}^{\text{cl}}$, and in particular $\bar{a}_0 \downarrow_{\bar{b}}^{\text{cl}}$. Since $\bar{a}_0 \equiv \bar{a}$, there exists $\bar{b}' \geq \bar{a}$ such that $\bar{a} \equiv \bar{b}'$, a contradiction. $\square$

urhood-abstract

Lemma 3.70. *Let $X \subseteq \mathbb{M}^n$ be definable with parameters $\bar{c}$ and $(U_{\bar{t}})_{\bar{t} \in I}$ be a family of subsets of $\mathbb{M}^n$, such that each $U_{\bar{t}}$ is definable with parameters $\bar{t}\bar{c}$. Let $d \leq n$, and assume that, for every $\bar{a} \in X$ there exists $\bar{b} \in I$ such that $\bar{a} \in U_{\bar{b}}$, $\bar{a} \downarrow_{\bar{c}}^{\text{M}} \bar{b}$, and $\dim^{\text{cl}}(X \cap U_{\bar{b}}) \leq d$. Then, $\dim^{\text{cl}}(X) \leq d$.*

Proof. Assume, for contradiction, that $\dim^{\text{cl}}(X) > d$; let $\bar{a} \in X$ such that $\text{rk}^{\text{cl}}(\bar{a}/\bar{c}) > d$. Choose $\bar{b}$ as in the hypothesis of the lemma; then, $\text{rk}^{\text{cl}}(\bar{a}/\bar{b}\bar{c}) > d$, absurd. $\square$

lem:cl-U

Lemma 3.71. *Let $I \subseteq \mathbb{M}^n$ be definable and $<$ be a definable linear ordering on I . Let $(X_{\bar{b}})_{\bar{b} \in I}$ be a definable increasing family of subsets of $\mathbb{K}^m$ and $X := \bigcup_{\bar{b} \in I} X_{\bar{b}}$. Let $d \leq m$, and assume that, for every $\bar{b} \in I$, $\dim^{\text{cl}}(X_{\bar{b}}) \leq d$. Then, $\dim^{\text{cl}}(X) \leq d$.*

Proof. Let $\bar{c}$ be the parameters used to define I , $<$, and $(X_{\bar{b}})_{\bar{b} \in I}$. Let $\bar{a} \in X$ such that $\text{rk}^{\text{cl}}(\bar{a}/\bar{c}) = \dim^{\text{cl}}(X)$. Let $\bar{b} \in I$ such that $\bar{a} \in X_{\bar{b}}$. Choose $\bar{a}', \bar{b}' \in \mathbb{M}$ such that $\bar{a}'\bar{b}' \equiv_{\bar{c}} \bar{a}\bar{b}$ and $\bar{a}'\bar{b}' \downarrow_{\bar{c}}^{\text{cl}} \bar{a}\bar{b}$. W.l.o.g., $\bar{b}' \geq \bar{b}$; hence, $\bar{a} \in X_{\bar{b}'}$ and

$$d \geq \dim^{\text{cl}}(X_{\bar{b}'}) \geq \text{rk}^{\text{cl}}(\bar{a}/\bar{c}\bar{b}') = \text{rk}^{\text{cl}}(\bar{a}/\bar{c}) = \dim^{\text{cl}}(X). \quad \square$$

We can extend the above lemma to directed families.

Lemma 3.72. *Let $\langle I, \leq \rangle$ be a definable directed set. Let $(X_{\bar{b}})_{\bar{b} \in I}$ be a definable increasing family of subsets of $\mathbb{M}^m$ and $X := \bigcup_{\bar{b} \in I} X_{\bar{b}}$. Let $d \leq m$, and assume that, for every $\bar{b} \in I$, $\dim^{\text{cl}}(X_{\bar{b}}) \leq d$. Then, $\dim^{\text{cl}}(X) \leq d$.*

Proof. W.l.o.g., $\langle I, \leq \rangle$ and the family $(X_{\bar{b}})_{\bar{b} \in I}$ are definable without parameters. Let $\bar{a} \in X$ such that $\text{rk}^{\text{cl}}(\bar{a}) = \dim^{\text{cl}}(X)$, and let $\bar{b}_0 \in I$ such that $\bar{a} \in X_{\bar{b}_0}$. By the Lemma 3.69, there exists $\bar{b} \in I$ such that $\bar{b} \geq \bar{b}_0$ and $\bar{a}\bar{b}_0 \downarrow^{\text{cl}} \bar{b}$. Hence, $\bar{a} \in X_{\bar{b}}$ and $\bar{a} \downarrow^{\text{cl}} \bar{b}$, and therefore

$$d \geq \dim^{\text{cl}}(X_{\bar{b}}) \geq \text{rk}(\bar{a}/\bar{b}) = \text{rk}(\bar{a}) = \dim^{\text{cl}}(X). \quad \square$$

Remark 3.73. The above lemma is not true if $(X_{\bar{b}})_{\bar{b} \in I}$ be a definable *decreasing* family of subsets of $\mathbb{M}^m$, instead of increasing. For instance, let $\mathbb{K}$ be a real closed field, $\equiv_{\text{acl}}$, $I := \mathbb{K}^{<0} \times \mathbb{K} \cup \{(0, 0)\}$; define $\langle x, y \rangle \leq \langle x', y' \rangle$ if $x \leq x'$ and $y = y'$, or $x = 0$. Let $I_{b_1, b_2} := \{\langle x, y \rangle \in I : \langle x, y \rangle \geq \langle b_1, b_2 \rangle\}$. Then, $\langle I, \leq \rangle$ is a directed set, $\dim^{\text{acl}}(I) = 2$, but $\dim^{\text{acl}}(I_{\bar{b}}) \leq 1$ for every $\bar{b} \in I$.

4 Matroids from dimensions

Van den Dries in [Dries89] gave a definition of dimension for definable sets; we will show that his approach is almost equivalent to ours. Let $\mathbb{K}$ be a first order structure.

Definition 4.1. A **dimension function** on $\mathbb{K}$ is a function d from definable sets in $\mathbb{K}$ to $\{-\infty\} \cup \mathbb{N}$, such that, for all $m \in \mathbb{N}$ and S, S_1 and S_2 definable subsets of $\mathbb{K}^m$, we have:

(D1m 1) $d(S) = -\infty$ iff $S = \emptyset$, $d(\{a\}) = 0$ for every $a \in \mathbb{K}$, $d(\mathbb{K}) = 1$.

(D2m 2) $d(S_1 \cup S_2) = \max(d(S_1), d(S_2))$.

(D3m 3) $d(S^\sigma) = d(S)$ for every permutation σ of the coordinates of $\mathbb{K}^m$.

(D4m 4) Let U be a definable subset of $\mathbb{K}^{m+1}$, and, for $i = 0, 1$, let $U(i) := \{x \in \mathbb{K}^m : d(U_x) = i\}$. Then, $U(i)$ is definable *with the same parameters as U* , and $d(U \cap \pi^{-1}(U(i))) = d(U(i)) + i$, $i = 0, 1$, where $\pi := \Pi_m^{m+1}$.

Notice that the axiom (Dim 4) is slightly stronger than the original axiom in [Dries89]; however, after expanding $\mathbb{K}$ by at most $|T|$ many constants, the situation in [Dries89] can be reduced to ours.

Definition 4.2. Given a dimension function d on $\mathbb{K}$, for every $A \subset \mathbb{K}$ and $b \in \mathbb{K}$ we define $b \in \text{cl}^d(A)$ iff there exists $X \subseteq \mathbb{K}$ definable with parameters in A , such that $d(X) = 0$ and $b \in X$.

Theorem 4.3. cl^d (more precisely, the extension of cl^d to a monster model) is an existential matroid with definable dimension. The dimension induced by cl^d is precisely d .

Conversely, if cl is an existential matroid with definable dimension, then $\dim^{\text{cl}}$ is a dimension function, and $\text{cl}^{\dim^{\text{cl}}} = \text{cl}$.

Proof. The only non-trivial facts are that, if d is a dimension function, then cl^d is definable, satisfies EP and the existence axiom.

Definability) Let $a \in \text{cl}(B)$. Let $X \subseteq \mathbb{K}$ be B -definable such that $d(X) = 0$ and $a \in X$. Let $\phi(x, \bar{b})$ be the B -formula defining X . By (Dim 4), w.l.o.g. $d(\phi(\mathbb{K}, \bar{y})) \leq 0$ for every $\bar{y}$.³ Hence, $\phi(x, \bar{y})$ is an x -narrow formula.

EP) Let $a \in \text{cl}(Bc) \setminus \text{cl}(B)$. Assume, for contradiction, that $c \notin \text{cl}(Ba)$. Let $X \subseteq \mathbb{K}^2$ be B -definable, such that $a \in X_c$ and $d(X_c) = 0$. Let $X' := X \cap \pi^{-1}(X(0))$, where $\pi := \Pi_1^2$. By assumption, $\langle c, a \rangle \in X'$ and, by (Dim 4), $\dim(X') \leq 1$; w.l.o.g., $X = X'$.

Let $Z := \{u \in \mathbb{K} : d(X^u) = 1\}$. Since $c \in X^a$ and $c \notin \text{cl}(Ba)$, $a \in Z$. Since $a \notin \text{cl}(B)$, $d(Z) = 1$. Hence, by (Dim 4) and (Dim 3), $d(X) = 2$, absurd.

Existence) Immediate from Lemma 3.20-5. □

³Here it is important that in (Dim 4) we asked that the parameters of $U(i)$ are the same as the parameters of U .

5 Expansions

Remember that $\mathbb{M}$ is a monster model of a complete $\mathcal{L}$ -theory T . We are interested in the behaviour of definable matroids under expansions of $\mathbb{M}$. In this section we assume that cl is a definable closure operator on the monster model $\mathbb{M}$.

def:cl-induced

Definition 5.1. Given $X \subseteq \mathbb{M}$, let the restriction $\text{cl}^X : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ and the relativisation $\text{cl}_X : \mathcal{P}(\mathbb{M}) \rightarrow \mathcal{P}(\mathbb{M})$ of cl be defined as $\text{cl}^X(Y) := \text{cl}(Y) \cap X$ and $\text{cl}_X(Y) := \text{cl}(XY)$.

Lemma 5.2. *Given $X \subseteq \mathbb{M}$, cl^X is a closure operator on X and cl_X is a closure operator on $\mathbb{M}$. If moreover cl is a matroid, then both cl^X and cl_X are matroids, $A \downarrow_B^{\text{cl}_X} C$ iff $A \downarrow_{XB}^{\text{cl}} C$, and $\downarrow^{\text{cl}^X}$ is the restriction of $\downarrow^{\text{cl}}$ to the subsets of X .*

Remark 5.3. Given $B \subset \mathbb{M}$ (with $|B| < \kappa$), let $\mathbb{M}_B$ be the expansion of $\mathbb{M}$ with all constants from B .

1. if cl is definable, then cl_B is also definable (see Remark 3.27).
2. if cl is a matroid, then cl_B is also a matroid;
3. if cl is definable and satisfies Existence, then cl_B satisfies Existence too;
4. if cl is an existential matroid, then cl_B is also an existential matroid, and $\dim^{\text{cl}}$ and $\dim^{\text{cl}_B}$ coincide (the definable sets of $\mathbb{M}$ and of $\mathbb{M}_B$ are the same).

Example 5.4. In the above Remark, it is not true that, if cl is a definable matroid, and cl_B satisfies Existence, then cl satisfies Existence. For instance, let B be any non-empty subset of $\mathbb{M}$ (of cardinality less than κ), and $\text{cl} = \text{cl}^1$; then, $\text{cl}_B = \text{cl}^0$ satisfies Existence, but cl does not.

lem:cl-pair

Lemma 5.5. *Let $X \subseteq \mathbb{M}$. Let $\mathbb{M}'$ be the expansion of $\mathbb{M}$ with a predicate P for X . Assume that $\mathbb{M}'$ is a monster model of $T(X)$, and denote by cl'_X the closure operator $\text{cl}'_X(Y) := \text{cl}(XY)$ on $\mathbb{M}'$ (cl'_X coincides with cl_X).*

1. *If cl is definable, then cl'_X is definable on $\mathbb{M}'$;*
2. *if cl is a matroid, then cl'_X is a matroid;*

Proof. Let $D \subseteq X$ such that $|D| < \kappa$ and $\text{cl}(X) = \text{cl}(D)$.

5. Expansions

1. $b \in \text{cl}'_X(A)$ iff $b \in \text{cl}(AX)$ iff $\mathbb{M} \models \phi(b, \bar{a}, \bar{c})$ for some x -narrow formula $\phi(x, \bar{y}, \bar{z})$ and some $\bar{c} \in X^n$. Define $\psi(x, \bar{y}) := \exists \bar{z} (P(\bar{z}) \ \& \ \phi(x, \bar{y}, \bar{z}))$. Notice that ψ is an $\mathcal{L}(P)$ -formula, and that, for every $\bar{a}' \subset \mathbb{M}$, $\psi(\mathbb{M}', \bar{a}') \subseteq \text{cl}'_X(\bar{a}')$.
2. Trivial. □

-small-relative

Lemma 5.6. *Let $\mathbb{M}$, X and $\mathbb{M}'$ be as in the above lemma. Let $\langle \mathbb{B}, Y \rangle \prec \langle \mathbb{M}, X \rangle$; assume moreover that cl is a definable closure operator on $\mathbb{M}$. Then, for every $A \subseteq \mathbb{B}$, $\text{cl}_Y(A) \cap \mathbb{B} = \text{cl}_X(A) \cap \mathbb{B}$.*

Hence, in the above situation, inside $\mathbb{B}$ we do not need to distinguish between cl_X and cl_Y .

air-independent

Lemma 5.7. *Let cl be a definable matroid (not necessarily existential) and X, Y, X^* , and Y^* be elementary substructures of $\mathbb{M}$, such that $X \subseteq X^* \cap Y$ and $X^* \cup Y \subseteq Y^*$. Let $\mathcal{L}^2$ be the expansion of $\mathcal{L}$ with a new unary predicate P , and consider $\langle Y, X \rangle$ and $\langle Y^*, X^* \rangle$ as $\mathcal{L}^2$ -structures. Assume that $(Y, X) \preceq (Y^*, X^*)$. Then, $X^* \downarrow_X^{\text{cl}} Y$.*

Proof. Let $\bar{x}^* \subset X^*$; it suffices to prove that $\bar{x}^* \downarrow_X^{\text{cl}} Y$. However, $\text{tp}_{\mathcal{L}}(\bar{x}^*/Y)$ is finitely satisfied in X , and we are done. □

Assume that $\mathbb{M}$ expands a ring without 0 divisors. Let $\mathbb{M}'$ be an expansion of $\mathbb{M}$ to a larger language $\mathcal{L}'$; assume that $\mathbb{M}'$ is also a monster model and that cl' is an existential matroid on $\mathbb{M}'$. We have seen that in this case cl' is the unique existential matroid on $\mathbb{M}'$, and that, for every X definable subset of $\mathbb{M}'$, $\dim'(X) = 0$ iff $F(X^4) \neq \mathbb{M}'$ (where $\dim'$ is the dimension induced by cl'). It is clear that cl' , in general, is not definable in $\mathbb{M}$. However, the dimension function $\dim'$ is definable in $\mathbb{M}$: Hence, we can restrict the dimension function $\dim'$ to the sets definable in $\mathbb{M}$ (with parameters), and get a function $\dim$.

em:cl-expansion

Lemma 5.8. *Let $\mathbb{M}$, $\mathbb{M}'$, $\dim'$ and $\dim$ be as above. Then, $\dim$ is a dimension function on $\mathbb{M}$ (i.e., it satisfies the axioms in Definition 3.28). The matroid cl induces by $\dim$ is characterized by For every A and b , $b \in \text{cl}(A)$ iff there exists $X \subseteq \mathbb{M}$, definable in $\mathbb{M}$ with parameters A , such that $\dim'(X) = 0$.*

Corollary 5.9. *Assume that $\mathbb{M}$ expands a ring without zero divisors. Let $\mathbb{M}'$ be an expansion of $\mathbb{M}$. If $\mathbb{M}'$ is geometric, then $\mathbb{M}$ is also geometric.*

Compare the above corollary with [Adler05, Corollary 2.38 and Example 2.40].

6 Extension to imaginary elements

sec:imaginary

Again, $\mathbb{M}$ is a monster model of a complete theory T , and cl is an existential matroid on $\mathbb{M}$. Let $\mathbb{M}^{\text{eq}}$ be the set of imaginary elements, and T^{eq} be the theory of $\mathbb{M}^{\text{eq}}$. Our aim is to extend to $\mathbb{M}^{\text{eq}}$ the matroid cl to a closure operator cl^{eq} and the rank rk^{cl} to a “rank function” rk (see Definition 6.6).

Notice that acl^{eq} is a closure operator on $\mathbb{M}^{\text{eq}}$ extending acl ; however, if $\text{cl} = \text{acl}$, then in general $\text{cl}^{\text{eq}} \neq \text{acl}^{\text{eq}}$; hence, when $\text{cl} = \text{acl}$, we will have to pay attention not to confuse the two possible extensions of cl to $\mathbb{M}^{\text{eq}}$ (cf. Remark 6.11).

On the other hand, by dcl^{eq} we will always denote the usual extension of dcl to imaginary element: $a \in \text{dcl}(b)$ if $\Xi(a/B) = \{a\}$.

We will start with the definition of $a \in \text{cl}^{\text{eq}}(B)$ when a is real and B is imaginary.

def:matt-1

Definition 6.1. Let B be a set of imaginary elements (of cardinality less than κ), and a be a real element. We say that $a \in \text{cl}^{\text{eq}}(B)$ iff $\Xi(a/B)$ has finite rk^{cl} .

Remark 6.2. If a and B are real, then $a \in \text{cl}^{\text{eq}}(B)$ iff $a \in \text{cl}(B)$.

Remark 6.3. If a is real and B is imaginary, then $a \in \text{cl}^{\text{eq}}(B)$ iff $\text{rk}^{\text{cl}}(\Xi(a/B)) < \kappa$.

Proof. Assume that $\text{rk}^{\text{cl}}(\Xi(a/B)) \geq \aleph_0$; we want to show that $\text{rk}^{\text{cl}}(\Xi(a/B)) \geq \kappa$. For every $\lambda < \kappa$, consider the type in λ variables $p(\bar{x})$ over B , saying that, for every $i < \lambda$, $x_i \equiv_B a$, and the x_i are cl -independent. By assumption, $p(\bar{x})$ is consistent, and hence satisfied in $\mathbb{M}$ by λ -tuple $(a_i)_{i < \lambda}$; thus, the a_i are λ -independent elements in $\Xi(a/B)$. Therefore, $\text{rk}^{\text{cl}}(\Xi(a/B)) > \lambda$ for every $\lambda < \kappa$; since κ is a limit cardinal, we are done. $\square$

Recall that $\mathbb{M}$ has geometric elimination of imaginaries if every for imaginary tuple $\bar{a}$ there exists a real tuple $\bar{b}$ such that $\bar{a}$ and $\bar{b}$ are inter-algebraic. If $\mathbb{M}$ had geometric elimination of imaginaries, we could define $\bar{a} \in \text{cl}^{\text{eq}}(B)$ iff there exists a real tuple $\bar{c}$ such that $\bar{a} \in \text{acl}^{\text{eq}}(\bar{c})$ and $\bar{c} \subset \text{cl}^{\text{eq}}(B)$. In the general case, we need a more involved definition and some preliminary lemmata.

