On the number of independent orders

Kota Takeuchi* and Akito Tsuboi†

Abstract

We investigate a model theoretic invariant $\kappa_{srd}^{m}(T)$, which was introduced by Shelah[1], and prove that $\kappa_{srd}^{m}(T)$ is sub-additive. When $\kappa_{srd}^{m}(T)$ is infinite, this gives the equality $\kappa_{srd}^{m}(T) = \kappa_{srd}^{1}(T)$, answering a question in [1]. We apply the same proof method to analyze another invariant $\kappa_{ird}^{m}(T)$, and show that it is also sub-additive, improving a result in [1].

1 Introduction

It is a basic fact that if a theory T is unstable then we can find an unstable 1-formula $\varphi(x, y)$ that witnesses the instability of T. (Recall that a formula $\varphi(x, y)$ is called a 1-formula if the length $|x|$ of x is 1.) Similar situations are true for some other properties of theories, such as TP, TP_1, TP_2, IP, IP_n and SOP. Namely, if a theory T has one of these properties, then we can find a 1-formula witnessing the property. So, it is of interest to know whether such a 1-formula exists as a witness for other important properties of T. The present paper deals with this kind of question, and we are concerned with the number of independent definable orders existing in the monster model $\mathcal{M}$ of T.

Shelah [1] defined three invariants $\kappa_{inp}^{m}(T)$, $\kappa_{srd}^{m}(T)$ and $\kappa_{ird}^{m}(T)$, where m is a positive integer. The first, second, and third invariants are concerning the number of independent partitions, independent orders, and independent

*Institute of Mathematics, University of Tsukuba. Partially supported by KAKENHI 19K20209i

†Institute of Mathematics, University of Tsukuba. Partially supported by KAKENHI 17K05342.

strict orders existing in $\mathcal{M}^m$, respectively. In [1], it was shown that $\kappa^m_{ird}(T)$ does not change its value as m varies (at least if it is an infinite regular cardinal). Then it was asked if a corresponding result holds for $\kappa^m_{inp}(T)$ and $\kappa_{srd}(T)$ ([1, Questions 7.5 and 7.9]). The question about $\kappa^m_{inp}(T)$ was solved in [4]. Although the terminology is different, Chernikov essentially proved the inequality $\kappa^{n+m}_{inp}(T) \leq \kappa^n_{inp}(T) \times \kappa^m_{inp}(T)$, which yields $\kappa^m_{inp}(T) = \kappa^1_{inp}(T)$ if $\kappa^m_{inp}(T)$ is infinite. Furthermore, he conjectured that the invariant is sub-additive, i.e. $\kappa^{n+m}_{inp}(T) + 1 \leq \kappa^n_{inp}(T) + \kappa^m_{inp}(T)$. This conjecture arose in connection with [6], in which it was shown that the dp-rank is sub-additive. It is known that dp-rank coincides with the rank counting the number of independent partitions under the assumption of NIP. Several other invariants (e.g. $\kappa_{cdt}, \kappa_{sct}$) introduced in [1] were studied in [5], and similar type of results were obtained.

It seems however that $\kappa^m_{srd}(T)$ has not been studied well, and there seems to be no answer to Shelah's question on $\kappa^m_{srd}(T)$. Since it has been shown that if T is NIP then $\kappa^m_{ird}(T) = \kappa^m_{srd}(T)$, there is no difference between those two invariants under the assumption of NIP. Under the assumption of NIP, in [2], the condition $\kappa^m_{ird}(T) < n$ was characterized by using the notion of collapse of indiscernible sequences. In this paper we examine how the value $\kappa^m_{srd}(T)$ changes as m changes without any assumption on T (such as NIP). We will prove that $\kappa^m_{srd}(T)$ is sub-additive, which gives a positive answer to a question by Shelah. The concept of mutually indiscernible sequences plays a central role in our proof technique. We will also see that the same technique can be applied when analyzing $\kappa^m_{ird}(T)$, and will prove that the invariant is also sub-additive. This gives an improvement of a result in [1] on $\kappa^m_{ird}(T)$, when it is finite.

