# INFINITESIMAL ANALYSIS WITHOUT THE AXIOM OF CHOICE 

KAREL HRBACEK AND MIKHAIL G. KATZ


#### Abstract

It is often claimed that analysis with infinitesimals requires more substantial use of the Axiom of Choice than traditional elementary analysis. The claim is based on the observation that the hyperreals entail the existence of nonprincipal ultrafilters over $\mathbb{N}$, a strong version of the Axiom of Choice, while the real numbers can be constructed in $\mathbf{Z F}$. The axiomatic approach to nonstandard methods refutes this objection. We formulate a theory SPOT in the st- $\in$-language which suffices to carry out infinitesimal arguments, and prove that SPOT is a conservative extension of ZF. Thus the methods of Calculus with infinitesimals are just as effective as those of traditional Calculus. The conclusion extends to large parts of ordinary mathematics and beyond. We also develop a stronger axiomatic system SCOT, conservative over $\mathbf{Z F}+\mathbf{A D C}$, which is suitable for handling such features as an infinitesimal approach to the Lebesgue measure. Proofs of the conservativity results combine and extend the methods of forcing developed by Enayat and Spector.


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## 1. Introduction

Many branches of mathematics exploit the Axiom of Choice (AC) to one extent or another. It is of considerable interest to gauge how much the Axiom of Choice can be weakened in the foundations of nonstandard analysis. Critics of analysis with infinitesimals often claim that nonstandard methods require more substantial use of AC than their standard counterparts. The goal of this paper is to refute such a claim.
1.1. Axiom of Choice in Mathematics. We begin by considering the extent to which AC is needed in traditional non-infinitesimal mathematics. Simpson 35 introduces a useful distinction between settheoretic mathematics and ordinary or non-set-theoretic mathematics. The former includes such disciplines as general topology, abstract algebra and functional analysis. It is well known that fundamental theorems in these areas require strong versions of AC. Thus

- Tychonoff's Theorem in general topology is equivalent to full AC (over Zermelo-Fraenkel set theory ZF).
- Prime Ideal Theorem asserts that every ring with unit has a (two-sided) prime ideal. PIT is an essential result in abstract algebra and is "almost" as strong as AC (it is equivalent over ZF to Tychonoff's Theorem for Hausdorff spaces). It is also
equivalent to the Ultrafilter Theorem: Every proper filter over a set $S$ (i.e., in $\mathcal{P}(S)$ ) can be extended to an ultrafilter.
- Hahn-Banach Theorem for general vector spaces is equivalent to the statement that every Boolean algebra admits a real-valued measure, a form of AC that is somewhat weaker than PIT.
Jech [20] and Howard and Rubin [15] are comprehensive references for the relationships between these and many other forms of AC.

Researchers in set-theoretic mathematics have to accept strong forms of AC as legitimate whether or not they use nonstandard methods. Our concern here is with ordinary mathematics, which, according to Simpson, includes fields such as the Calculus, countable algebra, differential equations, and real and complex analysis. It is often felt that results in these fields should be effective in the sense of not being dependent on AC. However, even these branches of mathematics cannot do entirely without AC. There is a number of fundamental classical results that rely on it; they include

- the equivalence of continuity and sequential continuity for realvalued functions on $\mathbb{R}$;
- the equivalence of the $\varepsilon-\delta$ definition and the sequential definition of closure points for subsets of $\mathbb{R}$;
- closure of the collection of Borel sets under countable unions and intersections;
- countable additivity of Lebesgue measure.

Without an appeal to $\mathbf{A C}$ one cannot even prove that $\mathbb{R}$ is not a union of countably many countable sets, or that a strictly positive function cannot have vanishing Lebesgue integral (Kanovei and Katz [23]). However, these results follow already from ACC, the Axiom of Choice for Countable collections, a weak version of AC that many mathematicians use without even noticing 1 Nevertheless, it is true enough that no choice is needed to define the real number system itself, or to develop the Calculus and much of ordinary mathematics.

It has to be emphasized that objections to $\mathbf{A C}$ are not a matter of ontology, but of epistemology. In other words, the issue is not the existence of objects, but proof techniques and procedures. For better or worse, many mathematicians nowadays believe that the objects of interest to them can be represented by set-theoretic structures in a universe that satisfies ZFC, Zermelo-Fraenkel set theory with the Axiom of Choice. Nevertheless, they may prefer results that are effective, that is, do not use AC. For the purposes of this discussion, mathematical

[^1]results are effective if they can be proved in ZF. Much of ordinary mathematics is effective in this sense.

We now consider whether nonstandard methods require anything more. A common objection to infinitesimal methods in the Calculus is the claim that the mere existence of the hyperreal. 2 implies the existence of a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. The proof is simple: Fix an infinitely large integer $\nu$ in ${ }^{*} \mathbb{N} \backslash \mathbb{N}$ and define $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ by $X \in \mathcal{U} \longleftrightarrow \nu \in^{*} X$, for $X \subseteq \mathbb{N}$. It is easy to see that $\mathcal{U}$ is a nonprincipal ultrafilter over $\mathbb{N}$. For example, if $X \cup Y \in \mathcal{U}$, then $\nu \in{ }^{*}(X \cup Y)=$ ${ }^{*} X \cup{ }^{*} Y$, where the last step is by the Transfer Principle. Hence either $\nu \in{ }^{*} X$ or $\nu \in{ }^{*} Y$, and so $X \in \mathcal{U}$ or $Y \in \mathcal{U}$. If $X$ is finite, then $X={ }^{*} X$, hence $\nu \not ⿻^{*} X$ and so $\mathcal{U}$ is nonprincipal.

By the well-known result of Sierpiński [34] (see also Jech [20], Problem 1.10), $\mathcal{U}$ is a non-Lebesgue-measurable set (when subsets of $\mathbb{N}$ are identified with real numbers in some natural way). In the celebrated model of Solovay [36], ZF + ACC holds (even the stronger ADC, the Axiom of Dependent Choice, holds there), but all sets of real numbers are Lebesgue measurable, hence there are no nonprincipal ultrafilters over $\mathbb{N}$ in this model. The existence of nonprincipal ultrafilters over $\mathbb{N}$ requires a strong version of AC such as PIT; it cannot be proved in $\mathbf{Z F}$ (or even $\mathbf{Z F}+\mathbf{A D C}$ ).
1.2. Countering the objection. How can such an objection be answered? As in the case of the traditional mathematics, the key is to look not at the objects but at the methods used. Currently there are two popular ways to practice Robinson's nonstandard analysis: the modeltheoretic approach and the axiomatic/syntactic approach. Analysis with infinitesimals does not have to be based on hyperreal structures in the universe of ZFC. It can be developed axiomatically; the monograph by Kanovei and Reeken [24] is a comprehensive reference for such approaches. Internal axiomatic presentations of nonstandard analysis, such as IST or BST, extend the usual $\in$-language of set theory by a unary predicate $\mathbf{s t}(\mathbf{s t}(x)$ reads $x$ is standard). For reference, the axioms of BST are stated in Section 7.

It is of course possible to weaken ZFC to ZF within BST or IST, but this move by itself does not answer the above objection. It is easily seen, by a modification of the argument given above for hyperreals, that the theory obtained from BST or IST by replacing ZFC by ZF proves

[^2]PIT (Hrbacek [16]). This argument uses the full strength of the principles of Idealization and Standardization (see Section 7). However, Calculus with infinitesimals can be fully carried out assuming much less. Examination of texts such as Keisler [26] and Stroyan [40] reveals that only very weak versions of these principles are ever used there. Of course one has to postulate that infinitesimals exist (Nontriviality), but stronger consequences of Idealization are not needed. As for Standardization, these textbooks only explicitly postulate a special consequence of it, namely, the following principle:

SP (Standard Part) Every limited real is infinitely close to a standard real;
see Keisler [26, 27], Axioms A - E. However, this is somewhat misleading. Keisler does not develop the Calculus from his axioms alone; they describe some properties of the hyperreals, but the hyperreals are considered to be an extension of the field $\mathbb{R}$ of real numbers in the universe of ZFC, and the principles of ZFC can be freely used. In particular, the principle of Standardization is not an issue; it is automatically satisfied for any formula. While Standardization for formulas about integers appears innocuous, Standardization for formulas about reals can lead to the existence of nonprincipal ultrafilters. On the other hand, some instances of Standardization over the reals are unavoidable, for example to prove the existence of the function $f^{\prime}$ (the derivative of $f$ ) defined in terms of infinitesimals for a given real-valued function $f$ on $\mathbb{R}$.
1.3. SPOT and SCOT. In the present text, we introduce a theory SPOT in the st- $\epsilon$-language, a subtheory of IST and BST, and we show that SPOT proves Countable Idealization and enough Standardization for the purposes of the Calculus. We use $\forall$ and $\exists$ as quantifiers over sets and $\forall^{\text {st }}$ and $\exists^{\text {st }}$ as quantifiers over standard sets. The axioms of SPOT are:

ZF (Zermelo - Fraenkel Set Theory)
$\mathbf{T}$ (Transfer) Let $\phi$ be an $\in$-formula with standard parameters. Then

$$
\forall^{\text {st }} x \phi(x) \rightarrow \forall x \phi(x)
$$

O (Nontriviality) $\quad \exists \nu \in \mathbb{N} \forall^{\text {st }} n \in \mathbb{N}(n \neq \nu)$.
$\mathbf{S P}^{\prime}$ (Standard Part)

$$
\forall A \subseteq \mathbb{N} \exists^{\text {st }} B \subseteq \mathbb{N} \forall^{\text {st }} n \in \mathbb{N}(n \in B \longleftrightarrow n \in A)
$$

Our main result is the following.

Theorem A The theory SPOT is a conservative extension of ZF.
Thus the methods used in the Calculus with infinitesimals do not require any appeal to the Axiom of Choice.

The result allows significant strengthenings. We let $\mathbf{S N}$ be the Standardization principle for st- $\in$-formulas with no parameters (see Section (6). The principle allows Standardization of much more complex formulas than SPOT alone.

Theorem B The theory $\mathbf{S P O T}+\mathbf{S N}$ is a conservative extension of ZF.

It is also possible to add some Idealization. We let $\mathbf{B I}^{\prime}$ be Bounded Idealization (see Section 7) for $\in$-formulas with standard parameters.

Theorem C The theory $\mathbf{S P O T}+\mathbf{B}+\mathbf{B I}^{\prime}$ is a conservative extension of $\mathbf{Z F}$.

This is the theory BST with ZFC replaced by ZF, Standardization weakened to SP and Bounded Idealization weakened to $\mathbf{B I}^{\prime}$; we denote it $\mathbf{B S P T}^{\prime}$. This theory enables the applicability of some infinitesimal techniques to arbitrary topological spaces. It also proves that there is a finite set $S$ containing all standard reals, a frequently used idea.

As noted above, some important results in elementary analysis and elsewhere in ordinary mathematics require the Axiom of Countable Choice. On the other hand, ACC entails no "paradoxical" consequences, such as the existence of Lebesgue-non-measurable sets, or the existence of an additive function on $\mathbb{R}$ different from $f_{a}: x \mapsto a x$ for all $a \in \mathbb{R}$. Many mathematicians find ACC acceptable. These considerations apply as well to the following stronger axiom.

ADC (Axiom of Dependent Choice) If $R$ is a binary relation on a set $A$ such that $\forall a \in A \exists a^{\prime} \in A\left(a R a^{\prime}\right)$, then for every $a \in A$ there exists a sequence $\left\langle a_{n} \mid n \in \mathbb{N}\right\rangle$ such that $a_{0}=a$ and $a_{n} R a_{n+1}$ for all $n \in \mathbb{N}$.

This axiom is needed for example to prove the equivalence of the two definitions of a well-ordering (Jech [21], Lemma 5.2):
(1) Every nonempty subset of a linearly ordered set $(A,<)$ has a least element.
(2) $A$ has no infinite decreasing sequence $a_{0}>a_{1}>\ldots>a_{n}>\ldots$

We denote by ZF $c$ ("ZFC Lite") the theory $\mathbf{Z F}+\mathbf{A D C}$. This theory is sufficient for axiomatizing ordinary mathematics (and many

|  | SPOT | SPOT+SN | SCOT | BSPT $^{\prime}$ | BSCT $^{\prime}$ | BST |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| -theory | ZF | ZF | ZF +ADC | ZF | ZF+ADC | ZFC |
| Transfer | yes | yes | yes | yes | yes | yes |
| Idealization | countable | countable | countable | standard <br> params | standard <br> params | full |
| Standardiz. | SP | SP; <br> standard <br> params | SC; <br> standard <br> params | SP | SC | full |
| Countable <br> st- $\epsilon$ <br> Choice | no | no | yes | no | yes | Standard <br> -size <br> Choice |

Table 1. Theories and their properties
results of set-theoretic mathematics as well). Let SCOT be the theory obtained from SPOT by strengthening ZF to ZF $c$, SP to Countable st- $\in$-Choice ( $\mathbf{C C}$ ), and adding $\mathbf{S N}$; see Section 3.

Theorem D The theory SCOT is a conservative extension of $\mathbf{Z F} c$.
In SCOT one can carry out most techniques used in infinitesimal treatments of ordinary mathematics. As examples, we give a proof of Peano's Existence Theorem and an infinitesimal construction of Lebesgue measure in Section 3, Thus the nonstandard methods used in ordinary mathematics do not require any more choice than is generally accepted in traditional ordinary mathematics.

Further related conservative extension theorems can be found in Sections 5, 6 and 7 .

## 2. Theory SPOT and Calculus with infinitesimals

2.1. Some consequences of SPOT. The axioms of SPOT were given in Section 1.3 ,

Lemma 2.1. The theory SPOT proves the following:

$$
\forall^{\text {st }} n \in \mathbb{N} \forall m \in \mathbb{N}(m<n \rightarrow \mathbf{s t}(m)) .
$$

Proof. Given a standard $n \in \mathbb{N}$ and $m<n$, let $A=\{k \in \mathbb{N} \mid k<m\}$. By $\mathbf{S P}^{\prime}$ there is a standard $B \subseteq \mathbb{N}$ such that for all standard $k, k \in B$ iff $k \in A$ iff $k<m$. The set $B \subseteq \mathbb{N}$ is bounded above by $n$ (Transfer), so it has a greatest element $k_{0}(<$ is a well-ordering of $\mathbb{N})$, which
is standard by Transfer. Now we have $k_{0}<m$ and $k_{0}+1 \nless m$, so $k_{0}+1=m$ and $m$ is standard.

Lemma 2.2. (Countable Idealization) Let $\phi$ be an $\in$-formula with arbitrary parameters. The theory SPOT proves the following:

$$
\forall^{\text {st }} n \in \mathbb{N} \exists x \forall m \in \mathbb{N}(m \leq n \rightarrow \phi(m, x)) \longleftrightarrow \exists x \forall^{\text {st }} n \in \mathbb{N} \phi(n, x)
$$

Proof. If $\forall^{\text {st }} n \in \mathbb{N} \phi(n, x)$, then, for every standard $n \in \mathbb{N}, \forall m \in$ $\mathbb{N}(m \leq n \rightarrow \phi(m, x))$, by Lemma 2.1.

Conversely, assume $\forall^{\text {st }} n \in \mathbb{N} \exists x \forall m \in \mathbb{N}(m \leq n \rightarrow \phi(m, x))$. By the Axiom of Separation of $\mathbf{Z F}$, there is a set

$$
S=\{n \in \mathbb{N} \mid \exists x \forall m \in \mathbb{N}(m \leq n \rightarrow \phi(m, x))\}
$$

and the assumption implies that $\forall^{\text {st }} n \in \mathbb{N}(n \in S)$.
Assume that $S$ contains standard integers only. Then $\mathbb{N} \backslash S \neq \emptyset$ by the axiom $\mathbf{O}$. Let $\nu$ be the least element of $\mathbb{N} \backslash S$. Then $\nu$ is nonstandard but $\nu-1$ is standard, a contradiction.

Let $\mu$ be some nonstandard element of $S$. We have $\exists x \forall m \in \mathbb{N}(m \leq$ $\mu \rightarrow \phi(m, x))$; as $n \leq \mu$ holds for all standard $n \in \mathbb{N}$, we obtain $\exists x \forall^{\text {st }} n \in \mathbb{N} \phi(n, x)$.

Countable Idealization easily implies the following more familiar form. We use $\forall^{\text {st fin }}$ and $\exists^{\text {st fin }}$ as quantifiers over standard finite sets.

Corollary 2.3. Let $\phi$ be an $\in$-formula with arbitrary parameters. The theory SPOT proves the following: For every standard countable set $A$

$$
\forall^{\text {st fin }} a \subseteq A \exists x \forall y \in a \phi(x, y) \longleftrightarrow \exists x \forall^{\text {st }} y \in A \phi(x, y)
$$

The axiom $\mathbf{S P}^{\prime}$ is often stated and used in the form

$$
\begin{equation*}
\forall x \in \mathbb{R}\left(x \text { limited } \rightarrow \exists^{\text {st }} r \in \mathbb{R}(x \approx r)\right) \tag{SP}
\end{equation*}
$$

where $x$ is limited iff $|x| \leq n$ for some standard $n \in \mathbb{N}$, and $x \approx r$ iff $|x-r| \leq 1 / n$ for all standard $n \in \mathbb{N}, n \neq 0$. The unique standard real number $r$ is called the standard part of $x$ or the shadow of $x$; notation $\operatorname{sh}(x)$.

We note that in the statement of $\mathbf{S P}^{\prime}, \mathbb{N}$ can be replaced by any countable standard set $A$.

Lemma 2.4. The statements $\mathbf{S P}^{\prime}$ and $\mathbf{S P}$ are equivalent (over $\mathbf{Z F}+$ $\mathbf{O}+\mathbf{T})$.
Proof of Lemma 2.4. $\mathbf{S P}^{\prime} \Rightarrow \mathbf{S P}$ : Assume $x \in \mathbb{R}$ is limited by a standard $n_{0} \in \mathbb{N}$. Let $A=\{q \in \mathbb{Q} \mid q \leq x\}$. Applying $\mathbf{S P}^{\prime}$ with $\mathbb{N}$ replaced by $\mathbb{Q}$, we obtain a standard set $B \subseteq \mathbb{Q}$ such that $\forall^{\text {st }} q \in \mathbb{Q}(q \in B \longleftrightarrow$ $q \in A)$. As $\forall^{\text {st }} q \in B\left(q \leq n_{0}\right)$ holds, the set $B$ is bounded above (apply

Transfer to the formula $q \in B \rightarrow q \leq n_{0}$ ) and so it has a supremum $r \in \mathbb{R}$, which is standard (Transfer again). We claim that $x \approx r$. If not, then $|x-r|>\frac{1}{n}$ for some standard $n$, hence either $x<r-\frac{1}{n}$ or $x>r+\frac{1}{n}$. In the first case $\sup B \leq r-\frac{1}{n}$ and in the second, $\sup B \geq r+\frac{1}{n}$; either way contradicts sup $B=r$.
$\mathbf{S P} \Rightarrow \mathbf{S P}^{\prime}$ : The obvious idea is to represent the characteristic function of a set $A \subseteq \mathbb{N}$ by the binary expansion of a real number in $[0,1]$. But some real numbers have two binary expansions and therefore correspond to two distinct subsets of $\mathbb{N}$. This is a source of technical complications that we avoid by using decimal expansions instead.

Given $A \subseteq \mathbb{N}$, let $\chi_{A}$ be the characteristic function of $A$. Define a real number $x_{A}=\sum_{n=0}^{\infty} \frac{\chi(n)}{10^{n+1}}$; as $0 \leq x_{A} \leq \frac{1}{9}$, there is a standard real number $r \approx x_{A}$. Let $r=\sum_{n=0}^{\infty} \frac{a_{n}}{10^{n+1}}$ be the decimal expansion of $r$ where for every $n$ there is $k>n$ such that $a_{k} \neq 9$. Note that if $n$ is standard, then there is a standard $k$ with this property, by Transfer. If $\chi(n)=a_{n}$ for all $n$, then $A$ is standard and we let $B=A$. Otherwise let $n_{0}$ be the least $n$ where $\chi(n) \neq a_{n}$. From $r \approx x_{A}$ it follows easily that $n_{0}$ is nonstandard. In particular, $a_{n} \in\{0,1\}$ holds for all standard $n$, hence, by Transfer, for all $n \in \mathbb{N}$. Let $B=\left\{n \in \mathbb{N} \mid a_{n}=1\right\}$. Then $B$ is standard and for all standard $n \in \mathbb{N}, n \in B$ iff $a_{n}=1$ iff $\chi(n)=1$ iff $n \in A$.