Lemma 6.4 (Exchange Principle [Gagelman05, 3.1]). *cl^{eq} satisfies the Exchange Principle for real points over imaginary parameters. That is, for a and b real elements and C imaginary, if $a \in \text{cl}^{\text{eq}}(bC) \setminus \text{cl}^{\text{eq}}(C)$, then $b \in \text{cl}^{\text{eq}}(aC)$.*

6. Extension to imaginary elements

Proof. Let a , b , and C as in the hypothesis, and assume, for contradiction, that $b \notin \text{cl}^{\text{eq}}(aC)$. Let B be a real set (of cardinality less than κ), such that $\Xi(a/bC) \subseteq \text{cl}(B)$. By enlarging B , w.l.o.g. we can assume that $C \subseteq \text{dcl}^{\text{eq}}(B)$. Since $a \notin \text{cl}^{\text{eq}}(C)$, $\Xi(a/C) \not\subseteq \text{cl}(B)$, and therefore there exists $\sigma \in \text{Aut}(\mathbb{M}/C)$ such that $a' := a^\sigma \notin \text{cl}(B)$. Since $b \notin \text{cl}^{\text{eq}}(aC)$, we have $b^\sigma \notin \text{cl}^{\text{eq}}(a'C)$, and therefore there exists $b' \equiv_{a'C} b^\sigma$ such that $b' \notin \text{cl}(a'B)$.

Notice that $a'b' \equiv_C ab$; let $\mu \in \text{Aut}(\mathbb{M}/C)$ such that $a' = a^\mu$ and $b' = b^\mu$, and define $B' := B^\mu$. Since $\Xi(a/bC) \subseteq \text{cl}(B)$, we have $\Xi(a'/b'C) \subseteq \text{cl}(B')$. Moreover, since $C \subseteq \text{dcl}^{\text{eq}}(B')$, we have $\Xi(a'/b'B') \subseteq \Xi(a'/b'C) \subseteq \text{cl}(B')$. Thus, $a' \in \text{cl}(b'B) \setminus \text{cl}(B)$; hence, since a' , b' , and B are real, $b' \in \text{cl}(a'B)$, absurd. $\square$

It is relatively easy to also prove the following:

Lemma 6.5 (Transitivity). *cl^{eq} is transitive for real sets over imaginary parameters: that is, if A is a imaginary, $\bar{b}$ is a tuple of reals, and c is real, such that $\bar{b} \subseteq \text{cl}^{\text{eq}}(A)$ and $c \in \text{cl}^{\text{eq}}(A\bar{b})$, then $c \in \text{cl}^{\text{eq}}(A)$.*

def:rank

Definition 6.6. Let X be a set and rk be a function from finite subsets of X to $\mathbb{N}$. Let $\bar{a}$ and $\bar{b}$ vary among the finite tuples of X and B , C among the subsets of X . rk is a rank function if it satisfies the following conditions:

Finite character: for all $\bar{a}$ and B there exists $B' \subseteq B$ finite, such that $\text{rk}(\bar{a}/B') = \text{rk}(\bar{a}/B)$.

Additivity: for every $\bar{a}$, $\bar{b}$, and B , $\text{rk}(\bar{a}\bar{b}/B) = \text{rk}(\bar{a}/\bar{b}C) + \text{rk}(\bar{b}/C)$.

Transitivity: for all $B \subseteq C$ and all a , $\text{rk}(\bar{a}/C) \leq \text{rk}(\bar{a}/B)$.

cor:rk-real-im

Corollary 6.7 ([Gagelman05]). • For any real tuple $\bar{b}$ and imaginary set A , any two maximally $A\text{-cl}^{\text{eq}}$ -independent sub-tuples of $\bar{b}$ have the same cardinality; thus, we may use the notation $\text{rk}^{\text{cl}}(-/-)$ accordingly, as long as the first argument is real.

- The function rk^{cl} defined above coincides with the usual one when both arguments are real.
- The function rk^{cl} has Finite Character, is Additive and Transitive, as long as the first argument is real.
- rk^{cl} satisfies Extension (for real first argument): if $\bar{a}$ is real and B and C are imaginary sets, then there exists $\bar{a}' \equiv_C \bar{a}$ such that $\text{rk}^{\text{cl}}(\bar{a}'/BC) = \text{rk}^{\text{cl}}(\bar{a}/B)$.
- If $\bar{a}$ is real, then $\bar{a} \in \text{cl}^{\text{eq}}(C)$ iff $\text{rk}^{\text{cl}}(\bar{a}/C) = 0$.

Definition 6.8. Let A and $\bar{a}$ be imaginary. Choose $\bar{b}$ real, such that $\bar{a} \subseteq \text{acl}^{\text{eq}}(A\bar{b})$. Define $\text{rk}^{\text{cl}}(\bar{a}/A) := \text{rk}^{\text{cl}}(\bar{b}/A) - \text{rk}^{\text{cl}}(\bar{b}/\bar{a}A)$.

Lemma 6.9 ([Gagelman05, 3.3]). *The above definition does not depend on the choice of $\bar{b}$. Moreover, $\text{rk}^{\text{cl}}(\bar{a}/A)$ is a natural number, and coincides with the one given in Corollary 6.7 when $\bar{a}$ is real. Finally, rk^{cl} is a rank function on $\mathbb{M}^{\text{eq}}$.*

Proof. Let's prove that rk^{cl} does not depend on the choice of $\bar{b}$. Let $\bar{b}$ and $\bar{b}'$ be real tuples, such that $\bar{a} \subseteq \text{acl}^{\text{eq}}(A\bar{b})$ and $\bar{a} \subseteq \text{acl}^{\text{eq}}(A\bar{b}')$. W.l.o.g., $\bar{b} \subseteq \bar{b}'$ and hence $\bar{b}' = \langle \bar{b}, \bar{b}'' \rangle$. We must prove that $\text{rk}^{\text{cl}}(\bar{b}\bar{b}''/A) - \text{rk}^{\text{cl}}(\bar{b}\bar{b}''/\bar{a}A) = \text{rk}^{\text{cl}}(\bar{b}/A) - \text{rk}^{\text{cl}}(\bar{b}/\bar{a}A)$. The above is equivalent to $\text{rk}^{\text{cl}}(\bar{b}\bar{b}''/A) - \text{rk}^{\text{cl}}(\bar{b}/A) = \text{rk}^{\text{cl}}(\bar{b}\bar{b}''/\bar{a}A) - \text{rk}^{\text{cl}}(\bar{b}/\bar{a}A)$. But the left hand side is equal to $\text{rk}^{\text{cl}}(\bar{b}\bar{b}''/\bar{b}A)$, while the right hand side is equal to $\text{rk}^{\text{cl}}(\bar{b}\bar{b}''/\bar{a}\bar{b}A)$. Since $\bar{a} \in \text{acl}^{\text{eq}}(\bar{b}A)$, the two sides are equal.

Let's prove additivity. Let $\bar{a}$ and $\bar{a}'$ be imaginary tuples and C be a set of imaginary elements. We want to prove that $\text{rk}^{\text{cl}}(\bar{a}\bar{a}'/C) = \text{rk}^{\text{cl}}(\bar{a}/\bar{a}'C) + \text{rk}^{\text{cl}}(\bar{a}'/C)$. Let $\bar{b}$ and $\bar{b}'$ be real tuples, such that $\bar{a} \subseteq \text{acl}^{\text{eq}}(C\bar{b})$ and $\bar{a}' \subseteq \text{acl}^{\text{eq}}(C\bar{b}')$. We have to show that $\text{rk}^{\text{cl}}(\bar{b}\bar{b}'/C) - \text{rk}^{\text{cl}}(\bar{b}\bar{b}'/\bar{a}\bar{a}'C) = \text{rk}^{\text{cl}}(\bar{b}/\bar{a}'C) - \text{rk}^{\text{cl}}(\bar{b}/\bar{a}\bar{a}'C) + \text{rk}^{\text{cl}}(\bar{b}'/C) - \text{rk}^{\text{cl}}(\bar{b}'/\bar{a}'C)$, that is

$$\begin{aligned} & - \text{rk}^{\text{cl}}(\bar{b}\bar{b}'/C) + \text{rk}^{\text{cl}}(\bar{b}\bar{b}'/\bar{a}\bar{a}'C) + \text{rk}^{\text{cl}}(\bar{b}/\bar{a}'C) - \text{rk}^{\text{cl}}(\bar{b}/\bar{a}\bar{a}'C) + \\ & + \text{rk}^{\text{cl}}(\bar{b}'/C) - \text{rk}^{\text{cl}}(\bar{b}'/\bar{a}'C) = 0. \end{aligned}$$

The above is equivalent to

$$- \text{rk}^{\text{cl}}(\bar{b}/\bar{b}'C) + \text{rk}^{\text{cl}}(\bar{b}'/\bar{b}\bar{a}\bar{a}'C) + \text{rk}^{\text{cl}}(\bar{b}/\bar{a}'C) - \text{rk}^{\text{cl}}(\bar{b}'/\bar{a}'C) = 0. \quad (1) \quad \boxed{\text{eq:rank-1}}$$

Let $C' := C\bar{a}'$. Since $\bar{a} \subseteq \text{acl}^{\text{eq}}(C\bar{b})$ and $\bar{a}' \subseteq \text{acl}^{\text{eq}}(C\bar{b}')$, (1) is equivalent to

$$- \text{rk}^{\text{cl}}(\bar{b}/\bar{b}'C') + \text{rk}^{\text{cl}}(\bar{b}'/\bar{b}C') + \text{rk}^{\text{cl}}(\bar{b}/C') - \text{rk}^{\text{cl}}(\bar{b}'/C') = 0.$$

Finally, $\text{rk}^{\text{cl}}(\bar{b}'/\bar{b}C') + \text{rk}^{\text{cl}}(\bar{b}/C') = \text{rk}^{\text{cl}}(\bar{b}\bar{b}'/C') = \text{rk}^{\text{cl}}(\bar{b}/\bar{b}'C') + \text{rk}^{\text{cl}}(\bar{b}'/C')$, and we are done. $\square$

Finally, we define $a \in \text{cl}^{\text{eq}}(B)$ iff $\text{rk}^{\text{cl}}(\bar{a}/B) = 0$, where a and B can be either real or imaginaries.

Lemma 6.10. *The operator cl^{eq} defined above is a closure operator, coincides with cl for real elements, and extends the operator defined in 6.1.*

$\boxed{\text{rem:acleq}}$

Remark 6.11. Assume that $\mathbb{M}$ is pregeometric structure and $\text{cl} = \text{acl}$. Given $\bar{b}$ a real or imaginary tuple, we have $\text{acl}^{\text{eq}}(\bar{b}) \subseteq \text{cl}^{\text{eq}}(\bar{b})$ and $\text{cl}^{\text{eq}}(\bar{b}) \cap \mathbb{M} =$

$\text{acl}^{\text{eq}}(\bar{b}) \cap \mathbb{M}$. However, it is not true in general that $\text{cl}^{\text{eq}} = \text{acl}^{\text{eq}}$: more precisely, $\text{cl}^{\text{eq}} = \text{acl}^{\text{eq}}$ iff $\mathbb{M}$ is surgical [Gagelman05]. For instance, if $\mathbb{M}$ is a model of the theory of p -adic fields, then $\mathbb{M}$ is geometric but not surgical; the imaginary sort Γ corresponding to the value group has dimension 0 but it is infinite; therefore, $\Gamma \subset \text{cl}^{\text{eq}}(\emptyset) \setminus \text{acl}^{\text{eq}}(\emptyset)$.

7 Density

sec:density

Again, $\mathbb{M}$ is a monster model of a complete theory T , and cl is an existential matroid on $\mathbb{M}$.

def:dense

Definition 7.1. Let $\mathbb{K} \preceq \mathbb{M}$, and $X \subseteq \mathbb{K}$. We say that X is **dense** in $\mathbb{K}$ if, for every $\mathbb{K}$ -definable subset U of $\mathbb{K}$, if $\dim^{\text{cl}}(U) = 1$, then $U \cap X \neq \emptyset$. We define $\text{cl}_{\mathbb{K}}(X) := \text{cl}(X) \cap \mathbb{K}$, and we say that X is **cl-closed** in $\mathbb{K}$ if $\text{cl}_{\mathbb{K}}(X) = X$.

Examples 7.2. 1. If $\mathbb{K}$ is geometric, then X is dense in $\mathbb{K}$ iff X intersects every infinite definable subset of $\mathbb{K}$.

2. If $\mathbb{K}$ is strongly minimal, then X is dense in $\mathbb{K}$ iff X is infinite.

3. If $\mathbb{K}$ is o-minimal and densely ordered, then X is dense in $\mathbb{K}$ in the sense of the above definition iff X is topologically dense in $\mathbb{K}$ (this is our motivation for the choice of the term “dense”). See also §9 for a generalisation of this example.

Remark 7.3. If $X \subset \mathbb{K}$ is dense (in $\mathbb{K}$), and $a \in X$, then $X \setminus \{a\}$ is also dense.

Proof. If $U \subseteq \mathbb{K}$ is definable and of dimension 1, then $U \setminus \{a\}$ is also definable and of dimension 1. $\square$

density-closure

Lemma 7.4. Let $X \subseteq \mathbb{K} \preceq \mathbb{M}$. If X is cl-closed and dense in $\mathbb{K}$, then $X \preceq \mathbb{K}$.

Proof. Robinson’s test. Let $A \subseteq \mathbb{K}$ be definable, with parameters from A : we must show that $A \cap X \neq \emptyset$. If $\dim^{\text{cl}}(A) = 1$, this is true because X is dense in $\mathbb{K}$. If $\dim^{\text{cl}}(A) = 0$, this is true because X is cl-closed in $\mathbb{K}$. $\square$

-consistent-GCH

Lemma 7.5. Let $\mathbb{K} \preceq \mathbb{M}$ be a saturated model of cardinality $\lambda > |T|$. Then, there exists $X \subset \mathbb{K}$ such that X is a cl-basis of $\mathbb{K}$ and X is dense in $\mathbb{K}$. Moreover, there exists $\mathbb{F} \prec \mathbb{K}$ such that $\mathbb{F}$ is cl-closed and dense in $\mathbb{K}$ and $\mathbb{F}$ is not equal to $\mathbb{K}$.

Proof. Let $(A_i)_{i < \lambda}$ be an enumeration of all subsets of $\mathbb{K}$ which are definable (with parameters from $\mathbb{K}$) and of dimension 1. Build a cl-independent sequence $(a_i)_{i < \lambda}$ inductively: for every $\mu < \lambda$, we make so that $(a_i)_{i < \mu}$ is cl-independent, and, for every $i < \mu$ there exists $j < \mu$ such that $a_j \in A_i$. Let i_μ be the smallest index such that A_{i_μ} does not contain any a_i for $i < \mu$.

Claim 3. i_μ exists.

Otherwise, $\mathbb{K}$ would have a basis of cardinality less than λ , contradicting the saturation hypothesis.

Claim 4. There exists $a_\mu \in A_\mu$ such that a_μ is cl-independent from $(a_i)_{i < \mu}$.

Otherwise, $\text{rk}^{\text{cl}}(A_\mu) < \lambda$, absurd.

Define a_μ as in the above claim.

By construction, $X' := \{a_i : i < \lambda\}$ is cl-independent and dense in $\mathbb{K}$; we can complete it to a cl-basis X , which is also dense.⁴

Choose $a \in X$, let $Y := X \setminus \{a\}$, and $\mathbb{F} := \text{cl}(Y)$. Since X is dense, Y is also dense, and therefore $\mathbb{F}$ is dense in $\mathbb{K}$. Moreover, since X is a cl-basis, $a \notin \mathbb{F}$. Finally, by Lemma 7.4, $\mathbb{F} \prec \mathbb{K}$. $\square$

The proof of the above lemma shows the following stronger results.

Corollary 7.6. *Let $\mathbb{K}$ be as in Lemma 7.5. Let $c \in \mathbb{K} \setminus \text{cl } \emptyset$. Then, there exists $\mathbb{F} \prec \mathbb{K}$ cl-closed and dense in $\mathbb{K}$, such that $c \notin \mathbb{F}$.*

Given $\mathbb{K} \models T$, and X, Y subsets of $\mathbb{K}$, we say that X is dense in $\mathbb{K}$ w.r.t. Y if for every subset U of $\mathbb{K}$ definable with parameters from X , if $\dim^{\text{cl}}(U) = 1$, then $U \cap X \neq \emptyset$.

lem:Td-consistent

Lemma 7.7. *There exists $\mathbb{F} \prec \mathbb{K} \models T$ such that $\mathbb{F}$ is a proper dense and cl-closed subset of $\mathbb{K}$.*

Proof. If T has a saturated model of cardinality $> |T|$, we can apply Lemma 7.5. Otherwise, let $\mathbb{K}_0 \prec \mathbb{K}_1 \prec \dots$ be an elementary chain of models of T , such that, for every $n \in \mathbb{N}$, $\mathbb{K}_{n+1}$ is $(|\mathbb{K}_n| + |T|)^+$ -saturated, and let $\mathbb{K} := \bigcup_{n \in \mathbb{N}} \mathbb{K}_n$. Proceeding as in Lemma 7.5, for every $n \in \mathbb{N}$ we build a cl-independent set $\mathcal{A}_n$ of elements in $\mathbb{K}_{n+1}$, such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ and $\mathcal{A}_n$ is dense in $\mathbb{K}_{n+1}$ w.r.t. $\mathbb{K}_n$. Let $\mathcal{A} := \bigcup_n \mathcal{A}_n$. Then, $\mathcal{A}$ is a cl-independent set of elements in $\mathbb{K}$, which is also dense in $\mathbb{K}$. Conclude as in Lemma 7.5. $\square$

⁴Is it true or not that X' is already a basis?

8 Dense pairs

Let $\mathbb{B}$ be a real closed field and $\mathbb{A}$ a proper dense subfield of $\mathbb{A}$, such that $\mathbb{A}$ is also real closed. We call $\langle \mathbb{B}, \mathbb{A} \rangle$ a dense pair of real closed fields, and we consider its theory, in the language of ordered fields expanded with a predicate for a (dense) subfield. Robinson [Robinson74] proved that the theory of dense pairs of real closed fields is complete. Van den Dries [Dries98] extended the results to o-minimal theories: if T is a complete o-minimal theory expanding the theory of (densely) ordered Abelian groups, then the theory of dense elementary pairs of models of T is complete. Macintyre [Macintyre75] introduced an abstract notion of density, in the context of geometric theories, which for o-minimal theories specialises to the usual topological notion, and proved various results; more recent work has been done in the context of so called “lovely pairs” of geometric or simple structures: see for instance [Berenstein07, BPV03].

