# Definably Topological Dynamics of p-Adic Algebraic Groups

Jiaqi Bao and Ningyuan Yao July 13, 2021

#### Abstract

We study the p-adic algebraic groups G from the definable topological-dynamical point of view. We consider the case that M is an arbitrary p-adic closed field and G an algebraic group over  $\mathbb{Q}_p$  admitting an Iwa-sawa decomposition G = KB, where K is open and definably compact over  $\mathbb{Q}_p$ , and B is a borel subgroup of G over  $\mathbb{Q}_p$ . Our main result is an explicit description of the minimal subflow and Ellis Group of the universal definable G(M)-flow  $S_G(M^{\text{ext}})$ . We prove that the Ellis group of  $S_G(M^{\text{ext}})$  is isomorphic to the Ellis group of  $S_B(M^{\text{ext}})$ , which is  $B/B^0$ .

As applications, we conclude that the Ellis groups corresponding to  $\mathrm{GL}(n,M)$  and  $\mathrm{SL}(n,M)$  are isomorphic to  $(\hat{\mathbb{Z}}\times\mathbb{Z}_p^*)^n$  and  $(\hat{\mathbb{Z}}\times\mathbb{Z}_p^*)^{n-1}$  respectively, generalizing the main result of Penazzi, Pillay, and Yao in [23].

# 1 Introduction

In this paper, we consider the topological dynamics of algebraic groups over a p-adically closed field. The model theoretic approach to topological dynamics was introduced by Newelski [19], then developed by a number of papers, including [20], [1] and [16], and now called definable topological dynamics. Definable topological dynamics studies the action of a group G definable in a structure M on its type space  $S_G(M)$  and tries to link the invariants suggested by topological dynamics with model-theoretic invariants. For example, in the case when Th(M) is stable, Newelski proved that the Ellis group of  $S_{G,ext}(M) = S_G(M)$  is isomorphic to the definable Bohr compactification  $G^*/G^{*00}$  of G, where  $G^*$  is the interpretation of G in a saturated elementary extension [21], which is another formulation of fundamental theorems

of stable group theory by replacing the generic types with the Ellis group. Newelski tried to generalize this result to some tame unstable context where generic types may not exist, he conjectured in [19] that such isomorphism holds ture for NIP context.

A considerable amount of work was motivated by the Ellis group conjecture. The first counterexample of this conjecture is found in [9], where the authors showed that Newelski's conjecture fails in the case of  $G = \mathrm{SL}(2,\mathbb{R})$ . G. Jagiella provided a range of counterexamples by extending results to groups over  $\mathbb{R}$  with compact-torsion-free decomposition in [12], and the second author of this paper showed that their results could be extended to any elementary extension M of an o-minimal extension of the reals [35]. Namely, the Ellis groups of  $S_{G,ext}(M)$  is isomorphic to the Ellis groups of  $S_G(\mathbb{R})$ . The study of  $\mathrm{SL}(2,\mathbb{Q}_p)$  in [23] provided another counterexample. Kirk showed in [15] that  $\mathrm{SL}(2,\mathbb{C}((t)))$  is also a counterexample. In fact, The main results of [9, 23, 15] showing that the Ellis group corresponding to  $\mathrm{SL}(2,M)$  is isomorphic to  $B^*/B^{*00}$ , where B is the Borel subgroup of  $\mathrm{SL}(2,-)$ , when M is  $\mathbb{R}$ ,  $\mathbb{Q}_p$ , or  $\mathbb{C}((t))$ . A recent result in [14] of G. Jagiella showed that there is a onto homomorphism from the Ellis groups of  $\mathrm{SL}_2(K)$  to  $B^*/B^{*00}$  if K has NIP.

As mentioned in [13], one may generalize these results to the case where G has a "nice" decomposition in the p-adic setting. In this paper, We provide a way of computing the Ellis groups for p-adic algebraic groups that admit a definable "Iwasawa decomposition". We finally showed that:

**Theorem.** Let G be a linear algebraic group defined in  $\mathbb{Q}_p$ . Suppose that  $G(\mathbb{Q}_p)$  admits a Iwasawa decomposition G = KB with B a borel subgroup of G, definable over  $\mathbb{Q}_p$ , and K an open compact subgroup of G. Then for any  $M \succ \mathbb{Q}_p$  we have that the Ellis group of  $S_{G,ext}(M)$  is isomorphic to  $B^*/B^{*00} = B^*/B^{*0}$ .