As explained in the Introduction, Standardization over uncountable sets such as $\mathbb{R}$, even for very simple formulas, implies the existence of nonprincipal ultrafilters over $\mathbb{N}$, and so it cannot be proved in SPOT (consider a standard set $U$ such that $\forall^{\text {st }} X(X \in U \longleftrightarrow X \subseteq \mathbb{N} \wedge \nu \in$ $X$ ), where $\nu$ is a nonstandard integer). But we need to be able to prove the existence of various subsets of $\mathbb{R}$ and functions from $\mathbb{R}$ to $\mathbb{R}$ that arise in the Calculus and may be defined in terms of infinitesimals. Unlike the undesirable example above, such uses generally involve Standardization for formulas with standard parameters.

An st- $\epsilon$-formula $\Phi\left(v_{1}, \ldots, v_{n}\right)$ is $\boldsymbol{\Delta}^{\text {st }}$ if it is of the form

$$
Q_{1}^{\text {st }} x_{1} \ldots Q_{m}^{\text {st }} x_{m} \psi\left(x_{1}, \ldots, x_{m}, v_{1}, \ldots, v_{n}\right)
$$

where $\psi$ is an $\in$-formula and $Q$ stands for $\exists$ or $\forall$.
Lemma 2.5. Let $\Phi\left(v_{1}, \ldots, v_{n}\right)$ be a $\boldsymbol{\Delta}^{\text {st }}$ formula with standard parameters. Then SPOT proves: $\quad \forall^{\text {st }} S \exists^{\text {st }} P \forall^{\text {st }} v_{1}, \ldots, v_{n}$

$$
\left(\left\langle v_{1}, \ldots, v_{n}\right\rangle \in P \longleftrightarrow\left\langle v_{1}, \ldots, v_{n}\right\rangle \in S \wedge \Phi\left(v_{1}, \ldots, v_{n}\right)\right) .
$$

Proof. Let $\Phi\left(v_{1} \ldots, v_{n}\right)$ be $Q_{1}^{\text {st }} x_{1} \ldots Q_{m}^{\text {st }} x_{m} \psi\left(x_{1}, \ldots, x_{m}, v_{1}, \ldots, v_{n}\right)$ and $\phi\left(v_{1} \ldots, v_{n}\right)$ be $Q_{1} x_{1} \ldots Q_{m} x_{m} \psi\left(x_{1}, \ldots, x_{m}, v_{1}, \ldots, v_{n}\right)$. By Transfer, $\Phi\left(v_{1} \ldots, v_{n}\right) \longleftrightarrow \phi\left(v_{1} \ldots, v_{n}\right)$ for all standard $v_{1} \ldots, v_{n}$. The set $P=$
$\left\{\left\langle v_{1}, \ldots, v_{n}\right\rangle \in S \mid \phi\left(v_{1}, \ldots, v_{n}\right)\right\}$ exists by the Separation Principle of ZF, and has the required property.

This result has twofold importance:

- The meaning of every predicate that for standard inputs is defined by a $Q_{1}^{\text {st }} x_{1} \ldots Q_{m}^{\text {st }} x_{m} \psi$ formula with standard parameters is automatically extended to all inputs, where it it given by the $\in$-formula $Q_{1} x_{1} \ldots Q_{m} x_{m} \psi$.
- Standardization holds for all $\in$-formulas with additional predicate symbols, as long as all these additional predicates are defined by $\Delta^{\text {st }}$ formulas with standard parameters.
In BST all st- $\epsilon$-formulas are equivalent to $\boldsymbol{\Delta}^{\text {st }}$ formulas (see Kanovei and Reeken [24], Theorem 3.2.3). In SPOT the equivalence is true only for certain classes of formulas, but they include definitions of all the basic concepts of the Calculus and much beyond.

We recall that $h \in \mathbb{R}$ is infinitesimal iff $0<|h|<\frac{1}{n}$ holds for all standard $n \in \mathbb{N}, n>0$. We use $\forall^{\text {in }}$ and $\exists^{\text {in }}$ for quantifiers ranging over infinitesimals and 0 . The basic concepts of the Calculus have infinitesimal definitions that involve a single alternation of such quantifiers. The following proposition strengthens a result in Vopěnka [41], p. 148. It shows that the usual infinitesimal definitions of Calculus concepts are $\boldsymbol{\Delta}^{\text {st }}$. The variables $x, y$ range over $\mathbb{R}$ and $m, n, \ell$ range over $\mathbb{N} \backslash\{0\}$.
Proposition 2.6. In SPOT the following is true: Let $\phi(x, y)$ be an $\in$-formula with arbitrary parameters. Then $\quad \forall^{\text {in }} h \exists^{\text {in }} k \phi(h, k) \longleftrightarrow$

$$
\forall^{\text {st }} m \exists^{\text {st }} n \forall x[|x|<1 / n \rightarrow \exists y(|y|<1 / m \wedge \phi(x, y))] .
$$

By duality, we also have: $\quad \exists^{\text {in }} h \forall^{\text {in }} k \phi(h, k) \longleftrightarrow$

$$
\exists^{\text {st }} m \forall^{\text {st }} n \exists x[|x|<1 / n \wedge \forall y(|y|<1 / m \rightarrow \phi(x, y))] .
$$

Proof. The formula $\forall^{\mathbf{i n}} h \exists^{\mathbf{i n}} k \phi(h, k)$ means:

$$
\forall x\left[\forall^{\text {st }} n(|x|<1 / n) \rightarrow \exists y \forall^{\text {st }} m(|y|<1 / m \wedge \phi(x, y))\right]
$$

where we assume that the variables $m, n$ do not occur freely in $\phi(x, y)$. Using Countable Idealization (Lemma 2.2), we rewrite this as

$$
\forall x\left[\forall^{\text {st }} n(|x|<1 / n) \rightarrow \forall^{\text {st }} m \exists y \forall \ell \leq m(|y|<1 / \ell \wedge \phi(x, y))\right]
$$

We now use the observation that $\forall \ell \leq m(|y|<1 / \ell)$ is equivalent to $|y|<1 / m$, and the rules $\left(\alpha \rightarrow \forall^{\text {st }} v \beta\right) \longleftrightarrow \forall^{\text {st }} v(\alpha \rightarrow \beta)$ and $\left(\forall^{\text {st }} v \beta \rightarrow \alpha\right) \longleftrightarrow \exists^{\text {st }} v(\beta \rightarrow \alpha)$, valid assuming that $v$ is not free in $\alpha$ (note that Transfer implies $\exists n \operatorname{st}(n)$ ). This enables us to rewrite the preceding formula as follows:

$$
\forall x \forall^{\text {st }} m \exists^{\text {st }} n[|x|<1 / n \rightarrow \exists y(|y|<1 / m \wedge \phi(x, y))] .
$$

After exchanging the order of the first two universal quantifiers, we obtain the formula

$$
\left.\forall^{\text {st }} m \forall x \exists^{\text {st }} n[|x|<1 / n \rightarrow \exists y|y|<1 / m \wedge \phi(x, y))\right]
$$

to which we apply (the dual form of) Countable Idealization to get

$$
\forall^{\mathbf{s t}} m \exists^{\text {st }} n \forall x \exists \ell \leq n[|x|<1 / \ell \rightarrow \exists y(|y|<1 / m \wedge \phi(x, y))]
$$

After rewriting $\exists \ell \leq n[|x|<1 / \ell \rightarrow \ldots]$ as $[\forall \ell \leq n(|x|<1 / \ell) \rightarrow \ldots]$ and replacing $\forall \ell \leq n(|x|<1 / \ell)$ by $|x|<1 / n$, we obtain

$$
\forall^{\text {st }} m \exists^{\text {st }} n \forall x[|x|<1 / n \rightarrow \exists y(|y|<1 / m \wedge \phi(x, y))]
$$

proving the proposition.
2.2. Mathematics in SPOT. We give some examples to illustrate how infinitesimal analysis works in SPOT.

Example 2.7. If $F$ is a standard real-valued function on an open interval $(a, b)$ in $\mathbb{R}$ and $a, b, c, d$ are standard real numbers with $c \in$ $(a, b)$, we can define

$$
\begin{equation*}
F^{\prime}(c)=d \longleftrightarrow \forall^{\mathbf{i n}} h \exists^{\mathbf{i n}} k\left(h \neq 0 \rightarrow \frac{F(c+h)-F(c)}{h}=d+k\right) \tag{1}
\end{equation*}
$$

Let $\Phi(F, c, d)$ be the formula on the right side of the equivalence in (1). Lemma 2.5 establishes that the formula $\Phi$ is equivalent to a $\Delta^{\text {st }}$ formula, and $\phi(F, c, d)$ provided by the proof of Lemma 2.5 is easily seen to be equivalent to the standard $\varepsilon-\delta$ definition of derivative. For any standard $F$, the set $F^{\prime}=\{\langle c, d\rangle \mid \phi(F, c, d)\}$ is standard; it is the derivative function of $F$.

Proposition [2.6 generalizes straightforwardly to all formulas that have the form $\mathrm{A} h_{1} \ldots \mathrm{~A} h_{n} \mathrm{E} k_{1} \ldots \mathrm{E} k_{m} \phi\left(h_{1}, \ldots, k_{1}, \ldots, v_{1}, \ldots\right)$ or $\mathrm{E} h_{1} \ldots \mathrm{E} h_{n} \mathrm{~A} k_{1} \ldots \mathrm{~A} k_{m} \phi\left(h_{1}, \ldots, k_{1}, \ldots, v_{1}, \ldots\right)$ where each A is either $\forall$ or $\forall^{\text {in }}$, and each E is either $\exists$ or $\exists^{\text {in }}$. All such formulas are equivalent to $\Delta^{\text {st }}$ formulas.

Formulas of the form $\mathrm{Q}_{1} h_{1} \ldots \mathrm{Q}_{n} h_{n} \phi\left(h_{1}, \ldots, h_{n}, v_{1}, \ldots, v_{k}\right)$ where each $Q$ is either $\forall$ or $\forall^{\text {in }}$ or $\exists$ or $\exists^{\text {in }}$, but all quantifiers over infinitesimals are of the same kind (all existential or all universal), are also $\boldsymbol{\Delta}^{\text {st }}$. As an example, $\exists^{\text {in }} h \forall y \exists^{\mathbf{i n}} k \phi(h, k, x, y)$ is equivalent to

$$
\exists h \forall y \exists k \forall^{\text {st }} m \forall^{\text {st }} n(|h|<1 / m \wedge|k|<1 / n \wedge \phi(h, k, x, y)) .
$$

The two quantifiers over standard elements of $\mathbb{N}$ can be replaced by a single one:

$$
\exists h \forall y \exists k \forall^{\text {st }} m(|h|<1 / m \wedge|k|<1 / m \wedge \phi(h, k, x, y)),
$$

and then moved to the front using Countable Idealization.

Klein and Fraenkel proposed two benchmarks for a useful theory of infinitesimals (see Kanovei et al. [22]):

- a proof of the Mean Value Theorem by infinitesimal techniques;
- a definition of the definite integral in terms of infinitesimals.

The theory SPOT easily meets these criteria. The usual nonstandard proof of the Mean Value Theorem (Robinson [31], Keisler [26, 27]) uses Standard Part and Transfer, and is easily carried out in SPOT. The familiar infinitesimal definition of the Riemann integral for standard bounded functions on a standard interval $[a, b]$ also makes sense in SPOT and can be expressed by a $\Delta^{\text {st }}$ formula. In the next example we outline a treatment inspired by Keisler's use of hyperfinite Riemann sums in [27].

## Example 2.8. Riemann Integral.

We fix a positive infinitesimal $h$ and the corresponding "hyperfinite time line" $\mathbb{T}=\left\{t_{i} \mid i \in \mathbb{Z}\right\}$ where $t_{i}=i \cdot h$. Let $f$ be a standard realvalued function continuous on the standard interval $[a, b]$. Let $i_{a}, i_{b}$ be such that $i_{a} \cdot h-h<a \leq i_{a} \cdot h$ and $i_{b} \cdot h<b \leq i_{b} \cdot h+h$. Then

$$
\begin{equation*}
\int_{a}^{b} f(t) d t=\operatorname{sh}\left(\sum_{i=i_{a}}^{i_{b}} f\left(t_{i}\right) \cdot h\right) . \tag{2}
\end{equation*}
$$

It is easy to show that the value of the integral does not depend on the choice of $h$. We thus have, for standard $f, a, b, r: \quad \int_{a}^{b} f(t) d t=r \quad$ iff $\forall^{\mathbf{i n}} h \exists^{\mathbf{i n}} k\left(\sum_{i=i_{a}}^{i_{b}} f\left(t_{i}\right) \cdot h=r+k\right)$ iff $\exists^{\mathbf{i n}} h \exists^{\mathbf{i n}} k\left(\sum_{i=i_{a}}^{i_{b}} f\left(t_{i}\right) \cdot h=r+k\right)$.

The formulas are of the form AE and EE respectively, and therefore equivalent to $\Delta^{\text {st }}$ formulas.

The approach generalizes easily to the Riemann integral of bounded functions on $[a, b]$. We say that $\mathcal{T}_{h}=\left\langle t_{i}^{\prime}\right\rangle_{i=i_{a}}^{i_{b}}$ is an $h$ - tagging on $[a, b]$ if $i \cdot h \leq t_{i}^{\prime} \leq(i+1) \cdot h$ for all $i=i_{a}, \ldots, i_{b}-1$ and $i_{b} \cdot h \leq t_{i_{b}}^{\prime} \leq b$. Then for standard $f, a, b, r$

- $f$ is Riemann integrable on $[a, b]$ and $\int_{a}^{b} f(x) d x=r$ iff
- $\forall^{\mathrm{in}} h \forall \mathcal{T}_{h} \exists^{\mathrm{in}} k\left(\sum_{i=i_{a}}^{i_{b}} f\left(t_{i}^{\prime}\right) \cdot h=r+k\right)$ iff
- $\exists^{\text {in }} h \forall \mathcal{T}_{h} \exists^{\text {in }} k\left(\sum_{i=i_{a}}^{i_{b}} f\left(t_{i}^{\prime}\right) \cdot h=r+k\right)$.

These formulas are again equivalent to $\boldsymbol{\Delta}^{\text {st }}$ formulas (the first one is of the form AAE and in the second one both quantifiers over standard sets are existential).

The tools available in SPOT enable nonstandard definitions and proofs in parts of mathematics that go well beyond the Calculus.

Example 2.9. Fréchet Derivative. Given standard normed vector spaces $V$ and $W$, a standard open subset $U$ of $V$, a standard function $f: U \rightarrow W$, a standard bounded linear operator $A: V \rightarrow W$ and a standard $x \in U ; A$ is the Fréchet derivative of $f$ at $x \in U$ iff

$$
\forall z \forall^{\text {in }} h \exists^{\text {in }} k\left(\|z\|_{V}=h>0 \rightarrow \frac{\|f(x+z)-f(x)-A \cdot z\|_{W}}{\|z\|_{V}}=k\right) .
$$

This definition is equivalent to a $\Delta^{\text {st }}$ formula.
In Section 6] we show that Standardization for arbitrary formulas with standard parameters can be added to SPOT and the resulting theory is still conservative over ZF. This result enables one to dispose of any concerns about the form of the defining formula.

## 3. Theory SCOT and Lebesgue measure

We recall (see Section 1.3) that SCOT is SPOT + ADC + SN + $\mathbf{C C}$, where the principle $\mathbf{C C}$ of Countable st- $\in$-Choice postulates the following.

CC Let $\phi(u, v)$ be an st- $\epsilon$-formula with arbitrary parameters. Then $\forall^{\text {st }} n \in \mathbb{N} \exists x \phi(n, x) \rightarrow \exists f\left(f\right.$ is a function $\wedge \forall^{\text {st }} n \in \mathbb{N} \phi(n, f(n))$.

The set $\mathbb{N}$ can be replaced by any standard countable set $A$. We consider also the principle $\mathbf{S C}$ of Countable Standardization.

SC (Countable Standardization) Let $\psi(v)$ be an $\mathbf{s t}$ - $\epsilon$-formula with arbitrary parameters. Then

$$
\exists^{\text {st }} S \forall^{\text {st }} n(n \in S \longleftrightarrow n \in \mathbb{N} \wedge \psi(n)) .
$$

Lemma 3.1. The theory $\mathbf{S P O T}+\mathbf{C C}$ proves $\mathbf{S C}$.
Proof. Let $\phi(n, x)$ be the formula " $(\psi(n) \wedge x=0) \vee(\neg \psi(n) \wedge x=1)$ ". If $f$ is a function provided by $\mathbf{C C}$, let $A=\{n \in \mathbb{N} \mid f(n)=0\}$. By SP there is a standard set $S$ such that, for all standard $n \in \mathbb{N}, n \in S$ iff $n \in A$ iff $\psi(n)$ holds.

We introduce an additional principle $\mathbf{C C}{ }^{\text {st }}$.
CC ${ }^{\text {st }}$ Let $\phi(u, v)$ be an st- $\in$-formula with arbitrary parameters. Then
$\forall^{\text {st }} n \in \mathbb{N} \exists^{\text {st }} x \phi(n, x) \rightarrow \exists^{\text {st }} F\left(F\right.$ is a function $\wedge \forall^{\text {st }} n \in \mathbb{N} \phi(n, F(n))$.
The principle $\mathbf{C C}_{\mathbb{R}}^{\text {st }}$ is obtained from $\mathbf{C C}{ }^{\text {st }}$ by restricting the range of the variable $x$ to $\mathbb{R}$.

Lemma 3.2. The theory $\mathbf{S P O T}+\mathbf{C C}$ proves $\mathbf{C C}_{\mathbb{R}}^{\text {st }}$.

Proof. First use the principle $\mathbf{C C}$ to obtain a function $f: \mathbb{N} \rightarrow \mathbb{R}$ such that $\forall^{s t} n \in \mathbb{N}(f(n) \in \mathbb{R} \wedge \boldsymbol{s t}(f(n)) \wedge \phi(n, f(n))$. Next define a relation $r \subseteq \mathbb{N} \times \mathbb{N}$ by $\langle n, m\rangle \in r$ iff $m \in f(n)$. By Lemma 3.1, SC holds. By SC there is a standard $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $\langle n, m\rangle \in R$ iff $\langle n, m\rangle \in r$ holds for all standard $\langle n, m\rangle$. Now define $F: \mathbb{N} \rightarrow \mathbb{R}$ by $F(n)=\{m \mid\langle n, m\rangle \in R\}$. The function $F$ is standard and, for every standard $n$, the sets $F(n)$ and $f(n)$ have the same standard elements. As they are both standard, it follows by Transfer that $F(n)=f(n)$.

The full principle $\mathbf{C C}^{\text {st }}$ can conservatively be added to SCOT; see Proposition 5.6.

A useful consequence of $\mathbf{S C}$ is the ability to carry out external induction.

Lemma 3.3. (External Induction) Let $\phi(v)$ be an $\mathbf{s t}-\in$-formula with arbitrary parameters. Then $\mathbf{S P O T}+\mathbf{S C}$ proves the following:

$$
\left[\phi(0) \wedge \forall^{\text {st }} n \in \mathbb{N}(\phi(n) \rightarrow \phi(n+1)) \rightarrow \forall^{\text {st }} n \phi(n)\right]
$$

Proof. SC yields a standard set $S \subseteq \mathbb{N}$ such that $\forall^{\text {st }} n \in \mathbb{N}(n \in S \longleftrightarrow$ $\phi(n))$. We have $0 \in S$ and $\forall^{\text {st }} n \in \mathbb{N}(n \in S \rightarrow n+1 \in S)$. Then $\forall n \in \mathbb{N}(n \in S \rightarrow n+1 \in S)$ by Transfer, and $S=\mathbb{N}$ by induction. Hence $\forall^{\text {st }} n \in \mathbb{N} \phi(n)$ holds.