In § 7 we also proposed an abstract notion of density, which for geometric theories specialises to the one given by Macintyre (and, independently, by others): see Remark 14.9. However, it is not true in general that the theory of dense pairs of models of T is complete (unless T is geometric): the main result of this section is that if T expands the theory of integral domains, and we add the additional condition that $\mathbb{A}$ is cl-closed in $\mathbb{B}$, we obtain a complete theory, which we denote by T^d . (if T is geometric, the condition is trivially true). We will also show that T^d admits an existential matroid (§8.4). For the proofs we will follow closely [Dries98].

$\mathbb{M}$ is a monster model of a complete theory T , and cl is an existential matroid on $\mathbb{M}$. For this section, we will write $\dim$ instead of $\dim^{\text{cl}}$, rk instead of rk^{cl} , and $\bigcup$ instead of $\bigcup^{\text{cl}}$.

Definition 8.1. Let $\mathcal{L}^2$ be the expansion of $\mathcal{L}$ by a new unary predicate P . Let T^2 be the $\mathcal{L}^2$ -expansion of T , whose models are the pairs $\langle \mathbb{K}, \mathbb{F} \rangle$, with $\mathbb{F} \prec \mathbb{K}$, $\mathbb{F} \neq \mathbb{K}$, and $\mathbb{F}$ cl-closed in $\mathbb{K}$.

Assume that $\dim$ is definable. Let T^d be the $\mathcal{L}^2$ -expansion of T saying that $\mathbb{F}$ is cl-closed and dense in $\mathbb{K}$ (we need definability of $\dim$ to express in a first-order way that $\mathbb{F}$ is dense in $\mathbb{K}$).

Notice that, by Lemma 7.4, T^d extends T^2 . Notice that if $\text{cl} = \text{acl}$, then T^2 is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle$, with $\mathbb{F} \prec \mathbb{K} \models T$; however, if $\text{cl} \neq \text{acl}$, then there exists $\mathbb{F} \prec \mathbb{M}$ with $\mathbb{F}$ not cl-closed in $\mathbb{M}$.

Lemma 8.2. T^d is consistent.

Proof. By Lemma 7.7. □

Proviso. For the remainder of this section, we assume that T expands the theory of integral domains (and therefore $\dim$ is definable) and that $\langle \mathbb{K}, \mathbb{F} \rangle \models T^d$.

thm:Td-complete

Theorem 8.3. T^d is complete.

Definition 8.4. An $\mathcal{L}^2$ formula $\phi(\bar{x})$ is **basic** if it is of the form

$$\exists \bar{y} (U(\bar{y}) \ \& \ \psi(\bar{x}, \bar{y})),$$

where ψ is an $\mathcal{L}$ -formula,⁵

thm:Td-eq

Theorem 8.5. Each $\mathcal{L}^2$ -formula $\psi(\bar{x})$ is equivalent, modulo T^d , to a Boolean combination of basic formulae, with the same parameters as ψ .

Theorems 8.3 and 8.5 will be proved in §8.2.

8.1 Small sets

In this subsection we will assume that $\langle \mathbb{K}, \mathbb{A} \rangle \models T^2$.

def:small

Definition 8.6. A subset X of $\mathbb{K}$ is **$\mathbb{A}$ -small** if $X \subseteq f(\mathbb{A}^n)$, for some $\mathbb{Z}$ -application $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ which is definable in $\mathbb{K}$.

Definition 8.7. Let $X \subseteq \mathbb{K}^n$. We say that X is **weakly dense** in $\mathbb{K}$ if, for every definable $U \subseteq \mathbb{K}^n$, if $X \subseteq U$, then $\dim(U) = n$.

For instance, if $\text{cl} = \text{acl}$, then X is a weakly dense subset of $\mathbb{K}$ iff X is infinite.

Remark 8.8. If X is a weakly dense subset of $\mathbb{K}$, then X^n is a weakly dense subset of $\mathbb{K}^n$.

lem:weakly-dense

Lemma 8.9. If $\mathbb{K} \models T$ and $\mathbb{K}' \preceq \mathbb{K}$, then $\mathbb{K}'$ is weakly dense in $\mathbb{K}$.

Proof. W.l.o.g., the pair $\langle \mathbb{K}, \mathbb{K}' \rangle$ is ω -saturated. Assume, for contradiction, that $U \subset \mathbb{K}$ is definable, with parameters $\bar{b} \in \mathbb{K}^n$, $\dim(U) = 0$, and $\mathbb{K}' \subseteq U$. By saturation, $\text{rk}(\mathbb{K}')$ is infinite; let $\bar{c} \in \mathbb{K}^{m+1}$ be independent elements. However, $\bar{c} \in U$, and therefore $\bar{c} \subset \text{cl}(\bar{b})$, absurd. $\square$

The following result is the most delicate one.

⁵Basic formulae were called “special” in [Dries98].

finite-dimension

Lemma 8.10 ([Dries98, 1.1]). *Let $f : \mathbb{K}^{n+1} \rightsquigarrow \mathbb{K}$ be a Z -application $\mathbb{A}$ -definable in $\mathbb{K}$, and let $b_0 \in \mathbb{K} \setminus \mathbb{A}$. For every $x \in \mathbb{K}$ and $\bar{y} = \langle y_0, \dots, y_n \rangle \in \mathbb{K}^{n+1}$ let $p(\bar{y}, x) := y_0 + y_1x + \dots + y_nx^n$. Then, there exists $\bar{a} \in \mathbb{A}^{n+1}$ such that*

$$p(\bar{a}, b_0) \notin f(\mathbb{A}^n \times \{b\}).$$

Proof. Otherwise there is for each $\bar{a} \in \mathbb{A}^{n+1}$ a tuple $\bar{c} \in \mathbb{A}^n$ such that $p(\bar{a}, b_0) \in f(\bar{c}, b_0)$. W.l.o.g., f is definable without parameters. For each $\bar{y} \in \mathbb{K}^{n+1}$ and $\bar{z} \in \mathbb{K}^n$ let $D(\bar{y}, \bar{z}) := \{x \in \mathbb{K} : p(\bar{y}, x) \in f(\bar{z}, x)\}$. Define $W := \{\langle \bar{y}, \bar{z} \rangle : \dim(D(\bar{y}, \bar{z})) = 1\}$, and $Y := \Pi_{n+1}^{2n+1}(W)$. Since $b_0 \notin \mathbb{A}$ and $\mathbb{A}$ is cl-closed, $\mathbb{A}^{n+1} \subseteq Y$. Since Y is definable, Lemma 8.9 implies that $\dim(Y) = n + 1$; therefore, $\dim(W) \geq n + 1$. Let $Z := \{\bar{z} \in \mathbb{K}^n : \dim(\{\bar{y} : \langle \bar{y}, \bar{z} \rangle \in W\}) \geq 1\}$. Since $\dim(W) \geq n + 1$ and $\dim(\mathbb{K}^n) = n$, we have that Z is non-empty.

Choose $\bar{c} \in Z$. Let $\bar{a} \in \mathbb{K}^{n+1}$ such that $\langle \bar{a}, \bar{c} \rangle \in W$ and $\text{rk}(\bar{a}/\bar{c}) \geq 1$. By definition of W , $\dim(D(\bar{a}, \bar{c})) = 1$; choose $b \in D(\bar{a}, \bar{c})$ such that $\text{rk}(b/\bar{c}\bar{a}) = 1$. Define $d := p(\bar{a}, b)$; remember that $d \in f(\bar{c}, b)$, and therefore $d \in \text{cl}(\bar{c}b)$. Let $\bar{a}' \in \mathbb{K}^{n+1}$ such that $\bar{a}' \equiv_{\bar{c}bd} \bar{a}$ and $\bar{a}' \perp_{\bar{c}bd} \bar{a}$. Since $d \in \text{cl}(\bar{c}, b)$, we have $\bar{a}' \perp_{\bar{c}b} \bar{a}$. Moreover, $p(\bar{a}', b) = d$; therefore, $p(\bar{a} - \bar{a}', b) = 0$.

If $\bar{a} \neq \bar{a}'$, this implies that b is algebraic over $\bar{a} - \bar{a}'$, and therefore $b \in \text{cl}(\bar{a}\bar{a}')$, contradicting the fact that $b \notin \text{cl}(\bar{a}\bar{c})$ and $\bar{a}' \perp_{\bar{c}b} \bar{a}$.

If instead $\bar{a} = \bar{a}'$, then $\bar{a}' \perp_{\bar{c}b} \bar{a}$ implies that $\bar{a} \subset \text{cl}(\bar{c}b)$, contradicting the facts that $b \notin \text{cl}(\bar{c}\bar{a})$ and $\text{rk}(\bar{a}/\bar{c}) \geq 1$. $\square$

Notice that the hypothesis of the above lemma can be weakened to:
 $\mathbb{K} \models T$, $\mathbb{A}$ is a proper cl-closed and weakly dense subset of $\mathbb{K}$.

rem:pair-small

Remark 8.11 ([Dries98, 1.3]). Each $\mathbb{A}$ -small subset of $\mathbb{K}$ is a proper subset of $\mathbb{K}$.

Proof. Same as [Dries98, Corollary 1.3]. $\square$

pair-small-union

Remark 8.12. A finite union of $\mathbb{A}$ -small subsets of $\mathbb{K}$ is also $\mathbb{A}$ -small.

Lemma 8.13. *Let $B \subseteq \mathbb{K}$ be a proper cl-closed subset. Then, B is co-dense in $\mathbb{K}$; that is, $\mathbb{K} \setminus B$ is dense in $\mathbb{K}$.*

Proof. Since B is cl-closed, $F(B^4) \subseteq B$. Assume, for contradiction, that there exists U definable in $\mathbb{K}$, such that $\dim(U) = 1$ and $U \subseteq B$. Then, $F(U^4) = \mathbb{K}$, and therefore $F(B^4) = \mathbb{K}$. However, since $\mathbb{K}$ is cl-closed, $F(B^4) \subseteq B$ contradicting the assumption that $B \neq \mathbb{K}$. $\square$

The hypothesis in the above lemma can be weakened to:
 B proper subset of $\mathbb{K} \models T$, and $F(B^4) \subseteq B$.

dense-cardinality

Lemma 8.14 ([Dries98, Lemma 1.5]). *If the pair $\langle \mathbb{K}, \mathbb{A} \rangle$ is λ -saturated, where λ is an infinite cardinal and $\lambda > |T|$, then $\dim(\mathbb{K}/\mathbb{A}) \geq \lambda$. Hence, if $|X| < \lambda$, then $\text{cl}(\mathbb{A}X)$ is co-dense in $\mathbb{K}$.*

Proof. Same as [Dries98, Lemma 1.5]. Let E be a generating set for $\mathbb{K}/\mathbb{A}$, and suppose that $|E| < \lambda$. Let $\Gamma(v)$ be the set of $\mathcal{L}^2$ -formulae of the form

$$\forall y_1 \dots \forall y_n (U(\bar{y}) \rightarrow v \notin f(\bar{y}, e_1, \dots, e_p)),$$

where $f(\bar{y}, \bar{z})$ is a Z -application $\emptyset$ -definable in $\mathbb{K}$, and $e_1, \dots, e_p$ are in E . By Remarks 8.11 and 8.12, $\Gamma(v)$ is a consistent set of formulae, with less than λ many parameters. By saturation, there exists $b \in \mathbb{K}$ realising the partial type $\Gamma(v)$. Thus $b \notin \text{cl}(\mathbb{A}E)$, absurd. $\square$

Notice that in the original [Dries98, Lemma 1.5], if T expands RCF, then van den Dries' assumption that $\mathcal{A}$ is dense in $\mathcal{B}$ is superfluous; density is used if however T expands only the theory of ordered Abelian groups.

8.2 Proof of Theorems 8.3 and 8.5

sec:Td-proof

The proof is similar to the one in [Boxall09].

Definition 8.15. Let $\langle \mathbb{B}, \mathbb{A} \rangle \models T^2$ and $C \subseteq \mathbb{B}$. Let $\bar{c}$ be a tuple of elements from $\mathbb{B}^{\text{eq}}$; the P -type of $\bar{c}$, denoted by $P\text{-tp}(\bar{c})$, is the information which tells us which members of $\bar{c}$ are in $\mathbb{A}$ (notice: the elements in $\bar{c}$ are real or imaginary, but only real elements can be in $\mathbb{A}$). We say that $\bar{c}$ is P -independent if $\bar{c} \downarrow_{\mathbb{A} \cap \bar{c}} \mathbb{A}$ (where, again, only the real element of $\bar{c}$ can be in $\mathbb{A} \cap \bar{c}$).

We will use a superscript 1 to denote model-theoretic notions for $\mathcal{L}$, and a superscript 2 to denote those notions for $\mathcal{L}^2$: for instance, we will write $a \equiv_C^1 a'$ if the $\mathcal{L}$ -type of a and a' over C is the same, and $a \equiv_C^2 a'$ the $\mathcal{L}^2$ -type of a and a' over C is the same.

Both theorems are an immediate consequence of the following proposition.

op:back-and-forth

Proposition 8.16. *Let $\langle \mathbb{B}, \mathbb{A} \rangle$ and $\langle \mathbb{B}', \mathbb{A}' \rangle$ be models of T^d . Let $\bar{c}$ be a (possibly infinite) P -independent tuple in $\mathbb{B}^{\text{eq}}$, and $\bar{c}'$ be an P -independent tuple in $(\mathbb{B}')^{\text{eq}}$ of the same length and the same sorts. If $\bar{c} \equiv^1 \bar{c}'$ and $P\text{-tp}(\bar{c}) = P\text{-tp}(\bar{c}')$, then $\bar{c} \equiv^2 \bar{c}'$.*

Proof. Back-and-forth argument. Let $\lambda < \kappa$ be a cardinal strictly greater than $|T|$ and the length of $\bar{c}$. W.l.o.g., we can assume that both $\langle \mathbb{B}, \mathbb{A} \rangle$ and

$\langle \mathbb{B}', \mathbb{A}' \rangle$ are λ -saturated. Let $\bar{e}$ (resp. $\bar{e}'$) be the subtuple of $\bar{c}$ (resp. of $\bar{d}'$) of non-real elements. Let

$$\begin{aligned} \Gamma := \{ & f : \tilde{c} \rightarrow \tilde{c}' : \quad \bar{c} \subset \tilde{c} \subset \mathbb{B}^{\text{eq}}, \quad \bar{c}' \subset \tilde{c}' \subset (\mathbb{B}')^{\text{eq}}, \\ & \tilde{c} \text{ \& } \tilde{c}' \text{ of the same length less than } \lambda \text{ and of the same sorts,} \\ & \text{with all non-real elements of } \tilde{c} \text{ in } \bar{e}, \\ & f \text{ is a bijection,} \\ & \tilde{c} \text{ \& } \tilde{c}' \text{ are } P\text{-independent, } \tilde{c} \equiv^1 \tilde{c}', \quad P\text{-tp}(\tilde{c}) = P\text{-tp}(\tilde{c}') \}. \end{aligned}$$

We want to prove that Γ has the back-and-forth property. So, let $f : \tilde{c} \rightarrow \tilde{c}'$ be in Γ , and $d \in \mathbb{B} \setminus \bar{c}$; we want to find $g \in \Gamma$ such that g extends f and d is in the domain of g . W.l.o.g., $\tilde{c} = \bar{c}$ and $\tilde{c}' = \bar{c}'$. Let $\bar{a} := \bar{c} \cap \mathbb{A}$, and $\bar{a}' := \bar{c}' \cap \mathbb{A}'$. Notice that $f(\bar{a}) = \bar{a}'$ and that $\mathbb{A} \cap \text{cl}(\bar{c}) = \mathbb{A} \cap \text{cl}(\bar{a})$, and similarly for $\bar{c}'$. We distinguish some cases.

CASE 1. $d \in \mathbb{A} \cap \text{cl}(\bar{c}) = \mathbb{A} \cap \text{cl}(\bar{a})$. Notice that $\bar{c}d \perp_{\bar{a}d} \mathbb{A}$, and therefore $\bar{c}d$ is P -independent. There is a x -narrow formula $\phi(x, \bar{y})$ such that $\mathbb{B} \models \phi(d, \bar{a})$. Choose $d' \in \mathbb{A}'$ such that $\bar{c}d \equiv^1 \bar{c}'d'$; therefore, $d' \in \text{cl}(\bar{a}')$, and thus $\bar{c}'d'$ is also P -independent and has the same P -type as $\bar{c}d$. Thus, we can extend f to $\bar{c}d$ setting $g(d) = d'$.

CASE 2. $d \in \mathbb{A} \setminus \text{cl}(\bar{c}) = \mathbb{A} \setminus \text{cl}(\bar{a})$. Since $\bar{c} \perp_{\bar{a}} \mathbb{A}$ and $c \in \mathbb{A}$, we have $\bar{c} \perp_{\bar{a}d} \mathbb{A}$, and therefore $\bar{c}d$ is P -independent. Let $q(x) := \text{tp}^1(d/\bar{c})$, and $q' := f(q) \in S_1^1(\bar{c}')$. Notice that $q \perp_{\bar{a}} \bar{c}$ (because $d \perp_{\bar{a}} \bar{c}$), and therefore $q' \perp_{\bar{a}'} \bar{c}'$. Since $\mathbb{A}'$ is dense in $\mathbb{B}'$ and $\langle \mathbb{B}', \mathbb{A}' \rangle$ is λ -saturated, there exists $d' \in \mathbb{A}'$ realizing q' . It is now easy to see that $\bar{c}'d'$ is P -independent, and that we can extend f to $\bar{c}d$ setting $g(d) = d'$.

CASE 3. $d \in \text{cl}(\bar{c}\mathbb{A}) \setminus \mathbb{A}$. Let $\bar{a}_0 \in \mathbb{A}^n$ such that $d \in \text{cl}(\bar{a}_0)$ ($\bar{a}_0$ exists because cl is finitary). By applying n times cases 1 or 2, we can extend f to $f' \in \Gamma$ such that $\bar{a}_0$ is a subset of the domain of f' . By substituting f with f' , we are reduced to the case that $d \in \text{cl}(\bar{c}) \setminus \mathbb{A}$. Hence, $\bar{c} \perp_{\bar{a}} \mathbb{A}$ and $d \in \text{cl}(\bar{c})$: therefore, $\bar{c}d \perp_{\bar{a}} \mathbb{A}$, and hence $\bar{c}d$ is P -independent. Let $d' \in \mathbb{B}'$ such that $d'\bar{c}' \equiv^1 d\bar{c}$. For the same reason as above, $\bar{c}'d'$ is also P -independent. It remains to show that $\bar{c}d$ and $\bar{c}'d'$ have the same P -type, that is that $d' \notin \mathbb{A}'$. If, for contradiction, $d' \in \mathbb{A}' \cap \text{cl}(\bar{c})$, then $d' \in \text{cl}(\bar{a})$; therefore, there would be a x -narrow-formula witnessing it, and thus $d \in \text{cl}(\bar{a}) \subseteq \mathbb{A}$, absurd.