By [34], any algebraic group trigonalizable over  $\mathbb{Q}_p$  has a global definable f-generic (dfg) type, and by [22] any definably compact group over  $\mathbb{Q}_p$  has a global finitly satisfiable generic (fsg) type. So the above decomposition is a kind of "fsg-dfg" decomposition in the model-theoritic view when B is split (trigonalizable) over  $\mathbb{Q}_p$ .

A split reductive algebraic group G over a local field F admits a Iwasawa decomposition G = KB with B a borel subgroup trigonalizable over F and K a maximal compact subgroup of G. We conclude directly from our main theorem that

**Corollary.** If G(M) is a split reductive algebraic group over  $\mathbb{Q}_p$ , then the Ellis group of  $S_{G,ext}(M)$  is isomorphic to  $(\hat{\mathbb{Z}} \times \mathbb{Z}_p^*)^m$  for some  $m \in \mathbb{N}$ . Partic-

ularly, if G is GL(n, M), then then Ellis group of the  $S_{G,ext}(M)$  is isomorphic to  $(\hat{\mathbb{Z}} \times \mathbb{Z}_p^*)^n$ .

In [13], Jagiella showed that if G is a definably connected group definable in an o-minimal expansion of a real closed field M, then the Ellis group of the flow  $S_{G,\text{ext}}(M)$  is abstractly isomorphic to a subgroup of a compact Lie group. Based on our result, we conjecture that

**Conjecture.** Let G be a group definable in p-adic closed field M. Ellis group of the flow  $S_{G,\text{ext}}(M)$  is abstractly isomorphic to a profinite group.

the paper is organized as follows. In the rest of this introduction we recall some notations, definitions and results, from earlier papers, relevant to our results.

In section 2.1, we will study the minimal subflow and the Ellis group of a definable group  $B = A \rtimes N$ , with A a fsg group and H a dfg group, in the NIP environment.

In Section 2.2 we will prove some general results for groups definable over  $\mathbb{Q}_p$  admiting compact-dfg decomposition.

In section 2.3, we prove the main results, on the minimal subflows and Ellis group of the action on G(M) on its type space over  $M^{\text{ext}}$ , where M is an arbitrary p-adic closed field and G a linear algebraic group admits Iwasawa decomposition, making use of the results of Section 2.1 and 2.2.

#### 1.1 Notations

We will assume a basic knowledge of model theory. Good references are |27| and [18]. Let  $\mathbb{T}$  be a complete theory with infinite models. Its language is L and M is the monster model, in which every type over a small subset  $A \subseteq M$ is realized, where "small" means  $|A| < |\mathbb{M}|$ . M, N, M', N' will denote small elementary submodels of M. By x, y, z we mean arbitrary n-variables and  $a, b, c \in \mathbb{M}$  the n-tuples in  $\mathbb{M}^n$  with  $n \in \mathbb{N}$ . Every formula is an  $L_{\mathbb{M}}$ -formula. For an  $L_M$ -formula  $\phi(x)$ ,  $\phi(M)$  denotes the definable subset of  $M^{|x|}$  defined by  $\phi$ , and a set  $X \subseteq M^n$  is definable if there is an  $L_M$ -formula  $\phi(x)$  such that  $X = \phi(M)$ . If  $\bar{X} \subseteq \mathbb{M}^n$  is definable, defined with parameters from M, then  $\bar{X}(M)$  will denote  $\bar{X} \cap M^n$ , the realizations from M, which is clearly a definable subset of  $M^n$ . Suppose that  $X \subseteq \mathbb{M}^n$  is a definable set, defined with parameters from M, then we write  $S_X(M)$  for the space of complete types concentrating on X(M). We use Def(X(M)) the denote the boolean algebra of all M-definable subset of X(M). We use freely basic notions of model theory such as definable type, heir, coheir, .... The book [27] is a possible source. Let A, B be subsets of  $\mathbb{M}$ , and  $p \in S(A)$ , by  $p \upharpoonright B$  we mean

the restriction of p to B if  $A \supseteq B$ , and p|B the unique heir of p over B if  $B \supseteq A$  with A a model and p definable.