In Example 3.6 it is convenient to use the language of external collections. Let $\phi(v)$ be an $\mathbf{s t}$ - $\in$-formula with arbitrary parameters. We use dashed curly braces to denote the external collection $!x \in A \mid \phi(x) \vdots$. We emphasize that this is merely a matter of convenience; writing $z \in\{x \in A \mid \phi(x)$; is just another notation for $\phi(z)$.

Standardization in BST implies the existence of a standard set $S$ such that $\forall^{\text {st }} z\left(z \in S \longleftrightarrow z \in{ }^{\prime} x \in A \mid \phi(x) \grave{\prime}\right)$. We do not have Standardization over uncountable sets in SCOT', but one important case can be proved.

Lemma 3.4. Let $\phi(v)$ be an $\mathbf{s t}-\epsilon$-formula with arbitrary parameters. Then SCOT proves that inf ${ }^{\text {st }} \mathfrak{i} r \in \mathbb{R} \mid \phi(r)$; exists.

The notation indicates the greatest standard $s \in \mathbb{R}$ such that $s \leq r$ for all standard $r$ with the property $\phi(r)(+\infty$ if there is no such $r)$.
Proof. Consider $\mathbf{S}=\left\{́ q \in \mathbb{Q} \mid \exists^{\text {st }} r \in \mathbb{R}(q \geq r \wedge \phi(r)) \vdots\right.$. As $\mathbb{Q}$ is countable, the principle $\mathbf{S C}$ implies that there is a standard set $S$ such that $\forall q \in \mathbb{Q}(q \in S \longleftrightarrow q \in \mathbf{S})$. Therefore $\inf S$ exists $(+\infty$ if $S=\emptyset)$ and it is what is meant above by $\inf ^{\text {st }}$ ! $r \in \mathbb{R} \mid \phi(r) \vdots$.

We give two examples of mathematics in SCOT.

Example 3.5. Peano Existence Theorem. Peano's Theorem asserts that every first-order differential equation of the form $y^{\prime}=f(x, y)$ has a solution (not necessarily a unique one) satisfying the initial condition $y(0)=0$, under the assumption that $f$ is continuous in a neighborhood of $\langle 0,0\rangle$. The infinitesimal proof begins by constructing the sequences

$$
\begin{gathered}
x_{0}=0, x_{k+1}=x_{k}+h \text { where } h>0 \text { is infinitesimal; } \\
y_{0}=0, y_{k+1}=y_{k}+h \cdot f\left(x_{k}, y_{k}\right) .
\end{gathered}
$$

One then shows that there is $N \in \mathbb{N}$ such that $x_{k}, y_{k}$ are defined for all $k \leq N, a=\operatorname{st}\left(x_{N}\right)>0$, and for some standard $M>0$, $\left|y_{k}\right| \leq M \cdot a$ holds for all $k \leq N$. The desired solution is a standard function $Y:[0, a] \rightarrow \mathbb{R}$ such that for all standard $x \in[0, a]$, if $x \approx x_{k}$, then $Y(x) \approx y_{k}$. On the face of it one needs Standardization over $\mathbb{R}$ to obtain this function, but in fact $\mathbf{S C}$ suffices. Consider the countable set $A=(\mathbb{Q} \times \mathbb{Q}) \cap([0, a] \times[-M \cdot a, M \cdot a])$. By SC, there is a standard $Z \subseteq A$ such that for all standard $\langle x, y\rangle,\langle x, y\rangle \in Z$ iff $\exists k \leq N\left(x \approx x_{k} \wedge\left(y \approx y_{k} \vee y \geq y_{k}\right)\right.$. Define a standard function $Y_{0}$ on $\overline{\mathbb{Q}} \cap[0, a]$ by $Y_{0}(x)=\inf \{y \mid\langle x, y\rangle \in Z\}$. It is easy to verify that $Y_{0}$ is continuous on $\mathbb{Q} \cap[0, a]$ and that its extension $Y$ to a continuous function on $[0, a]$ is the desired solution.

Example 3.6. Lebesgue measure. In a seminal paper [29] Loeb introduced measures on the external power set of ${ }^{*} \mathbb{R}$ which became known as Loeb measures, and used them to construct the Lebesgue measure on $\mathbb{R}$. Substantial use of external collections is outside the scope of this paper (see Subsection 8.7), but it is possible to eliminate the intermediate step and give an infinitesimal definition à la Loeb of the Lebesgue measure in internal set theory. We outline here how to construct the Lebesgue outer measure on $\mathbb{R}$ in SCOT.

Let $\mathbb{T}$ be a hyperfinite time line (see Example (2.8) and let $E \subseteq \mathbb{R}$ be standard. A finite set $A \subseteq \mathbb{T}$ covers $E$ if

$$
\forall t \in \mathbb{T}\left(\exists^{\text {st }} x \in E(t \approx x) \rightarrow t \in A\right)
$$

We define $\boldsymbol{\mu}$ by setting

$$
\begin{equation*}
\boldsymbol{\mu}(E)=\inf ^{\text {st }} \stackrel{r}{ } \quad r \in \mathbb{R}|r \approx| A \mid \cdot h \text { for some } A \text { that covers } E \tag{3}
\end{equation*}
$$

The collection whose infimum needs to be taken is external, but the existence of the infimum is justified by Lemma 3.4. It is easy to see that the value of $\boldsymbol{\mu}(E)$ is independent of the choice of the infinitesimal $h$ in the definition of $\mathbb{T}$. Thus the external function $\boldsymbol{\mu}$ can be defined for standard $E \subseteq \mathcal{P}(\mathbb{R})$ by an $\mathbf{s t}$ - $\epsilon$-formula with no parameters (preface
the formula on the right side of (3) by $\forall^{\text {in }} h$ or $\exists^{\text {in }} h$ ). The principle SN (see Subsection 1.3 and Section (6) yields a standard function $m$ on $\mathcal{P}(\mathbb{R})$ such that $m(E)=\boldsymbol{\mu}(E)$ for all standard $E \subseteq \mathbb{R}$. We prove that $m$ is $\sigma$-subadditive.

Let $E=\bigcup_{n=0}^{\infty} E_{n}$ where $E$ and the sequence $\left\langle E_{n} \mid n \in \mathbb{N}\right\rangle$ are standard. If $\sum_{n=0}^{\infty} m\left(E_{n}\right)=+\infty$ the claim is trivial, so we assume that $m\left(E_{n}\right)=r_{n} \in \mathbb{R}$ for all $n$. Fix a standard $\varepsilon>0$. For every standard $n \in \mathbb{N}$ there exists $A$ such that $\phi(n, A)$ : " $A$ covers $E_{n} \wedge$ $|A| \cdot h<r_{n}+\varepsilon / 2^{n+1}$ " holds. By Countable st- $\epsilon$-Choice there is a sequence $\left\langle A_{n} \mid n \in \mathbb{N}\right\rangle$ such that for all standard $n \phi\left(n, A_{n}\right)$ holds. By Countable Idealization ("Overspill") there is a nonstandard $\nu \in \mathbb{N}$ such that $\left|A_{n}\right| \cdot h<r_{n}+\varepsilon / 2^{n+1}$ holds for all $n \leq \nu$. We let $A=\bigcup_{n=0}^{\nu} A_{n}$. Clearly $A$ is finite and covers $E$. Thus for $r=\operatorname{sh}(|A| \cdot h)$ we obtain $m(E) \leq r$ and

$$
|A| \cdot h \leq \sum_{n=0}^{\nu}\left|A_{n}\right| \cdot h<\sum_{n=0}^{\nu} r_{n}+\varepsilon .
$$

Since the sequence $\sum_{n=0}^{\infty} r_{n}$ converges, we have $\operatorname{sh}\left(\Sigma_{n=0}^{\nu} r_{n}\right)=\Sigma_{n=0}^{\infty} r_{n}$ and $m(A) \leq \sum_{n=0}^{\infty} r_{n}+\varepsilon$. As this is true for all standard $\varepsilon>0$, we conclude that $m(E) \leq \Sigma_{n=0}^{\infty} r_{n}=\sum_{n=0}^{\infty} m\left(E_{n}\right)$.

For closed intervals $[a, b], m([a, b])=b-a$ : Compactness of $[a, b]$ implies that $\forall t \in \mathbb{T} \cap[a, b] \exists^{\text {st }} x \in E(t \approx x)$. Thus if $A$ covers $[a, b]$ then $A \supseteq \mathbb{T} \cap[a, b]$; and for $A=\mathbb{T} \cap[a, b]$ one sees easily that $|A| \cdot h \approx$ $(b-a)$. With more work, one can show that $m(E)$ coincides with the conventionally defined Lebesgue outer measure of $E$ for all standard $E \subseteq \mathbb{R}$. See Hrbacek [17] Section 3 for more details and other equivalent nonstandard definitions of the Lebesgue outer measure $3^{3}$ One can define Lebesgue measurable sets from $m$ in the usual way. One can also define Lebesgue inner measure for standard $E$ by

$$
\begin{gathered}
\mu^{-}(E)=\sup ^{\text {st }}\{r \in \mathbb{R}|r \approx| A \mid \cdot h \text { for some } A \text { such that } \\
\\
\forall t \in \mathbb{T}\left(t \in A \rightarrow \exists^{\text {st }} x \in E(t \approx x)\right) \vdots
\end{gathered}
$$

and prove that a standard bounded $E \subseteq \mathbb{R}$ is Lebesgue measurable iff $m(E)=m^{-}(E)$, and the common value is the Lebesgue measure of $E$; see Hrbacek [18].

[^3]
## 4. Conservativity of SPOT over ZF

In this section we apply forcing techniques to prove conservativity of SPOT over ZF.

Theorem 4.1. The theory SPOT is a conservative extension of ZF: If $\theta$ is an $\in$-sentence, then ( $\mathbf{S P O T} \vdash \theta$ ) implies that $(\mathbf{Z F} \vdash \theta)$.

Theorem 4.1 = Theorem $\mathbf{A}$ is an immediate consequence of the following proposition.

Proposition 4.2. Every countable model $\mathfrak{M}=\left(M, \in^{\mathfrak{M}}\right)$ of $\mathbf{Z F}$ has a countable extension $\mathfrak{M}^{*}=\left(M^{*}, \in^{*}\right.$, st) to a model of SPOT in which $M$ is the class of all standard sets.

Proof of Theorem 4.1. Suppose SPOT $\vdash \theta$ but ZF $\nvdash \theta$, where $\theta$ is an $\epsilon$-sentence. Then the theory $\mathbf{Z F}+\neg \theta$ is consistent, therefore it has a countable model $\mathfrak{M}$, by Gödel's Completeness Theorem. Using Proposition 4.2 one obtains its extension $\mathfrak{M}^{*} \vDash$ SPOT, so in particular $\mathfrak{M}^{*} \vDash \theta$ and, by Transfer in $\mathfrak{M}^{*}, \mathfrak{M} \vDash \theta$. This is a contradiction.

The rest of this section is devoted to the proof of Proposition 4.2,
4.1. Forcing according to Enayat and Spector. We combine the forcing notion used by Enayat [8] to construct end extensions of models of arithmetic, with the one used by Spector in [38] to produce extended ultrapowers of models $\mathfrak{M}$ of $\mathbf{Z F}$ by an ultrafilter $\mathcal{U} \in \mathfrak{M}$.

In this subsection we work in $\mathbf{Z F}$, define our forcing notion and prove its basic properties. The next subsection deals with generic extensions of countable models of $\mathbf{Z F}$ and the resulting extended ultrapowers. The general reference to forcing and generic models in set theory is Jech [21].

The set of all natural numbers is denoted $\mathbb{N}$ and letters $m, n, k, \ell$ are reserved for variables ranging over $\mathbb{N}$. The index set over which the ultrapowers will eventually be constructed is denoted $I$. In this section we assume $I=\mathbb{N}$. A subset $p$ of $\mathbb{N}$ is called unbounded if $\forall m \exists n \in$ $p(n \geq m)$ and bounded if it is not unbounded. Of course unbounded is the same as infinite, and bounded is the same as finite. We use this terminology with a view to Section 7, where the construction is generalized to $I=\mathcal{P}^{\text {fin }}(A)$ for any infinite set $A$. The notation $\forall^{\text {aad }} i \in p$ (for almost all $i \in p$ ) means $\forall i \in p \backslash c$ for some bounded $c$.

As usual, the symbol $\mathbb{V}$ denotes the universe of all sets, and $V_{\alpha}(\alpha$ ranges over ordinals) are the ranks of the von Neumann cumulative hierarchy. We let $\mathbb{F}$ be the class of all functions with domain $I$. The notation $\emptyset_{k}$ stands for the $k$-tuple $\langle\emptyset, \ldots, \emptyset\rangle$.

Definition 4.3. Let $\mathbb{P}=\{p \subseteq I \mid p$ is unbounded $\}$. For $p, p^{\prime} \in \mathbb{P}$ we say that $p^{\prime}$ extends $p$ (notation: $p^{\prime} \leq p$ ) iff $p^{\prime} \subseteq p$.

Let $\mathbb{Q}=\left\{q \in \mathbb{F} \mid \exists k \in \mathbb{N} \forall i \in I\left(q(i) \subseteq \mathbb{V}^{k} \wedge q(i) \neq \emptyset\right)\right\}$. The number $k$ is the rank of $q$. We note that $q(i)$ for each $i \in I$, and $q$ itself, are sets, but $\mathbb{Q}$ is a proper class. We let $\overline{1}=q$ where $q(i)=\{\emptyset\}$ for all $i \in I ; \overline{1}$ is the only $q \in \mathbb{Q}$ of $\operatorname{rank} 0$.

The forcing notion $\mathbb{H}$ is defined as follows: $\mathbb{H}=\mathbb{P} \times \mathbb{Q}$ and $\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathbb{H}$ extends $\langle p, q\rangle \in \mathbb{H}$ (notation: $\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\langle p, q\rangle$ ) iff $p^{\prime}$ extends $p$, $\operatorname{rank} q^{\prime}=$ $k^{\prime} \geq k=\operatorname{rank} q$, and for almost all $i \in p^{\prime}$ and all $\left\langle x_{0}, \ldots, x_{k^{\prime}-1}\right\rangle \in q^{\prime}(i)$, $\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)$. Every $\langle p, q\rangle \in \mathbb{H}$ extends $\langle p, \overline{1}\rangle$.

The poset $\mathbb{P}$ is used to force a generic filter over $I$ as in Enayat [8], and $\mathbb{H}$ forces an extended ultrapower of $\mathbb{V}$ by the generic filter $\mathcal{U}$ forced by $\mathbb{P}$. It is a modification of the forcing notion from Spector [38], with the difference that in [38] $\mathcal{U}$ is not forced but assumed to be a given ultrafilter in $\mathbb{V}$.

A set $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is $p^{\prime} \in D$ such that $p^{\prime}$ extends $p$. We note that for any set $S \subseteq I$, the set $D_{S}=\{p \in \mathbb{P} \mid p \subseteq S \vee p \subseteq I \backslash S\}$ is dense in $\mathbb{P}$.

Similarly, a class $E \subseteq \mathbb{H}$ is dense in $\mathbb{H}$ if for every $\langle p, q\rangle \in \mathbb{H}$ there is $\left\langle p^{\prime}, q^{\prime}\right\rangle \in E$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\langle p, q\rangle$.

The forcing language $\mathfrak{L}$ has a constant symbol $\check{z}$ for every $z \in \mathbb{V}$ (which we identify with $z$ when no confusion threatens), and a constant symbol $\dot{G}_{n}$ for each $n \in \mathbb{N}$. Given an $\in$-formula $\phi\left(w_{1}, \ldots, w_{r}, v_{1}, \ldots, v_{s}\right)$, we define the forcing relation $\langle p, q\rangle \Vdash \phi\left(\check{z}_{1}, \ldots, \check{z}_{r}, \dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ for $\langle p, q\rangle \in \mathbb{H}$ by meta-induction on the logical complexity of $\phi$. We use $\neg, \wedge$ and $\exists$ as primitives and consider the other logical connectives and quantifiers as defined in terms of these. Usually, we suppress the explicit listing in $\phi$ of the constant symbols $\check{z}$ for the elements of $\mathbb{V}$.

Definition 4.4. (Forcing relation.)
(1) $\langle p, q\rangle \Vdash \check{z}_{1}=\check{z}_{2}$ iff $z_{1}=z_{2}$.
(2) $\langle p, q\rangle \Vdash \check{z}_{1} \in \check{z}_{2}$ iff $z_{1} \in z_{2}$.
(3) $\langle p, q\rangle \Vdash \dot{G}_{n_{1}}=\dot{G}_{n_{2}}$ iff $\operatorname{rank} q=k>n_{1}, n_{2}$ and $\forall^{\mathbf{a a}} i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\left(x_{n_{1}}=x_{n_{2}}\right)$.
(4) $\langle p, q\rangle \Vdash \dot{G}_{n_{1}} \in \dot{G}_{n_{2}}$ iff $\operatorname{rank} q=k>n_{1}, n_{2}$ and $\forall^{\text {aa }} i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\left(x_{n_{1}} \in x_{n_{2}}\right)$.
(5) $\langle p, q\rangle \Vdash \dot{G}_{n}=\check{z}$ iff $\langle p, q\rangle \Vdash \check{z}=\dot{G}_{n}$ iff $\operatorname{rank} q=k>n$ and $\forall \forall^{\mathbf{a} \mathbf{a}} i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\left(x_{n}=z\right)$.
(6) $\langle p, q\rangle \Vdash \check{z} \in \dot{G}_{n}$ iff $\operatorname{rank} q=k>n$ and $\forall^{\text {aa }} i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\left(z \in x_{n}\right)$.
(7) $\langle p, q\rangle \Vdash \dot{G}_{n} \in \check{z} \mathrm{iff} \operatorname{rank} q=k>n$ and $\forall^{\mathbf{a x}} i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\left(x_{n} \in z\right)$.
(8) $\langle p, q\rangle \Vdash \neg \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ iff $\operatorname{rank} q=k>n_{1}, \ldots, n_{s}$ and there is no $\left\langle p^{\prime}, q^{\prime}\right\rangle$ extending $\langle p, q\rangle$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$.
(9) $\langle p, q\rangle \Vdash(\phi \wedge \psi)\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ iff
$\langle p, q\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ and $\langle p, q\rangle \Vdash \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$.
(10) $\langle p, q\rangle \Vdash \exists v \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, v\right)$ iff $\operatorname{rank} q=k>n_{1}, \ldots, n_{s}$ and for every $\left\langle p^{\prime}, q^{\prime}\right\rangle$ extending $\langle p, q\rangle$ there exist $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle$ extending $\left\langle p^{\prime}, q^{\prime}\right\rangle$ and $m \in \mathbb{N}$ such that $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \Vdash \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, \dot{G}_{m}\right)$.

Lemma 4.5. (Basic properties of forcing)
(1) If $\langle p, q\rangle \Vdash \phi$ and $\left\langle p^{\prime}, q^{\prime}\right\rangle$ extends $\langle p, q\rangle$, then $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi$.
(2) No $\langle p, q\rangle$ forces both $\phi$ and $\neg \phi$.
(3) Every $\langle p, q\rangle$ extends to $\left\langle p^{\prime}, q^{\prime}\right\rangle$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi$ or $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \neg \phi$.
(4) If $\langle p, q\rangle \Vdash \phi$ and $p^{\prime} \backslash p$ is bounded, then $\left\langle p^{\prime}, q\right\rangle \Vdash \phi$.

Proof. (1) - (3) are immediate from the definition of forcing and (4) can be proved by induction on the complexity of $\phi$.

The following proposition establishes a relationship between this forcing and ultrapowers.

Proposition 4.6. ("Eos's Theorem") Let $\phi\left(v_{1}, \ldots, v_{s}\right)$ be an $\in$-formula with parameters from $\mathbb{V}$.
Then $\langle p, q\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ iff $\operatorname{rank} q=k>n_{1}, \ldots, n_{s}$ and $\forall^{\mathbf{a a}} i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right)$.

Proof. For atomic formulas (cases (1) - (7)) the claim is immediate from the definition. Case (9) is also trivial (union of two bounded sets is bounded).

Case (8): Let $c=\left\{i \in p \mid \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \neg \phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right)\right\}$. We need to prove that $\langle p, q\rangle \Vdash \neg \phi$, iff $p \backslash c$ is bounded.