CASE 4. $d \notin \text{cl}(\bar{c}\mathbb{A})$. Let $\bar{a}_0 \subset \mathbb{A}$ be of cardinality less than λ such that $d \perp_{\bar{a}_0\bar{a}} \mathbb{A}$ ($\bar{a}_0$ exists because $\perp$ satisfies Local Character). By applying cases 2 and 3 sufficiently many times, we can extend f to $f' \in \Gamma$ such that $\bar{a}_0$ is contained in the domain of f' ; thus, w.l.o.g., $d \perp_{\bar{a}} P$. Let $d' \in \mathbb{A}m'$ such that $d'\bar{c}' \equiv^1 d\bar{c}$; moreover, by Lemma 8.14 we can also assume that $d' \perp_{\bar{a}'} \mathbb{A}'$. We

need only to show that $d' \notin \mathbb{A}'$. Assume, for contradiction, that $d' \in \mathbb{A}'$ and $d' \perp_{\bar{a}'} \mathbb{A}'$; then, $d' \perp_{\bar{a}'} d'$, thus $d' \in \text{cl}(\bar{a}')$, and hence $d \in \text{cl}(\bar{a})$, absurd. $\square$

8.3 Additional facts

Reasoning as in [Dries98, 2.6–2.9], from Theorems 8.3 and 8.5, and Proposition 8.16, we can deduce the following facts. We are still assuming that T expands an integral domain. We will say that A is T^2 -algebraically closed if A is a subset of some pair $\langle \mathbb{B}, \mathbb{C} \rangle \models T^d$ and A is algebraically closed w.r.t. the language $\mathcal{L}^2$, and similarly for T^2 -definably closed. We denote by acl^1 the algebraic closure in T , and by acl^2 the algebraic closure in T^2 . We denote by $\equiv^1$ elementary equivalence w.r.t. the language $\mathcal{L}$, and by $\equiv^2$ elementary equivalence w.r.t. the language $\mathcal{L}^2$. Similarly, $\text{tp}^1(\bar{b}/X)$ will denote the $\mathcal{L}$ -type of $\bar{b}$ over X , while $\text{tp}^2(\bar{b}/X)$ will denote the $\mathcal{L}^2$ type.

cor:vdd-26

Corollary 8.17 ([Dries98, 2.6]). *Let $\langle \mathbb{B}, \mathbb{A} \rangle$ be a model of T^d . Suppose $Y \subseteq \mathbb{B}^n$ is A_0 -definable in $\langle \mathbb{B}, \mathbb{A} \rangle$, for some $A_0 \subset \mathbb{A}$. Then $Y \cap \mathbb{A}^n$ is A_0 -definable in $\mathbb{A}$.*

cor:vdd-27

Corollary 8.18 ([Dries98, 2.7]). *Let $\langle \mathbb{B}, \mathbb{A} \rangle$ and $\langle \mathbb{B}', \mathbb{A}' \rangle$ be models of T^d , such that $\langle \mathbb{B}', \mathbb{A}' \rangle \subseteq \langle \mathbb{B}, \mathbb{A} \rangle$ and $\mathbb{B}'$ and $\mathbb{A}$ are cl-independent over $\mathbb{A}'$. Then, $\langle \mathbb{B}', \mathbb{A}' \rangle \preceq \langle \mathbb{B}, \mathbb{A} \rangle$. In particular, if $\mathbb{A} \prec \mathbb{B}' \preceq \mathbb{B}$, with $\mathbb{A} \neq \mathbb{B}'$, then $\langle \mathbb{B}', \mathbb{A} \rangle \preceq \langle \mathbb{B}, \mathbb{A} \rangle$.*

cor:Td-2extensions

Corollary 8.19 ([Dries98, 2.8]). *Let $A \subseteq B \subset \mathbb{M}$ be substructures. Assume that $\langle B, A \rangle$ have extensions $\langle \mathbb{B}_1, \mathbb{A}_1 \rangle \models T^d$ and $\langle \mathbb{B}_2, \mathbb{A}_2 \rangle \models T^d$, such that $B \perp_A \mathbb{A}_k$ and $B \cap \mathbb{A}_k = A$, $k = 1, 2$. Then, $\langle \mathbb{B}_1, \mathbb{A}_1 \rangle \equiv_B^2 \langle \mathbb{B}_2, \mathbb{A}_2 \rangle$, that is $\langle \mathbb{B}_1, \mathbb{A}_1 \rangle$ and $\langle \mathbb{B}_2, \mathbb{A}_2 \rangle$ satisfy the same $\mathcal{L}^2$ -formulae with parameters from B . More generally, for every $\bar{a}_1 \in (\mathbb{A}_1)^n$ and $\bar{a}_2 \in (\mathbb{A}_2)^n$, if $\bar{a}_1 \equiv_B^1 \bar{a}_2$, then $\bar{a}_1 \equiv_B^2 \bar{a}_2$; that is, if $\bar{a}_1$ and $\bar{a}_2$ realise the same $\mathcal{L}$ -types over B in $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively, then they realise the same $\mathcal{L}^2$ -type over B in $\langle \mathbb{B}_1, \mathbb{A}_1 \rangle$ and $\langle \mathbb{B}_2, \mathbb{A}_2 \rangle$ respectively.*

Notice that the hypothesis of the above Corollary implies that A is cl-closed (but not necessarily dense) in B .

Proof. Let $\bar{c}_k := B\bar{a}_k$. $\bar{c}_1$ and $\bar{c}_2$ have the same P -type, they are both P -independent, and $\bar{c}_1 \equiv^1 \bar{c}_2$; the conclusion follows from Proposition 8.16. $\square$

cor:Td-type-1

Corollary 8.20 ([Dries98, 2.9]). *Let $\langle \mathbb{B}_1, \mathbb{A}_1 \rangle \models T^d$ and $\langle \mathbb{B}_2, \mathbb{A}_2 \rangle \models T^d$, and let A be a common subset of $\mathbb{A}_1$ and $\mathbb{A}_2$. Suppose that $b_1 \in \mathbb{B}_1 \setminus \mathbb{A}_1$ and $b_2 \in \mathbb{B}_2 \setminus \mathbb{A}_2$ realise the same $\mathcal{L}$ -types over A in $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively, that is $b_1 \equiv_A^1 b_2$. Then, they realise the same $\mathcal{L}^2$ -types over A in $\langle \mathbb{B}_1, \mathbb{A}_1 \rangle$ and $\langle \mathbb{B}_2, \mathbb{A}_2 \rangle$ respectively, that is $b_1 \equiv_A^2 b_2$.*

Proof. Let $\bar{c}_i := b_i \mathbb{A}_i$, $i = 1, 2$. By assumption, $\bar{c}_1 \equiv^1 \bar{c}_2$, they have the same P -type, and they are both P -independent. The conclusion follows from Proposition 8.16. $\square$

For the remainder of this section, we will assume that $\langle \mathbb{B}, \mathbb{A} \rangle$ is a model of T^d , and that λ is a cardinal number such that $\kappa > \lambda > \max(|T|, |\mathbb{B}|)$.

lem:thm-2

Lemma 8.21 ([Dries98, Theorem 2]). *Let $\bar{b} \subset \mathbb{B}$ be P -independent. Given a set $Y \subset \mathbb{A}^n$, t.f.a.e.:*

1. Y is T^2 -definable over $\bar{b}$;
2. $Y = Z \cap \mathbb{A}^n$ for some set $Z \subseteq \mathbb{B}^n$ that is T -definable over $\bar{b}$.

Proof. (1 $\Rightarrow$ 2) is as in [Dries98, Theorem 2]. (2 $\Rightarrow$ 1) is obvious. $\square$

Lemma 8.22 ([Dries98, 3.1]). $\mathbb{A}$ is T^2 -algebraically closed in $\langle \mathbb{B}, \mathbb{A} \rangle$.

Proof. As in [Dries98, 3.1]: let $b \in \mathbb{B} \setminus \mathbb{A}$. Let $\langle \mathbb{B}^*, \mathbb{A}^* \rangle \succeq \langle \mathbb{B}, \mathbb{A} \rangle$ be a monster model. Since cl is existential, and $b \notin \text{cl}(\mathbb{A})$, there exists infinitely many distinct $b' \in \mathbb{B}^*$ such that $b \equiv_{\mathbb{A}}^1 b'$. By Corollary 8.20, $b \equiv_{\mathbb{A}}^2 b'$. Thus, b is not T^2 - $\mathbb{A}$ -algebraic in $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$, and therefore not T^2 - $\mathbb{A}$ -algebraic in $\langle \mathbb{B}, \mathbb{A} \rangle$. $\square$

lem:3.2

Lemma 8.23 ([Dries98, 3.2]). *Let $A_0 \subseteq \mathbb{A}$ be T -algebraically closed (T -definably closed). Then A_0 is T^2 -algebraically closed (resp. T^2 -definably closed).*

Proof. Assume A_0 is T -algebraically closed. Let $c \in \text{acl}^2(A_0)$, and $C := \{c_1, \dots, c_n\}$ be the set of $\mathcal{L}^2$ -conjugates of c/A_0 . By definition, C is A_0 -definable in $\langle \mathbb{B}, \mathbb{A} \rangle$, and, by the above Lemma, $C \subset \mathbb{A}$. Hence, by Corollary 8.17, C is A_0 -definable in $\mathbb{A}$. The case when A_0 is T -definably closed is similar. $\square$

lem:orbit-dense

Lemma 8.24. *Let $\langle \mathbb{B}, \mathbb{A} \rangle$ be a κ -saturated model of T^d . Let $D \subset \mathbb{B}$ such that $|D| < \lambda$, and $c \in \mathbb{B} \setminus \text{cl}(D)$. Define $C := \{c' \in \mathbb{B} : c' \equiv_D^1 c\} \cap \mathbb{A}$. Then, $|C| \geq \lambda$.*

Proof. Consider the following partial $\mathcal{L}^2$ -type over D :

$$p(x_i : i < \lambda) := \left(\bigwedge_i x_i \equiv_D^1 c \right) \& \left(\bigwedge_i U(x_i) \right) \& \left(\bigwedge_{i < j} x_i \neq x_j \right).$$

Claim 5. p is consistent.

If not, there exist $\bar{d} \subset D$, $\bar{b} \subset \mathbb{B}$, $\phi(x, \bar{d}) \in \text{tp}^1(c/D)$, such that $\phi(\mathbb{B}, \bar{d}) \setminus \mathbb{A} = \bar{b}$. Let $X := \phi(\mathbb{B}, \bar{d}) \setminus \bar{b}$: notice that X is definable in $\mathbb{B}$, and $X \subseteq \mathbb{A}$. Hence, since $\mathbb{A}$ is co-dense in $\mathbb{B}$, we conclude that $\dim(X) \leq 0$, and therefore $\dim(\phi(\mathbb{B}, \bar{d})) \leq 0$. Thus, $c \in \text{cl}(\bar{d}) \subseteq \text{cl}(D)$, absurd.

The conclusion follows immediately from the claim. $\square$

prop:3.3

Proposition 8.25 ([Dries98, 3.3]). *Let $\bar{b} \subset \mathbb{B}$ be P -independent. Then, $\text{dcl}^2(\bar{b}) = \text{dcl}^1(\bar{b})$, and similarly for the algebraic closure. Let $c \in \mathbb{B}^{\text{eq}}$ (i.e., c is an imaginary element for the structure $\mathbb{B}$). Then, $c \in \text{dcl}^2(\bar{b})$ iff $c \in \text{dcl}^1(\bar{b})$, and similarly for the algebraic closure.*

Idea for proof. W.l.o.g., we can assume that $\langle \mathbb{B}, \mathbb{A} \rangle$ is λ -saturated and that $\bar{b}$ has finite length. So, let c be a $\mathbb{B}$ -imaginary such that $c \in \text{acl}^2(\bar{b})$. We have to prove that $c \in \text{acl}^1(\bar{b})$.

Let $\mathbb{B}_1 := \text{cl}^{\mathbb{B}}(\mathbb{A}\bar{b})$; by Corollary 8.18, $\langle \mathbb{B}_1, \mathbb{A} \rangle \preceq \langle \mathbb{B}, \mathbb{A} \rangle$, and in particular $\mathbb{B}_2$ is T^2 -algebraically closed in $\langle \mathbb{B}, \mathbb{A} \rangle$, and therefore $c \in \mathbb{B}_1^{\text{eq}}$. Let $n \geq 0$ minimal and $\bar{a} \in \mathbb{A}^n$ such that $c \in \text{cl}(\bar{b}\bar{a})$.

cl:33-2

Claim 6. $c \in \text{cl}(\bar{b})$, i.e. $n = 0$.

If $n > 0$, by substituting $\bar{b}$ with $\bar{b}a_1, \dots, a_{n-1}$, and proceeding by induction on n , we can reduce to the case $n = 1$; let $a := a_1$. Consider the following partial $\mathcal{L}$ -type over $\bar{b}a$:

$$q(x) := (x \equiv_{\bar{b}}^1 a) \ \& \ (x \downarrow_{\bar{b}} a).$$

Since $\downarrow$ satisfies Existence, q is consistent. Let $d \in \mathbb{B}$ be any realisation of q . Since $d \downarrow_{\bar{b}} a$, we conclude that either $d \notin \text{cl}(\bar{b}a)$ or $d \in \text{cl}(\bar{b})$. However, the latter cannot happen, since $d \equiv_{\bar{b}}^1 a \notin \text{cl}(\bar{b})$; thus, $d \notin \text{cl}(\bar{b}a)$, and therefore $\dim(q) = 1$. Hence, since $\mathbb{A}$ is dense in $\mathbb{B}$ and $\langle \mathbb{B}, \mathbb{A} \rangle$ is ω -saturated, there exists $a' \in \mathbb{A}$ satisfying q . Reasoning in the same way, we can show that there exists $(a_2, a_3, a_4, \dots)$ a Morley sequence in q contained in $\mathbb{A}$. By Corollary 8.19, $a_i \equiv_{\bar{b}}^2 a$ for every i . Let $c_1, c_2, \dots, c_m$ be all the $\mathcal{L}^2$ -conjugates of c over $\bar{b}$ (there are finitely many of them), and let $\phi(x, y, \bar{z})$ be an x -narrow $\mathcal{L}$ -formula without parameters such that $\mathbb{B} \models \phi(c, a, \bar{b})$.

The $\mathcal{L}$ -formula (in y , with parameters in $\bar{b}c_1, \dots, c_m$) $\bigvee_i \phi(c_i, y, \bar{b})$ is equivalent to an $\mathcal{L}^2$ -formula in y with parameters $\bar{b}$; hence, every a_i satisfies it (because $a_i \equiv_{\bar{b}}^2 a$). Hence, w.l.o.g. $c_1 \in \text{cl}(\bar{b}a_2) \cap \text{cl}(\bar{b}a_3) = \text{cl}(\bar{b})$ (because $a_2 \downarrow_{\bar{b}} a_3$). Therefore, $c \in \text{cl}(\bar{b})$.

It remains to show that $c \in \text{acl}^1(\bar{b})$. Let $c_2 \in \mathbb{B}^{\text{eq}}$ such that $c_2 \equiv_{\bar{b}}^1 c$. Since $\mathbb{B}$ is ω -saturated, it suffices to prove that there are only finitely many such c_2 . Since $c \in \text{acl}^2(\bar{b})$, it suffices to prove that $c_2 \equiv_{\bar{b}}^2 c$. Let $\bar{b}_1 := \bar{b}c$, $\bar{b}_2 := \bar{b}c_2$, and $\bar{a} := \bar{b} \cap \mathbb{A}$. By assumption, $\bar{b}_1 \equiv^1 \bar{b}_2$. By Claim 6, we have $\bar{b}_1 \subseteq \text{cl}(\bar{b})$, and therefore, since $\bar{b} \downarrow_{\bar{a}} \mathbb{A}$, $\bar{b}_1$ is P -independent. Claim 6 also implies that $\bar{b}_2 \subseteq \text{cl}(\bar{b})$, and hence $\bar{b}_2$ is also P -independent. It remains to show that $\bar{b}_1$ and $\bar{b}_2$ have the same P -type. Assume e.g. that $c \in \mathbb{A}$. Since $\bar{b} \downarrow_{\bar{a}} \mathbb{A}$, we have that $c \in \text{cl}(\bar{a})$, and therefore $c_2 \in \text{cl}^{\mathbb{B}}(\bar{a}) \subseteq \mathbb{A}$.

The other assertions are proved in the same way. $\square$

c:small-closure

8.4 The small closure

We will be still assuming that T expands an integral domain. Let $\mathbb{M}^* := \langle \mathbb{B}^*, \mathbb{A}^* \rangle$ be a κ -saturated and strongly κ -homogeneous monster model of T^d , and $\langle \mathbb{B}, \mathbb{A} \rangle \prec \mathbb{M}^*$, with $|\mathbb{B}| < \kappa$. Notice that $\text{rk}(\mathbb{B}^*/\mathbb{A}^*) \geq \kappa$.

Definition 8.26. For every $X \subseteq \mathbb{B}^*$ we define the **small closure** of X as

$$\text{Scl}(X) := \text{cl}(X\mathbb{A}^*).$$

For lovely pairs (e.g., dense pairs of o-minimal structures), the small closure was already defined in [Berenstein07].

Remark 8.27. Scl is a definable matroid (on $\mathbb{M}^*$).

Proof. Scl coincides with the operator $\text{cl}_{\mathbb{A}^*}$ in Lemma 5.5. $\square$

Notice that we can apply Lemma 5.6: $\text{Scl}^{\mathbb{B}} = (\text{cl}^{\mathbb{B}})_{\mathbb{A}}$: we can “compute” the small closure of a subset of $\mathbb{B}$ inside $\mathbb{B}$ using $\mathbb{A}$ instead of $\mathbb{A}^*$.

We want to prove that Scl is existential; we will need a preliminary lemma.

Lemma 8.28. *Let $b \in \mathbb{B}^* \setminus \mathbb{A}^*$. Define $\mathbb{M}_b^*$ the expansion of $\mathbb{M}^*$ with a constant for b , and $\text{Scl}_b(X) := \text{Scl}(bX) = \text{cl}(X\mathbb{A}^*b)$. Then, Scl_b is an existential matroid on $\mathbb{M}_b^*$.*

Proof. That Scl_b is a definable matroid follows from Lemma 5.5, applied to Scl . Let $X \subseteq \mathbb{M}^*$, and $Y := \text{Scl}_b(X)$.