### 1.2 Definable topological dynamics

Assume that G is a group. By a (point-transitive) G-flow we mean a compact Hausdorff space X together with a left action of G on X by homeomorphism that contains a dense orbit. A set Y of X is called a subflow if it is a closed subspace of X which is closed under the action of G. A subflow flow is minimal if it has no proper subflows. The minimal subflows are "dynamically indecomposable" and considered to be the most fundamental G-flows. It is easy to see that  $Y \subseteq X$  is minimal iff  $Y = \operatorname{cl}(G \cdot y)$  for each  $y \in Y$ . For each  $g \in G$ , we consider the homeomorphism  $\pi_g : X \to X$  induced by the group action. Let  $X^X$  be the collection of all maps from X to itself, equiped with the product topology, which is a compact Hausdorff space by Tychonoff's theorem. Let E(X) be the closure of the set  $\{\pi_g | g \in G\}$  in  $X^X$ . Then E(X) together with the operation \* of the function composition is a semigroup, and the group action of G on E(X) given by  $g \cdot x = \pi_g * x$  makes E(X) a G-flow. It is natural isomorphism to its own Ellis semigroup. For every  $x \in X$  the closure of its G-orbit is exactly  $E(X)(x) = \{f(x) : f \in E(X)\}$ .

Every minimal subflow of E(X) is a minimal left ideal of the semigroup E(X), and homeomorphic to each other as G-flows. We sometimes use the phrase "minimal subflow of E(X)" to denote the homeomorphism class of minimal subflows of E(X). A minimal subflow I is the closure of the G-orbit of every  $p \in I$ , hence is E(X) \* p. We call  $u \in I$  an idempotent if u \* u = u. We denote the collection of all idempotents of I by J(I). For any  $u \in J(I)$ , (u \* I, \*) is a group with u as its identity. We have I is a disjoint union of u \* I's with  $u \in J(I)$ . All those groups are isomorphic to each other, even for different minimal left ideals. We call these groups the ideal groups and call their isomorphism class the Ellis group of the flow X. For more details, readers need to see Refs.[3, 8].

Now we consider the topological dynamics in the model-theoretic context. Let M be an L-structure. Take a saturated elementary extension  $\mathbb{M}$  of M. If  $U \subseteq M^n$  is definable, by an externally definable subset X of U we mean a subset of U of the form  $Y \cap U$  with Y an  $\mathbb{M}$ -definable subset of  $\mathbb{M}^n$ . By  $X \subseteq_{\text{ext}} U$  we mean X is an externally definable subset of U. We write  $\text{Def}^{\text{ext}}(U)$  for the the boolean algebra of all externally definable subset of U, and  $S_{U,\text{ext}}(M)$  the space of all ultrafilters of  $\text{Def}^{\text{ext}}(U)$ . In model theory, we consider a definable group  $G \subseteq M^n$  acting on its type space  $S_G(M)$ . Clearly  $S_G(M)$  is a G-flow. By [19], the Ellis semigroup of  $S_G(M)$  is  $S_{G,\text{ext}}(M)$ , and the semigroup operation of  $S_{G,\text{ext}}(M)$  can be explicitly described. We call

 $S_{G,\text{ext}}(M)$  the universial definable flow of G over M.

Let  $M^{\mathrm{ext}}$  be an expansion of M by adding predicates for all externally definable subsets of  $M^n$  with  $n \in \mathbb{N}^+$ , and  $L_M^{\mathrm{ext}}$ , the associated language of  $M^{\mathrm{ext}}$ , is a nature expansion of the language L. If  $\mathrm{Th}(M)$  has NIP (see [30] for the details of NIP), then  $\mathrm{Th}(M^{\mathrm{ext}})$  also has NIP, admits quantifier elimination, and all types over  $M^{\mathrm{ext}}$  are definable [29]. So we can identify  $S_{ext}(M)$  with  $S(M^{\mathrm{ext}})$  in NIP context. Let  $S_{M,\mathrm{fs}}(\mathbb{M})$  be the space of global types which is finitely satisfiable in M, then the trace of p in M, denoted by  $\mathrm{Tr}_M(p) = \{\phi(M) | \phi(x) \in p\}$  is in  $S_{\mathrm{ext}}(M)$ , and it is easy to see that  $p \mapsto \mathrm{Tr}_M(p)$  is a homeomorphim between  $S_{M,\mathrm{fs}}(\mathbb{M})$  and  $S_{\mathrm{ext}}(M)$ . In NIP theories, replacing  $S_{ext}(M)$  by  $S_G(M^{\mathrm{ext}})$ , we use  $p \mapsto \mathrm{Tr}_M(p)$  to denote the homeomorphism from  $S_{M,\mathrm{fs}}(\mathbb{M})$  to  $S(M^{\mathrm{ext}})$ , and  $q \mapsto q^{\mathbb{M}}$  to denote the inverse map.