Assume that $\langle p, q\rangle \Vdash \neg \phi$ and $p \backslash c$ is unbounded. We let $p^{\prime}=p \backslash c$ and $q^{\prime}(i)=\left\{\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \mid \phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right)\right\}$ for $i \in p^{\prime}, q^{\prime}(i)=\left\{\emptyset_{k}\right\}$ for $i \in I \backslash p^{\prime}$. Then $\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathbb{H}$ extends $\langle p, q\rangle$ and, by the inductive assumption, $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi$, a contradiction.

Conversely, assume $\langle p, q\rangle \nVdash \neg \phi$ and $p \backslash c$ is bounded. Then there is $\left\langle p^{\prime}, q^{\prime}\right\rangle$ of rank $k^{\prime}$ extending $\langle p, q\rangle$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi$. By the inductive assumption, there is a bounded set $d$ such that

$$
\forall i \in\left(p^{\prime} \backslash d\right) \forall\left\langle x_{0}, \ldots, x_{k^{\prime}-1}\right\rangle \in q^{\prime}(i) \phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right) .
$$

But $(p \backslash c) \cup d$ is a bounded set, so there exist $i \in\left(p^{\prime} \cap c\right) \backslash d$. For such $i$ and $\left\langle x_{0}, \ldots, x_{k^{\prime}-1}\right\rangle \in q^{\prime}(i)$ one has both $\neg \phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right)$ and $\phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right)$, a contradiction.

Case (10):
Let $c=\left\{i \in p \mid \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \exists v \psi\left(x_{n_{1}}, \ldots, x_{n_{s}}, v\right)\right\}$. We need to prove that $\langle p, q\rangle \Vdash \exists v \psi$ iff $p \backslash c$ is bounded.

Assume that $\langle p, q\rangle \Vdash \exists v \psi$ and $p \backslash c$ is unbounded. We let $p^{\prime}=p \backslash c$ and $q^{\prime}(i)=\left\{\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \mid \neg \exists v \psi\left(x_{n_{1}}, \ldots, x_{n_{s}}, v\right)\right\}$ for $i \in p^{\prime}$; $q^{\prime}(i)=\left\{\emptyset_{k}\right\}$ for $i \in I \backslash p^{\prime}$. Then $\left\langle p^{\prime}, q^{\prime}\right\rangle$ extends $\langle p, q\rangle$ and, by the definition of $\Vdash$, there exist $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle$ extending $\left\langle p^{\prime}, q^{\prime}\right\rangle$ with $\operatorname{rank} q^{\prime \prime}=k^{\prime \prime}$, and $m<k^{\prime \prime}$ such that $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \Vdash \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, \dot{G}_{m}\right)$. By the inductive assumption, there is a bounded set $d$ such that

$$
\forall i \in\left(p^{\prime \prime} \backslash d\right) \forall\left\langle x_{0}, \ldots, x_{k^{\prime \prime}-1}\right\rangle \in q^{\prime \prime}(i) \psi\left(x_{n_{1}}, \ldots x_{n_{s}}, x_{m}\right)
$$

Hence

$$
\forall i \in\left(p^{\prime \prime} \backslash d\right) \forall\left\langle x_{0}, \ldots, x_{k^{\prime \prime}-1}\right\rangle \in q^{\prime \prime}(i) \exists v \psi\left(x_{n_{1}}, \ldots x_{n_{s}}, v\right) .
$$

But $i \in p^{\prime \prime} \backslash d$ implies $i \in p^{\prime}$; this contradicts the definition of $q^{\prime}$.
Assume that $p \backslash c$ is bounded. By the Reflection Principle in ZF there is a least von Neumann rank $V_{\alpha}$ such that for all $i \in c$ and all $\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)$ there exists $v \in V_{\alpha}$ such that $\psi\left(x_{n_{1}}, \ldots x_{n_{s}}, v\right)$. Let $\left\langle p^{\prime}, q^{\prime}\right\rangle$ be any condition extending $\langle p, q\rangle$ and let $k^{\prime}=\operatorname{rank} q^{\prime}$. We let $p^{\prime \prime}=p^{\prime} \cap c$ and

$$
\begin{aligned}
q^{\prime \prime}(i)= & \left\{\left\langle x_{0}, \ldots, x_{k^{\prime}-1}, x_{k^{\prime}}\right\rangle \mid\right. \\
& \left.\left\langle x_{0}, \ldots, x_{k^{\prime}-1}\right\rangle \in q^{\prime}(i) \wedge \psi\left(x_{n_{1}}, \ldots x_{n_{s}}, x_{k^{\prime}}\right) \wedge x_{k^{\prime}} \in V_{\alpha}\right\}
\end{aligned}
$$

for $i \in p^{\prime \prime}, q^{\prime \prime}(i)=\left\{\emptyset_{k^{\prime}}\right\}$ otherwise. Then $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle$ extends $\left\langle p^{\prime}, q^{\prime}\right\rangle$ and, by the inductive assumption, $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \Vdash \psi\left(G_{n_{1}}, \ldots, G_{n_{s}}, G_{k^{\prime}}\right)$. This proves that $\langle p, q\rangle \Vdash \exists v \psi$.

We observe that if $q$ is in $\mathbb{Q}$ and $\ell<k=\operatorname{rank} q$, then $q \upharpoonright \ell$ defined by $(q \upharpoonright \ell)(i)=\left\{\left\langle x_{0}, \ldots, x_{\ell-1}\right\rangle \mid \exists x_{\ell}, \ldots, x_{k-1}\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\right\}$ is in $\mathbb{Q}$.

Corollary 4.7. If $\operatorname{rank} q=k>n_{1}, \ldots, n_{s},\left\langle p^{\prime}, q^{\prime}\right\rangle$ extends $\langle p, q\rangle$ and $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$, then $\left\langle p^{\prime}, q^{\prime} \upharpoonright k\right\rangle$ extends $\langle p, q\rangle$ and $\left\langle p^{\prime}, q^{\prime} \upharpoonright k\right\rangle \Vdash$ $\phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$.

Lemma 4.8. Let $z \in \mathbb{V}$. For every $\langle p, q\rangle$ there exist $\left\langle p, q^{\prime}\right\rangle$ extending $\langle p, q\rangle$ and $m<k^{\prime}=\operatorname{rank} q^{\prime}$ such that $\left\langle p, q^{\prime}\right\rangle \Vdash \check{z}=\dot{G}_{m}$.

Proof. Let $q^{\prime}(i)=\left\{\left\langle x_{0}, \ldots, x_{k-1}, x_{k}\right\rangle \mid\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \wedge x_{k}=z\right\}$, and $m=k, k^{\prime}=k+1$.

We write $\langle p, q\rangle \Vdash \check{z}=\dot{G}_{m}$ as $\langle p, q\rangle \Vdash z=\dot{G}_{m}$, and $\langle p, q\rangle \Vdash \check{z} \in \dot{G}_{m}$ as $\langle p, q\rangle \Vdash z \in \dot{G}_{m}$. We say that $\langle p, q\rangle$ decides $\phi$ if $\langle p, q\rangle \Vdash \phi$ or $\langle p, q\rangle \Vdash \neg \phi$. The following lemma is needed for the proof that the extended ultrapower satisfies SP.

Lemma 4.9. For every $\langle p, q\rangle$ and $m<k=\operatorname{rank} q$ there is $\left\langle p^{\prime}, q^{\prime}\right\rangle$ that extends $\langle p, q\rangle$ and is such that for every $n \in \mathbb{N},\left\langle p^{\prime}, q^{\prime}\right\rangle$ decides $n \in \dot{G}_{m}$.

Proof. We first construct a sequence $\left\langle\left\langle p_{n}, q_{n}\right\rangle \mid n \in \mathbb{N}\right\rangle$ such that $\left\langle p_{0}, q_{0}\right\rangle=\langle p, q\rangle$ and for each $n, \operatorname{rank} q_{n}=k, p_{n+1} \subset p_{n},\left\langle p_{n+1}, q_{n+1}\right\rangle$ extends $\left\langle p_{n}, q_{n}\right\rangle$, and $\left\langle p_{n+1}, q_{n+1}\right\rangle$ decides $n \in \dot{G}_{m}$.

Given $p_{n}$, let $c=\left\{i \in p_{n} \mid \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q_{n}(i)\left(n \in x_{m}\right)\right\}$. If $c$ is unbounded, we let $p_{n+1}^{\prime}=c$ and $q_{n+1}=q_{n}$. Otherwise $p_{n} \backslash c$ is unbounded and we let $p_{n+1}^{\prime}=p_{n} \backslash c$ and $q_{n+1}(i)=\left\{\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in\right.$ $\left.q_{n}(i) \mid n \notin x_{m}\right\}$ for $i \in p_{n+1}^{\prime}, q_{n+1}(i)=\left\{\emptyset_{k}\right\}$ otherwise. We obtain $p_{n+1}$ from $p_{n+1}^{\prime}$ by omitting the least element of $p_{n+1}^{\prime}$. Proposition 4.6 implies that $\left\langle p_{n+1}, q_{n+1}\right\rangle$ decides $n \in \dot{G}_{m}$.

Let $i_{n}$ be the least element of $p_{n}$. We define $\left\langle p^{\prime}, q^{\prime}\right\rangle$ as follows: $i \in p^{\prime}$ iff $i \in p_{n}$ and $q^{\prime}(i)=q_{n}(i)$, where $i_{n} \leq i<i_{n+1} ; q^{\prime}(i)=\left\{\emptyset_{k}\right\}$ otherwise. It is clear from the construction and Proposition 4.6 that $\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathbb{H}$, it extends $\langle p, q\rangle$, and it decides $n \in \dot{G}_{m}$ for every $n \in \mathbb{N}$.
4.2. Extended ultrapowers. In this subsection we define the extended ultrapower of a countable model of $\mathbf{Z F}$ by a generic filter $\mathcal{U}$, prove some fundamental properties of this structure, and conclude that it is a model of SPOT.

We take Zermelo-Fraenkel set theory as our metatheory, but the proof employs very little of its powerful machinery. Subsection 8.5 explains how the proof given below can be converted into a finitistic proof.

We use $\omega$ for the set of natural numbers in the metatheory, and $r, s$ as variables ranging over $\omega$. A set $S$ is countable if there is a mapping of $\omega$ onto $S$.

Let $\mathfrak{M}=\left(M, \in^{\mathfrak{M}}\right)$ be a countable model of ZF. Concepts defined in Subsection 4.1 make sense in $\mathfrak{M}$ and all results of 4.1 hold in $\mathfrak{M}$. When $\mathfrak{M}$ is understood, we use the notation and terminology from 4.1 for the concepts in the sense of $\mathfrak{M}$; thus $\mathbb{N}$ for $\mathbb{N}^{\mathfrak{M}}$, "unbounded" for "unbounded in the sense of $\mathfrak{M} ", \mathbb{P}$ for $\mathbb{P}^{\mathfrak{M}}, \Vdash$ for " $\Vdash$ in the sense of $\mathfrak{M} "$, etc. The model $\mathfrak{M}$ need not be well-founded externally, and $\omega$ is isomorphic to an initial segment of $\mathbb{N}$ which may be proper.

Definition 4.10. $\mathcal{U} \subseteq M$ is a filter on $\mathbb{P}$ if
(1) $\mathfrak{M} \vDash$ " $p \in \mathbb{P}$ " for every $p \in \mathcal{U}$;
(2) If $p \in \mathcal{U}$ and $\mathfrak{M} \vDash$ " $p$ ' $\in \mathbb{P} \wedge p$ extends $p^{\prime \prime}$, then $p^{\prime} \in \mathcal{U}$;
(3) For eny $p_{1}, p_{2} \in \mathcal{U}$ there is $p \in \mathcal{U}$ such that $\mathfrak{M} \vDash$ " $p$ extends $p_{1} \wedge$ $p$ extends $p_{2}$ ".
A filter $\mathcal{U}$ on $\mathbb{P}$ is $\mathfrak{M}$-generic if for every $D \in M$ such that $\mathfrak{M} \vDash$ " $D$ is dense in $\mathbb{P}$ " there is $p \in \mathcal{U}$ for which $p \in^{\mathfrak{M}} D$.

Since $\mathbb{P}$ has only countably many dense subsets in $\mathfrak{M}$, $\mathfrak{M}$-generic filters are easily constructed by recursion. Let $\left\langle p_{r} \mid r \in \omega\right\rangle$ be an enumeration of $\mathbb{P}$ and $\left\langle D_{r} \mid r \in \omega\right\rangle$ be an enumeration of all dense subsets of $\mathbb{P}$ in $\mathfrak{M}$. Let $q_{0}=p_{0}$ and for each $s \in \omega$ let $q_{s+1}=p_{r}$ for the least $r$ such that $\mathfrak{M} \vDash$ " $p_{r}$ extends $q_{s} \wedge p_{r} \in \mathcal{D}_{s}$ ". Then let $\mathcal{U}=\left\{p \in M \mid \mathfrak{M} \vDash " p \in \mathbb{P} \wedge q_{s}\right.$ extends $p$ " for some $\left.s \in \omega\right\}$.
$\mathfrak{M}$-generic filters $\mathcal{G} \subseteq M \times M$ on $\mathbb{H}$ are defined and constructed analogously.
Lemma 4.11. If $\mathcal{G}$ is an $\mathfrak{M}$-generic filter on $\mathbb{H}$, then $\mathcal{U}=\{p \in \mathbb{P} \mid$ $\exists q\langle p, q\rangle \in \mathcal{G}\}$ is an $\mathfrak{M}$-generic filter on $\mathbb{P}$.

Proof. In $\mathfrak{M}$ : if $D$ is dense in $\mathbb{P}$, then $\{\langle p, q\rangle \mid p \in D \wedge q \in \mathbb{Q}\}$ is dense in $\mathbb{H}$.

We now define the extended ultrapower of $\mathfrak{M}$ by $\mathcal{U}$; we follow closely the presentation in Spector [38].

Let $\Omega=\{m \in M \mid \mathfrak{M} \vDash$ " $m \in \mathbb{N}$ " $\}$. We define binary relations =* and $\in^{*}$ on $\Omega$ as follows:
$m={ }^{*} n$ iff there exists $\langle p, q\rangle \in \mathcal{G}$ such that $\operatorname{rank} q=k>m, n$ and $\langle p, q\rangle \Vdash \dot{G}_{m}=\dot{G}_{n} ;$
$m \in^{*} n$ iff there exists $\langle p, q\rangle \in \mathcal{G}$ such that $\operatorname{rank} q=k>m, n$ and $\langle p, q\rangle \Vdash \dot{G}_{m} \in \dot{G}_{n}$.

It is easily seen from the definition of forcing and Proposition 4.6 that $=*$ is an equivalence relation on $\Omega$, and a congruence relation with respect to $\in^{*}$. We denote the equivalence class of $m \in \Omega$ in the relation $=^{*}$ by $G_{m}$, define $G_{m} \in^{*} G_{n}$ iff $m \in^{*} n$, and let $N=\left\{G_{m} \mid m \in \Omega\right\}$. The extended ultrapower of $\mathfrak{M}$ by $\mathcal{U}$ is the structure $\mathfrak{N}=\left(N, \in^{*}\right)$.

There is a natural embedding $j$ of $\mathfrak{M}$ into $\mathfrak{N}$ defined as follows: By Lemma 4.8 for every $z \in M$ there exist $\langle p, q\rangle \in \mathcal{G}$ and $m<\operatorname{rank} q$ such that $\langle p, q\rangle \Vdash z=\dot{G}_{m}$. We let $j(z)=G_{m}$ and often identify $j(z)$ with $z$. It is easy to see that the definition is independent of the choice of representative from $G_{m}$, and that $j$ is an embedding of $\mathfrak{M}$ into $\mathfrak{N}$.

Proposition 4.12. (The Fundamental Theorem of Extended Ultrapowers) Let $\phi\left(v_{1}, \ldots, v_{s}\right)$ be an $\in$-formula with parameters from $M$.
If $G_{n_{1}}, \ldots, G_{n_{s}} \in N$, then the following statements are equivalent:
(1) $\mathfrak{N} \vDash \phi\left(G_{n_{1}}, \ldots, G_{n_{s}}\right)$.
(2) There is some $\langle p, q\rangle \in \mathcal{G}$ such that $\langle p, q\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ holds in $\mathfrak{M}$.
(3) There exists some $\langle p, q\rangle \in \mathcal{G}$ with $\operatorname{rank} q=k>n_{1}, \ldots, n_{s}$ such that $\mathfrak{M} \vDash \forall i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right)$.
Proof. Statement (3) is just a reformulation of (2) using Proposition 4.6 plus the fact that if $d$ is bounded, then $\langle p, q\rangle \in \mathcal{G}$ implies $\langle(p \backslash d), q\rangle \in \mathcal{G}$. (Observe that $D=\left\{\left\langle p^{\prime}, q^{\prime}\right\rangle \mid\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\langle p, q\rangle \wedge p^{\prime} \leq(p \backslash d)\right\}$ is dense in $\langle p, q\rangle$; for the definition of "dense in" see the sentence preceding Lemma 5.3.)

The equivalence of (1) and (2) is the Forcing Theorem. It is proved as usual, by induction on the logical complexity of $\phi$. The cases $v_{1}=v_{2}$ and $v_{1} \in v_{2}$ follow immediately from the definitions of $=^{*}$ and $\in^{*}$, and the conjunction is immediate from (3) in the definition of a filter.

We consider next the case where $\phi$ is of the form $\neg \psi$. First assume that $\mathfrak{N} \vDash \neg \psi\left(G_{n_{1}}, \ldots, G_{n_{s}}\right)$. Lemma 4.5 (3), implies that there exists $\langle p, q\rangle \in \mathcal{G}$ such that either $\langle p, q\rangle \Vdash \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ or $\langle p, q\rangle \Vdash$ $\neg \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$. In the first case $\mathfrak{N} \vDash \psi\left(G_{n_{1}}, \ldots, G_{n_{s}}\right)$ by the inductive assumption; a contradiction. Hence $\langle p, q\rangle \Vdash \neg \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$.

Assume that $\mathfrak{N} \not \models \neg \psi\left(G_{n_{1}}, \ldots, G_{n_{s}}\right)$; then $\mathfrak{N} \vDash \psi\left(G_{n_{1}}, \ldots, G_{n_{s}}\right)$ and the inductive assumption yields $\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathcal{G}$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash$ $\psi\left(G_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$. There can thus be no $\langle p, q\rangle \in \mathcal{G}$ such that $\langle p, q\rangle \Vdash$ $\neg \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$.

Finally, we assume that $\phi\left(u_{1}, \ldots, v_{s}\right)$ is of the form $\exists w \psi\left(u_{1}, \ldots, v_{s}, w\right)$. If $\mathfrak{N} \vDash \phi\left(G_{n_{1}}, \ldots, G_{n_{s}}\right)$ then $\mathfrak{N} \vDash \psi\left(G_{n_{1}}, \ldots, G_{n_{s}}, G_{m}\right)$ for some $m \in$ $\Omega$. By the inductive assumption there is some $\langle p, q\rangle \in \mathcal{G}$ such that $\langle p, q\rangle \Vdash \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, \dot{G}_{m}\right)$ and hence, by the definition of forcing, $\langle p, q\rangle \Vdash \exists w \psi\left(G_{n_{1}}, \ldots, G_{n_{s}}, w\right)$.

Conversely, if $\langle p, q\rangle \in \mathcal{G}$ and $\langle p, q\rangle \Vdash \exists w \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, w\right)$, then by the definition of forcing there are $\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathcal{G}$ and $m \in \Omega$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, \dot{G}_{m}\right)$. By the inductive assumption, $\mathfrak{N} \vDash$ $\psi\left(G_{n_{1}}, \ldots, G_{n_{s}}, G_{m}\right)$ holds, and hence $\mathfrak{N} \vDash \phi\left(G_{n_{1}}, \ldots, G_{n_{s}}\right)$.
Corollary 4.13. The embedding $j$ is an elementary embedding of $\mathfrak{M}$ into $\mathfrak{N}$.

Corollary 4.14. The structure $\mathfrak{N}$ satisfies $\mathbf{Z F}$.
Proposition 4.15. The structure $\widehat{\mathfrak{N}}=\left(N, \in^{*}, M\right)$ satisfies the principles of Transfer, Nontriviality and Standard Part.