Now, we explain some details of $\kappa^m_{srd}(T)$. A complete theory T is said to have the strict order property if there is a formula $\varphi(x_0, \dots, x_{m-1}, y_0, \dots, y_{n-1})$ and parameters $b_i \in \mathcal{M}^n$ $(i \in \omega)$ such that $\Phi := \{\varphi(\mathcal{M}, b_i) : i \in \omega\}$ becomes a strictly increasing sequence of uniformly defined definable sets of $\mathcal{M}^m$, where $\varphi(\mathcal{M}, b_i) = \{a \in \mathcal{M}^m : \mathcal{M} \models \varphi(a, b_i)\}$. Let $\Phi_0 = \{D_i : i \in \omega\}$ and $\Phi_1 = \{E_i : i \in \omega\}$ be two such strictly increasing sequences consisting of subsets of $\mathcal{M}^n$. We say Φ_0 and Φ_1 are independent if $(D_{i+1} \setminus D_i) \cap (E_{j+1} \setminus E_j) \neq \emptyset$, for any $(i, j) \in \omega^2$. We can naturally define the independence among a larger number of Φ_i's. Then $\kappa^n_{\mathrm{srd}}(T)$ is defined as the minimum cardinal κ for which there is no family $\{\Phi_i : i < \kappa\}$ of such independent sequences. (See Definition 2, for a more precise definition.) We put $\kappa_{\mathrm{srd}}(T) = \sup_{n \in \omega} \kappa^n_{\mathrm{srd}}(T)$. If there is a (non-trivial) definable order $<$ on $\mathcal{M}$, then clearly T has the strict order property, and $\kappa^n_{\text{srd}}(T) \geq n+1$. Indeed, for an increasing sequence $a_0 < a_1 < \dots \in \mathcal{M}$, if we let $X_{i,j} = \{(b_0, \dots, b_{n-1}) \in \mathcal{M}^n : b_i < a_j\}$, then $\Psi_i = \{X_{i,j} : j \in \omega\}$ $(i < n)$ will witness $\kappa^n_{\text{srd}}(T) \geq n+1$.

We investigate the invariants $\kappa^n_{\text{srd}}(T)$ and $\kappa^m_{\text{ird}}(T)$, and prove the following:

Theorem A. $\kappa^{m+n}_{\text{srd}}(T) + 1 \leq \kappa^m_{\text{srd}}(T) + \kappa^n_{\text{srd}}(T)$.

Theorem B. Suppose $\kappa^m_{\text{srd}}(T) \geq \omega$. Then $\kappa_{\text{srd}}(T) = \kappa^1_{\text{srd}}(T) = \kappa^m_{\text{srd}}(T)$.

Theorem C. $\kappa^{m+n}_{\text{ird}}(T) + 1 \leq \kappa^m_{\text{ird}}(T) + \kappa^n_{\text{ird}}(T)$.

2 Preliminaries

Let L be a language and T a complete L-theory with an infinite model. We work in a monster model $\mathcal{M} \models T$ with a very big saturation. For a set $A \subset \mathcal{M}$, $L(A)$ denotes the language obtained from L by augmented by the constants for elements in A. Finite tuples in $\mathcal{M}$ are denoted by $a, b, \dots$. The letters $x, y, \dots$ are used to denote finite tuples of variables. The length of x is denoted by $|x|$. Formulas are denoted by $\varphi, \psi, \dots$. For a formula φ and a condition $(*)$, we write $\varphi^{\text{if } (*)}$ to denote the formula φ if $(*)$ is true, and $\neg\varphi$ if $(*)$ is false. In this paper, we are mainly interested in formulas of the form $\varphi(x, b)$, where b is a parameter from $\mathcal{M}$. If $|x| = m$, this formula $\varphi(x, b)$ (or $\varphi(x, y)$) will be called an m-formula. The definable set defined by $\varphi(x, b)$ in $\mathcal{M}$ is denoted by $\varphi(\mathcal{M}, b)$.