We assume NIP throughout this paper. Now we use the notation  $S_G(M^{\text{ext}})$  instead of  $S_{G,\text{ext}}(M)$ . The semigroup operatorn of  $S_G(M^{\text{ext}})$  is defined as follows: For any  $p, q \in S_G(M^{\text{ext}})$ ,  $p * q = \{U \subseteq_{\text{ext}} G | \{g \in M | g^{-1}U \in q\} \in p\}$ 

Note that every type over  $M^{\text{ext}}$  is definable, and thus has a unique heir. By [19], p\*q can also be computered as follows: let  $a \models p$  and  $b \models q \mid (M^{\text{ext}}, a)$ , then  $p*q = \text{tp}(ab/M^{\text{ext}})$ .

# 1.3 NIP, definable amenablity, and connected components

Let  $G = G(\mathbb{M}^n)$  be a definable group. Recall that a type-definable over A subgroup  $H \subseteq G$  is a type-definable subset of G over A, and also a subgroup of G. We say that H has bounded index if  $|G/H| < 2^{|A|+|T|}$ . If  $\mathbb{M}$  has NIP, then there is a smallest type-definable subgroup of bounded index (see [28]), we call it the type-definable connected component of G, and denote it by  $G^{00}$ . We call the intersection of all  $\mathbb{M}$ -definable subgroups of G of finite index the definable connected component, and denote it by  $G^0$ . Clearly, both  $G^{00}$  and  $G^0$  are normal subgroups of G and  $G^{00} \leq G^0$ . Note that by [5],  $G^{00}$  is the same whether computed in T or in  $Th(M^{ext})$  if T has NIP.

In [19], Newelski conjectured that  $G/G^{00}$  is isomorphic to the Ellis group of G in NIP theories. Chernikov and Simon showed that the conjecture holds when G is definably amenable and NIP. Briefly, a group is definably amenable if it admits a global (left) G-invariant Keilser measure, where a global Keisker on G is a finitely additive probabilistic measure on the algebra of all  $\mathbb{M}$ -definable subsets of G.

We now recall the stability-theoretic notion of dividing: A type  $p(x) \in S(B)$  divides over a set  $A \subseteq B$  if there is a formula  $\phi(x, b) \in p$  and infinite A-

indiscernible sequence  $b_0, b_1, b_2, ...$  such that  $\{\phi(x, b_i) : i < \omega\}$  is inconsistent. A nice result of [6] showing that:

**Fact 1.3.1.** G is definably amenable iff there exists  $p \in S_G(\mathbb{M})$  such that for every  $g \in G = G(\mathbb{M})$ , gp does not divide over M. Following the notation of [6] we call a type p as in the right hand side a (global) strongly f-generic, over M, type of G.

Given a definable subset X of G, we define X to be f-generic if for some/any model M over which X is defined any left translate gX of X does not divide over M. As the notation suggests, the property does not depend on the model M chosen. Call a complete type p (over some set of parameters) f-generic iff every formula in p is f-generic. In [6], the authors showed that in NIP theories:

- Fact 1.3.2. If  $\mathbb{M}$  has the NIP and  $G \subseteq \mathbb{M}^n$  is a A-definable group, then G is definably amenable iff it admits a global f-generic type. Moreover, when G is definably amenable, we have:
  - (i)  $p \in S_G(\mathbb{M})$  is f-generic if and only if it is  $G^{00}$ -invariant;
  - (ii) A type-definable subgroup H fixing a global f-generic type is exactly  $G^{00}$ :
- (iii) Any global strongly f-generic type is f-generic;
- (iv) For any  $M \prec M$  containing A, if  $E_G^{\text{ext}} \subseteq S_G(M^{\text{ext}})$  is an Ellis group, then the map  $\sigma: E_G^{\text{ext}} \to G/G^{00}$  defined by  $p \mapsto p/G^{00}$  is an isomorphism.