Proof. Transfer is Corollary 4.13.
Working in $\mathfrak{M}$, for every $\langle p, q\rangle$ with $\operatorname{rank} q=k$ define $q^{\prime}$ of $\operatorname{rank} k+1$ by $q^{\prime}(i)=\left\{\left\langle x_{0}, \ldots, x_{k-1}, x_{k}\right\rangle \mid\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \wedge x_{k}=i\right\}$ and
note that $\left\langle p, q^{\prime}\right\rangle$ extends $\langle p, q\rangle$. By $\mathfrak{M}$-genericity of $\mathcal{G}$ some such $\left\langle p, q^{\prime}\right\rangle$ belongs to $\mathcal{G}$. It is easily seen from the Fundamental Theorem that $G_{k}$ is an integer in $\mathfrak{N}$ and that $\mathfrak{N} \vDash " G_{k} \neq n$ " for all $n \in \Omega$. Hence $\mathbf{O}$ holds in $\widehat{\mathfrak{N}}$.

It remains to prove the Standard Part principle. Let $G_{m} \in N$. By Lemma 4.9 there is $\langle p, q\rangle \in \mathcal{G}$ which decides $n \in \dot{G}_{m}$ for all $n \in \Omega$. The set $E=\left\{n \in \Omega \mid\langle p, q\rangle \Vdash n \in \dot{G}_{m}\right\}$ is definable in $\mathfrak{M}$, hence there is $e \in M$ such that $\mathfrak{M} \vDash$ " $n \in e$ " iff $n \in E$ iff $\mathfrak{N} \vDash$ " $n \in G_{m}$ ". Thus $\widehat{\mathfrak{N}} \vDash " \operatorname{st}(e) \wedge \forall^{\text {st }} \nu \in \mathbb{N}\left(\nu \in e \longleftrightarrow \nu \in G_{m}\right)$ ".

This proves Proposition 4.2, and hence Theorem $\boldsymbol{A}$.

## 5. Conservativity of SCOT over ZF $c$

In this section we show that if $\mathbf{Z F}$ is replaced by $\mathbf{Z F} c$, the Standard Part principle can be strengthened to Countable st- $\epsilon$-Choice.

We recall that $\mathbf{Z F} c$ implies $\mathbf{A C C}$; this provides enough choice to prove that the ordinary ultrapower of a countable model $\mathfrak{M}$ of $\mathbf{Z F} c$ by an $\mathfrak{M}$-generic filter $\mathcal{U}$ on $\mathbb{P}$ satisfies Loś's Theorem and thus yields an elementary extension of $\mathfrak{M}$. A proof that every countable model $\mathfrak{M}=\left(M, \in^{\mathfrak{M}}\right)$ of $\mathbf{Z F} c$ has an extension to a model of SPOT $+\mathbf{S C}$ in which $M$ is the class of all standard sets can be obtained by a straightforward adaptation of the arguments in Enayat's paper [8], in particular, of the proofs of Theorems B and C there. Essentially, all one has to do is to replace countable models of second-order arithmetic by countable models of $\mathbf{Z F} c$. We followed this approach in an early version of the present paper.

Another proof of conservativity of SCOT over ZF $c$ was suggested to us by Kanovei in a private communication. Its basic idea is to use forcing to add to $\mathfrak{M}$ a mapping of $\omega_{1}$ onto $\mathbb{R}$ without adding any reals. In the resulting generic extension there are nonprincipal ultrafilters over $\mathbb{N}$, and one can take an ultrapower of $\mathfrak{M}$ by one of them to obtain an extension that satisfies SCOT. However, this method does not seem adequate for handling Idealization over uncountable sets in Section 7 .

We first outline the simplification of the forcing that is possible in the presence of $\mathbf{A C C}$, and then use it to prove that, assuming $\mathfrak{M}$ is a countable model of $\mathbf{Z F} c$, the structure $\widehat{\mathfrak{N}}$ from Proposition4.15 satisfies also CC and SN. The proof can be viewed as a warm-up for similar but more complex arguments of the following sections.

We work in ZFc. Given $I=\mathbb{N}$ and $q \in \mathbb{Q}$ of rank $k$, ACC guarantees the existence of a function $f$ such that $\forall i \in I(f(i) \in q(i))$; let $\widehat{q}(i)=\{f(i)\}$ where $f(i)=\left\langle f_{0}(i), \ldots, f_{k-1}(i)\right\rangle$. For any $\langle p, q\rangle \in \mathbb{H}$
the condition $\langle p, \widehat{q}\rangle$ extends $\langle p, q\rangle$. We could replace $\mathbb{Q}$ by $\widetilde{\mathbb{Q}}=\{\widetilde{q} \in$ $\left.\mathbb{F} \mid \exists k \forall i \in I\left(\widetilde{q}(i) \in \mathbb{V}^{k}\right)\right\}$. But there is no need at all for the symbols $\dot{G}_{k}, k \in \mathbb{N}$, and Spector's component $\mathbb{Q}$ of the forcing notion $\mathbb{H}$, if our forcing language allows names for all $f \in \mathbb{F}$ (see below for details).

On the other hand, Standardization and Countable st- $\in$-Choice, unlike Transfer and Idealization, deal with st- $\epsilon$-formulas, so we need to extend our definition of the forcing relation to such formulas.

The forcing notion we use in this section is $\mathbb{P}$. The forcing language $\widetilde{\mathfrak{L}}$ has a constant symbol $\check{f}$ for every function $f \in \mathbb{F}$ (the check is usually suppressed). Forcing is defined for arbitrary st- $\epsilon$-formulas. Only the following clauses in the definiton of the forcing relation are necessary:

Definition 5.1. (Simplified forcing.)
(1') $p \Vdash f_{1}=f_{2}$ iff $\forall^{\text {aa }} i \in p\left(f_{1}(i)=f_{2}(i)\right)$.
(2') $p \Vdash f_{1} \in f_{2}$ iff $\forall^{\mathbf{a x}} i \in p\left(f_{1}(i) \in f_{2}(i)\right)$.
(8) $p \Vdash \neg \phi$ iff there is no $p^{\prime}$ extending $p$ such that $p^{\prime} \Vdash \phi$.
(9) $p \Vdash(\phi \wedge \psi)$ iff $p \Vdash \phi$ and $p \Vdash \psi$.
$\left(10^{\prime}\right) p \Vdash \exists v \psi$ iff for every $p^{\prime}$ extending $p$ there exist $p^{\prime \prime}$ extending $p^{\prime}$ and a function $f \in \mathbb{F}$ such that $p^{\prime \prime} \Vdash \psi(f)$.
$\langle p, q\rangle \Vdash \mathbf{s t}(f)$ iff $\exists x \forall^{\mathbf{a x}} i \in p(f(i)=x)$ iff $\forall^{\mathbf{a x}} i, i^{\prime} \in p\left(f(i)=f\left(i^{\prime}\right)\right)$.
The basic properties of forcing from Lemma 4.5 remain valid, but Proposition4.6("Loś's Theorem") of course holds only for $\in$-formulas, in the form $\quad p \Vdash \phi\left(f_{1}, \ldots, f_{r}\right)$ iff $\forall^{\text {aa }} i \in p \phi\left(f_{1}(i), \ldots, f_{r}(i)\right)$.

We now take Zermelo-Fraenkel set theory as our metatheory. Let $\mathfrak{M}$ be a countable model of $\mathbf{Z F} c, \mathcal{U}$ an $\mathfrak{M}$-generic filter on $\mathbb{P} \in M$, and $\mathfrak{N}=\left(N, \in^{*}\right)$ the ultrapower of $\mathfrak{M}$ by $\mathcal{U}$. If $\mathfrak{M} \vDash$ " $f \in \mathbb{F} "$, then $[f]_{\mathcal{U}}$ is the equivalence class of $f$ modulo $\mathcal{U}$. We identify $x \in M$ with $\left[c_{x}\right]_{\mathcal{U}}$ where $c_{x}$ is the constant function on $I$ with value $x$ in the sense of $\mathfrak{M}$, and let $\widehat{\mathfrak{N}}=\left(N, \in^{*}, M\right)$.

Proposition 4.12 takes the following form.
Proposition 5.2. Let $\phi\left(u_{1}, \ldots, u_{r}\right)$ be an $\mathbf{s t}-\in$-formula with parameters from $M$. If $\mathfrak{M} \vDash " f_{1}, \ldots, f_{r} \in \mathbb{F}$ ", then the following statements are equivalent:
(1) $\widehat{\mathfrak{N}} \vDash \phi\left(\left[f_{1}\right]_{\mathcal{U}}, \ldots,\left[f_{r}\right]_{\mathcal{U}}\right)$.
(2) There is some $p \in \mathcal{U}$ such that $p \Vdash \phi\left(f_{1}, \ldots, f_{r}\right)$ holds in $\mathfrak{M}$.

Corollaries 4.13 and 4.14 and Proposition 4.15 remain valid in this modified setting. For $\in$-formulas Proposition 5.2 is just a fancy way to state the ordinary Loś's Theorem, but for st- $\epsilon$-formulas it provides a useful handle on the behavior of $\widehat{\mathfrak{N}}$.

We need the following corollary (which can also be proved more tediously directly from the definition of the forcing relation). The statement " $D \subseteq \mathbb{P}$ is dense in $p$ " means that $\forall p^{\prime \prime} \leq p \exists p^{\prime} \leq p^{\prime \prime}\left(p^{\prime} \in D\right)$.

Lemma 5.3. If $p \Vdash \forall^{\text {st }} m \in \mathbb{N} \phi(m)$, then the set $D_{m}=\left\{p^{\prime} \in \mathbb{P} \mid p^{\prime} \Vdash\right.$ $\phi(m)\}$ is dense in $p$ for every $m \in \mathbb{N}$.

Proof. We show that the claim holds in every model $\mathfrak{M}$ of $\mathbf{Z F} c$. For every $p^{\prime \prime} \leq p$ in $\mathfrak{M}$ there is an $\mathfrak{M}$-generic filter $\mathcal{U}$ such that $p^{\prime \prime} \in \mathcal{U}$. By Proposition $5.2 \widehat{\mathfrak{N}} \vDash \forall^{\text {st }} m \phi(m)$, so $\mathfrak{M} \vDash " m \in \mathbb{N}$ " implies $\widehat{\mathfrak{N}} \vDash \phi(m)$. By 5.2 again, there exists $p^{\prime} \in \mathcal{U}$ such that $p^{\prime} \Vdash \phi(m)$. We can take $p^{\prime} \leq p^{\prime \prime}$.
Lemma 5.4. Let $\phi(u, v)$ be an $\mathbf{s t}-\in$-formula with parameters from $\widetilde{\mathfrak{L}}$. Then $\mathbf{Z F}$ c proves the following: If $p \Vdash \forall^{\text {st }} m \exists v \phi(m, v)$, then there is $p^{\prime} \in \mathbb{P}$ and a sequence $\left\langle f_{m} \mid m \in \mathbb{N}\right\rangle$ such that $p^{\prime}$ extends $p$ and $p^{\prime} \Vdash \phi\left(m, f_{m}\right)$ for every $m \in \mathbb{N}$.

Proof. By Lemma 5.3, the set $D_{m}=\left\{p^{\prime} \in \mathbb{P} \mid p^{\prime} \Vdash \exists v \phi(m, v)\right\}$ is dense in $p$ for each $m \in \mathbb{N}$. Clause ( $10^{\prime}$ ) in the definition of simplified forcing implies that also the set $E_{m}=\left\{p^{\prime} \in \mathbb{P} \mid \exists f \in \mathbb{F} p^{\prime} \Vdash \phi(m, f)\right\}$ is dense in $p$. We let
$\left\langle m^{\prime}, p^{\prime}\right\rangle \mathbf{R}\left\langle m^{\prime \prime}, p^{\prime \prime}\right\rangle$ iff $p^{\prime \prime} \subset p^{\prime} \subseteq p \wedge m^{\prime \prime}=m^{\prime}+1 \wedge p^{\prime} \in E_{m^{\prime}} \wedge p^{\prime \prime} \in E_{m^{\prime \prime}}$.
Applying ADC to the relation $\mathbf{R}$ we obtain a sequence $\left\langle p_{m} \mid m \in \mathbb{N}\right\rangle$ such that $p_{0} \subseteq p$ and, for each $m, p_{m+1} \subset p_{m}$ and $\exists f \in \mathbb{F}\left(p_{m} \Vdash\right.$ $\phi(m, f))$. We next use ACC to obtain a sequence $\left\langle f_{m} \mid m \in \mathbb{N}\right\rangle$ such that $p_{m} \Vdash \phi\left(m, f_{m}\right)$. Note that the Reflection Principle of $\mathbf{Z F}$ provides a set $A=V_{\alpha}$ such that for all $m$,

$$
\left(\exists f \in \mathbb{F} p_{m} \Vdash \phi(m, f)\right) \rightarrow\left(\exists f \in \mathbb{F} \cap A p_{m} \Vdash \phi(m, f)\right) .
$$

As in the proof of Lemma 4.9, let $i_{m}$ be the least element of $p_{m}$ and let $p^{\prime}=\bigcup_{m=0}^{\infty} p_{m} \cap\left(i_{m+1} \backslash i_{m}\right)$. Then for every $m$ the set $p^{\prime} \backslash p_{m}$ is bounded, hence $p^{\prime} \Vdash \phi\left(m, f_{m}\right)$.

Proposition 5.5. If $\mathfrak{M}$ satisfies $\mathbf{Z F}$ c, then $\widehat{\mathfrak{N}}$ satisfies $\mathbf{C C}$.
Proof. Assume that $\widehat{\mathfrak{N}} \vDash \forall^{\text {st }} m \exists v \phi(m, v)$. Then there is $p \in \mathcal{U}$ such that $p \Vdash \forall^{\text {st }} m \exists v \phi(m, v)$. By Lemma 5.4 there is $p^{\prime} \in \mathcal{U}$ and a sequence $\left\langle f_{m} \mid m \in \mathbb{N}\right\rangle$ such that $p^{\prime} \Vdash \phi\left(m, f_{m}\right)$ for every $m \in \mathbb{N}$.

We define a function $g$ on $I$ by $g(i)=\left\{\left\langle m, f_{m}(i)\right\rangle \mid m \in \mathbb{N}\right\}$. Recall that $\check{g}$ is the name for $g$ in the forcing language. By Łos's Theorem, $p^{\prime} \Vdash$ "g is a function with domain $\mathbb{N}$ ", and, for every $m \in \mathbb{N}$, $p^{\prime} \Vdash \check{g}(m)=f_{m}$. We conclude that $\widehat{\mathfrak{N}} \vDash " \check{g}$ is a function with domain $\mathbb{N}^{\prime \prime}$, and $\widehat{\mathfrak{N}} \vDash \forall^{\text {st }} m \in \mathbb{N} \phi(m, \check{g}(m))$.

Proposition 5.6. If $\mathfrak{M}$ satisfies $\mathbf{Z F} c$, then $\hat{\mathfrak{N}}$ satisfies $\mathbf{C C}^{\text {st }}$.
Proof. The assumption $\hat{\mathfrak{N}} \vDash \forall^{\text {st }} m \exists^{\text {st }} v \phi(m, v)$ implies that we can take $f_{m}=c_{x_{m}}$ in Lemma 5.4 and the proof of Proposition 5.5. For the function $g$ on $I$ defined by $g(i)=\left\{\left\langle m, x_{m}\right\rangle \mid m \in \mathbb{N}\right\}$ we then have $\widehat{\mathfrak{N}} \vDash " \check{g}$ is standard".

Let $p, p^{\prime} \in \mathbb{P}$ and let $\gamma$ be an increasing mapping of $p^{\prime}$ onto $p$; we extend $\gamma$ to $I=\mathbb{N}$ by defining $\gamma(a)=0$ for $a \in I \backslash p$.
Lemma 5.7. $p \Vdash \phi\left(f_{1}, \ldots, f_{r}\right)$ iff $p^{\prime} \Vdash \phi\left(f_{1} \circ \gamma, \ldots, f_{r} \circ \gamma\right)$.
Proof. This follows by induction on the cases in the definition of simplified forcing, using the observation that the mapping $p^{\prime \prime} \rightarrow p^{\prime} \cap \gamma^{-1}\left[p^{\prime \prime}\right]$ is an isomorphism of the posets $\left(\left\{p^{\prime \prime} \in \mathbb{P} \mid p^{\prime \prime} \leq p\right\}, \leq\right)$ and $\left(\left\{p^{\prime \prime} \in \mathbb{P} \mid\right.\right.$ $\left.\left.p^{\prime \prime} \leq p^{\prime}\right\}, \leq\right)$.
Corollary 5.8. Let $\phi$ be an st- $\epsilon$-formula with parameters from $\mathbb{V}$. Then $p \Vdash \phi$ iff $p^{\prime} \Vdash \phi$.
Proposition 5.9. The structure $\widehat{\mathfrak{N}}$ satisfies the principle of Standardization for $\mathbf{s t}-\epsilon$-formulas with no parameters.
Proof. For standard $x, \widehat{\mathfrak{N}} \vDash \phi(x)$ iff $p \Vdash \phi(x)$ for every $p \in \mathbb{P}$. The right side is expressible by an $\in$-formula.

This completes the proof of Theorem $\boldsymbol{D}$.

Another principle that can be added to SCOT is Dependent Choice for $\mathbf{s t}$ - $\epsilon$-formulas.

DC Let $\phi(u, v)$ be an st- $\in$-formula with arbitrary parameters.
If $B$ is a set, $b \in B$ and $\forall x \in B \exists y \in B \phi(x, y)$, then there is a sequence $\left\langle b_{n} \mid n \in \mathbb{N}\right\rangle$ such that $b_{0}=b$ and $\forall^{\text {st }} n \in \mathbb{N}\left(b_{n} \in B \wedge \phi\left(b_{n}, b_{n+1}\right)\right)$.

Theorem 5.10. SCOT $+\mathbf{D C}$ is a conservative extension of $\mathbf{Z F} c$.
Proof. We show that DC holds in the structure $\hat{\mathfrak{N}}$.
Let $\mathfrak{M} \vDash " b, B \in \mathbb{F} "$ and $p \Vdash b \in B \wedge \forall x \in B \exists y \in B \phi(x, y)$. We now let $A=\left\{\left\langle p^{\prime}, f^{\prime}\right\rangle \mid p^{\prime} \leq p \wedge p^{\prime} \Vdash f^{\prime} \in B\right\}$, note that $\langle p, b\rangle \in A$, and define $\mathbf{R}$ on $A$ by $\left\langle p^{\prime}, f^{\prime}\right\rangle \mathbf{R}\left\langle p^{\prime \prime}, f^{\prime \prime}\right\rangle$ iff $p^{\prime \prime} \leq p^{\prime} \wedge p^{\prime \prime} \Vdash \phi\left(f^{\prime}, f^{\prime \prime}\right)$.

It is clear from the properties of forcing that for every $\left\langle p^{\prime}, f^{\prime}\right\rangle \in A$ there is $\left\langle p^{\prime \prime}, f^{\prime \prime}\right\rangle \in A$ such that $\left\langle p^{\prime}, f^{\prime}\right\rangle \mathbf{R}\left\langle p^{\prime \prime}, f^{\prime \prime}\right\rangle$. Using ADC we obtain a sequence $\left\langle\left\langle p_{n}, f_{n}\right\rangle \mid n \in \mathbb{N}\right\rangle$ such that $p_{0} \leq p, f_{0}=b$, and for all $n \in \mathbb{N}$ $\left\langle p_{n}, f_{n}\right\rangle \in A, p_{n+1} \leq p_{n}$, and $p_{n+1} \Vdash \phi\left(f_{n}, f_{n+1}\right)$.

The rest of the proof imitates the arguments in the last paragraphs of the proofs of Lemma 5.4 and Proposition 5.5.

## 6. STANDARDIZATION FOR PARAMETER-FREE FORMULAS

In this section we prove that the structure $\widehat{\mathfrak{N}}=\left(N, \in^{*}, M\right)$ satisfies the principle of Standardization for parameter-free formulas assuming only that the model $\mathfrak{M}$ satisfies $\mathbf{Z F}$. Explicitly, the principle postulates:

SN Let $\phi(v)$ be an st- $\epsilon$-formula with no parameters. Then

$$
\forall^{\text {st }} A \exists^{\text {st }} S \forall^{\text {st }} x(x \in S \longleftrightarrow x \in A \wedge \phi(x)) .
$$

Lemma 6.1. The principle $\mathbf{S N}$ is equivalent to Standardization for st- $\epsilon$-formulas with standard parameters.