Standard set-theoretic notation will be used.

Definition 1. Let κ be a (finite or infinite) cardinal. Let $(\varphi_i(x; y_i))_{i \in \kappa}$ be a sequence of formulas, and $(b_{i,j})_{i \in \kappa, j \in \omega}$ a sequence of tuples, where $|b_{i,j}| = |y_i|$ for all i, j.

1. The pair $\langle (\varphi_i(x; y_i))_{i \in \kappa}, (b_{i,j})_{i \in \kappa, j \in \omega} \rangle$ will be called an ird-pattern of width κ, if it satisfies:
   (a) for any $\eta \in \omega^\kappa$, $\{\varphi_i(x, b_{i,j})^{\text{if } (j \geq \eta(i))} : i \in \kappa, j \in \omega\}$ is consistent.
2. The pair $\langle (\varphi_i(x; y_i))_{i \in \kappa}, (b_{i,j})_{i \in \kappa, j \in \omega} \rangle$ will be called an srd-pattern of width κ, if it satisfies:
   (a) for any $\eta \in \omega^\kappa$, $\{\varphi_i(x; b_{i,j})^{\text{if}(j \geq \eta(i))} : i \in \kappa, j \in \omega\}$ is consistent,

(b) for each $i \in \kappa$ and $j \in \omega$, $\varphi(\mathcal{M}, b_{i,j}) \subsetneq \varphi(\mathcal{M}, b_{i,j+i})$.

Definition 2. Let $* \in \{\text{ird}, \text{srd}\}$. $\kappa_*^m(T)$ is the minimum cardinal κ such that (in T) there is no $*$-pattern of width κ witnessed by m-formulas $\varphi_i(x; y_i)$ $(i \in \kappa)$. We write $\kappa_*^m(T) = \infty$, if there is no such κ. Also $\kappa_*(T)$ is defined as $\sup_{m \in \omega} \kappa_*^m(T)$.

Remark 3. *1. $\kappa_{\text{ird}}^m(T) > 1$ if and only if there is an unstable formula $\varphi(x, y)$ with $|x| = m$. $\kappa_{\text{ird}}(T) > 1$ if and only if T is unstable.*

2. $\kappa_{\text{srd}}^m(T) > 1$ if and only if there is a $\varphi(x, y)$ with $|x| = m$ having the strict order property. $\kappa_{\text{srd}}(T) > 1$ if and only if T has the strict order property.

If $\kappa < \kappa_{srd}^m(T)$, then there are κ-many φ_i's and a set $B = (b_{i,j})_{i \in \kappa, j \in \omega}$ satisfying the conditions (a) and (b) of the item 2 in Definition 1. The condition (b) states that each φ_i defines a strict order on $\mathcal{M}^m$, and the condition (a) states that the orders defined by φ_i's are independent. If $\kappa_{\text{srd}}^m(T) = \infty$, then there is a set $\{\varphi_i(x, y_i) : i < |T|^+\}$ witnessing the conditions. So, by choosing an infinite subset of $|T|^+$, we can assume $\varphi_i = \varphi$ for all $i < \omega$. Conversely, if $\kappa_{srd}(T) \geq \omega$ and if the witnessing formulas satisfy $\varphi_i = \varphi$ $(i < \omega)$, then by compactness, we see that there are arbitrarily many independent strict orders. Notice also that if $\kappa_{srd}^m(T) = \infty$ then T has the independence property.

Example 4. Let T be the theory of $\mathbb{N} = (\mathbb{N}, 0, 1, +, \cdot)$. Let $\varphi(x, y_0, y_1)$ be the formula asserting that the exponent of the y_0-th prime in the prime factorization of x is smaller than y_1. Then, for each i, $\Phi_i := \{\varphi(\mathcal{M}, i, j)\}_j$ forms an increasing sequence of definable sets. Moreover, Φ_i's are independent, so we have $\kappa_{srd}^1(T) = \infty$.