Among the strongly f-generics, there are two extreme case:

- (1) There is a small submodel M such that every left G-translate of  $p \in S_G(\mathbb{M})$  is finitely satisfiable in M, we call such types the fsg (finitely satisfiable generic);
- (2) There is a small submodel M such that every left G-translate of  $p \in S_G(\mathbb{M})$  is definable over M, we call such types the dfg (definable f-generic).

Clearly, both fsg and dfg groups are definably amenable. We now discuss these two cases. Let  $\operatorname{Stab}_l(p)$  denotes the stabilizer of p with respect to the left group action, and  $\operatorname{Stab}_r(p)$  the stabilizer of p with respect to the right group action. By [10] we have:

Fact 1.3.3. Let G be an A-definable fsg group witnessed by a global type p and a small model M. Then:

(i) Any left (right) translate of p is a global generic type and is finitely satisfiable in any small model  $N \subseteq A$ .

- (ii)  $G^{00} = \operatorname{Stab}_{l}(p) = \operatorname{Stab}_{r}(p)$ .
- (iii) Let q be a global generic type, then q is finitely satisfiable in any small model  $N \subseteq A$  hence can be considered as a type in  $S_G(N^{\text{ext}})$ . Moreover q is a generic type in the G(N)-flow  $S_G(N^{\text{ext}})$ .
- (iv) If  $A \subseteq N$ , then G(N)-flow  $S_G(N^{\text{ext}})$  has a unique minimal subflow Gen(G(N)), which where is the space of all generic types in  $S_G(N^{\text{ext}})$ .
- (v) For any  $A \subseteq N, N' \prec M$ ,  $\operatorname{Gen}(G(N))$  is homeomorphic to  $\operatorname{Gen}(G(N'))$  via  $p \mapsto \operatorname{Tr}_N(p^M)$  (So we simply call it the generic type space of G, and denote it by  $\operatorname{Gen}(G)$ ).
- (vi)  $\mathcal{I} = \operatorname{Gen}(G)$  is a two-sided ideal.

**Lemma 1.3.4.** Suppose that  $G \subseteq \mathbb{M}^n$  is a group definable over M admitting fsg and  $\mathcal{I} \subseteq S_G(M^{\text{ext}})$  is the minimal subflow. Then

- (i) For any  $q \in \mathcal{I}$ , the Ellis group contains q is  $q * \mathcal{I} = q * S_G(M^{\text{ext}})$ .
- (ii) For any  $q \in \mathcal{I}$  and  $\operatorname{tp}(a/M^{\operatorname{ext}}), \operatorname{tp}(b/M^{\operatorname{ext}}) \in S_G(M^{\operatorname{ext}})$ , we have that  $q * \operatorname{tp}(a/M^{\operatorname{ext}}) = q * \operatorname{tp}(b/M^{\operatorname{ext}})$  iff  $a/G^{00} = b/G^{00}$ .
- (iii) For each  $q \in \mathcal{I}$  and  $r \in S_G(M^{ext})$ , there is  $s \in q * \mathcal{I}$  such that q = s \* r.

*Proof.* (i): let  $u \in J(\mathcal{I})$  be such that  $q \in u * \mathcal{I}$ . Since  $u * \mathcal{I}$  is a group, we have  $u * \mathcal{I} = q * u * \mathcal{I} = q * \mathcal{I}$  is the Ellis group containing q. Clearly, we have  $q * \mathcal{I} \subseteq q * S_G(M^{\text{ext}})$ . Since  $\mathcal{I}$  is a two-sided ideal,  $u * S_G(M^{\text{ext}}) \subseteq \mathcal{I}$ , thus

$$q * S_G(M^{\text{ext}}) = q * u * S_G(M^{\text{ext}}) \subseteq q * \mathcal{I}.$$

So  $q * \mathcal{I} = q * S_G(M^{\text{ext}})$  as required.