Proof. Given $\phi\left(v, p_{1}, \ldots, p_{\ell}\right)$ where $p_{1}, \ldots, p_{\ell}$ are standard, we let $P=$ $\left\{\left\langle p_{1}, \ldots, p_{\ell}\right\rangle\right\}$ and apply $\mathbf{S N}$ to the formula $\psi(w)$ with no parameters expressing " $\exists w_{1}, \ldots, w_{\ell}\left(w=\left\langle v, w_{1}, \ldots, w_{\ell}\right\rangle \wedge \phi\left(v, w_{1}, \ldots, w_{\ell}\right)\right)$ " and to the standard set $A \times P$. We get a standard $S$ such that for all standard inputs $\left\langle x, y_{1}, \ldots, y_{\ell}\right\rangle \in S \longleftrightarrow\left\langle x, y_{1}, \ldots, y_{\ell}\right\rangle \in A \times P \wedge \phi\left(x, y_{1}, \ldots, y_{\ell}\right)$ holds. The set $T=\left\{x \in A \mid\left\langle x, p_{1}, \ldots, p_{\ell}\right\rangle \in S\right\}$ standardizes $\phi\left(v, p_{1}, \ldots, p_{\ell}\right)$.

With the exception of the last proposition, in this section we work in ZF. As in Section [5, we bin with extending forcing to $\mathbf{s t}$ - $\epsilon$-formulas. We add the following clauses to Definition 4.4:
(11) $\langle p, q\rangle \Vdash \operatorname{st}(z)$ for every $z \in \mathbb{V}$.
(12) $\langle p, q\rangle \Vdash \operatorname{st}\left(\dot{G}_{n}\right)$ iff $\operatorname{rank} q=k>n$ and
$\exists x \forall^{\text {aad }} i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\left(x_{n}=x\right)$ or, equivalently,
$\forall^{\mathbf{a a}} i, i^{\prime} \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \forall\left\langle x_{0}^{\prime}, \ldots, x_{k-1}^{\prime}\right\rangle \in q\left(i^{\prime}\right)\left(x_{n}=x_{n}^{\prime}\right)$.
The basic properties of forcing from Lemma 4.5 in Subsection 4.1 remain valid, but "Łośs Theorem" does not hold for st- $\epsilon$-formulas. However, Proposition 4.6 is instrumental in the proof of Lemma 4.9, The technical lemma that follows is a simple consequence of Proposition 4.6 for forcing of $\in$-formulas, but it remains valid even for forcing of $\mathbf{s t}$ - $\epsilon$-formulas.

Definition 6.2. For $\sigma=\left\langle n_{1}, \ldots, n_{s}\right\rangle$ where $n_{1}, \ldots, n_{s}<k$ are mutually distinct let $\pi_{\sigma}^{k}: \mathbb{V}^{k} \rightarrow \mathbb{V}^{s}$ be the "projection" of $\mathbb{V}^{k}$ onto $\mathbb{V}^{s}$ :

$$
\pi_{\sigma}^{k}\left(\left\langle x_{0}, \ldots, x_{k-1}\right\rangle\right)=\left\langle x_{n_{1}}, \ldots, x_{n_{s}}\right\rangle
$$

For $q \in \mathbb{Q}$ of rank $k, q \upharpoonright \sigma$ of rank $s$ is defined by $(q \upharpoonright \sigma)(i)=\pi_{\sigma}^{k}[q(i)]$.
Lemma 6.3. Let $\phi$ be an $\mathbf{s t}-\in$-formula with parameters from $\mathbb{V}$. Assume that $\operatorname{rank} q_{1}=k_{1}, \sigma_{1}=\left\langle n_{1}, \ldots, n_{s}\right\rangle$ where $n_{1}, \ldots, n_{s}<k_{1}$, and $\operatorname{rank} q_{2}=k_{2}, \sigma_{2}=\left\langle m_{1}, \ldots, m_{s}\right\rangle$ with $m_{1}, \ldots, m_{s}<k_{2}$. If
$\left(q_{1} \upharpoonright \sigma_{1}\right)(i)=\left(q_{2} \upharpoonright \sigma_{2}\right)(i)$ for all $i \in p$ (we write $q_{1} \upharpoonright \sigma_{1}={ }_{p} q_{2} \upharpoonright \sigma_{2}$ ), then $\left\langle p, q_{1}\right\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ if and only if $\left\langle p, q_{2}\right\rangle \Vdash \phi\left(\dot{G}_{m_{1}}, \ldots, \dot{G}_{m_{s}}\right)$.

Proof. The proof is by induction on the logical complexity of $\phi$.
For $\in$-formulas the assertion follows immediately from Proposition 4.6. In particular, it holds for all atomic formulas involving $\in$ and $=$ (cases (1) - (7) in the definition of forcing). It is also clear for st (cases (11) and (12)) and for conjunction (case (9)).

Case (8):
Assume that the statement is true for $\phi,\left\langle p, q_{1}\right\rangle \Vdash \neg \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$, and $\left\langle p, q_{2}\right\rangle \nVdash \neg \phi\left(\dot{G}_{m_{1}}, \ldots, \dot{G}_{m_{s}}\right)$. Then there exists a condition $\left\langle p^{\prime}, q_{2}^{\prime}\right\rangle \leq$ $\left\langle p, q_{2}\right\rangle$ such that $\left\langle p^{\prime}, q_{2}^{\prime}\right\rangle \Vdash \phi\left(\dot{G}_{m_{1}}, \ldots, \dot{G}_{m_{s}}\right)$. From the inductive assumption it follows that $\left\langle p^{\prime}, q_{2}^{\prime} \upharpoonright \sigma_{2}\right\rangle \Vdash \phi\left(\dot{G}_{0}, \ldots, \dot{G}_{s-1}\right)$ (recall that $\left.q_{2}^{\prime} \upharpoonright \sigma_{2} \subseteq \mathbb{V}^{s}\right)$. Let $\bar{q}=q_{2}^{\prime} \upharpoonright \sigma_{2} \leq q_{2} \upharpoonright \sigma_{2}={ }_{p} q_{1} \upharpoonright \sigma_{1}$. We define $q_{1}^{\prime}=\left(\pi_{\sigma_{1}}^{k_{1}}\right)^{-1}[\bar{q}] \cap q_{1} \leq q_{1}$. Now $\left\langle p^{\prime}, q_{1}^{\prime}\right\rangle \leq\left\langle p, q_{1}\right\rangle$ and $q_{2}^{\prime} \upharpoonright \sigma_{2}={ }_{p^{\prime}} q_{1}^{\prime} \upharpoonright \sigma_{1}$, so by the inductive assumption $\left\langle p^{\prime}, q_{1}^{\prime}\right\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$. This is a contradiction with $\left\langle p, q_{1}\right\rangle \Vdash \neg \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$. The reverse implication follows by exchanging the roles of $\left\langle p, q_{1}\right\rangle$ and $\left\langle p, q_{2}\right\rangle$.

Case (10):
Assume that $\left\langle p, q_{1}\right\rangle \Vdash \exists v \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, v\right)$, Let $\left\langle p^{\prime}, q_{2}^{\prime}\right\rangle \leq\left\langle p, q_{2}\right\rangle$, where $\operatorname{rank} q_{2}^{\prime}=k_{2}^{\prime}=k_{2}+r$. We need to find $\left\langle p^{\prime \prime}, q_{2}^{\prime \prime}\right\rangle$ extending $\left\langle p^{\prime}, q_{2}^{\prime}\right\rangle$ and $m$ such that $\left\langle p^{\prime \prime}, q_{2}^{\prime \prime}\right\rangle \Vdash \psi\left(\dot{G}_{m_{1}}, \ldots, \dot{G}_{m_{s}}, \dot{G}_{m}\right)$. This will prove that $\left\langle p, q_{2}\right\rangle \Vdash \exists v \psi\left(\dot{G}_{m_{1}}, \ldots, \dot{G}_{m_{s}}, v\right)$.

We let $\bar{q}=\pi_{\sigma_{2}}^{k_{2}^{\prime}}\left[q_{2}^{\prime}\right] \subseteq \pi_{\sigma_{2}}^{k_{2}}\left[q_{2}\right]={ }_{p} \pi_{\sigma_{1}}^{k_{1}}\left[q_{1}\right]$ and $\overline{\bar{q}}=\left(\pi_{\sigma_{1}}^{k_{1}}\right)^{-1}[\bar{q}] \cap q_{1} \leq q_{1}$. We define $q_{1}^{\prime}$ of rank $k_{1}^{\prime}=k_{1}+r$ by
$q_{1}^{\prime}(i)=\left\{\left\langle x_{0}, \ldots, x_{k_{1}-1}, y_{0}, \ldots, y_{r-1}\right\rangle \mid\left\langle x_{0}, \ldots, x_{k_{1}-1}\right\rangle \in \overline{\bar{q}} \wedge\right.$
$\left\langle x_{n_{1}}, \ldots, x_{n_{s}}\right\rangle=\pi_{\sigma_{2}}^{k_{2}^{\prime}}\left(\left\langle x_{0}^{\prime}, \ldots, x_{k_{2}-1}^{\prime}, y_{0}, \ldots, y_{r-1}\right\rangle\right)$ for some $\left.\left\langle x_{0}^{\prime}, \ldots, x_{k_{2}-1}^{\prime}, y_{0}, \ldots, y_{r-1}\right\rangle \in q_{2}^{\prime}\right\}$
for $i \in p^{\prime} ; q_{1}^{\prime}(i)=\left\{\emptyset_{k_{1}^{\prime}}\right\}$ otherwise. We observe that $q_{1}^{\prime}(i) \neq \emptyset$ and $\pi_{\sigma_{1}}^{k_{1}^{\prime}}\left[q_{1}^{\prime}\right]={ }_{p^{\prime}} \pi_{\sigma_{2}}^{k_{2}^{\prime}}\left[q_{2}^{\prime}\right]$.

We have $\left\langle p^{\prime}, q_{1}^{\prime}\right\rangle \leq\left\langle p, q_{1}\right\rangle$; hence there are $\left\langle p^{\prime \prime}, q_{1}^{\prime \prime}\right\rangle \leq\left\langle p^{\prime}, q_{1}^{\prime}\right\rangle$ with $\operatorname{rank} q_{1}^{\prime \prime}=k_{1}^{\prime \prime}$ and $n$ such that $\left\langle p^{\prime \prime}, q_{1}^{\prime \prime}\right\rangle \Vdash \psi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}, \dot{G}_{n}\right)$. Finally we construct $q_{2}^{\prime \prime}$ of rank $k_{2}^{\prime \prime}=k_{2}^{\prime}+1$ such that $\left\langle p^{\prime \prime}, q_{2}^{\prime \prime}\right\rangle \leq\left\langle p^{\prime \prime}, q_{2}^{\prime}\right\rangle$ and, for some $m, \pi_{\sigma_{1} \# n}^{k_{1}^{\prime \prime}}\left(q_{1}^{\prime \prime}\right)={ }_{p^{\prime \prime}} \pi_{\sigma_{2} \# m}^{k_{2}^{\prime \prime}}\left(q_{2}^{\prime \prime}\right)$, where $\sigma_{1} \# n=\left\langle n_{1}, \ldots, n_{s}, n\right\rangle$ and $\sigma_{2} \# m=\left\langle m_{1}, \ldots, m_{s}, m\right\rangle$. By the inductive assumption, this establishes $\left\langle p^{\prime \prime}, q_{2}^{\prime \prime}\right\rangle \Vdash \psi\left(\dot{G}_{m_{1}}, \ldots, \dot{G}_{m_{s}}, \dot{G}_{m}\right)$.

We start with $\widehat{q}=\pi_{\sigma_{1}}^{k_{1}^{\prime \prime}}\left[q_{1}^{\prime \prime}\right] \subseteq \pi_{\sigma_{1}}^{k_{1}^{\prime}}\left[q_{1}^{\prime}\right]={ }_{p^{\prime}} \pi_{\sigma_{2}}^{k_{2}^{\prime}}\left[q_{2}^{\prime}\right]$ and define

$$
\begin{aligned}
& q_{2}^{\prime \prime}(i)=\left\{\left\langle x_{0}, \ldots, x_{k_{2}^{\prime}-1}, z\right\rangle \mid\left\langle x_{0}, \ldots, x_{k_{2}^{\prime}-1}\right\rangle \in\left(\pi_{\sigma_{2}}^{k_{2}^{\prime}}\right)^{-1}[\widehat{q}] \cap q_{2}^{\prime} \wedge\right. \\
& \left.\left\langle x_{n_{1}}, \ldots, x_{n_{s}}, z\right\rangle \in \pi_{\sigma_{1} \# n}^{k_{1}^{\prime \prime}}\left(q_{1}^{\prime \prime}\right)\right\}
\end{aligned}
$$

for $i \in p^{\prime \prime} ; q_{2}^{\prime \prime}(i)=\left\{\emptyset_{k_{2}^{\prime}+1}\right\}$ otherwise.
We have $\left\langle p^{\prime \prime}, q_{2}^{\prime \prime}\right\rangle \leq\left\langle p^{\prime \prime}, q_{2}^{\prime}\right\rangle$ and $\operatorname{rank} q_{2}^{\prime \prime}=k_{2}^{\prime \prime}=k_{2}^{\prime}+1$. Let $m=k_{2}^{\prime}$. It follows from the construction that $\pi_{\sigma_{1} \# n}^{k_{1}^{\prime \prime}}\left(q_{1}^{\prime \prime}\right)={ }_{p^{\prime \prime}} \pi_{\sigma_{2} \# m}^{k_{2}^{\prime \prime}}\left(q_{2}^{\prime \prime}\right)$.
Corollary 6.4. Let $\phi$ be an st-e-sentence with parameters from $\mathbb{V}$. Then $\langle p, q\rangle \Vdash \phi$ iff $\langle p, \overline{1}\rangle \Vdash \phi$.

As in Section 5, let $p, p^{\prime} \in \mathbb{P}$ and let $\gamma$ be an increasing mapping of $p^{\prime}$ onto $p$; we extend $\gamma$ to $I$ by defining $\gamma(a)=0$ for $a \in I \backslash p$.
Lemma 6.5. Let $\phi\left(v_{1}, \ldots, v_{s}\right)$ be an st- $\in$-formula with parameters from $\mathbb{V}$. Then $\langle p, q\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ iff $\left\langle p^{\prime}, q \circ \gamma\right\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$.
Proof. As for Lemma 5.7.
Corollary 6.6. Let $\phi\left(v_{1}, \ldots, v_{r}\right)$ be an $\mathbf{s t - \epsilon - f o r m u l a . ~ F o r ~} z_{1}, \ldots, z_{r} \in \mathbb{V}$ $\langle p, q\rangle \Vdash \phi\left(z_{1}, \ldots, z_{r}\right)$ iff $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi\left(z_{1}, \ldots, z_{r}\right)$.

Proposition 6.7. The structure $\widehat{\mathfrak{N}}=\left(N, \in^{*}, M\right)$ satisfies the principle of Standardization for $\mathbf{s t}-\in$-formulas with no parameters.
Proof. For standard $z, \widehat{\mathfrak{N}} \vDash \phi(z)$ iff $\langle p, q\rangle \Vdash \phi(z)$ for every $\langle p, q\rangle \in \mathbb{H}$. The right side is expressible by an $\in$-formula.

This completes the proof of Theorem $\boldsymbol{B}$.
There is yet another principle that can be added to SPOT and keep it conservative over ZF. One of its important consequences is the impossibility to uniquely specify an infinitesimal.

UP (Uniqueness Principle) Let $\phi(v)$ be an st- $\in$-formula with standard parameters. If there exists a unique $x$ such that $\phi(x)$, then this $x$ is standard.

Theorem 6.8. SPOT + UP is a conservative extension of ZF.
Proof. If $\widehat{\mathfrak{N}} \vDash \exists x[\phi(x) \wedge \forall y(\phi(y) \rightarrow y=x) \wedge \neg \operatorname{st}(x)]$, then there is $\langle p, q\rangle \in \mathcal{G}$ and $m<k=\operatorname{rank} q$ such that $\langle p, q\rangle \Vdash \phi\left(\dot{G}_{m}\right) \wedge \neg \operatorname{st}\left(\dot{G}_{m}\right) \wedge$ $\forall y\left(\phi(y) \rightarrow y=\dot{G}_{m}\right)$.

Let $r=q \upharpoonright\{m\}$, i.e., for all $i \in I, x \in r(i) \longleftrightarrow \exists\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in$ $q(i)\left(x_{m}=x\right)$.

Claim 1. For every $x \in \bigcup_{i \in p} r(i)$ the set $p_{x}=\{i \in p \mid x \in r(i)\}$ is bounded; we let $i_{x}$ denote its greatest element.

Proof of Claim 1. Let $x$ be such that $p_{x}$ is unbounded. Define

$$
\begin{aligned}
q_{x}(i) & =\left\{\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \mid\left(x_{k}=x\right)\right\} \text { for } i \in p_{x} ; \\
& =\left\{\emptyset_{k}\right\} \text { otherwise } .
\end{aligned}
$$

Then $\left\langle p_{x}, q_{x}\right\rangle \leq\langle p, q\rangle$ and $\left\langle p_{x}, q_{x}\right\rangle \Vdash \dot{G}_{m}=x$, i.e., $\left\langle p_{x}, q_{x}\right\rangle \Vdash \mathbf{s t}\left(\dot{G}_{m}\right)$, a contradiction.

Claim 2. There exist unbounded mutually disjoint sets $p_{1}, p_{2} \subset p$ and nonempty sets $s(i) \subseteq r(i)$ for all $i \in p_{1} \cup p_{2}$ such that $\left(\bigcup_{i \in p_{1}} s(i)\right) \cap\left(\bigcup_{i \in p_{2}} s(i)\right)=\emptyset$.

We postpone the proof of Claim 2 and complete the proof of the theorem.

We let $\widetilde{q}(i)=\left\{\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \mid x_{k} \in s(i)\right\}$ for $i \in p_{1} \cup p_{2}$, $\widetilde{q}(i)=\left\{\emptyset_{k}\right\}$ otherwise. We have $\left\langle p_{1}, \widetilde{q}\right\rangle \leq\langle p, q\rangle,\left\langle p_{2}, \widetilde{q}\right\rangle \leq\langle p, q\rangle$, and consequently $\left\langle p_{1}, \widetilde{q}\right\rangle \Vdash \phi\left(G_{m}\right),\left\langle p_{2}, \widetilde{q}\right\rangle \Vdash \phi\left(G_{m}\right)$.

Let $\gamma$ be an increasing mapping of $p_{1}$ onto $p_{2}$ extended by $\gamma(i)=0$ for $i \in I \backslash p_{1}$. By Lemma $6.5\left\langle p_{1}, \widetilde{q} \circ \gamma\right\rangle \Vdash \phi\left(\dot{G}_{m}\right)$. We "amalgamate" $\left\langle p_{1}, \widetilde{q}\right\rangle$ and $\left\langle p_{1}, \widetilde{q} \circ \gamma\right\rangle$ to form a condition of rank $2 k$ as follows: $\left\langle x_{0}, \ldots, x_{k-1}, y_{0}, \ldots, y_{k-1}\right\rangle \in \widehat{q}(i) \longleftrightarrow$

$$
\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in \widetilde{q}(i) \wedge\left\langle y_{0}, \ldots, y_{k-1}\right\rangle \in \widetilde{q}(\gamma(i))
$$

and observe that $\left\langle p_{1}, \widehat{q}\right\rangle \leq\langle p, \widetilde{q}\rangle$. Let $\sigma_{1}=k$ and $\sigma_{2}=\{k+\ell \mid \ell<k\}$. We have $\widehat{q} \upharpoonright \sigma_{1}=\widetilde{q}$ and $\widehat{q} \upharpoonright \sigma_{2}=\widetilde{q} \circ \gamma$. By Lemma 6.3 $\left\langle p_{1}, \widehat{q}\right\rangle \Vdash \phi\left(\dot{G}_{m}\right) \wedge$ $\phi\left(\dot{G}_{k+m}\right)$. But $x_{m} \neq y_{m}$ holds for all $\left\langle x_{0}, \ldots, x_{k-1}, y_{0}, \ldots, y_{k-1}\right\rangle \in \widehat{q}(i)$ and all $i \in p_{1}$, so $\left\langle p_{1}, \widehat{q}\right\rangle \Vdash \dot{G}_{m} \neq \dot{G}_{k+m}$. This contradicts $\left\langle p_{1}, \widehat{q}\right\rangle \Vdash$ $\forall y\left(\phi(y) \rightarrow y=\dot{G}_{m}\right)$.