Indiscernibility is a substantial concept in modern model theory. In our paper [3], a couple of results concerning the existence of an indiscernible tree are presented. Here in this paper, the notion of mutual indiscernibility is important.

Definition 5. A set $\{B_i : i < \kappa\}$ of indiscernible sequences is said to be mutually indiscernible over A if for every $i < \kappa$, the sequence B_i is indiscernible over $A \cup \bigcup_{i \neq j < \kappa} B_j$.

The following proposition is simple to prove, but plays an important role in our argument.

Proposition 6. *For each $i < \kappa$, let $B_i = (b_{i,j})_{j\in\omega}$ be an infinite sequence of tuples of the same length. Let $\Gamma((X_i)_{i<\kappa})$ be a set of formulas, where $X_i = (x_{i,j})_{j\in\omega}$ $(i < \kappa)$ and $|x_{i,j}| = |b_{i,j}|$. We assume the following property for Γ:*

- *(*) if B'_i is an infinite subsequence of B_i $(i < \kappa)$ then $(B'_i)_{i<\kappa}$ realizes $\Gamma((X_i)_{i<\kappa})$.*

Then, for any set A, we can find $\{C_i : i < \kappa\} \models \Gamma((X_i)_{i<\kappa})$ that is mutually indiscernible over A.

The following observation, shown by Proposition 6, is a key in our proof of Theorem 9.

Remark 7. Let Z denote $\mathbb{Z}$ or $\mathbb{Z} \cup \{\pm\infty\}$. Then, there is an srd-pattern of width κ witnessed by a sequence $(\varphi_i(x; y_i))_{i\in\kappa}$ of formulas if and only if there are tuples a and $b_{i,j}$ $(i \in \kappa, j \in Z)$ with the following properties:

1. For all $i \in \kappa$ and $j \leq k \in Z$, $\varphi_i(\mathcal{M}, b_{i,j}) \subset \varphi_i(\mathcal{M}, b_{i,k})$;
2. $\{B_i : i \in \kappa\}$ is mutually indiscernible, where $B_i = (b_{i,j})_{j\in Z}$;
3. For all $i \in \kappa$ and $j \in Z$, $\mathcal{M} \models \varphi_i(a, b_{i,j})$ if and only if $j \geq 0$.

In the equivalence above, we can also assume the following condition in addition to 1 –3.

4. $\{B_{i,+} : i \in \kappa\} \cup \{B_{i,-} : i \in \kappa\}$ is mutually indiscernible over a, i.e. $B_{i,+}$ is indiscernible over $\{a\} \cup B_{i,-} \cup \bigcup_{i'\neq i} B_{i'}$ and $B_{i,-}$ is indiscernible over $\{a\} \cup B_{i,+} \cup \bigcup_{i'\neq i} B_{i'}$, where $B_{i,+} = (b_{i,j})_{j\geq 0}$ and $B_{i,-} = (b_{i,j})_{j<0}$.

Remark 8. *Let $(D_i)_{i\in I}$ be an increasing sequence of sets in $\mathcal{M}^n$, where I is a linearly ordered set. Then the following sequences are also increasing:*

1. *$(D_i \cap D)_{i\in I}$, where D is a subset of $\mathcal{M}^n$;*
2. *$(\pi(D_i))_{i\in I}$, where $\pi : \mathcal{M}^n \to \mathcal{M}^m$ is the projection $(x_0, \dots, x_{n-1}) \mapsto (x_{i_0}, \dots, x_{i_{m-1}})$.*

3 Main Results

In the following theorem, κ, κ_0 and κ_1 are arbitrary cardinals, but the interesting case is when they are finite.