Claim 7. $Y \prec \mathbb{M}^*$ (as an $\mathcal{L}^2$ -structure).

By Lemma 7.4, Y is an elementary $\mathcal{L}$ -substructure of $\mathbb{B}^*$. By Theorems 8.3, and 8.5 and, again, Lemma 7.4, it suffices to show that $\mathbb{A}^*$ is a cl -closed, dense, and proper subset of Y , which is trivially true.

The lemma then follows from the above claim and Lemma 3.22; non-triviality follows from the fact that $\text{Zrk}(\mathbb{B}^*/\mathbb{A}^*) \geq \kappa$. $\square$

lem:scl

Lemma 8.29. *Scl is an existential matroid.*

Proof. The only thing that needs proving is Existence. Define $\Xi^2(a/C)$ as the set of conjugates of a over C in $\mathbb{M}$. Assume that $\Xi^2(a/C) \subseteq \text{Scl}(CD)$. We want to prove that $a \in \text{Scl}(C)$. Choose $b, b' \in \mathbb{B}^*$ which are cl -independent over $\mathbb{A}^*C$. By applying the previous lemma to Scl_b and $\text{Scl}_{b'}$, we see that

$$a \in \text{Scl}_b(C) \cap \text{Scl}_{b'}(C) = \text{cl}(\mathbb{A}^*Cb) \cap \text{cl}(\mathbb{A}^*Cb') = \text{cl}(\mathbb{A}^*C) = \text{Scl}(C). \quad \square$$

Hence, we can define the dimension induced by Scl , and denote it by Sdim . Notice that, by Theorem 3.47, Scl is the only existential matroid on T^d .

Lemma 8.30. *Let $X \subseteq (\mathbb{B}^*)^n$ be definable in $\mathbb{B}^*$. Then $\text{Sdim}(X) = \dim(X)$.*

Proof. From $\text{cl} \subseteq \text{Scl}$ follows immediately that $\text{Sdim}(X) \leq \dim(X)$. For the opposite inequality, we proceed by induction on $k := \dim(X)$. Assume, for contradiction, that $\text{Sdim}(X) < k$. W.l.o.g., $\dim(\Pi_k^n(X)) = k$; therefore, w.l.o.g. $k = n$. If $k = 1$, then $\text{Sdim}(X) = 0$, and therefore $F^4(X) \neq \mathbb{B}^{*n}$, contradicting $\dim(X) = 1$. For the inductive step, assume $k = n > 1$, and let $U := \{a \in \mathbb{B}^n : \dim(X_a) = n - 1\}$. U is definable in $\mathbb{B}$, and therefore, by inductive hypothesis, $\text{Sdim}(U) = \dim(U) = n - 1$. By the case $k = 1$, for every $a \in \mathbb{K}^{n-1}$, $\dim(X_a) = \text{Sdim}(X)$, and therefore $\text{Sdim}(X_a) = 1$ for every $a \in U$. Thus, $\text{Sdim}(X) = n$. $\square$

Definition 8.31. Let $X \subseteq (\mathbb{B}^*)^n$ be definable in $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$. We say that X is small if $\text{Sdim}(X) = 0$. Let $Y \subseteq \mathbb{B}^n$ be definable in $\langle \mathbb{B}, \mathbb{A} \rangle$. We say that Y is small if $\text{Sdim}(Y^*) = 0$, where Y^* is the interpretation of Y inside $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$.

Notice that, if $X \subset \mathbb{B}^n$ is $\mathbb{A}$ -small (in the sense of Definition 8.6), then X is also small in the above sense. The next lemma shows that the converse is also true.

lem:pair-small

Lemma 8.32. *Let $\langle \mathbb{B}, \mathbb{A} \rangle \preceq \langle \mathbb{B}^*, \mathbb{A}^* \rangle$ and $X \subseteq \mathbb{B}^n$ be definable in $\langle \mathbb{B}, \mathbb{A} \rangle$. Let X^* be the interpretation of X inside $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$. Let $\bar{c} \in \mathbb{B}^k$ be the parameters of definition of X . T.f.a.e.:*

1. X is small;
2. X^* is small;
3. $X^* \subseteq \text{Scl}(\bar{b})$ for some finite tuple $\bar{b} \subset \mathbb{B}^*$;
4. $X^* \subseteq \text{Scl}(\bar{c})$;
5. $X^* \subseteq \text{cl}(\bar{c}\mathbb{A}^*)$;
6. X is $\mathbb{A}$ -small: that is, there exists a Z -application $f^* : (\mathbb{B}^*)^m \rightsquigarrow (\mathbb{B}^*)^n$, definable in $\mathbb{B}^*$ with parameters, such that $f^*(\mathbb{A}^{*m}) \supseteq X^*$;
7. X^* is $\mathbb{A}^*$ -small: that is, there exists a Z -application $f : \mathbb{B}^m \rightsquigarrow \mathbb{B}^n$, definable in $\mathbb{B}$ with parameters $\bar{c}$, such that $f^*(\mathbb{A}^{*m}) \supseteq X^*$, where f^* is the interpretation of f in $\mathbb{B}^*$;
8. there exists a Z -application $g : \mathbb{B}^{m+k} \rightsquigarrow \mathbb{B}^n$, definable in $\mathbb{B}$ without parameters, such that $g^*(\mathbb{A}^{*m} \times \{\bar{c}\}) \supseteq X^*$;
9. there exists a Z -application $f : \mathbb{B}^m \rightsquigarrow \mathbb{B}^n$, definable in $\mathbb{B}$ without parameters, such that $f(\mathbb{A}^m \times \{\bar{c}\}) \supseteq X$.

Proof. The only non-trivial implication is $(5 \Rightarrow 7)$, which is proved by a compactness argument using Remark 3.57. $\square$

Conjecture 8.33 ([Dries98, 3.6]). *Let $f : \mathbb{A}^n \rightarrow \mathbb{A}$ be T^2 -definable with parameters $\bar{b}$. Let $\bar{a} \subset \mathbb{A}^m$ such that $\bar{b} \downarrow_{\bar{a}} \mathbb{A}$ and $\text{dcl}^1(\bar{b}\bar{a}) \cap \mathbb{A} = \text{dcl}^1(\bar{a})$. Then, f is given piecewise by functions definable in $\mathbb{A}$ with parameters $\bar{a}$.*

Idea for proof. By compactness, it suffices to show that, given an elementary extension $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$ of $\langle \mathbb{B}, \mathbb{A} \rangle$ and a point $a^* \in (\mathbb{A}^*)^n$, we have $f(a^*) \in \mathbb{A}' := \text{dcl}^1(\bar{a}a^*)$. Let $\mathbb{B}' := \text{dcl}^1(\bar{b}\bar{a}a^*)$. Since $\bar{b} \downarrow_{\bar{a}} \mathbb{A}$, we have $\mathbb{B}' \downarrow_{\mathbb{A}'} \mathbb{A}$. Hence, by Proposition 8.25, $\mathbb{B}'$ is dcl^2 -closed, and therefore $f(a^*) \in \mathbb{B}'$. Hence,

$$f(a^*) \in \text{dcl}^1(\bar{b}\bar{a}a^*) \cap \mathbb{A}^* = \text{cl}(\bar{b}\bar{a}a^*) \cap \mathbb{A}^* \cap \text{dcl}^1(\bar{b}\bar{a}a^*) = \text{cl}(\bar{a}a^*) \cap \text{dcl}^1(\bar{b}\bar{a}a^*).$$

It remains to show that $\text{cl}(\bar{a}a^*) \cap \text{dcl}^1(\bar{b}\bar{a}a^*) = \text{dcl}^1(\bar{a}a^*)$. $\square$

Lemma 8.34 ([Boxall09, 6.1.3]). *Let $f : \mathbb{A}^n \rightarrow \mathbb{B}$ be T^2 -definable with parameters $\bar{b}$. Assume that $\bar{b}$ is P -independent. Then, there exists $g : \mathbb{B}^n \rightarrow \mathbb{B}$ which is T -definable with parameters $\bar{b}$, and such that $f = g \upharpoonright \mathbb{A}^n$.*

Proof. Let $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$ be an elementary extension of $\langle \mathbb{B}, \mathbb{A} \rangle$ and $a^* \in (\mathbb{A}^*)^n$. By Proposition 8.25, there exists a function $g_i : \mathbb{B}^n \rightarrow \mathbb{B}$ which is T -definable with parameters $\bar{b}$, such that $f(a) = g_i(a)$. By compactness, finitely many g_i will suffice. The conclusion then follows from Lemma 8.21. $\square$

prop:3.5

Proposition 8.35 ([Dries98, 3.5]). *Let $\bar{b} \in \mathbb{B}^k$ and $\bar{a} \in \mathbb{B}^{k'}$ such that $\bar{b} \downarrow_{\bar{a}} \mathbb{A}$ and $\bar{b} \cap \mathbb{A} \subseteq \bar{a}$. Let $X \subseteq \mathbb{B}^n$ be T -definable (possibly, inside an imaginary sort) with parameters $\bar{b}$, such that $\dim(X) = d$. Let $Y \subseteq X$ be T^2 -definable, with parameters $\bar{b}$. Then, there exist $S \subset X$ which is T^2 -definable with parameters $\bar{b}$, and $Z \subseteq X$ which is T^2 -definable with parameters $\bar{b}\bar{a}$, such that $Z \Delta Y \subseteq S$ and $\text{Sdim}(S) < d$.*

In particular, if $\dim(X) = 0$, then every T^2 -definable subset of X is already T -definable.

Proof. The proof is a variant of Beth's definability theorem, using Proposition 8.16. W.l.o.g., $\langle \mathbb{B}, \mathbb{A} \rangle$ is λ -saturated for some cardinal λ such that $|T| < \lambda < \kappa$.

Let $W := \{p \in S_X^2(\bar{a}\bar{b}) : \text{Sdim}(p) = d\}$. Notice that W is a closed subset of $S_X^2(\bar{a}\bar{b})$ (the Stone space of T^2 -types over $\bar{a}\bar{b}$ containing the formula " $x \in X$ "). Let $\theta : S_X^2(\bar{a}\bar{b}) \rightarrow S_X^1(\bar{a}\bar{b})$ be the restriction map: notice that θ is a continuous homomorphism, and therefore $V := \theta(W)$ is closed in $S_X^1(\bar{a}\bar{b})$. Let $\rho := \theta \upharpoonright W$.

Claim 8. ρ is injective (and therefore ρ is a homeomorphism between W and V).

We have to prove that, for every $\bar{c}$ and $\bar{c}' \in X$, if $\text{Sr}k(\bar{c}/\bar{a}\bar{b}) = \text{Sr}k(\bar{c}'/\bar{a}\bar{b}) = d$ and $\bar{c} \equiv_{\bar{a}\bar{b}}^1 \bar{c}'$, then $\bar{c} \equiv_{\bar{a}\bar{b}}^2 \bar{c}'$. Let $\bar{d} := \bar{a}\bar{b}\bar{c}$ and $\bar{d}' := \bar{a}\bar{b}\bar{c}'$. By Proposition 8.16, it suffices to prove that $\bar{d}$ and $\bar{d}'$ are both P -independent and have the same P -type. Since $\text{Sr}k(\bar{c}/\bar{a}\bar{b}) = d$ and $\bar{c} \in X$, we have that $\text{Sr}k(\bar{c}/\bar{a}\bar{b}) = \text{rk}(\bar{c}/\bar{a}\bar{b})$, which is equivalent to $\bar{c} \perp_{\bar{a}\bar{b}} \mathbb{A}$, and hence (since $\bar{b} \perp_{\bar{a}} \mathbb{A}$) $\bar{d} \perp_{\bar{a}} \mathbb{A}$, that is $\bar{d}$ is P -independent, and similarly for $\bar{d}'$. It remains to show that $\bar{d}$ and $\bar{d}'$ have the same P -type. Let $d_i \in \mathbb{A}$; we have to prove that $d'_i \in \mathbb{A}$. Since $\bar{d} \perp_{\bar{a}} \mathbb{A}$, we have $d_i \in \text{cl}(\bar{a})$, and hence $d'_i \in \text{cl}^{\mathbb{B}}(\bar{a}') \subseteq \mathbb{A}$.

Let $U := S_Y^2(\bar{a}\bar{b}) \cap W$; since Y is definable, U is clopen in W , and since ρ is a homeomorphism, $\rho(U)$ is clopen in V . Hence, there exists Z subset of X , such that Z is T -definable over $\bar{a}\bar{b}$ and $V \cap S_{Z_1}^1(\bar{a}\bar{b}) = \rho(U)$.

Claim 9. There exists $S \subset X$ which is T_2 -definable over $\bar{b}$, such that $\text{Sdim}(S) < d$ and $Y \Delta Z \subseteq S$.

Assume not. Then, the following partial type over $\bar{a}\bar{b}$ is consistent:

$$\Phi(\bar{x}) := \bar{x} \in X \ \& \ \bar{x} \in Y \Delta Z \ \& \ \bar{x} \notin S,$$

where S varies among the subsets of X which are T^2 -definable over $\bar{b}$, with $\text{Sdim}(S) < d$. Let $\bar{c} \in X$ be a realization of Φ and $p := \text{tp}^2(\bar{c}/\bar{a}\bar{b}) \in S_X^2(\bar{a}\bar{b})$. By assumption, $\text{Sdim}(\bar{c}/\bar{a}\bar{b}) = d$, and therefore $p \in W$. Hence, $\rho(p) = \text{tp}^1(\bar{c}/\bar{a}\bar{b}) \in V$. Since ρ is injective, we have

$$\rho(p) \in \rho(S_Y^2(\bar{a}\bar{b}) \cap W) \Delta \rho(S_Z^2(\bar{a}\bar{b}) \cap W) \subseteq S_Z^1(\bar{a}\bar{b}) \Delta S_Y^1(\bar{a}\bar{b}) = \emptyset,$$

absurd. $\square$

In general, given $\bar{b} \in \mathbb{B}^n$, it is always possible to find $\bar{a} \in \mathbb{A}^{n'}$ such that $\bar{b} \perp_{\bar{a}} \mathbb{A}$. However, there are some examples when $\mathbb{B}$ is o-minimal, but $\bar{a}$ cannot be found inside $\text{dcl}^2(\bar{b})$.⁶

cor:3.4

Corollary 8.36 ([Dries98, 3.4]). *Let $\bar{b}$ and $\bar{a}$ be as in the above Proposition. Let Γ be a T -definable set (possibly, in some imaginary sort) over $\bar{b}$, and let the function $f : \mathbb{B}^n \rightarrow \Gamma$ be T^2 -definable with parameters $\bar{b}$. Then, there exist $S \subseteq \mathbb{B}^n$, which is T^2 -definable over $\bar{b}$ and with $\text{Sdim}(S) < n$, and $\hat{f} : \mathbb{B}^m \rightarrow \Gamma$, which is T -definable over $\bar{b}\bar{a}$, such that f agrees with $\hat{f}$ outside S .*

Proof. W.l.o.g., $\langle \mathbb{B}, \mathbb{A} \rangle$ is κ -saturated.

Claim 10. There exists a set $S \subset \mathbb{B}^n$ which is T^2 -definable with parameters $\bar{b}$, with $\text{Sdim}(S) < n$, and finitely many functions $g_1, \dots, g_k : \mathbb{B}^n \rightarrow \Gamma$ that are T -definable with parameters $\bar{b}\bar{a}$, such that f agrees off S with some of the g_i .

⁶Thanks to J. Ramakrishnan for pointing this out.

Assume that the claim does not hold. Hence, for every S and every g as in the claim, there exists $\bar{c} \in \mathbb{B}^n$ such that $\bar{c} \notin S$ and $f(\bar{c}) \neq g(\bar{c})$. Thus, the following partial $\mathcal{L}^2$ -type over $\bar{b}\bar{a}$ is consistent:

$$p(\bar{x}) := (\bar{x} \in \mathbb{B}^n \setminus \text{Scl}(\bar{b})) \ \& \ (f(\bar{x}) \neq g(\bar{x})),$$

where we let $g : \mathbb{B}^n \rightarrow \Gamma$ vary among the functions that are T -definable with parameters $\bar{b}\bar{a}$. Let $\bar{c}$ be a realisation of p . Notice that the choice of $\bar{a}$ and the fact that $\text{Sr}(\bar{c}/\bar{a}\bar{b}) = n$ imply that $\bar{c}\bar{b}\bar{a} \downarrow_{\bar{a}} \mathbb{A}$. Hence, by Proposition 8.25, $f(\bar{c}) \in \text{dcl}^1(\bar{c}\bar{b}\bar{a})$. Hence $f(\bar{c}) = g(\bar{c})$ for some function $g : \mathbb{B}^n \rightarrow \mathbb{B}$ which is T -definable with parameters $\bar{b}\bar{a}$, absurd.

The above Claim plus Proposition 8.35 imply the conclusion. $\square$

The above Corollary gives a way to find the parameters of definition of $\hat{f}$ (and of S) starting from the parameters $\bar{b}$ of f .

Example 8.37. In general, $\hat{f}$ cannot be defined using only $\bar{b}$ as parameters. Consider a_1 and a_2 in $\mathbb{A}$ which are independent over the empty set, $b_1 \in \mathbb{B} \setminus \mathbb{A}$, and $b_2 := a_1 + b_1 a_2 \in \mathbb{B} \setminus \mathbb{A}$. Let $\bar{a} := \langle a_1, a_2 \rangle$ and $\bar{b} := \langle b_1, b_2 \rangle$. Notice that $\text{rk}(\bar{a}\bar{b}) = 3$, while $\text{Sr}(\bar{a}\bar{b}) = 2$. Let f be the constant function a_1 . Then, f is T^2 -definable over $\bar{b} := \langle b_1, b_2 \rangle$, but is not T -definable over $\bar{b}$.

tion:dmin-scl-1

Question 8.38. Assume that T is d-minimal (see §9). Is it true that, for every $X \subseteq \mathbb{B}^*$, $\text{Scl}(X) = \text{acl}(\mathbb{A}^*X)$?

Conjecture 8.39 (J. Ramakrishnan). *Assume that T is o-minimal. Then, for every $X \subset \mathbb{B}$,*

$$\text{acl}^2(X) = \text{acl}^1(X \cup (\text{acl}^2(X) \cap \mathbb{A})).$$

8.5 Elimination of imaginaries

Let cl be an existential matroid on $\mathbb{M}$. Remember that element $e \in \mathbb{M}^{\text{eq}}$ is an equivalence class $X \subseteq \mathbb{M}^n$ for some $\emptyset$ -definable equivalence relation E on $\mathbb{M}^n$. If $\bar{c} \in X$ we say that $\bar{c}$ **represents** e .