Proof of Claim 2. W.l.o.g. we can assume $p=I=\mathbb{N}$ (map $p$ onto $\mathbb{N}$ in an increasing way). Define sequences $\left\langle n_{\ell} \mid \ell \in \mathbb{N}\right\rangle,\left\langle\alpha_{\ell} \mid \ell \in \mathbb{N}\right\rangle$ and $\left\langle s\left(n_{\ell}\right) \mid \ell \in \mathbb{N}\right\rangle$ by recursion as follows:

Let $n_{0}=0, \alpha_{0}=\min \left\{i_{x} \mid x \in r(0)\right\}$ and $s\left(n_{0}\right)=s(0)=\{x \in r(0) \mid$ $\left.i_{x}=\alpha_{0}\right\}$.

At stage $\ell+1$ let $n_{\ell+1}=\alpha_{\ell}+1, \alpha_{\ell+1}=\min \left\{i_{x} \mid x \in r\left(n_{\ell+1}\right)\right\}$ and $s\left(n_{\ell+1}\right)=\left\{x \in r\left(n_{\ell+1}\right) \mid i_{x}=\alpha_{\ell+1}\right\}$.

We observe that $s\left(n_{\ell}\right) \cap s\left(n_{\ell^{\prime}}\right)=\emptyset$ for all $\ell \neq \ell^{\prime}$. It remains to let $p_{1}=\left\{n_{2 \ell} \mid \ell \in \mathbb{N}\right\}$ and $p_{2}=\left\{n_{2 \ell+1} \mid \ell \in \mathbb{N}\right\}$.

## 7. Idealization

We recall the axioms of the theory BST; see the references Kanovei and Reeken [24] and Fletcher et al. [9] for motivation and more detail. In addition to the axioms of $\mathbf{Z F C}$, they are:

B (Boundedness) $\quad \forall x \exists^{\text {st }} y(x \in y)$.
$\mathbf{T}$ (Transfer) Let $\phi(v)$ be an $\in$-formula with standard parameters. Then

$$
\forall^{\text {st }} x \phi(x) \rightarrow \forall x \phi(x) .
$$

$\mathbf{S}$ (Standardization) Let $\phi(v)$ be an st- $\epsilon$-formula with arbitrary parameters. Then

$$
\forall^{\text {st }} A \exists^{\text {st }} S \forall^{\text {st }} x(x \in S \longleftrightarrow x \in A \wedge \phi(x))
$$

BI (Bounded Idealization) Let $\phi$ be an $\in$-formula with arbitrary parameters. For every set $A$

$$
\forall^{\text {st fin }} a \subseteq A \exists y \forall x \in a \phi(x, y) \longleftrightarrow \exists y \forall^{\text {st }} x \in A \phi(x, y)
$$

7.1. Idealization over uncountable sets. In order to obtain models with Bounded Idealization, the construction of Subsections 4.1 and 4.2 needs to be generalized from $I=\mathbb{N}$ to $I=\mathcal{P}^{\text {fin }}(A)$, where $A$ is any infinite set. The key is the right definition of "unbounded" subsets of $I$. We work in ZF.

We use the notation $\mathcal{P}^{\leq m}(A)$ for $\left\{a \in \mathcal{P}^{\text {fin }}(A)| | a \mid \leq m\right\}$.
Definition 7.1. A set $p \subseteq \mathcal{P}^{\mathrm{fin}}(A)$ is thick if

$$
\forall m \in \mathbb{N} \exists n \in \mathbb{N} \forall a \in \mathcal{P}^{\leq m}(A) \exists b \in p \cap \mathcal{P}^{\leq n}(A)(a \subseteq b)
$$

We let $\nu_{p}(m)$ denote the least $n$ with this property. The set $p$ is thin if it is not thick.

Clearly $\left\{a \in \mathcal{P}^{\text {fin }}(A) \mid x \in a\right\}$ is thick for every $x \in A$ (with $\nu_{p}(m)=$ $m+1)$. We now carry out the developments of Subsections 4.1 and 4.2 with unbounded and bounded replaced by thick and thin, respectively. The definition of forcing and proofs of Lemmas 4.5 and 4.8 are as before. The following observation enables the proof of Proposition 4.6 to go through as well.

Lemma 7.2. If $p$ is thick and $S \subseteq \mathcal{P}^{\text {fin }}(A)$, then either $p \cap S$ or $p \backslash S$ is thick.

Proof. Otherwise there is $m_{1}$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N} \exists a_{1} \in \mathcal{P}^{\leq m_{1}}(A) \forall b \in(p \cap S) \cap \mathcal{P}^{\leq n}(A)\left(a_{1} \nsubseteq b\right) \tag{*}
\end{equation*}
$$

and there is $m_{2}$ such that
$\left({ }^{* *}\right) \quad \forall n \in \mathbb{N} \exists a_{2} \in \mathcal{P}^{\leq m_{2}}(A) \forall b \in(p \backslash S) \cap \mathcal{P}^{\leq n}(A)\left(a_{2} \nsubseteq b\right)$.
Let $m_{1}, m_{2}$ be as above, and let $m=m_{1}+m_{2}$. Since $p$ is thick,
$(* * *) \quad \exists n \in \mathbb{N} \forall a \in \mathcal{P}^{\leq m}(A) \exists b \in p \cap \mathcal{P}^{\leq n}(A)(a \subseteq b)$.
Fix such $n$; for $a_{1} \in \mathcal{P} \leq m_{1}(A)$ such that $\forall b \in(p \cap S) \cap \mathcal{P} \leq n(A)\left(a_{1} \nsubseteq b\right)$ and $a_{2} \in \mathcal{P} \leq m_{2}(A)$ such that $\forall b \in(p \backslash S) \cap \mathcal{P} \leq n(A)\left(a_{2} \nsubseteq b\right)$ we have $a_{1} \cup a_{2} \in \mathcal{P} \leq m(A)$. By $\left({ }^{* * *}\right)$ there is $b \in p \cap \mathcal{P} \leq n(A)$ such that $a_{1} \cup a_{2} \subseteq b$. Depending on whether $b \in p \cap S$ or $b \in p \backslash S$, this contradicts $\left(^{*}\right)$ or $\left.{ }^{* *}\right)$.

The next lemma enables a generalization of Lemma 4.9,
Lemma 7.3. Let $\left\langle p_{n} \mid n \in \mathbb{N}\right\rangle$ be such that, for all $n, p_{n} \in \mathbb{P}$ and $p_{n} \supseteq p_{n+1}$. Then there is $p \in \mathbb{P}$ with the property that for every $n$ there is $k \in \mathbb{N}$ such that $\forall a \in\left(p \backslash p_{n}\right)(|a| \leq k)$. In particular, $p \backslash p_{n}$ is thin.
Proof. Define $p=\bigcup_{m=0}^{\infty}\left\{a \in p_{m}| | a \mid \leq \nu_{p_{m}}(m)\right\}$. Since $p \backslash p_{n} \subseteq$ $\bigcup_{m=0}^{n-1}\left\{a \in p_{m}| | a \mid \leq \nu_{p_{m}}(m)\right\}, a \in p \backslash p_{n}$ implies $|a| \leq k$ for $k=$ $\max \left\{\nu_{p_{0}}(0), \ldots, \nu_{p_{n-1}}(n-1)\right\}$.

We show that $p$ is thick. Given $m \in \mathbb{N}$, we let $n=\nu_{p_{m}}(m)$. If $a \in \mathcal{P}^{\mathrm{fin}}(A)$ and $|a| \leq m$, then there is $b \in p_{m}$ such that $a \subseteq b$ and $|b| \leq n$. By the definition of $p, b \in p$. So $n$ has the required property.

With these changes, the rest of the development of Subsections 4.1 and 4.2 goes through and establishes the following strengthening of Proposition 4.15.
Proposition 7.4. Assume that $\mathfrak{M} \vDash$ " $I=\mathcal{P}^{\mathrm{fin}}(A) \wedge A$ is infinite". The structure $\widehat{\mathfrak{N}}_{A}=\left(N_{A}, \in^{*}, M\right)$ constructed for this I satisfies the principles of Transfer, Nontriviality, Boundedness, Standard Part, and Bounded Idealization over A for $\in$-formulas with standard parameters.
Proof. To prove that $\hat{\mathfrak{N}}_{A} \vDash \mathbf{O}$ one can take $d \in M$ such that $\mathfrak{M} \vDash$ " $d$ is a function on $I=\mathcal{P}^{\mathrm{fin}}(A) \wedge \forall a \in \mathcal{P}^{\mathrm{fin}}(A)(d(a)=|a|)$ ". Nontriviality also follows from Bounded Idealization.

Let $G_{m} \in N$ and let $\langle p, q\rangle \in \mathcal{G}$ have $\operatorname{rank} q=k>m$. There is some $X \in M$ such that $\mathfrak{M} \vDash " \forall i \in I \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i)\left(x_{m} \in X\right)$." By Proposition $4.6\langle p, q\rangle \Vdash \forall v\left(v \in \dot{G}_{m} \rightarrow v \in \tilde{X}\right)$, so $\forall v \in G_{m}(v \in X)$ holds in $\mathfrak{N}_{A}$. This proves Boundedness in $\widehat{\mathfrak{N}}_{A}$.

It remains to prove that Bounded Idealization over $A$ holds in $\widehat{\mathfrak{N}}_{A}$. Let $\phi(u, v)$ be an $\in$-formula with parameters from $M$. Assume that $\mathfrak{M} \vDash \forall a \in \mathcal{P}^{\mathrm{fin}}(A) \exists y \forall x \in a \phi(x, y)$. By the Reflection Principle in $\mathbf{Z F}, \mathfrak{M} \vDash " \exists$ an ordinal $\alpha \forall a \in \mathcal{P}^{\text {fin }}(A) \exists y \in V_{\alpha} \forall x \in a \phi(x, y)$ ". We work in the model $\mathfrak{M}$.

Let $\langle p, q\rangle \in \mathbb{H}$ be a forcing condition and $k=\operatorname{rank} q$. For $i=a \in p$ we let $q^{\prime}(a)=$
$\left\{\left\langle x_{0}, \ldots, x_{k-1}, x_{k}\right\rangle \mid\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(a) \wedge x_{k} \in V_{\alpha} \wedge \forall x \in a \phi\left(x, x_{k}\right)\right\} ;$ $q^{\prime}(a)=\left\{\emptyset_{k+1}\right\}$ otherwise. Then $\left\langle p, q^{\prime}\right\rangle$ extends $\langle p, q\rangle$. For every $x \in A$ the set $c=p \cap\left\{a \in \mathcal{P}^{\mathrm{fin}}(A) \mid x \in a\right\}$ is in $\mathbb{P}$ because $p \backslash c$ is thin (Lemma 7.2), and so, by Proposition 4.6, $\left\langle p, q^{\prime}\right\rangle \Vdash \phi\left(x, \dot{G}_{k}\right)$.

By the genericity of $\mathcal{G}$ there exist $\langle p, q\rangle \in \mathcal{G}$ and $k \in \Omega$ such that $\langle p, q\rangle \Vdash \phi\left(x, \dot{G}_{k}\right)$ for all $x \in A$. By the Fundamental Theorem 4.12
$\mathfrak{N}_{A} \vDash \phi\left(x, G_{k}\right)$ for all $x \in A$. This means that the $\rightarrow$ implication of Idealization over $A$ for $\in$-formulas with standard parameters is satisfied in $\widehat{\mathfrak{N}}_{A}$. The opposite implication follows from $\mathbf{S P}$; see Lemma 2.1 in Section 2 ,
7.2. Further theories. Nelson's IST postulates a form of Idealization that is even stronger than Bounded Idealization (but it contradicts Boundedness).

I (Idealization) Let $\phi$ be an $\in$-formula with arbitrary parameters.

$$
\forall^{\text {st fin }} a \exists y \forall x \in a \phi(x, y) \longleftrightarrow \exists y \forall^{\text {st }} x \phi(x, y)
$$

The theory $\mathbf{B S P T}^{\prime}=\mathbf{Z F}+\mathbf{T}+\mathbf{S P}^{\prime}+\mathbf{B}+\mathbf{B I}^{\prime}$ is introduced in Subsection 1.3. We let ISPT ${ }^{\prime}$ be the theory obtained from $\mathbf{B S P T}^{\prime}$ by deleting Boundedness and replacing Bounded Idealization for formulas with standard parameters by $\mathbf{I}^{\prime}$, Nelson's Idealization for formulas with standard parameters. In other words, $\mathbf{I S P T}^{\prime}=\mathbf{Z F}+\mathbf{T}+\mathbf{S P}^{\prime}+\mathbf{I}^{\prime}$.

Principle $\mathbf{I}$ implies the existence of a finite set that contains all standard sets as elements, and has certain undesirable consequences from the metamathematical point of view. Kanovei and Reeken [24], Theorem 4.6.23, prove that there are countable models $\mathfrak{M}=\left(M, \in^{\mathfrak{M}}\right)$ of ZFC that cannot be extended to a model of IST in which $M$ would be the class of all standard sets (assuming ZFC is consistent). We do not know whether the same is the case for ISPT ${ }^{\prime}$. Nevertheless we have the following result.
Theorem 7.5. ISPT $^{\prime}$ is a conservative extension of $\mathbf{Z F}$.
Proof. Let us assume that $\mathbf{I S P T}^{\prime} \vdash \theta$ but $\mathbf{Z F} \nvdash \theta$, for some $\in$-sentence $\theta$. Let $\mathbf{Z F}_{r}$ be $\mathbf{Z F}$ with the Axiom Schema of Replacement restricted to $\Sigma_{r}$-formulas, and let $\mathbf{I S P T}_{r}^{\prime}$ be $\mathbf{I S P T}^{\prime}$ with $\mathbf{Z F}$ replaced by $\mathbf{Z F}_{r}$. There is $r \in \omega$ for which $\mathbf{I S P T}_{r}^{\prime} \vdash \theta$.

Let $\mathfrak{M}=\left(M, \in^{\mathfrak{M}}\right)$ be a model of $\mathbf{Z F}+\neg \theta$. By the Reflection Principle of $\mathbf{Z F}$, valid in $\mathfrak{M}$, there is $\alpha \in M$ such that $\mathfrak{M} \vDash$ " $\alpha$ is a limit ordinal", $\mathfrak{M} \vDash \phi^{V_{\alpha}}$ for every axiom $\phi$ of $\mathbf{Z F}_{r}$, and $\mathfrak{M} \vDash(\neg \theta)^{V_{\alpha}}$.

We let $A=\left(V_{\alpha}\right)^{\mathfrak{M}}$ and use Proposition [7.4 to extend $\mathfrak{M}$ to a model $\widehat{\mathfrak{N}}_{A}$. We define $N_{\alpha}=\left\{x \in N_{A} \mid \widehat{\mathfrak{N}}_{A} \vDash x \in V_{\alpha}\right\}$. It is easy to verify that $\widehat{\mathfrak{N}}_{\alpha}=\left(N_{\alpha}, \in^{*} \upharpoonright N_{\alpha}, M \cap N_{\alpha}\right)$ is a model of $\mathbf{I S P T}_{r}^{\prime}$; hence $\theta$ holds in $\widehat{\mathfrak{N}}_{\alpha}$. On the other hand, $(\neg \theta)^{V_{\alpha}}$ holds in $\mathfrak{M}$ and hence, by Transfer, $\neg \theta$ holds in $\widehat{\mathfrak{N}}_{\alpha}$. A contradiction.

Kanovei and Reeken [24], Theorem 3.4.5, showed that the class of bounded sets in IST satisfies the axioms of BST. This result holds also for $\mathbf{I S P T}^{\prime}$ and $\mathbf{B S P T}^{\prime}$, respectively, and establishes the following theorem.

Theorem 7.6. The theory $\mathbf{B S P T}^{\prime}$ is a conservative extension of $\mathbf{Z F}$.
This concludes the proof of Theorem $\boldsymbol{C}$.
Finally, we prove that if $\mathfrak{M}$ satisfies ADC, then the model $\widehat{\mathfrak{N}}_{A}$ constructed in Proposition 7.4 satisfies CC. We note that the definition of forcing for st- $\epsilon$-formulas in Section 6, and Lemma 6.3, extend to $I=\mathcal{P}^{\text {fin }}(A)$.

Proposition 7.7. If $\mathfrak{M}$ is a countable model of $\mathbf{Z F} c$, then the extended ultrapower $\widehat{\mathfrak{N}}_{A}$ satisfies Countable st- $\epsilon$-Choice (both $\mathbf{C C}$ and $\mathbf{C C}{ }^{\text {st }}$ ).

Proof. We work in ZF $c$.
CC: Let us assume that $\langle p, q\rangle$ has rank $k$ and $\langle p, q\rangle \Vdash \forall^{\text {st }} m \in$ $\mathbb{N} \exists v \phi(m, v)$. Let $E_{m}=$

$$
\left\{\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathbb{H} \mid\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\langle p, q\rangle \wedge\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi\left(m, \dot{G}_{n}\right) \text { for some } n>k\right\} .
$$

By an argument like the one in Lemma 5.3 it follows that for every $m$ and every $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \leq\langle p, q\rangle$ there is $\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \in$ $E_{m}$. [We note that the $E_{m}$ may be proper classes, but by the Reflection Principle there is a set $S$ such that $\langle p, q\rangle \in S$ and for every $m$ and every $\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \in S$ there is $\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \in S \cap E_{m}$. The classes $E_{m}$ can be replaced by the sets $S \cap E_{m}$ in the argument below.]

We let $\left\langle m^{\prime},\left\langle p^{\prime}, q^{\prime}\right\rangle\right\rangle \mathbf{R}\left\langle m^{\prime \prime},\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle\right\rangle$ iff
$\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \leq\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\langle p, q\rangle \wedge m^{\prime \prime}=m^{\prime}+1 \wedge\left\langle p^{\prime}, q^{\prime}\right\rangle \in E_{m^{\prime}} \wedge\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \in E_{m^{\prime \prime}}$.
Applying ADC to the relation $\mathbf{R}$ we obtain a sequence $\left\langle\left\langle p_{m}, q_{m}\right\rangle\right|$ $m \in \mathbb{N}\rangle$ such that $\left\langle p_{0}, q_{0}\right\rangle \leq\langle p, q\rangle$ and, for each $m,\left\langle p_{m+1}, q_{m+1}\right\rangle \leq$ $\left\langle p_{m}, q_{m}\right\rangle$ and $\left\langle p_{m}, q_{m}\right\rangle \Vdash \phi\left(m, \dot{G}_{n}\right)$ for some $n \in \mathbb{N}, n>k$. Let $\operatorname{rank} q_{m}=\ell_{m}$ and let $n_{m}<\ell_{m}, n_{m}>k$, be the least such $n$.

As in the proof of Lemma 7.3, let $p_{\infty}=\bigcup_{m=0}^{\infty} C_{m}$ where $C_{m}=\{a \in$ $\left.p_{m}| | a \mid \leq \nu_{p_{m}}(m)\right\}$. We recall that $p_{\infty} \backslash p_{m}$ is thin for every $m$; hence $\left\langle p_{\infty}, q_{m}\right\rangle \Vdash \phi\left(m, \dot{G}_{n_{m}}\right)$. We define a function $q_{\infty} \in \mathbb{Q}$ of rank $k+1$ as follows: If $a \in C_{m} \backslash \bigcup_{j=0}^{m-1} C_{j}$ then

$$
\begin{aligned}
& q_{\infty}(a)=\left\{\left\langle x_{0}, \ldots,, x_{k}\right\rangle \mid x_{k} \text { is a function } \wedge \operatorname{dom} x_{k}=\mathbb{N} \wedge \forall j \leq m\right. \\
& \exists y_{k}, \ldots, y_{\ell_{j}-1}\left(\left\langle x_{0}, \ldots, x_{k-1}, y_{k}, \ldots, y_{\ell_{j}-1}\right\rangle \in q_{j}(a) \wedge x_{k}(j)=y_{n_{j}}\right) \wedge \\
& \left.\forall j>m\left(x_{k}(j)=0\right)\right\} .
\end{aligned}
$$

By "Loś's Theorem", $\left\langle p_{\infty}, q_{\infty}\right\rangle \Vdash$ " $\dot{G}_{k}$ is a function with dom $\dot{G}_{k}=$ N."