Theorem 9. *Let κ, κ_0 and κ_1 be cardinals such that $\kappa+1 = \kappa_0+\kappa_1$. Suppose that there is an srd-pattern of width κ with formulas $\varphi_i(x; y_i)$ $(i \in \kappa)$, where $x = x_0x_1$. Then, there is $l \in \{0,1\}$ for which we can find formulas $\psi_i(x_l; y_i')$ $(i \in \kappa_l)$ witnessing the definition of srd-pattern of width κ_l.*

Proof. Let $Z = \mathbb{Z} \cup \{\pm\infty\}$ and choose $b_{i,j}$ $(i \in \kappa, j \in Z)$ and a satisfying the conditions 1 – 4 in Remark 7. We write a in the form $a = a_0a_1$, where $|a_0| = |x_0|$ and $|a_1| = |x_1|$. For $\eta \in \mathbb{Z}^\kappa$, let

$$\Delta_\eta(x_0, a_1) := \{\varphi_i(x_0, a_1, b_{i,j})^{\text{if } j \geq \eta(i)} : i \in \kappa, j \in Z\}.$$

Then choose a maximal $F \subset \kappa$ satisfying the following property:

(*) For any $\eta \in \mathbb{Z}^\kappa$ with $\mathrm{supp}(\eta) \subset F$ (i.e., $\eta(i) = 0$ if $i \notin F$), $\Delta_\eta(x_0, a_1)$ is consistent.

There are two complementary cases:

Case 1: Suppose $|F| \geq \kappa_0$. In this case the proof is straightforward for $l = 0$, since the formulas $\varphi_i(x_0; x_1y_i)$ $(i \in F)$ and the tuples $c_{i,j} = a_1b_{i,j}$ $(i \in F, j \in \omega)$ form an srd-pattern of width κ_0e.

Case 2: Suppose $|F| < \kappa_0$. Then the set $\kappa \setminus F$ has the cardinality $\geq \kappa_1$. Without loss of generality, we can assume $\kappa_1 \subset \kappa \setminus F$. In this case, for any $\alpha \in \kappa_1$, the extension $F \cup \{\alpha\} \supset F$ does not satisfy $(*)$. Namely, there is η with $\mathrm{supp}(\eta) \subset F \cup \{\alpha\}$, for which the set $\Delta_\eta(x_0, a_1)$ is inconsistent. Fix $\alpha \in \kappa_1$ for a while. Since $\{\varphi_i(\mathcal{M}, a_1, b_{i,j}) : j \in Z\}$ is a strictly increasing sequence for each i, we can choose $\eta_0 \in \mathbb{Z}^F$ and $m \in Z \setminus \{0\}$ such that the subset

$$\begin{aligned}
&\{\varphi_i(x_0, a_1, b_{i,\eta_0(i)}), \neg\varphi_i(x_0, a_1, b_{i,\eta_0(i)-1}) : i \in F\} \\
&\quad \cup \{\varphi_\alpha(x_0, a_1, b_{\alpha,m}), \neg\varphi_\alpha(x_0, a_1, b_{\alpha,m-1})\} \\
&\quad \cup \{\varphi_i(x_0, a_1, b_{i,0}), \neg\varphi_i(x_0, a_1, b_{i,-1}) : i \in \kappa \setminus (F \cup \{\alpha\})\}
\end{aligned}$$

of Δ_η is inconsistent. Since the other case is similar and in fact easier, we assume $m > 0$. Then, by compactness, and since $\{B_{i,+} : i \in \kappa\} \cup \{B_{i,-} :$

$i \in \kappa\}$ is mutually indiscernible over a, we can find finite sets $F_0 \subset F$ and $F_1 \subset \kappa \setminus (F \cup \{\alpha\})$ such that

$$\begin{aligned}\Sigma_\alpha(x_0) := \{&\varphi_i(x_0, a_1, b_{i,\eta_0(i)}), \neg\varphi_i(x_0, a_1, b_{i,\eta_0(i)-1}) : i \in F_0\} \\ &\cup \{\varphi_\alpha(x_0, a_1, b_{\alpha,\infty}), \neg\varphi_\alpha(x_0, a_1, b_{\alpha,0})\} \\ &\cup \{\varphi_i(x_0, a_1, b_{i,\infty}), \neg\varphi_i(x_0, a_1, b_{i,-\infty}) : i \in F_1\}\end{aligned}$$