- (ii): Since  $q * S_G(M^{\text{ext}})$  is the Ellis group generated by q, we see that (ii) can be concluded directly from Fact 1.3.2(iv).
- (iii): Suppose that  $r \vdash k/G^{00}$  and  $q \vdash t/G^{00}$ . By Fact 1.3.2 there is  $s \in q * \mathcal{I}$  such that  $s \vdash (tk^{-1}/G^{00})$ . Now  $s * r \in s * \mathcal{I} = q * \mathcal{I}$  and  $(s * r)/K^{00} = q/K^{00}$ , so s \* r = q as required.

We now discuss the dfg groups.

**Fact 1.3.5.** [2] Let  $B \subseteq \mathbb{M}^m$  be a group definable over M, and  $p \in S_B(\mathbb{M})$  is a global f-generic type. If p is definable over M, then

- $(i)\ \ \textit{Every left $G$-translate of $p$ is definable over $M$;}$
- (ii)  $G^{00} = G^0 = \text{Stab}_l(p);$
- (iii)  $G \cdot p$  is closed, and hence a minimal subflow of  $S_G(\mathbb{M})$ .

**Fact 1.3.6.** [5] Suppose that  $G \subseteq \mathbb{M}^n$  is a dfg group definable over M. Let  $\mathcal{J} \subseteq S_G(M^{\text{ext}})$  be a minimal subflow. Then

(i) G has a global dfg type with respect to  $M^{\rm ext}$  in  ${\rm Th}(M^{\rm ext})$ .

(ii) Let  $N^*$  be a extension of  $M^{\text{ext}}$  and  $p \in \mathcal{J}$ , then the unique heir  $\bar{p} \in S_G(N^*)$  of p is an f-generic type. Moreover any  $G(N^*)$ -translate of  $\bar{p}$  is an global heir of some  $q \in S_G(M^{\text{ext}}) * p$ .

**Lemma 1.3.7.** Suppose that  $G \subseteq \mathbb{M}^n$  is a dfg group definable over M. Let  $\mathcal{J} \subseteq S_G(M^{\mathrm{ext}})$  be a minimal subflow. Then  $\mathcal{J}$  is an Ellis subgroup of  $S_B(M^{\mathrm{ext}})$ .

Proof. Let  $\mathbb{M}^* \succ M^{\text{ext}}$  be a saturated extension. Let  $p \in \mathcal{J}$  and  $\bar{p} \in S_G(\mathbb{M}^*)$  the unique heir of p over  $\mathbb{M}^*$ . By Fact 1.3.6, the G-orbit  $G \cdot \bar{p}$  is homeomorphic to  $\mathcal{J}$  via the map  $\bar{q} \mapsto \bar{q} \upharpoonright M^{\text{ext}}$ . Since  $\bar{p}$  is  $G^{00}$ -invariant,  $G \cdot \bar{p}$  is isomorphic to  $G/G^{00}$  via  $\bar{q} \mapsto \bar{q}/G^{00}$ . So  $q \mapsto q/G^{00}$  is also a ismomorphism from  $\mathcal{J}$  to  $G/G^{00}$ . By Fact 1.3.2, we see that  $\mathcal{J}$  is an Ellis group.

One could conclude directly from the above Fact that

Corollary 1.3.8. If G has dfg, then for any  $q_1, q_2 \in S_G(M^{\text{ext}})$  and  $p \in \mathcal{J}_M^{\text{ext}}$ ,  $q_1 * p = q_2 * p$  iff  $q_1/G^0 = q_2/G^0$ .

# **1.4** Groups definable in $(\mathbb{Q}_p, +, \times, 0, 1)$

We first give our notations for p-adics. By "the p-adics", we mean the field  $\mathbb{Q}_p$ .  $M_0$  denotes the structure  $(\mathbb{Q}_p, +, \times, 0, 1)$ ,  $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$  is the multiplicative group.  $\mathbb{Z}$  is the ordered additive group of integers, the value group of  $\mathbb{Q}_p$ . The group homomorphism  $\nu: \mathbb{Q}_p^* \longrightarrow \mathbb{Z}$  is the valuation map.  $\mathbb{M}$  denotes a very saturated elementary extension  $(\mathbb{K}, +, \times, 0, 1)$  of  $M_0$ . Similarly,  $\mathbb{K}^* = \mathbb{Q}_p \setminus \{0\}$  is the multiplicative group. We sometimes write  $\mathbb{Q}_p$  for  $M_0$  and  $\mathbb{K}$  for  $\mathbb{M}$ .