Now assume that $\widehat{\mathfrak{N}}_{A} \vDash \forall^{\text {st }} m \exists v \phi(m, v)$. Then there is $\langle p, q\rangle \in \mathcal{G}$ such that $\langle p, q\rangle \Vdash \forall^{\text {st }} m \exists v \phi(m, v)$. By the above discussion, there is a condition of the form $\left\langle p_{\infty}, q_{\infty}\right\rangle$ such that $\left\langle p_{\infty}, q_{\infty}\right\rangle \in \mathcal{G}$; hence
$\widehat{\mathfrak{N}}_{A} \vDash " G_{k}$ is a function with $\operatorname{dom} G_{k}=\mathbb{N}$." Fix $m \in M$; let $\widehat{\mathfrak{N}}_{A} \vDash$ $G_{k}(m)=G_{\ell} ;$ we can assume $\ell>k$. Then there is some $\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathcal{G}$, $\left\langle p^{\prime}, q^{\prime}\right\rangle \leq\left\langle p_{\infty}, q_{\infty}\right\rangle$, such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \dot{G}_{k}(m)=\dot{G}_{\ell}$.

Let $\sigma_{m}=k \cup\left\{n_{m}\right\}$ and $\sigma^{\prime}=k \cup\{\ell\}$.
From $\left\langle p_{\infty}, q_{m}\right\rangle \Vdash \phi\left(m, \dot{G}_{n_{m}}\right)$ and Lemma 6.3 it follows that the condition $\left\langle p_{\infty}, q_{m}\left\lceil\sigma_{m}\right\rangle \Vdash \phi\left(m, \dot{G}_{k}\right)\right.$. We see from the construction that $\left\langle p^{\prime} \cap p_{m}, q^{\prime} \upharpoonright \sigma^{\prime}\right\rangle \leq\left\langle p^{\prime} \cap p_{m}, q_{m} \upharpoonright \sigma_{m}\right\rangle$. As $p^{\prime} \backslash p_{m}$ is thin, we have $\left\langle p^{\prime}, q^{\prime} \upharpoonright \sigma^{\prime}\right\rangle \Vdash \phi\left(m, \dot{G}_{k}\right)$, and by Lemma 6.3 again, $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \phi\left(m, \dot{G}_{\ell}\right)$. From $\left\langle p^{\prime}, q^{\prime}\right\rangle \in \mathcal{G}$, we conclude that $\widehat{\mathfrak{N}}_{A} \vDash \phi\left(m, G_{\ell}\right)$ and hence also $\widehat{\mathfrak{N}}_{A} \vDash \phi\left(m, G_{k}(m)\right)$.
$\mathbf{C C}^{\text {st }}$ : Assuming $\langle p, q\rangle \Vdash \forall^{\text {st }} m \in \mathbb{N} \exists^{\text {st }} v \phi(m, v)$, we can require in the definition of $E_{m}$ that $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \mathbf{s t}\left(\dot{G}_{n}\right)$, i.e., $\left\langle p^{\prime}, q^{\prime}\right\rangle \Vdash \dot{G}_{n}=c_{z_{n}}$ for a uniquely determined $z_{n}$. In the definition of $q_{\infty}(a)$ we can let $x_{k}=\left\langle z_{n_{j}} \mid j \in \mathbb{N}\right\rangle$. Then $\left\langle p_{\infty}, q_{\infty}\right\rangle \Vdash \operatorname{st}\left(\dot{G}_{k}\right)$ and $\widehat{\mathfrak{N}}_{A} \vDash \boldsymbol{s t}\left(G_{k}\right)$.

Let $\mathbf{B S C T}^{\prime}$ be the theory obtained from $\mathbf{B S P T}^{\prime}$ by adding ADC and strengthening $\mathbf{S P}$ to $\mathbf{C C}$; analogously for $\mathbf{I S C T}^{\prime}$. The last two theorems of this section follow by the same arguments as those used to prove Theorems 7.5 and 7.6.
Theorem 7.8. The theory $\mathbf{I S C T}^{\prime}$ is a conservative extension of $\mathbf{Z F} c$.
Theorem 7.9. The theory $\mathbf{B S C T}^{\prime}$ is a conservative extension of $\mathbf{Z F}$ c.

## 8. Final Remarks

### 8.1. Open problems.

(1) Are the theories $\mathbf{B S C T}^{\prime}+\mathbf{S N}$ and $\mathbf{I S C T}^{\prime}+\mathbf{S N}$ (defined above) conservative extensions of $\mathbf{Z F} c$ ?

We do not know whether $\widehat{\mathfrak{N}}_{A}$ for $I=\mathcal{P}^{\text {fin }}(A)$ with uncountable $A$ satisfies $\mathbf{S N}$. In the absence of AC, a way to formulate and prove a suitable analog of Lemma 5.7 is not obvious.
(2) Are the theories BSPT and ISPT conservative extensions of ZF? Are the theories BSCT and ISCT conservative extensions of $\mathbf{Z F} c$ ?

Here BSPT is obtained from $\mathbf{B S P T}^{\prime}$ by strengthening (Bounded) Idealization to allow arbitrary parameters; similarly for the other theories. The likely answer is yes; the obvious approach is to iterate the forcing used to prove the primed versions. Spector develops iterated extended ultrapowers in [39]. His method would require nontrivial adaptations in our framework, but it is likely to work provided the answer to problem (1) is yes. The ultimate result would be that $\mathbf{B S C T}+\mathbf{S N}$ and $\mathbf{I S C T}+\mathbf{S N}$ are conservative extensions of $\mathbf{Z F} c$.
(3) Does every countable model of $\mathbf{Z F}$ have an extension to a model of $\mathrm{BSPT}^{\prime}$ ?

The likely answer is again yes, using a suitable iteration of extended ultrapowers.
(4) Is SPOT $+\mathbf{S C}$ a conservative extension of $\mathbf{Z F}$ ?
8.2. Forcing with filters. A more elegant and potentially more powerful notion of forcing is obtained by replacing $\mathbb{P}$ with

$$
\widetilde{\mathbb{P}}=\{\mathcal{P} \mid \mathcal{P} \text { is a filter of unbounded subsets of } I\}
$$

where $\mathcal{P}^{\prime}$ extends $\mathcal{P}$ iff $\mathcal{P} \subseteq \mathcal{P}^{\prime}$. "Lośs Theorem" 4.6 then takes the form: $\langle\mathcal{P}, q\rangle \Vdash \phi\left(\dot{G}_{n_{1}}, \ldots, \dot{G}_{n_{s}}\right)$ iff $\operatorname{rank} q=k>n_{1}, \ldots, n_{s}$ and $\exists p \in \mathcal{P} \forall i \in p \forall\left\langle x_{0}, \ldots, x_{k-1}\right\rangle \in q(i) \phi\left(x_{n_{1}}, \ldots, x_{n_{s}}\right)$.
The forcing notion $\mathbb{P}$ we actually use amounts to restricting oneself to principal filters.
8.3. Zermelo set theory. Similar results can be obtained for theories weaker than $\mathbf{Z F}$. Let $\mathbf{Z}=\mathbf{Z F}$ - Replacement be the Zermelo set theory, and let BT denote Transfer for bounded formulas. In the proof of Proposition 4.2 the extended ultrapower can be replaced by the extended bounded ultrapower (see Chang and Keisler [6], Sec. 4.4, for a discussion of ordinary bounded ultrapowers). This proves that $\mathbf{S P O T}^{-}=\mathbf{Z}+\mathbf{O}+\mathbf{B T}+\mathbf{S P}$ is a conservative extension of $\mathbf{Z}$. With some modifications, this theory can be taken as an axiomatization of the internal part of nonstandard universes of Keisler [6, 27] (the superstructure framework for nonstandard analysis). Analogous results can be obtained for $\mathbf{S C O T}^{-}, \mathbf{B S P T}^{-}$and $\mathbf{B S C T}^{-}$.
8.4. Weaker theories. Reverse Mathematics has as its goal the calibration of the exact set-theoretic strength of the principal results in ordinary mathematics. One of its chief accomplishments is the discovery that, with a few exceptions, every theorem in ordinary mathematics is logically equivalent (over $\mathbf{S}_{1}$ ) to one of the five subtheories $\mathbf{S}_{1}-\mathbf{S}_{5}$ of second-order arithmetic $\mathbf{Z}_{2}$ (Simpson [35, p. 33]), known collectively as "The Big Five." Here $\mathbf{S}_{1}$ is the weakest of the five theories, the second-order arithmetic with recursive comprehension axiom, also denoted $\mathbf{R C A}_{0}$, and $\mathbf{S}_{2}$, known as $\mathbf{W K L}_{0}$, is obtained by adding the Weak König's Lemma to the axioms of $\mathbf{R C A}_{0}$. We refer to Simpson 35 for a comprehensive introduction to Reverse Mathematics.

Keisler and others extended the ideas of Reverse Mathematics to the nonstandard realm. In Keisler's paper [28] it is established that if $\mathbf{S}$ is any of the "Big Five" theories above, then $\mathbf{S}$ has a conservative extension ${ }^{*} \mathbf{S}$ to a theory in the language with an additional unary
predicate st; the axioms of ${ }^{*} \mathbf{S}$ include $\mathbf{O}, \mathbf{S P}$ and, for the theories stronger than $\mathbf{W K L}_{0}$, also FOT (First-Order Transfer).

In a somewhat different direction, there is an extensive body of work by van den Berg, Sanders and others (see [5], [32] and the references therein) devoted to determining the exact proof-theoretic strength of particular results in infinitesimal ordinary mathematics. Substantial parts of it can be carried out in these and other elementary systems for nonstandard mathematics, for example Nelson 30 and Sommer and Suppes [37]. However, these systems do not enable the natural reasoning as practiced in analysis. They are usually formalized in the language of second-order arithmetic or type theory. Basic objects of ordinary analysis, such as real numbers, continuous functions and separable metric spaces, have to be represented in these theories via suitable codes, and the results may have to be presented "up to infinitesimals," because the full strength of the Transfer principle or the Standard Part principle is not available. The focus of this paper is on theories like SPOT or SPOT ${ }^{-}$, which axiomatize both the traditional and the nonstandard methods of ordinary mathematics in the way they are customarily practiced. Rather than looking for the weakest principles that enable a proof of a given mathematical theorem, we formulate theories that are as strong as possible while still effective (conservative over $\mathbf{Z F}$ ) or semi-effective (conservative over $\mathbf{Z F} c$ ).
8.5. Finitistic proofs. The model-theoretic proof of Proposition 4.2 as given here is carried out in ZF. Using techniques from Simpson [35], Chapter II, esp. II. 3 and II.8, it can be verified that the proof goes through in $\mathbf{R C A}_{0}$ (w.l.o.g. one can assume that $M \subseteq \omega$ ).

The proof of Theorem A from Proposition 4.2 requires the Gödel Completeness Theorem and therefore $\mathbf{W K L}_{0}$; see 35, Theorem IV.3.3. We conclude that Theorem A can be proved in $\mathbf{W K L}_{0}$.

Theorem A, when viewed as an arithmetical statement resulting from identifying formulas with their Gödel numbers, is $\Pi_{2}^{0}$. It is well-known that $\mathbf{W K L}_{0}$ is conservative over PRA (Primitive Recursive Arithmetic) for $\Pi_{2}^{0}$ sentences ([35], Theorem IX.3.16); therefore Theorem A is provable in PRA. The theory PRA is generally considered to correctly capture finitistic reasoning as envisioned by Hilbert [12] (see e.g., Simpson [35], Remark IX.3.18) and Hilbert-Bernays [13, 14] (see Zach [42, p. 417). We conclude that Theorem A has a finitistic proof.

These remarks apply equally to Theorems B - D.
8.6. SPOT and CH. Connes (see for example [7], pp. 20-21) objects to the use of ultrafilters but approves of the Continuum Hypothesis
( $\mathbf{C H}$ ). In the absence of full $\mathbf{A C}$, it is important to distinguish (at least) two versions of $\mathbf{C H}$.

CH: Every infinite subset of $\mathbb{R}$ is either countable or equipotent to $\mathbb{R}$.
$\mathrm{CH}^{+}: \quad \mathbb{R}$ is equipotent to $\aleph_{1}$ (often written $2^{\aleph_{0}}=\aleph_{1}$ ).
The axioms $\mathbf{C H}$ and $\mathbf{C H}^{+}$are equivalent over ZFC, but not over $\mathbf{Z F} c$. It is known that $\mathbf{Z F} c+\mathbf{C H}$ does not imply the existence of any nonprincipal ultrafilters over $\mathbb{N}(\mathbf{C H}$ holds in the Solovay model). We have:

Proposition 8.1. The theory $\mathbf{S P O T}+\mathbf{C H}$ is a conservative extension of $\mathbf{Z F}+\mathbf{C H}$.

Proof. Let $\phi$ be an $\in$-sentence. Then SPOT $+\mathbf{C H} \vdash \phi$ iff SPOT $\vdash$ $(\mathbf{C H} \rightarrow \phi)$ iff $\mathbf{Z F} \vdash(\mathbf{C H} \rightarrow \phi)$ iff $\mathbf{Z F}+\mathbf{C H} \vdash \phi$.

However, it seems clear that Connes has $\mathbf{C H}^{+}$in mind. But $\mathbf{C H}^{+}$ implies that $\mathbb{R}$ has a well-ordering (of order type $\aleph_{1}$ ). From this it easily follows that there exist nonprincipal ultrafilters over $\mathbb{N}$ (for example, Jech [21], p. 478 proves a much stronger result). Thus Connes's position on this matter is incoherent.

Apart from the issue of $\mathbf{C H}$, Connes's repeated criticisms of Robinson's framework starting in 1994 are predicated on the premise that infinitesimal analysis requires ultrafilters on $\mathbb{N}$ (which are incidentally freely used in some of the same works where Connes criticizes Robinson). Our present article shows that Connes's premise is erroneous from the start. $7^{4}$
8.7. External sets. This paper employs only definable external sets, and only in Subsection 3. It is sometimes claimed that the axiomatic approach is inferior to the model-theoretic one because substantial use of external sets is essential for some of the most important new contributions of Robinsonian nonstandard analysis to mathematics, such as the constructions of nonstandard hulls and Loeb measures. The following observations are relevant:

- Except in some very special cases, nonstandard hulls and Loeb measures fall in the scope of set-theoretic mathematics, and the use of $\mathbf{A C}$ in their construction is not an issue.
- Hrbacek and Katz [19] demonstrate that nonstandard hulls and Loeb measures can be constructed in internal-style nonstandard set theories such as BST and IST.

[^4]- The theory BST can seamlessly be extended to HST, a nonstandard set theory that axiomatizes also external sets (see Kanovei and Reeken [24]). In HST the constructions of nonstandard hulls and Loeb measures can be carried out in ways analogous to those familiar from the model-theoretic approach.


## 9. Conclusion

In this paper we establish that infinitesimal methods in ordinary mathematics require no Axiom of Choice at all, or only those weak forms of AC that are routinely used in the traditional treatments. This conclusion follows from the fact that the theory SPOT and its various strengthenings, which do not imply the existence of nonprincipal ultrafilters over $\mathbb{N}$, or other strong forms of $\mathbf{A C}$, are sufficient to carry out infinitesimal arguments in ordinary mathematics (and beyond).

But most users of nonstandard analysis work with hyperreals, and the existence of hyperreals does imply the existence of nonprincipal ultrafilters over $\mathbb{N}$. So it would seem that ultrafilters are needed, after all. However, this view implicitly assumes that set theory like ZFC, based exclusively on the membership predicate $\in$, is the only correct framework for the Calculus.

Historically ${ }^{5}$ the Calculus of Newton and Leibniz was first made rigorous by Dedekind, Weierstrass and Cantor in the 19th century using the $\varepsilon-\delta$ approach. It was eventually axiomatized in the $\in$-language as ZFC. After Robinson's development of nonstandard analysis it was realized that Calculus with infinitesimals also admits a rigorous formulation, closer to the ideas of Leibniz, Bernoulli, Euler (see [1]) and Cauchy (see [3). It can be axiomatized in a set theory using the st- $\in$ language. The primitive predicate st can be thought of as a formalization of the Leibnizian distinction between assignable and inassignable quantities. Such theories are obtained from ZFC by adding suitable versions of Transfer, Idealization and Standardization.

Now that it has been established that the infinitesimal methods do not carry a heavier foundational burden than their traditional counterparts, one can ask the following question. Which foundational framework constitutes a more faithful formalization of the techniques of the 17-19 century masters? For all the achievements of Cantor, Dedekind and Weierstrass in streamlining analysis, built into the transformation they effected was a failure to provide a theory of infinitesimals which were the bread and butter of 17-19 century analysis, until Weierstrass.

[^5]By the yardstick of success in formalization of classical analysis, arguably SPOT, SCOT and other theories developed in the present text are more successful than $\mathbf{Z F}$ and $\mathbf{Z F}+\mathbf{A D C}$.

One can learn to work in the universe of an st- $\in$-set theory intuitively. This universe can be viewed as an extension of the standard set-theoretic universe, either by a (soritical) predicate st (the internal picture) or by new ideal objects (the standard picture); see Fletcher et al. [9, Sec. 5.5, for a detailed discussion ${ }^{6}$ This extended universe has a unique set of real numbers, constructed in the usual way, and containing both standard and nonstandard elements. It also of course has choice functions and ultrafilters over $\mathbb{N}$, just as the universe of ZFC does. Mathematicians concerned about AC can analyze their methods of proof and determine whether a particular result can be carried out in one of the theories considered in this paper. For most if not all of ordinary mathematics, both traditional and infinitesimal, the answer is likely to be affirmative. It then follows from Theorems A-D that these results are just as effective as those provable in respectively $\mathbf{Z F}$ or $\mathbf{Z F}+\mathbf{A D C}$.

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${ }^{6}$ The internal picture fits well with the multiverse philosophy of Hamkins 11]. One of his postulates is Well-foundedness Mirage: Every universe $V$ appears to be illfounded from the point of view of some better universe ([11, Sec. 9). The internal picture proposes something stronger but in the same spirit: the ill-foundedness is witnessed by a predicate st for which $(V, \in, \mathbf{s t})$ satisfies BST. This point is elaborated in (9, Section 7.3.
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Department of Mathematics, City College of CUNY, New York, NY 10031,

Email address: khrbacek@icloud.com
Department of Mathematics, Bar Ilan University, Ramat Gan 5290002 IsRAEL,

Email address: katzmik@math.biu.ac.il


[^0]:    Date: February 9, 2021.
    2020 Mathematics Subject Classification. Primary 26E35, Secondary 03A05, 03C25, 03C62, 03E70, 03H05.
    Key words and phrases. nonstandard analysis; axiom of choice; ultrafilter; forcing; extended ultrapower.

[^1]:    ${ }^{1}$ For example Halmos [10], p. 42; see [4], Sec. 5.7 for further discussion.

[^2]:    ${ }^{2}$ By the hyperreals we mean a proper elementary extension of the reals, i.e., a proper extension that satisfies Transfer. The definite article is used merely for grammatical correctness. Subsets of $\mathbb{N}$ can be identified with real numbers; see $\mathbf{S P} \Rightarrow \mathbf{S P}^{\prime}$ in the proof of Lemma 2.4 for one way to do that.

[^3]:    ${ }^{3}$ In [17] Remark (3) on page 22 it is erroneously claimed that the statement $m_{1}(A)=$ $r$ is equivalent to an internal formula. The existence of the function $m_{1}$ there follows from Standardization, just as in the case of $m$ above.

[^4]:    ${ }^{4}$ A more detailed analysis of Connes's views can be found in Sanders ([33, 2020) and references therein.

[^5]:    ${ }^{5}$ See for example Katz and Sherry [25] and Bair et al. 2].