Now, let

$$B^* := \{b_{i,j}\}_{i \in F, j \in \mathbb{Z}} \cup \{b_{i,-\infty}\}_{i \in \kappa \setminus F} \cup \{b_{i,\infty}\}_{i \in \kappa \setminus F}$$

Then the parameters appearing in $\Sigma_\alpha(x_0)$, other than B^*, are a_1 and $b_{\alpha,0}$. (The definition of B^* does not depend on α and hereafter we work with the language $L(B^*)$.) So we write Σ_α as $\Sigma_\alpha(x_0, a_1, b_{\alpha,0})$. By preparing a variable z_α with $|z_\alpha| = |b_{\alpha,j}|$, let $\psi'_\alpha(x_0, x_1, z_\alpha)$ be the formula $\bigwedge \Sigma_\alpha(x_0, x_1, z_\alpha)$. Recall that the set $\Sigma_\alpha(x_0, a_1, b_{\alpha,0})$ is inconsistent. However, the set $\Sigma_\alpha(x_0, a_1, b_{\alpha,-1})$ is consistent, by our choice of F and the condition $(*)$. By the condition 4 in Remark 7, this means that $\psi'_\alpha(x_0, a_1, b_{\alpha,j})$ is consistent if and only if $j < 0$. So, if we define

$$\begin{aligned}\psi_\alpha(x_1, z_\alpha) &:= (\exists x_0)\, \psi'_\alpha(x_0, x_1, z_\alpha), \\ c_{\alpha,j} &:= b_{\alpha,-j-1},\end{aligned}$$

then we have

$$\mathcal{M} \models \psi_\alpha(a_1, c_{\alpha,j}) \iff j \geq 0.$$

Since this is true for all $\alpha \in \kappa_1$, it follows that $\langle(\psi_\alpha)_{\alpha\in\kappa_1}, (c_{\alpha,j})_{\alpha\in\kappa_1, j\in\mathbb{Z}}\rangle$ satisfies the condition 3 in Remark 7. The condition 2 is easily shown, since the sequences $(c_{\alpha,j})_{j\in\mathbb{Z}}$ $(\alpha \in \kappa_1)$ are mutually indiscernible over B^*. Finally the condition 1 follows from Remark 8. Hence, $\langle(\psi_\alpha)_{\alpha\in\kappa_1}, (c_{\alpha,j})_{\alpha\in\kappa_1, j\in\mathbb{Z}}\rangle$ is an srd-pattern of width κ_1. □

Corollary 10. *1.* $\kappa^{m+n}_{\mathrm{srd}}(T) + 1 \leq \kappa^m_{\mathrm{srd}}(T) + \kappa^n_{\mathrm{srd}}(T)$.

2. If $\kappa^m_{\mathrm{srd}}(T)$ *is infinite, then* $\kappa^m_{\mathrm{srd}}(T) = \kappa^1_{\mathrm{srd}}(T) = \kappa_{\mathrm{srd}}(T)$.

Proof. We only prove the first item. We can assume $\kappa^{m+n}_{\mathrm{srd}}(T)$ is finite, since the infinite case is easier. By way of a contradiction, we assume $\kappa^{m+n}_{\mathrm{srd}}(T) + 1 > \kappa^m_{\mathrm{srd}}(T) + \kappa^n_{\mathrm{srd}}(T)$. Then there must be an srd-pattern of width $\kappa := \kappa^m_{\mathrm{srd}}(T) + \kappa^n_{\mathrm{srd}}(T) - 1$ witnessed by $(m+n)$-formulas. By Theorem 9, using the equation $\kappa + 1 = \kappa^m_{\mathrm{srd}}(T) + \kappa^n_{\mathrm{srd}}(T)$, we would have (i) the existence of an srd-pattern of width $\kappa^m_{\mathrm{srd}}(T)$ by m-formulas, or (ii) the existence of an srd-pattern of width $\kappa^n_{\mathrm{srd}}(T)$ by n-formulas. In either case, we reach a contradiction. $\square$