For convenience, we use  $\mathbb{G}_a$  and  $\mathbb{G}_m$  denote the additive group and multiplicative group of field  $\mathbb{M}$  respectively. So  $\mathbb{G}_a(M_0)$  and  $\mathbb{G}_m(M_0)$  (or  $\mathbb{G}_m(\mathbb{Q}_p)$ ) are  $(\mathbb{Q}_p, +)$  and  $(\mathbb{Q}_p^*, \times)$  respectively.

We will be referring a lot to the comprehensive survey [4] for the basic model theory of the p-adics. A key point is Macintyre's theorem [17] that  $\operatorname{Th}(\mathbb{Q}_p, +, \times, 0, 1)$  has quantifier elimination in the language of rings  $L_{ring}$  together with new predicates  $P_n(x)$  for the n-th powers for each  $n \in \mathbb{N}^+$ . Moreover, for any polynormals  $f, g \in \mathbb{Q}_p[\bar{x}]$ , the relation  $v(f(\bar{x})) \leq v(g(\bar{x}))$  is quantifier-free definable in the Macintyre's language  $L_{ring} \cup \{P_n | n \in \mathbb{N}^+\}$ , in particular it is definable in the language of rings. (See Section 3.2 of [4].) By [7], every type over  $\mathbb{Q}_p$  is definable. A p-adically closed field is a model of pCF :=  $\operatorname{Th}(\mathbb{Q}_p)$ , which has NIP (see [4] for details). The theory pCF also has definable Skolem functions [32].

The p-adic field  $\mathbb{Q}_p$  is a locally compact topological field, with basis given by the sets

$$\mathcal{B}(a,n) = \{ x \in \mathbb{Q}_p \mid x \neq a \land v(x-a) \ge n \}$$

for  $a \in \mathbb{Q}_p$  and  $n \in \mathbb{Z}$ . The valuation ring  $\mathbb{Z}_p$  is compact. The topology is given by a definable family of a definable sets, so it extends to any p-adically closed field M, making M be a topological field (usually not locally compact).

For any  $X \subseteq \mathbb{Q}_p^n$ , the "topological dimension", denoted by  $\dim(X)$ , is the greatest  $k \leq n$  such that the image of X under some projection from  $M_0^n$  to  $M_0^k$  contains an open subset of  $\mathbb{Q}_p^k$ . On the other side, as model-theoretic algebraic closure coincides with field-theoretic algebraic closure ([11], Proposition 2.11), we see that for any model M of pCF the algebraic closure satisfies exchange (so gives a so-called pregeometry on M) and there is a finite bound on the sizes of finite sets in uniformly definable families. If a is a finite tuple from  $M \models p$ CF and B a subset of M then the algebraic dimension of a over B, denoted by  $\dim(a/B)$ , is the size of a maximal subtuple of a which is algebraically independent over B.

When  $X \subseteq \mathbb{Q}_p^n$  is definable, the algebraic dimension of X, denoted by  $\operatorname{alg-dim}(X)$ , is the maximal  $\operatorname{dim}(a/B)$  such that  $a \in X(\mathbb{M})$  and B contains the parameters over which X is defined. It is important to know that when  $X \subseteq \mathbb{Q}_p^n$  is definable, then its algebraic-dimension coincides with its "topological dimension", namely  $\dim(X) = \operatorname{alg-dim}(X)$ . As a conclusion, for any definable  $X \subseteq \mathbb{Q}_p^n$ ,  $\dim(X)$  is exactly the algebraic geometric dimension of its Zariski closure.

By a definable manifold  $X \subseteq \mathbb{Q}_p^n$  over a subset  $A \subseteq \mathbb{Q}_p$ , we mean a topological space X with a covering by finitely many open subsets  $U_1, ..., U_m$ , and homeomorphisms of  $U_i$  with some definable open  $V_i \subseteq \mathbb{Q}_p^n$  for i = 1, ..., m, such that the transition maps are A-definable and continuous. If the transition maps are  $C^k$ , then we call X a definable  $C^k$  manifold over  $\mathbb{Q}_p$  of dimension n. A definable group  $G \subseteq \mathbb{Q}_p^n$  can be equipped uniquely with the structure of a definable manifold over K such that the group operation is  $C^{\infty}$  (see [24] and [22]). The facts described above work for any  $M \models p$ CF.