Definition 8.40. We say that $\mathbb{M}$ has **cl-elimination of imaginaries** if, for every $e \in \mathbb{M}^{\text{eq}}$, there exists $\bar{c}$ representing e , such that $\bar{c} \in \text{cl}^{\text{eq}}(e)$. Given $\bar{b} \subset \mathbb{M}$, we say that $\mathbb{M}$ has cl-elimination of imaginaries **modulo** $\bar{b}$ if, for every $e \in \mathbb{M}^{\text{eq}}$, there exists $\bar{c}$ representing e , such that $\bar{c} \in \text{cl}^{\text{eq}}(e\bar{b})$.

If $\mathbb{K} \preceq \mathbb{M}$ we say that $\mathbb{K}$ has cl-elimination of imaginaries (modulo some $\bar{b} \subset \mathbb{K}$) if $\mathbb{M}$ has it.

Compare the above notion with weak elimination of imaginaries (see [CF04]).

top:cl-elimination

Proposition 8.41. *Let $\bar{b} \subset \mathbb{M}$. Assume that $\text{cl}(\bar{b})$ is dense in $\mathbb{M}$. Then, $\mathbb{M}$ has cl-elimination of imaginaries modulo $\bar{b}$.*

Corollary 8.42. *Let $\mathbb{M}$ be geometric. Assume that $\text{acl}(\emptyset)$ is acl-dense in $\mathbb{M}$ (e.g., $\mathbb{M}$ is an algebraically closed field). Then, $\mathbb{M}$ has weak elimination of imaginaries. If moreover $\mathbb{M}$ expands a field, then $\mathbb{M}$ has elimination of imaginaries.*

Corollary 8.43. *Assume that $\mathbb{M}$ expands an integral domain. Let $\langle \mathbb{B}, \mathbb{A} \rangle \models T^d$. Let $b \in \mathbb{B} \setminus \mathbb{A}$. Then, $\langle \mathbb{B}, \mathbb{A} \rangle$ has Scl-elimination of imaginaries modulo b .*

Proof. $\text{Scl}(b)$ is Scl-dense in $\langle \mathbb{B}, \mathbb{A} \rangle$ for every $b \in \mathbb{B} \setminus \mathbb{A}$. □

In the situation of the above corollary, it is not true that $\langle \mathbb{B}, \mathbb{A} \rangle$ has Scl-elimination of imaginaries (modulo $\emptyset$). For instance, let $X := \mathbb{B} \setminus \mathbb{A}$. Then, $X \cap \text{Scl}^{\text{eq}}(\ulcorner X \urcorner) = \emptyset$.

Before proving the Proposition 8.41, we need some preliminaries. Let $X \subseteq \mathbb{M}^n$ be a subset definable with parameters $\bar{b}$. Let $\mathbb{M}'$ be the expansion of $\mathbb{M}$ with a new predicate denoting X . Notice that $\mathbb{M}$ and $\mathbb{M}'$ have the same definable sets. However, cl is no longer an existential matroid on $\mathbb{M}'$: for instance, if $X = \{b\}$ is a singleton, and $b \notin \text{cl}(\emptyset)$, then $b \in \text{acl}_{\mathbb{M}'}(\emptyset) \setminus \text{cl}(\emptyset)$, and therefore cl is not existential on $\mathbb{M}'$. Notice that $\downarrow^{\text{cl}}$ satisfies all the axioms of a symmetric independence relation on $\mathbb{M}'$, except the Extension axiom.

Let $e := \ulcorner X \urcorner \in \mathbb{M}^{\text{eq}}$ be the canonical parameter for X . For every $Z \subseteq \mathbb{M}$, define $\text{cl}_e(Z) := \text{cl}^{\text{eq}}(eZ) \cap \mathbb{M}$ (notice that, if $e = \emptyset$, then $\text{cl}_e = \text{cl}$).

Lemma 8.44. *cl_e is an existential matroid on $\mathbb{M}'$.*

Proof. We only need to check that cl_e satisfies Existence. Let B and C be subsets of $\mathbb{M}$ such that $a \notin \text{cl}_e(B)$, that is $a \notin \text{cl}^{\text{eq}}(eB)$. Let $a' \equiv_{eB}^{\mathbb{M}} a$, such that $a' \downarrow_e^{\text{cl}} BC$. Then, $a' \equiv_B^{\mathbb{M}'} a$ and $a' \notin \text{cl}^{\text{eq}}(eBC) = \text{cl}_e(BC)$. □

Proof of Proposition 8.41. W.l.o.g., $\bar{b} = \emptyset$. Let $e \in \mathbb{M}^{\text{eq}}$. Let E be a $\emptyset$ -definable equivalence relation on $\mathbb{M}$ and X be an equivalence class of E , such that $e = X$. Let cl_e be defined as above. Since $\text{cl}(\emptyset)$ is dense in $\mathbb{M}$ and $\text{cl} \subseteq \text{cl}_e$, $\mathbb{K} := \text{cl}_e(\emptyset)$ is cl_e -dense in $\mathbb{M}'$. Hence, by Lemma 7.4, $\mathbb{K} \preceq \mathbb{M}'$. Thus, since X is $\emptyset$ -definable in $\mathbb{M}'$, there exists $\bar{c} \in X \cap \mathbb{K}$. □

9 D-minimal topological structures

sec:dmin

In this section we will introduce d-minimal structures. They are topological structures whose definable sets are particularly simple from the topological

point of view; they generalise o-minimal structures. We will show that for d-minimal structure the topology induces a canonical existential matroid, which we denote by Zcl . Moreover, the abstract notion of density introduced in §7 coincides with the usual topological notion. Finally, if T is a complete d-minimal theory expanding the theory of fields, then in T^d the condition that the smaller structure is cl-closed is superfluous. Our definition of d-minimality extends an older definition by C. Miller [Miller05], that applied only to linearly ordered structures.

Let $\mathbb{K}$ be a first-order topological structure in the sense of [Pillay87]. That is, $\mathbb{K}$ is a structure with a topology, such that a basis of the topology is given by $\{\Phi(\mathbb{K}, \bar{a}) : \bar{a} \in \mathbb{K}^m\}$ for a certain formula without parameters $\Phi(x, \bar{y})$; fix such a formula $\Phi(x, \bar{y})$, and denote $B_{\bar{a}} := \Phi(\mathbb{K}, \bar{a})$. Examples of topological structures are valued fields, or ordered structures. On $\mathbb{K}^n$ we put the product topology.⁷ Let $\mathbb{M} \succeq \mathbb{K}$ be a monster model. Given $X \subseteq \mathbb{K}^n$, we will denote by $\overline{X}$ the topological closure of X inside $\mathbb{K}^n$.

Definition 9.1. $\mathbb{K}$ is d-minimal if:

1. it is T_1 (i.e., its points are closed);
2. it has no isolated points;
3. for every $X \subseteq \mathbb{M}$ definable subset (with parameters in $\mathbb{M}$), if X has empty interior, then X is a finite union of discrete sets.
4. for every $X \subset \mathbb{K}^n$ definable and discrete, $\Pi_1^n(X)$ has empty interior;
5. given $X \subseteq \mathbb{K}^2$ and $U \subseteq \Pi_1^2(X)$ definable sets, if U is open and non-empty, and X_a has non-empty interior for every $a \in U$, then X has non-empty interior.

Lemma 9.2. *Assume that $\mathbb{K}$ is d-minimal. Let $Z \subset \mathbb{K}^2$ be definable, such that $\Pi_1^2(Z)$ has empty interior, and Z_x has empty interior for every $x \in \mathbb{K}$. Then, $\theta(Z)$ has empty interior, where θ is the projection onto the second coordinate.*

Proof. By assumption, w.l.o.g. $\Pi_1^2(Z)$ is discrete and, for every $x \in \mathbb{K}$, Z_x is also discrete. Therefore, Z is discrete, and hence $\theta(Z)$ has empty interior. $\square$

Definition 9.3. Given $A \subset \mathbb{M}$ and $b \in \mathbb{M}$, we say that $b \in \text{Zcl}(A)$ if there exists $X \subset \mathbb{M}$ A -definable such that $b \in X$ and X has empty interior (or, equivalently, X is discrete).

⁷Allowing a different topology (e.g. Zariski topology) might be a better choice.

Examples 9.4.

1. p -adic fields and algebraically closed valued fields are d-minimal;
2. densely ordered o-minimal structures are d-minimal.

ex:dmin

Example 9.5. A structure $\mathbb{K}$ is **definably complete** if it expands a linear order $\langle K, < \rangle$, and every $\mathbb{K}$ -definable subset of K has a supremum in $K \sqcup \{\pm\infty\}$. C. Miller defines a d-minimal structure as a definably complete structure $\mathbb{K}$ such that, given $\mathbb{K}'$ an $\aleph_0$ -saturated elementary extension of $\mathbb{K}$, every $\mathbb{K}'$ -definable subset of $\mathbb{K}'$ is the union of an open set and finitely many discrete sets. In particular, o-minimal structures and ultra-products of o-minimal structures are d-minimal in Miller's sense. If $\mathbb{K}$ expands a field and is a d-minimal structures in the sense of Miller, then $\mathbb{K}$ is d-minimal in our sense (the proof will be given elsewhere). Conversely, any definably complete structure which is d-minimal in our sense is also d-minimal in Miller's sense.

Proviso. *For the remainder of this section, we assume that $\mathbb{K}$ is d-minimal, and T is the theory of $\mathbb{K}$.*

Remark 9.6. If A and B are definable subsets of $\mathbb{K}$ with empty interior, then $A \cup B$ has empty interior. Hence, for every $C \subseteq \mathbb{K}$ definable, $\text{bd}(C)$ has empty interior.

lem:open-orbit

Lemma 9.7. *If $c \notin \text{Zcl}(A)$, then $\Xi(c/A)$ has non-empty interior.*

Proof. Let $X \subseteq \mathbb{M}$ be an A -definable set containing c . Since $c \notin \text{Zcl}(A)$, $c \in \overset{\circ}{X}$. Consider the partial type over cA

$$\Gamma(\bar{y}) := \{\Phi(c, \bar{y}) \ \& \ \Phi(\mathbb{M}, \bar{y}) \subseteq X\},$$

where X varies among the A -definable sets containing c . By the above consideration, Γ is consistent; let $\bar{b} \subset \mathbb{M}$ be a realisation of Γ .

Claim 11. $c \in \Phi(\mathbb{M}, \bar{b}) \subseteq \Xi(c/A)$.

In fact, if $c \in X$, where X is A -definable, then, by definition, $c \in \Phi(\mathbb{M}, \bar{b}) \subseteq X$, and therefore any $x' \in \Phi(\mathbb{M}, \bar{b})$ satisfies all the A -formulae satisfied by c . $\square$

Theorem 9.8. *Zcl is an existential matroid.*

Proof. Finite character, extension and monotonicity are obvious.

The fact that Zcl is definable is also obvious.

Idempotence) Let $\bar{b} := \langle b_1, \dots, b_n \rangle$, $a \in \text{Zcl}(\bar{b}\bar{c})$ and $\bar{b} \subset \text{Zcl}(\bar{c})$. We must prove that $a \in \text{Zcl}(\bar{c})$. Let $\phi(x, \bar{y}, \bar{z})$ and $\psi_i(y, \bar{z})$ be formulae, $i = 1, \dots, n$,

such that $\phi(\mathbb{K}, \bar{y}, \bar{z})$ and $\psi_i(\mathbb{K}, \bar{z})$ are discrete for every $\bar{y}$ and $\bar{z}$, and $\mathbb{K} \models \phi(a, \bar{b}, \bar{c})$ and $\mathbb{K} \models \psi_i(b_i, \bar{c})$, $i = 1, \dots, n$. Let

$$Z := \{ \langle x, \bar{y} \rangle : \mathbb{K} \models \phi(x, \bar{y}, \bar{c}) \ \& \ \bigvee_{i=1}^n \psi_i(y_i, z_i) \},$$

and $W := \Pi_1^{n+1} Z$. By hypothesis, Z is a discrete subset of $\mathbb{K}^{n+1}$, and therefore, by Assumption (4), W has empty interior. Moreover, W is $\bar{c}$ -definable and $a \in W$, and hence $a \in \text{Zcl}(\bar{c})$. Notice that weak d-minimality suffices.

EP) Let $a \in \text{Zcl}(b\bar{c}) \setminus \text{Zcl}(\bar{c})$. We must prove that $b \in \text{Zcl}(a\bar{c})$. Assume not. Let $Z \subset \mathbb{K}^2$ be $\bar{c}$ -definable, such that $\langle a, b \rangle \in Z$ and Z^y is discrete for every $y \in \mathbb{K}$. Since $b \in Z_a$ and $b \notin \text{Zcl}(a\bar{c})$, $b \in \text{int}(Z_a)$; hence, w.l.o.g. Z_x is open for every $x \in \mathbb{K}$. Let $U := \Pi_1^2(Z)$. Since $a \in U$ and $a \notin \text{Zcl}(\bar{c})$, $a \in \overset{\circ}{U}$. Hence, by Condition (5), Z has non-empty interior; but this contradicts the fact Z^y is discrete for every $y \in \mathbb{K}$.

Existence follows from Lemma 9.7.

Non-triviality) Consider the following partial type over the empty set:

$$\Lambda(x) := \{x \notin Y\},$$

where Y varies among the discrete $\emptyset$ -definable sets. Since $\mathbb{K}$ has no isolated points, Λ is finitely satisfiable; if $a \in \mathbb{M}$ is a realisation of Λ , then $a \notin \text{Zcl}(\emptyset)$. $\square$

We will denote by Zrk and $\dim$ the rank and dimension on $\mathbb{M}$ according to Zcl . We will say Z -closed instead of Zcl -closed.

Remark 9.9. Let $X \subseteq \mathbb{K}^n$ be definable. If X has non-empty interior, then $\dim(X) = n$. If $\Pi_d^n(X)$ has non-empty interior, then $\dim(X) \geq d$.

Conjecture 9.10. Let $X \subseteq \mathbb{K}^n$ be definable. Then, $\dim(X) = d$ iff, after a permutation of variables, $\Pi_d^n(X)$ has non-empty interior.

Conjecture 9.11. For every $X \subseteq \mathbb{K}^n$ definable, $\dim(\overline{X}) = \dim X$.

However, it is not true that $\dim(\partial X) < \dim(X)$ if X is definable and non-empty.

Lemma 9.12. X is dense in $\mathbb{K}$ according to definition 7.1 (w.r.t. Zcl) iff X is topologically dense in $\mathbb{K}$.

Proof. Assume that X is dense in $\mathbb{K}$ according to Zcl . Let $A \subseteq \mathbb{K}$ be an open definable set; thus, $\dim(A) = 1$, and therefore $A \cap X \neq \emptyset$. Conversely, if X is topologically dense in $\mathbb{K}$, let $A \subseteq \mathbb{K}$ be definable and of dimension 1. Thus, A has non-empty interior, and therefore $A \cap X \neq \emptyset$. $\square$

Proviso. For the remainder of this section, will assume that $\mathbb{K}$ is d -minimal and expands an integral domain, that $+$ and $-$ are continuous (and therefore $\langle \mathbb{K}, + \rangle$ is a topological group), and that T is the theory of $\mathbb{K}$.

Notice that an algebraically closed field with the Zariski topology is not a topological group, because $+$ is not continuous. Notice also that, since we required that points are closed, $\mathbb{K}$ is a regular topological space.

rem:dense-sum

Remark 9.13. Let $X \subseteq \mathbb{K}$ be dense (but not necessarily definable). Then, for every $b \in \mathbb{K}$ and every V neighbourhood of 0, there exists $a \in X$ such that $b \in a + V$.

Proof. Since $-$ is continuous, there exists V' neighbourhood of 0 such that $V' = -V'$ and $V' \subseteq V$. Since X is dense, there exists $a \in X$ such that $a \in b + V'$. Hence, $b \in a - V' \subseteq a + V$. $\square$

or:dense-dminimal

Corollary 9.14. T^d is complete. Besides, T^d is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle$ such that $\mathbb{F} \prec \mathbb{K} \models T$ and $\mathbb{F}$ is a (topologically) dense proper subset of $\mathbb{K}$.

Proof. We must show that if $\mathbb{F} \preceq \mathbb{K}$ is topologically dense in $\mathbb{K}$, then $\mathbb{F}$ is cl-closed in $\mathbb{K}$. W.l.o.g., the pair $\langle \mathbb{K}, \mathbb{F} \rangle$ is ω -saturated. Let $b \in \text{cl}^{\mathbb{K}}(\mathbb{F})$; we must prove that $b \in \mathbb{F}$. Let $Z \subset \mathbb{K}$ be $\mathbb{F}$ -definable and discrete, such that $b \in Z$. Let U' be a definable neighbourhood of b , such that $Z \cap U' = \{b\}$. Define $U := U' - b$; since $\mathbb{K}$ is a topological group, U is a neighbourhood of 0 in $\mathbb{K}$, and there exists V open neighbourhood of 0 definable in $\mathbb{K}$, such that $V = -V$ and $V + V \subseteq U$.

Claim 12. There exists an $\mathbb{F}$ -definable open neighbourhood W of 0 such that $W \subseteq V$.

Suppose the claim is not true. Since $\mathbb{F}$ is a regular space, there exists X definable open neighbourhood of 0 such that $\overline{X} \subseteq V$. Let $X_{\mathbb{F}} := X \cap \mathbb{F}$. $X_{\mathbb{F}}$ is a neighbourhood of 0 in $\mathbb{F}$; thus, since the topology has a definable basis, there exists $W_{\mathbb{F}} \subseteq X_{\mathbb{F}}$ such that $W_{\mathbb{F}}$ is $\mathbb{F}$ -definable and $W_{\mathbb{F}}$ is an open neighbourhood of 0. Let W be the interpretation of $W_{\mathbb{F}}$ in $\mathbb{K}$. Since W is open and $\mathbb{F}$ is dense in $\mathbb{K}$, $W_{\mathbb{F}}$ is dense in W ; therefore, $W \subseteq \overline{W_{\mathbb{F}}} \subseteq \overline{X} \subseteq V$.

By Remark 9.13, there exists $a \in \mathbb{F}$ such that $b \in W'$, where $W' := a + W$.

Claim 13. $W' \subseteq U'$.

The claim is equivalent to $a + W \subseteq b + U$, that is $W + (a - b) \subseteq U$. By assumption, $b \in a + W$, and therefore $a - b \in -W$. Hence, $W + (a - b) \subseteq W - W \subseteq V - V \subseteq U$.