The above argument can be applied to show the corresponding result for $\kappa^m_{\mathrm{ird}}(T)$. The following theorem on $\kappa^m_{\mathrm{ird}}(T)$ gives an improvement of [1, Theorem 7.10]. (In that book he investigated $\kappa^m_{\mathrm{ird}}(T)$ when it is infinite.) In the following theorem, κ, κ_0 and κ_1 are any cardinals as before.

Theorem 11. *Assume $\kappa + 1 = \kappa_0 + \kappa_1$. Suppose that there is an ird-pattern of width κ with formulas $\varphi_i(x; y_i)$ $(i \in \kappa)$, where $x = x_0x_1$. Then, there is $l \in \{0, 1\}$ for which we can find formulas $\psi_i(x_l; y'_i)$ $(i \in \kappa_l)$ witnessing an ird-pattern of width κ_l.*

Proof. The outline of the proof is quite similar to that of Theorem 9. However, for completeness, the details of the proof are provided. In the present proof, our linear order Z has the form $Z = \mathbb{Z}_- + \mathbb{Z} + \mathbb{Z}_+$, where both $\mathbb{Z}_-$ and $\mathbb{Z}_+$ are copies of $\mathbb{Z}$, and the order is defined so that $\mathbb{Z}_- < \mathbb{Z} < \mathbb{Z}_+$.

Choose $b_{i,j}$ $(i \in \kappa, j \in Z)$ and $a = a_0a_1$ satisfying the conditions 1 – 4 in Remark 7. Then for $\eta \in \mathbb{Z}^\kappa$, consider the set $\Delta_\eta(x_0, a_1)$, which is defined in the same way as in the proof of previous theorem. Again, choose a maximal $F \subset \kappa$ satisfying the following property:

(**) For any $\eta \in Z^\kappa$ with $\mathrm{supp}(\eta) \subset F$, $\Delta_\eta(x_0, a_1)$ is consistent.

Case 1: Suppose $|F| \geq \kappa_0$. The proof is straightforward as the previous theorem so we skip this case.

Case 2: Suppose $|F| < \kappa_0$. Without loss of generality, we can assume $\kappa_1 \subset \kappa \setminus F$. In this case, for any $\alpha \in \kappa_1$, there is η with $\mathrm{supp}(\eta) \subset F \cup \{\alpha\}$, for which the set $\Delta_\eta(x_0, a_1)$ is inconsistent. By compactness, we can choose finite sets $F_0 \subset F$, $F_1 \subset \kappa \setminus (F \cup \{\alpha\})$, and $U_i, O_i \subset \mathbb{Z}$ $(i \in F_0 \cup F_1 \cup \{\alpha\})$ with the following properties:

1. $U_i < O_i$, for any i;

2. $U_i < 0 \leq O_i$, if $i \in F_1$;

3. The following set $\Sigma_\alpha(x_0)$ is inconsistent:

$$\begin{gathered}\{\neg\varphi_i(x_0, a_1, b_{i,j}) : i \in F_0, j \in U_i\} \cup \{\varphi_i(x_0, a_1, b_{i,j}) : i \in F_0, j \in O_i\} \\ \cup\{\neg\varphi_\alpha(x_0, a_1, b_{i,j}) : j \in U_\alpha\} \cup \{\varphi_\alpha(x_0, a_1, b_{i,j}) : j \in O_\alpha\} \\ \cup\{\neg\varphi_i(x_0, a_1, b_{i,j}) : i \in F_1, j \in U_i\} \cup \{\varphi_i(x_0, a_1, b_{i,j}) : i \in F_1, j \in O_i\}.\end{gathered}$$

If $U_\alpha < 0 \leq O_\alpha$ holds, then Σ_α must be consistent, by our choice of F. So, since the other case is similarly proven, we can assume $U_\alpha^+ := \{j \in U_\alpha : 0 \leq j\} \neq \emptyset$. Moreover U_α is assumed to be chosen so that $|U_\alpha^+|$ is minimum.

Since $\{B_{i,+} : i \in \kappa\} \cup \{B_{i,-} : i \in \kappa\}$ is mutually indiscernible over a, we can assume

- $U_i, O_i \subset \mathbb{Z}$ $(i \in F)$;
- $U_i \subset \mathbb{Z}_-,\ O_i \subset \mathbb{Z}_+$ $(i \in \kappa \setminus (F \cup \{\alpha\}))$;
- $U_\alpha^- := \{j \in U_\alpha : j < 0\} \subset \mathbb{Z}^-$, $U_\alpha^+ = \{0, \ldots, k-2, k-1\} \subset \mathbb{Z}$;
- $O_\alpha \subset \mathbb{Z}_+$.

Now, let

$$B^* := \{b_{i,j} : i \in F, j \in \mathbb{Z}\} \cup \{b_{i,j} : i \in \kappa \setminus F, j \in \mathbb{Z}_- \cup \mathbb{Z}_+\}$$

Then the parameters appearing in $\Sigma_\alpha(x_0)$, other than B^*, are a_1 and $(b_{\alpha,j})_{j \in U_\alpha^+}$. So we write Σ_α as $\Sigma_\alpha(x_0, a_1, (b_{\alpha,j})_{j\in k})$. Let $\psi'_\alpha(x_0, x_1, z_\alpha)$ be the formula $\bigwedge \Sigma_\alpha(x_0, x_1, z_\alpha)$. Recall that the set $\Sigma_\alpha(x_0, a_1, (b_{\alpha,j})_{j\in k})$ is inconsistent. However, the set $\Sigma(x_0, a_1, (b_{\alpha,j})_{j\in\{-k,\ldots,-1\}})$ is consistent, by the choice of F. By the condition 4, if we set $c_{\alpha,l} = (b_{\alpha,j})_{j\in\{lk,lk+1,\ldots,lk+(k-1)\}}$, this means that $\psi'_\alpha(x_0, a_1, c_{\alpha,j})$ is consistent if and only if $j < 0$. The rest of the proof is almost identical with that of *srd*-case. □

From this theorem we deduce the following corollary. The item 2 is essentially shown in [1].

Corollary 12. *1.* $\kappa_{\mathrm{ird}}^{m+n}(T) + 1 \leq \kappa_{\mathrm{ird}}^m(T) + \kappa_{\mathrm{ird}}^n(T)$.

2. If $\kappa_{\mathrm{ird}}^m(T)$ *is infinite, then* $\kappa_{\mathrm{ird}}^m(T) = \kappa_{\mathrm{ird}}^1(T) = \kappa_{\mathrm{ird}}(T)$.

References

[1] Shelah, S., *Classification theory and the numbers of non-isomorphic models.* Amsterdam: North-Holland, (1990).

[2] Guingona, V., Hill, C. and Scow, L., "Characterizing model-theoretic dividing lines via collapse of generalized indiscernibles." Annals of Pure and Applied Logic, 168(5), pp.1091-1111, (2017).

[3] Takeuchi, K. and Tsuboi, A. , "On the existence of indiscernible trees." Annals of Pure and Applied Logic, 163(12), pp.1891-1902, (2012).

[4] Chernikov, Artem. "Theories without the tree property of the second kind." Annals of Pure and Applied Logic 165.2 (2014): 695-723.

[5] Chernikov, Artem, and Nicholas Ramsey. "On model-theoretic tree properties." Journal of Mathematical Logic 16.02 (2016): 1650009.

[6] Kaplan, Itay, Alf Onshuus, and Alexander Usvyatsov. "Additivity of the dp-rank." Transactions of the American Mathematical Society 365.11 (2013): 5783-5804.