However, W' is $\mathbb{F}$ -definable, and $b \in W' \cap Z \subseteq V \cap Z = \{b\}$. Hence, b is $\mathbb{F}$ -definable, and therefore $b \in \mathbb{F}$. $\square$

9. D -minimal topological structures

Denote $\downarrow := \downarrow^{\text{Zcl}}$. Given $\bar{a} := \langle \bar{a}_1, \dots, \bar{a}_n \rangle \in \mathbb{M}^{n \times m}$ and $\bar{b} \in \mathbb{M}^n$, denote

$$B_{\bar{a}} + \bar{b} := (B_{\bar{a}_1} + b_1) \times \dots \times (B_{\bar{a}_n} + b_n) \subseteq \mathbb{M}^n.$$

neighbourhood-0

Lemma 9.15. *Let $d \in \mathbb{M}$, V be a definable neighbourhood of d , and $C \subset \mathbb{M}$. Then, there exists $\bar{a} \in \mathbb{M}^m$ such that $\bar{a} \downarrow_d C$ and $d \in B_{\bar{a}} \subseteq V$.*

Proof. Let $X := \{\bar{a} \in \mathbb{M}^m : d \in B_{\bar{a}}\}$. Let $\leq$ be the quasi-ordering on X given by reverse inclusion: $\bar{a} \leq \bar{a}'$ if $B_{\bar{a}} \supseteq B_{\bar{a}'}$. Fix $\bar{b} \in X$ such that $B_{\bar{b}} \subseteq V$. Since $(X, \leq)$ is a directed set, by Lemma 3.69, there exists $\bar{a} \in X$ such that $\bar{a} \downarrow_d^{\text{cl}} C$ and $B_{\bar{a}} \subseteq B_{\bar{b}} \subseteq V$. $\square$

neighbourhood-1

Lemma 9.16. *Let $\bar{d} \in \mathbb{M}^n$, V be a definable neighbourhood of $\bar{d}$, and $C \subset \mathbb{M}$. Then, there exist $\bar{a} \in \mathbb{M}^{m \times n}$ and $\bar{b} \in \mathbb{M}^n$ such that $\bar{d} \in B_{\bar{a}} + \bar{b} \subseteq V$ and $\bar{a}\bar{b} \downarrow C\bar{d}$.*

Proof. Proceeding by induction on n , it suffices to treat the case $n = 1$. Let $V_0 := V - d$; it is a definable neighbourhood of 0. Since $\mathbb{M}$ is a topological group, there exists V_1 definable and open, such that $0 \in V_1$, $V_1 = -V_1$, and $V_1 + V_1 \subseteq V_0$. By Lemma 9.15, there exists $\bar{a} \in \mathbb{M}^m$ such that $\bar{a} \downarrow C d$ and $0 \in B_{\bar{a}} \subseteq V_1$. Let $W := d - B_{\bar{a}}$. Since $\dim(W) = 1$, there exists $b \in W$ such that $b \notin \text{Zcl}(C\bar{a}d)$.

Claim 14. $d \in B_{\bar{a}} + b$.

In fact, $b \in -B_{\bar{a}} + d$, and therefore $d - b \in B_{\bar{a}}$

Claim 15. $\bar{a}\bar{b} \downarrow C d$.

By construction, $b \downarrow C\bar{a}d$, and therefore $b \downarrow_{\bar{a}} C d$, and hence $\bar{a}b \downarrow_{\bar{a}} C d$. Together with $\bar{a} \downarrow C d$, this implies the claim. $\square$

Corollary 9.17. *Let $X \subseteq \mathbb{M}^n$ be a definable set, and $k \in \mathbb{N}$. Assume that, for every $\bar{x} \in X$, there exists $V_{\bar{x}}$ definable neighbourhood of $\bar{x}$, such that $\dim(V_{\bar{x}} \cap X) \leq k$. Then, $\dim(X) \leq k$.*

Proof. Let C be the set of parameters of X . By Lemma 9.16, for every $x \in X$ there exist $\bar{a} \in \mathbb{K}^{n \times m}$ and $\bar{b} \in \mathbb{K}^n$ such that $\bar{a}\bar{b} \downarrow C x$ and $\bar{x} \in B_{\bar{a}} + \bar{b} \subseteq V_{\bar{x}}$; notice that $\dim(X \cap (B_{\bar{a}} + \bar{b})) \leq k$. Hence, by Lemma 3.70, $\dim(X) \leq k$. $\square$

We do not know if the above Corollary remains true if we drop the assumption that $\mathbb{M}$ expands a group.

Corollary 9.18. *Let $C \subset \mathbb{M}$ and $p \in S_n(C)$. Then, p is stationary iff p is realised in $\text{dcl}(C)$.*

Proof. Assume for contradiction, that p is stationary, but $\dim(p) > 0$. Let $\bar{a}_0$ and $\bar{a}_1$ be realisations of p independent over C . Since $\dim(p) > 0$, $\bar{a}_0 \neq \bar{a}_1$. Since $\mathbb{M}$ is Hausdorff, Lemma 9.16 implies that there exists V open neighbourhood of $\bar{a}_0$, definable with parameters $\bar{b}$, such that $\bar{a}_1 \notin V$ and $\bar{b} \perp C\bar{a}_0\bar{a}_1$. Hence, by Lemma 3.10, $\bar{a}_0 \perp_{C\bar{b}} \bar{a}_1$. Since p is stationary, Lemma 3.65 implies $\bar{a}_0 \equiv_{\bar{b}} \bar{a}_1$, contradicting the fact that $\bar{a}_0 \in V$, while $\bar{a}_1 \notin V$. $\square$

10 Cl-minimal structures

Let $\mathbb{M}$ be a monster model, T be the theory of $\mathbb{M}$, and cl be an existential matroid on $\mathbb{M}$. We denote by $\dim$ and rk the dimension and rank induced by cl .

Definition 10.1. $p \in S_n(A)$ is a **generic type** if $\dim(p) = n$. $\mathbb{M}$ is **cl-minimal** if, for every $A \subset \mathbb{M}$, there exists only one generic 1-type over A .

Lemma 10.2. *For every $0 < n \in \mathbb{N}$ and $A \subset \mathbb{M}$, there exists at least one generic type in $S_n(A)$. If $\mathbb{M}$ is cl-minimal, then for every n and A there exists exactly one generic type in $S_n(A)$.*

Lemma 10.3. *If $\mathbb{M}$ is cl-minimal, then $\dim$ is definable.*

Proof. Notice that, given $\bar{x} := \langle x_1, \dots, x_n \rangle$ and a formula $\phi(\bar{x}, \bar{y})$, the set $U_\phi^n := \{\bar{a} : \dim(\phi(\mathbb{K}, \bar{a})) = n\}$ is always type-definable (Lemma 3.42). By the above Lemma, $\mathbb{K}^n \setminus U_\phi^n = U_{\neg\phi}^n$, and therefore U_ϕ^n is both type-definable and ord-definable, and hence definable. $\square$

Remark 10.4. $\mathbb{M}$ is cl-minimal iff, for every $n > 0$ and every X definable subset of $\mathbb{K}^n$, exactly one among X and $\mathbb{K}^n \setminus X$, has dimension n .

m:clminimal-fixed

Remark 10.5. If $\mathbb{K} \preceq \mathbb{M}$ and $\dim$ is definable, then $\mathbb{K}$ is cl-minimal iff, for every X definable subset of $\mathbb{K}$, either $\dim(X) = 0$, or $\dim(\mathbb{K} \setminus X) = 0$; that is, we can check cl-minimality directly on $\mathbb{K}$.

Examples 10.6. 1. $\mathbb{M}$ is strongly minimal iff acl is a matroid and $\mathbb{M}$ is acl -minimal.

2. Consider Example 3.54. In that context, a type is generic in our sense iff it is generic in the sense of stable groups. Hence, $\mathbb{G}$ is cl-minimal iff it has only one generic type iff it is connected (in the sense of stable groups).

em:minimal-pair

Lemma 10.7. *Assume that T is cl-minimal. Then, T^d is also cl-minimal. Moreover, T^d coincides with T^2 .*

Proof. Let $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$ be a monster model of T^d . Let $C \subset \mathbb{B}^*$ with $|C| < \kappa$. Define $\mathbb{A} := \text{cl}(\mathbb{A}^*C)$, and $q_C(x)$ the partial $\mathcal{L}^2$ -type over C given by

$$q_C(x) := x \notin \mathbb{A}.$$

It is clear that every generic $1-T^d$ -type over C expands q_C . Hence, it suffices to prove that q_C is complete. Let b and $b' \in \mathbb{B}^*$ satisfy q_C . By Corollary 8.18, $\langle \mathbb{B}^*, \mathbb{A}^* \rangle \preceq \langle \mathbb{B}^*, \mathbb{A} \rangle$. By assumption, b and b' are not in $\mathbb{A}$; hence, since T is cl-minimal, they satisfy the same generic $1-T$ -type $p_{\mathbb{A}}$; thus, by Corollary 8.20, $b \equiv_{\mathbb{A}}^2 b'$. $\square$

11 Connected groups

Let $\mathbb{M}$ be a monster model, and cl be an existential matroid on it. Denote $\dim := \dim^{\text{cl}}$, $\text{rk} := \text{rk}^{\text{cl}}$, and $\downarrow := \downarrow^{\text{cl}}$.

Definition 11.1. Let $X \subseteq \mathbb{M}^n$ be definable (with parameters). Assume that $m := \dim(X) > 0$. We say that X is *connected* if, for every Y definable subset of X , either $\dim(Y) < n$, or $\dim(X \setminus Y) < n$.

For instance, if $\mathbb{M}$ is cl-minimal and $X = \mathbb{M}$, then X is connected.

Remark 11.2. If X is connected, then, for every $l \geq 0$, X^l is also connected.

Remark 11.3. Let $X \subseteq \mathbb{M}^n$ be definable, of dimension $m > 0$. Then, X is connected iff for every $A \subset \mathbb{M}$ containing the parameters of definition of X , there exists exactly one n -type over A in X which is generic (i.e., of dimension m).

lem:semigroup

Lemma 11.4. *Let $G \subseteq \mathbb{M}^n$ be definable and connected. Assume that G is a semigroup with left cancellation. Assume moreover that G has either right cancellation or right identity. Then G is a group.*

Cf. [Poizat87, 1.1].

Proof. Assume not. Let $m := \dim(G)$. W.l.o.g., G is definable without parameters. For every $a \in G$, let $a \cdot G := \{a \cdot x : x \in G\}$. Since G has left cancellation, we have $\dim(a \cdot \mathbb{M}) = m$.

Let $F := \{a \in G : a \cdot G = G\}$. Our aim is to prove that $F = G$.

It is easy to see that G is multiplicatively closed.

First, assume that G has a right identity element 1. The following claim is true for any abstract semigroup with left cancellation and right identity 1.

Claim 16. 1 is also the left identity.

In fact, for every $a, b \in G$, $a \cdot b = (a \cdot 1) \cdot b = a \cdot (1 \cdot b)$. Since we have left cancellation, we conclude that $b = 1 \cdot b$ for every b , and we are done.

Obviously, $1 \in F$. For every $a \in F$, denote by a^{-1} the (unique) element of G such that $a \cdot a^{-1} = 1$.

Claim 17. $a^{-1} \cdot a = 1$.

In fact, $a \cdot (a^{-1} \cdot a) = 1 \cdot a = a \cdot 1$, and the claim follows from left cancellation.

Claim 18. F is a group.

We have already seen that F is multiplicatively closed and $1 \in F$. Let $a \in F$. Then, for every $g \in G$, $a^{-1} \cdot (a \cdot g) = g$, and therefore $a^{-1} \in F$.

Claim 19. $\dim(F) < m$.

Assume, for contradiction, that $\dim(F) = m$. Let $a \in G \setminus F$. Then, $F \cap (a \cdot F) \neq \emptyset$; let $u, v \in F$ such that $u = a \cdot v$.

Since $u \in F$ and F is a group, there exists $w \in F$ such that $v \cdot w = 1$; hence, $u \cdot w = a \cdot 1 = a$, and therefore $a \in F$, absurd.

Choose $a, b \in G$ independent (over the empty set). Since $\dim(a \cdot G) = \dim(b \cdot G) = m$, we have $a \in b \cdot G$ and $b \in a \cdot G$. Let $u, v \in G$ such that $b = a \cdot u$ and $a = b \cdot v$. Hence, $a = a \cdot u \cdot v$.

Since $a \cdot 1 = a \cdot u \cdot v$, we have $1 = u \cdot v$. Hence, both u and v are in F . However, since $\dim(F) < m$ and $b = a \cdot u$, we have $\text{rk}(b/a) \leq \text{rk}(u) < m$, absurd.

If instead G has right cancellation, it suffices, by symmetry, to show that G has a left identity. Reasoning as above, we can show that there exists a and b in G such that $a \cdot b = a$. We claim that b is a left identity. In fact, for every $c \in G$, we have $a \cdot b \cdot c = a \cdot c$, and therefore $b \cdot c = c$, and we are done. $\square$

Proviso. For the remainder of this section, $\langle G, \cdot \rangle$ is a definable connected group, of dimension $m > 0$, with identity 1.

If G is Abelian, we will also use $+$ instead of $\cdot$ and 0 instead of 1.

Hence, if G expands a ring without zero divisors, then, by applying the above lemma to the multiplicative semigroup of G , we obtain that G is a division ring.

Remark 11.5. Let $X \subseteq G$ be definable, such that $X \cdot X \subseteq X$. Then, $\dim(X) = m$ iff $X = G$.

Proof. Assume that $\dim(X) = m$. Let $a \in G$. Then, $X \cap (a \cdot X^{-1}) \neq \emptyset$; choose $u, v \in X$ such that $u = a \cdot v^{-1}$. Hence, $a = u \cdot v \in X \cdot X = X$. $\square$

lem:homo **Lemma 11.6.** *Let $f : G \rightarrow G$ be a definable homomorphism. If $\dim(\ker f) = 0$, then f is surjective.*

Cf. [Poizat87, 1.7].

Proof. Let $H := f(G)$ and $K := \ker(f)$; notice that $H < G$ and $K < G$. Moreover, by additivity of dimension, $m = \dim(H) + \dim(K)$. Hence, if $\dim(K) = 0$, then $\dim(H) = m$, therefore $H = G$ and f is surjective. $\square$

Example 11.7. $\langle \mathbb{Z}, + \rangle$ cannot be cl-minimal, because the homomorphism $x \mapsto 2x$ has trivial kernel but is not surjective.

lem:quotient **Lemma 11.8.** *Let $H < G$ be definable, with $\dim(H) = k < m$. Then, G/H is connected, and $\dim(G/H) = m - k$.*

Proof. That $\dim(G/H) = m - k$ is obvious. Let $X \subseteq G/H$ be definable of dimension $m - k$. We must prove that $\dim(G/H \setminus X) < m$. Let $\pi : G \rightarrow G/H$ be the canonical projection, and $Y := \pi^{-1}(X)$. Then, $\dim(Y) = m$, and therefore $\dim(G \setminus Y) < m$. Thus, $\dim(G/H \setminus X) = \dim(\pi(Y)) < m - k$. $\square$

Conjecture 11.9. *If $m = 1$, then G is Abelian. Cf. Reineke's Theorem [Poizat87, 3.10].*

Idea for proof. Assume for contradiction that G is not Abelian. Let $Z := Z(G)$ and $\overline{G} := G/Z$. Since $Z < G$ and $Z \neq G$, we have $\dim(Z) = 0$, and therefore $\overline{G}$ is also connected and of dimension 1.

For every $a \in G$, let U_a be the set of conjugates of a .

cl:conjugate *Claim 20.* If $a \notin Z$, then $\dim(U_a) = 1$.

By general group theory, $U_a \equiv G/C(a)$, where $C(a)$ is the centraliser of a . Since $a \notin Z$, $C(a)$ is not all of G ; moreover, $C(a) < G$, therefore $\dim(C(a)) = 0$, and thus $\dim(U_a) = 1$, and similarly for U_b .

cl:conjugate-2 *Claim 21.* For every $a, b \in G \setminus Z$, a is a conjugate of b .

In fact, by connectedness and the above claim, $U_a \cap U_b \neq \emptyset$, and thus $U_a = U_b$.

Claim 22. For every $x, y \in \overline{G} \setminus \{1\}$, x is a conjugate of y .

In fact, $x = \bar{a}$ and $y = \bar{b}$ for some $a, b \in G \setminus Z$. By Claim 21, a and b are conjugate in G , and thus x and y are conjugate in $\overline{G}$.

Thus, $\overline{G}$ is a definable (in the imaginary sorts) connected group of dimension 1, such that any two elements different from the identity are in the same conjugacy class (and therefore $\overline{G}$ is torsion-free and has trivial centre).

One should now prove that such a group $\overline{G}$ cannot exist. $\square$

Notice that the above conjecture is false if $m > 1$.

Lemma 11.10. *Assume that $m = 1$ and G is Abelian. Let p be a prime number. Then, either $pG = 0$, or G is divisible by p .*

Proof. Let $H := pG$ and $K := \{x \in G : px = 0\}$. If $\dim(H) = 1$, then $G = H$ and therefore G is p -divisible. If $\dim(H) = 0$, then $\dim(K) = 1$, thus $G = K$ and $pG = 0$. $\square$

Notice that the above lemma needs the hypothesis that $m = 1$. For instance, let $\mathbb{M}$ be the algebraic closures of $\mathbb{F}_p$, and let $G := \mathbb{M} \times \mathbb{M}^*$ (where $\mathbb{M}^*$ is the multiplicative group of $\mathbb{M}$).

Theorem 11.11. *Assume that G expands an integral domain (and $\langle G, + \rangle$ is connected). Then, G is an algebraically closed field.*

Cf. Macintyre's Theorem [Poizat87, 3.1, 6.11].

Proof. Let $\langle G^*, \cdot \rangle$ be the multiplicative semigroup of G . By Lemma 11.4, $\langle G^*, \cdot \rangle$ is a group, and therefore G is a field. For every $n \in \mathbb{N}$, consider the map $f_n : G^* \rightarrow G^*$ $x \mapsto x^n$. Since f_n has finite kernel, Lemma 11.6 implies that f_n is surjective, and therefore every element of G has an n th root in G . In particular, G is perfect.

Let $p := \text{char}(G)$. If $p > 0$, consider the map $h : G \rightarrow G$, $x \mapsto x^p + x$. Notice that h is an additive homomorphism with finite kernel; hence, h is surjective.

Since G^l is also connected for every $0 < l \in \mathbb{N}$, the above is true not only for G , but also for every finite-degree extension G_1 of G .

The rest of the proof is the same as in [Poizat87, 3.1]: G contains all roots of 1 (because G^* is divisible), and, if G were not algebraically closed, there would exist a finite extension G_1 and normal finite extension L of G_1 , such that the Galois group of L/G_1 is cyclic and of prime order q . If $q \neq p$, then L/G_1 is a Kummer extension, absurd. If $q = p$, then L/G_1 is an Artin-Schreier extension, also absurd. $\square$

In the above theorem it is essential that G is connected. For instance, if $\mathbb{M}$ is a formally p -adic field, then $\mathbb{M}$ itself is a non-algebraically closed field (of dimension 1). Notice also that the first step in the proofs of [Poizat87, 3.1, 6.11] is showing that G is connected.

Question 11.12. Can we weaken the hypothesis in the above theorem from “ G expands an integral domain” to “ G expands a ring without zero divisors”?

12 Ultraproducts

Let I be an infinite set, and μ be an ultrafilter on I . For every $i \in I$, let $\langle \mathbb{K}_i, \text{cl}_i \rangle$ be a pair given by first-order $\mathcal{L}$ -structure $\mathbb{K}_i$ and an existential matroid cl_i on $\mathbb{K}_i$. Let $\mathcal{K}$ be the family $(\langle \mathbb{K}_i, \text{cl}_i \rangle)_{i \in I}$, and $\mathbb{K} := \Pi_i \mathbb{K}_i / \mu$ be the corresponding ultraproduct.

We will give some sufficient condition on the family $\mathcal{K}$, such that there is an existential matroid on $\mathbb{K}$ induced by the family of cl_i . Denote by d_i the dimension induced by cl_i .

Definition 12.1. We say that the dimension is uniformly definable (for the family $\mathcal{K}$) if, for every formula $\phi(\bar{x}, \bar{y})$ without parameters, $\bar{x} = \langle x_1, \dots, x_n \rangle$, $\bar{y} = \langle y_1, \dots, y_m \rangle$, and for every $l \leq n$, there is a formula $\psi(\bar{y})$, also without parameters, such that, for every $i \in I$,

$$\{\bar{y} \in \mathbb{K}_i^m : d_i(\phi(\mathbb{K}_i, \bar{y})) = l\} = \psi(\mathbb{K}_i).$$

We denote by d_ϕ^l the formula ψ .

Remark 12.2. The dimension is uniformly definable if, for every formula $\phi(x, \bar{y})$ without parameters, $\bar{y} = \langle y_1, \dots, y_m \rangle$, there is a formula $\psi(\bar{y})$, also without parameters, such that, for every $i \in I$,

$$\{\bar{y} \in \mathbb{K}_i^m : d_i(\phi(\mathbb{K}_i, \bar{y})) = 1\} = \psi(\mathbb{K}_i).$$

For instance, if every $\mathbb{K}_i$ expands a ring without zero divisors, then the dimension is uniformly definable: given $\psi(x, \bar{y})$, define $\psi(\bar{y})$ by

$$\forall z \exists x_1, \dots, x_4 (z = F^4(x_1, \dots, x_4) \ \& \ \bigwedge_{i=1}^4 \phi(x_i, \bar{y})).$$

For the remainder of this section, we assume that the dimension is uniformly definable for $\mathcal{K}$.

Definition 12.3. Let d be the function from definable sets in $\mathbb{K}$ to $\{-\infty\} \cup \mathbb{N}$ defined in the following way:

given a $\mathbb{K}$ -definable set $X = \Pi_{i \in I} X_i / \mu$ and $l \in \mathbb{N}$, $d(X) = l$ if, for μ -almost every $i \in I$, $d(X_i) = l$.

Theorem 12.4. d is a dimension function on $\mathbb{K}$. Let cl be the existential matroid induced by d . Then, $a \in \text{cl}(\bar{b})$ implies that, for μ -almost every $i \in I$, $a_i \in \text{cl}_i(\bar{b}_i)$, but the converse is not true.

Remark 12.5. Let $X \subseteq \mathbb{K}^n$ be definable with parameters $\bar{c}$; let $\phi(\bar{x}, \bar{c})$ be the formula defining X . Given $l \in N$, $d(X) = l$ iff, for μ -almost every $i \in I$, $\mathbb{K}_i \models d_\phi^l(\bar{c}_i)$.

Lemma 12.6. *If each $\mathbb{K}_i$ is cl-minimal, then $\mathbb{K}$ is also cl-minimal.*

Proof. By Remark 10.5. □

Examples 12.7. The ultraproduct $\mathbb{K}$ of strongly minimal structures is not strongly minimal in general (it will not even be a pregeometric structure), but if each structure expands a ring without zero divisors, then $\mathbb{K}$ will have a (unique) existential matroid, and will be cl-minimal.

It is easy to find a family $\mathcal{K} = (\mathbb{K}_i)_{i \in \mathbb{N}}$ of strongly minimal structures expanding a field, such that any non-principal ultraproduct of $\mathbb{K}$ $\mathcal{K}$ is not pregeometric, does satisfy the Independence Property, and has an infinite definable subset with a definable linear ordering. Moreover, one can also impose that the trivial chain condition for uniformly definable subgroups of $\langle \mathbb{K}, + \rangle$ fails in $\mathbb{K}$ [Poizat87, 1.3]. However, $\mathbb{K}$ will satisfy the following conditions:

1. Every definable associative monoid with left cancellation is a group [Poizat87, 1.1];
2. Given G a definable group acting in a definable way on a definable set E , if E is a definable subset of A and $g \in G$ such that $g \cdot A \subseteq A$, then $g \cdot A = A$ [Poizat87, 1.2].

We do not know if conditions (1) and (2) in the above example are true for an arbitrary cl-minimal structure expanding a field.

Remark 12.8. Assume that each $\mathbb{K}_i$ is a first-order topological structure, and that the definable basis of the topology of each $\mathbb{K}_i$ is given by the same function $\Phi(x, \bar{y})$. Then, $\mathbb{K}$ is also a first-order topological structure, and $\Phi(x, \bar{y})$ defines a basis for the topology of $\mathbb{K}$. If each $\mathbb{K}_i$ is d-minimal, then $\mathbb{K}$ has an existential matroid, but it needs not be d-minimal.

Assume that each $\mathbb{K}$ is d-minimal and satisfies the additional condition

- (*) Every definable subset of $\mathbb{K}_i$ of dimension 0 is discrete.

Then, $\mathbb{K}$ is also d-minimal and satisfies condition (*).

Example 12.9. The ultraproduct of o-minimal structures is not necessarily o-minimal, but it is d-minimal, and satisfies condition (*).

13 Dense tuples of structures

In this section we assume that T expands the theory of integral domains. We will extend the results of §8 to dense tuples of models of T .

Definition 13.1. Fix $n \geq 1$. Let $\mathcal{L}^n$ be the expansion of $\mathcal{L}$ by $(n-1)$ new unary predicates $P_1, \dots, P_{n-1}$. Let T^n be the $\mathcal{L}^n$ -expansion of T , whose models are sequences $\mathbb{K}_1 \prec \dots \prec \mathbb{K}_{n-1} \prec \mathbb{K}_n \models T$, where each $\mathbb{K}_i$ is a proper cl-closed elementary substructure of $\mathbb{K}_{i+1}$. Let T^{nd} be the expansion of T^{n+1} saying that $\mathbb{K}_1$ is dense in $\mathbb{K}_n$. We also define $T^{0d} := T$.

For instance, $T^1 = T$, T^2 is the theory we already defined in §8, and $T^{1d} = T^d$.

Lemma 13.2. *If T is cl-minimal, then T^n is complete for every $n \geq 1$ (and therefore coincides with $T^{(n-1)d}$). Moreover, T^n has a (unique) existential matroid cl^n : given $\langle \mathbb{K}_n, \dots, \mathbb{K}_1 \rangle \models T^n$, we have $b \in \text{cl}^n(A)$ iff $b \in \text{cl}(A\mathbb{K}_{n-1})$. Finally, T^n is cl^n -minimal.*

Proof. By induction on n : iterate n times Lemma 10.7. $\square$

Corollary 13.3. *Assume that T is strongly minimal. Then, T^n is complete, and coincides with the theory of tuples $\mathbb{K}_1 \prec \dots \prec \mathbb{K}_n \models T$.*

Proof. One can use either the above Lemma, or reason as in [Keisler64], using Lemma 8.10. $\square$

Remark 13.4. Let $\langle \mathbb{B}, \mathbb{A} \rangle$ be a κ -saturated model of T^d . Let $U \subseteq \mathbb{B}$ be $\mathbb{B}$ -definable and of dimension 1. Then $\text{rk}(U \cap \mathbb{A}) \geq \kappa$.

Theorem 13.5. *T^{nd} is complete. There is a (unique) existential matroid on T^{nd} .*

Proof. By induction on n , we will prove T^{nd} coincides with $(\dots (T^d)^d \dots)^d$ iterated n times. This implies both that T^{nd} is complete, and that it has an existential matroid.

It suffices to treat the case $n = 2$. Notice that $\langle \mathbb{K}_2, \mathbb{K}_1 \rangle \prec \langle \mathbb{K}_3, \mathbb{K}_1 \rangle \models T^d$. It suffices to show that $\mathbb{K}_2$ is Scl-dense in $\langle \mathbb{K}_3, \mathbb{K}_1 \rangle$. W.l.o.g., we can assume that $\langle \mathbb{K}_3, \mathbb{K}_2, \mathbb{K}_1 \rangle$ is κ -saturated.

Let $X \subseteq \mathbb{K}_3$ be definable in $\langle \mathbb{K}_3, \mathbb{K}_1 \rangle$ (with parameters from $\mathbb{K}_3$), such that $\text{Sdim}(X) = 1$. We need to show that X intersects $\mathbb{K}_2$. By Corollary 8.35, there exists U and S subsets of $\mathbb{K}_3$, such that X is definable in $\mathbb{K}_3$, S is definable in $\langle \mathbb{K}_3, \mathbb{K}_1 \rangle$ and small, and $X = U \Delta S$. Therefore, $\dim(U) = 1$. If, by contradiction, $X \cap \mathbb{K}_2 = \emptyset$, then $\mathbb{K}_2 \cap U \subseteq S$; therefore, $\text{SrK}(\mathbb{K}_2 \cap U) < \omega$ (where SrK is the rank induced by Scl), contradicting Remark 13.4. $\square$

Corollary 13.6. *Assume that T is d -minimal. Then, T^{nd} coincides with the theory of $(n+1)$ -tuples $\mathbb{K}_1 \prec \cdots \prec \mathbb{K}_n \prec \mathbb{K}_{n+1} \models T$, such that $\mathbb{K}_1$ is (topologically) dense in $\mathbb{K}_{n+1}$.*

Proof. Notice that if $\langle \mathbb{K}_n, \dots, \mathbb{K}_1 \rangle$ satisfy the assumption, then, by Corollary 9.14, each $\mathbb{K}_i$ is cl-closed in $\mathbb{K}_n$. $\square$

13.1 Dense tuples of topological structures

Assume that T expands the theory of integral domains. Assume that $\mathbb{M}$ has both an existential matroid cl and a definable topology (in the sense of [Pillay87]). Let $\phi(x, \bar{y})$ be a formula such that the family of sets

$$B_{\bar{b}} := \phi(\mathbb{M}, \bar{b}),$$

as $\bar{b}$ varies in $\mathbb{M}^k$, is a basis of the topology of $\mathbb{M}$. If $\bar{b} = \langle \bar{b}_1, \dots, \bar{b}_m \rangle$, we denote by $B_{\bar{b}}^n := B_{\bar{b}_1} \times \cdots \times B_{\bar{b}_m} \subseteq \mathbb{M}^m$.

Assume the following conditions:

- Hypothesis.** I. *Every definable non-empty open subset of $\mathbb{M}$ has dimension 1.*
- II. *For every $m \in \mathbb{N}$, every U open subset of $\mathbb{M}^m$, and every $\bar{a} \in U$, the set $\{\bar{b} : \bar{a} \in B_{\bar{b}}\}$ has non-empty interior.*

Remark 13.7. Assumption I implies that a definable subset of $\mathbb{M}^m$ with non-empty interior has dimension m (but the converse is not true: there can be definable subsets of dimension m but with empty interior). Moreover, it implies that a cl-dense subset of $\mathbb{M}^m$ is also topologically dense (but, again, the converse is not true).

Example 13.8. 1. If $\mathbb{M}$ is either a valued field (with the valuation topology) or a linearly ordered field (with the order topology), then it satisfies Assumption II.

2. If $\mathbb{M}$ is a d -minimal structure, then it satisfies Assumption I.

3. Let $\mathbb{M}$ be either a formally p -adic field, or an algebraically closed valued field, or a d -minimal expansion of a linearly ordered definably complete field (cf. Example 9.5). Then, $\mathbb{M}$ satisfies both assumptions.

We have two notions of closure and of density on $\mathbb{M}$: the ones given by the topology and the ones given by the matroid; to distinguish them, we will

speak about topological closure and cl-closure respectively (and similarly for density).

The following theorem follows easily from [BH10] (we are assuming that the Hypothesis holds).

Theorem 13.9 ([BH10, Corollary 3.4]). *Let $\mathbb{C} := \langle \mathbb{B}, \mathbb{A}_{n-1}, \dots, \mathbb{A}_1 \rangle \models T^{nd}$. Let $\bar{c} \subset \mathbb{B}$ be cl-independent over $\bar{c} \cap \mathbb{A}_{n-1}$. Let $U \subseteq \mathbb{B}^m$ be open and definable in $\mathbb{C}$, with parameters $\bar{c}$. Then, U is definable in $\mathbb{B}$, with parameters $\bar{c}$.*

In the terminology of [DMS08], the above theorem proves that $\mathbb{B}$ is the **open core** of $\mathbb{C}$.

Proof. By induction on n , it suffices to do the case when $n = 2$, i.e. when $\mathbb{D} = \langle \mathbb{B}, \mathbb{A} \rangle$. W.l.o.g., $\mathbb{D}$ is λ -saturated and λ -homogeneous, for some $|T| < \lambda < \kappa$. We want to verify that the hypothesis of [BH10, Corollary 3.1] is satisfied. Let $D_m := \{\bar{b} \in \mathbb{B}^m : \text{Sr}(\bar{b}/\bar{c}) = m\}$.

1. If $V \subseteq \mathbb{B}^n$ is $\mathbb{B}$ -definable and of dimension m , then $V \cap D_m$ is non-empty: therefore, D_m is topologically dense in $\mathbb{B}^m$.
2. Let $\bar{d} \in D_m$, and $U \subseteq \mathbb{B}^m$ be open, such that $p := \text{tp}^1(\bar{d}/\bar{c})$ is realized in U . We have to show that p is realized in $U \cap D_m$. Let $\bar{d}' \in U$ be such a realization, and let $\bar{b} \subset \mathbb{B}$ such that $\bar{d}' \in B_{\bar{b}}$. Since $\bar{d}' \equiv_{\bar{c}}^1 \bar{d}$, we have that $\bar{d}'$ is cl-independent over $\bar{c}$. By changing $\bar{b}$ if necessary, we can also assume that $\bar{d}' \perp \bar{b}\bar{c}$ (cf. the proof of Lemma 9.16), and thus $\bar{d}'$ is cl-independent over $\bar{b}\bar{c}$. Finally, since $\mathbb{A}$ is cl-dense in $\mathbb{B}$, there exists $\bar{d}'' \equiv_{\bar{b}\bar{c}}^1 \bar{d}$ such that $\bar{d}''$ is cl-independent over $\bar{b}\bar{c}\mathbb{A}$.
3. By Proposition 8.16, for every $dv \in D_m$, $\text{tp}^2(\bar{d}/\bar{c})$ is determined by $\text{tp}^1(\bar{d}/\bar{c})$ in conjunction with “ $\bar{d} \in D_m$ ”.

Hence, we can apply [BH10, Corollary 3.1] and we are done. $\square$

14 The (pre)geometric case

Remember that $\mathbb{M}$ is a pregeometric structure if acl satisfies EP. If moreover $\mathbb{M}$ eliminates the quantifier $\exists^\infty$, then $\mathbb{M}$ is geometric.

In this section we gather various results about (pre)geometric structures, mainly in order to clarify and motivate the general case of structures with an existential matroid.

Remember that $\mathbb{M}$ has geometric elimination of imaginaries if every for imaginary tuple $\bar{a}$ there exists a real tuple $\bar{b}$ such that $\bar{a}$ and $\bar{b}$ are inter-algebraic.

Remark 14.1. T is pregeometric iff T is a real-rosy theory of real $\mathfrak{p}$ -rank 1. Moreover, if T is pregeometric and has geometric elimination of imaginaries, then $\downarrow^{\mathfrak{p}} = \downarrow^{\text{acl}}$; and $\dim^{\text{acl}}$ is equal to the $\mathfrak{p}$ -rank: see [EO07] for definitions and proofs.

Remark 14.2. The model-theoretic algebraic closure acl is a definable closure operator.

Fact. *Let $\mathbb{M}$ be a definably complete and d -minimal expansion of a field. Then, $\mathbb{M}$ has elimination of imaginaries and definable Skolem functions; moreover, $\mathbb{M}$ is rosy iff it is o-minimal. In particular, an ultraproduct of o-minimal structures expanding a field is rosy iff it is o-minimal.*

The proof of the above fact will be given elsewhere.

For the remainder of this section, $\mathbb{M}$ is pregeometric (and T is its theory).

Remark 14.3. acl is an existential matroid on $\mathbb{M}$. The induced independence relation $\downarrow^{\text{acl}}$ coincides with real $\mathfrak{p}$ -independence $\downarrow^{\mathfrak{p}}$ and with the M -dividing notion $\downarrow^M$ of [Adler05]. A formula is x -narrow (for acl) iff it is algebraic in x .

Remark 14.4. Let $X \subseteq \mathbb{M}^n$ be definable. $\dim^{\text{acl}}(X) = 0$ iff X is finite.

Remark 14.5. $\mathbb{M}$ is geometric iff $\dim^{\text{acl}}$ is definable.

Remark 14.6. $\mathbb{M}$ is acl -minimal iff it is strongly minimal.

In §6 we defined $\tilde{\text{acl}}$, the extension of acl to the imaginary sorts.

Remark 14.7. If a is real and B is imaginary, then $a \in \tilde{\text{acl}}(B)$ iff $a \in \text{acl}^{\text{eq}}(B)$.

Remark 14.8. T.f.a.e.:

1. acl^{eq} coincides with the extension of acl defined in §6;
2. T is superrosy of $\mathfrak{p}$ -rank 1 [EO07];
3. T is surgical [Gagelman05].

geometric-density

Remark 14.9. X is dense in $\mathbb{M}$ iff for every U infinite definable subset of $\mathbb{M}$, $U \cap X \neq \emptyset$. If $\mathbb{F} \preceq \mathbb{K}$, then $\mathbb{F}$ is acl -closed in $\mathbb{K}$.

Remark 14.10. Assume that T is geometric. Then, T^2 is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle$, with $\mathbb{F} \prec \mathbb{K} \models T$, and T^d is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle \models T^2$, such that $\mathbb{F}$ is dense in $\mathbb{K}$. For every $X \subseteq \mathbb{K}$, $\text{Scl}(X) = \text{acl}(\mathbb{F}X)$ (cf. Question 8.38).

For more on the theory T^d in the case when T is geometric, and in particular when T is o-minimal, see [Berenstein07].

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