By [22], a group  $K \subseteq \mathbb{M}^l$  definable over  $M_0$  has fsg iff it is definably compact over  $M_0$ . The type-definable connected component  $K^{00}$  coincides with its definable connected component  $K^0$ , which is also the kernel of the standard part map st :  $K \to K(M_0)$ . Namely,  $K^0$  is the set of infinitesimals of K over  $M_0$ .

By [25], a group  $H \subseteq \mathbb{M}^k$  definable over  $M_0$  has dfg iff there is a trigonalizable algebraic group A over  $M_0$  and a definable homomorphic  $f: H \to A$  such that both ker(f) and A/im(f) are finite. In particular, any trigonalizable algebraic group over  $M_0$  has dfg.

### 2 Main Results

### 2.1 Semi-product of a fsg group and a dfg group

We now consider a group  $B = B(\mathbb{M})$  definable in a NIP structure M, which could be decomposed into a semi-product  $B = A \rtimes H$ , where A has fsg, H has dfg, and both of the definable over M. Clearly, B is definably amenable. We will study the minimal subflow and the Ellis group of  $S_B(M^{\text{ext}})$  in this section.

**Lemma 2.1.1.**  $B^{00} = A^{00} \times H^0$ .

*Proof.* Since  $A/A^{00}$  and  $H/H^0$  is bounded, we see that  $B/(A^{00} \rtimes H^0)$  is bounded. So  $B^{00} \subseteq A^{00} \rtimes H^0$ . On the other side,  $A/(B^{00} \cap A) \cong AB^{00}/B^{00}$  and  $H/(B^{00} \cap H) \cong HB^{00}/B^{00}$ . So  $B^{00} \cap A$  has bounded index in A and  $B^{00} \cap H$  has bounded index in H, which imples that  $A^{00} \subseteq B^{00}$  and  $H^0 = H^{00} \subseteq B^{00}$ . So  $A^{00} \rtimes H^0 \subseteq B^{00}$ .

**Lemma 2.1.2.** Let  $\mathcal{I}_A^M$  be the (unique) minimal subflow of  $S_A(M^{\mathrm{ext}})$  and  $\mathcal{J}_H^M$  be a minimal subflow of  $S_H(M^{\mathrm{ext}})$ , then  $\mathcal{I}_A^M * \mathcal{J}_H^M$  is a minimal subflow of  $S_B(M^{\mathrm{ext}})$ . Moreover if  $u \in \mathcal{I}_A^M$  and  $v \in \mathcal{J}_H^M$  are idempotents, then u \* v is an idempotent.

*Proof.* Let  $N^* \succ M^{\text{ext}}$  be an  $|M|^+$ -saturated model. Clearly,  $u \vdash A^{00}$  and  $v \vdash H^0$  since they are idempotents. Let  $a_1, a_2, h_1 \in B(N^*)$  such that  $a_1 \models u$ ,  $h_1 \models v | (M^{\text{ext}}, a_1)$ , and  $a_2 \models u | (M^{\text{ext}}, a_1, h_1)$ , let  $h_2 \models v | N^*$ , then

$$u * v * u * v = \text{tp} (a_1 h_1 a_2 h_2 / M^{\text{ext}}) = \text{tp} (a_1 a_2 h_1^{a_2} h_2 / M^{\text{ext}})$$

Since  $H^0$  is a normal subgroup of B, we see that  $h_1^{a_2} \in B^{00}(N^*)$ . By Fact 1.3.6, tp  $(h_2/N^*)$  is  $B^{00}(N^*)$ -invariant, so tp  $(h_2/N^*)$  = tp  $(h_1^{a_2}h_2/N^*)$ , which implies that

$$\operatorname{tp}(a_1 a_2 h_1^{a_2} h_2 / M^{\operatorname{ext}}) = \operatorname{tp}(a_1 a_2 / M^{\operatorname{ext}}) * \operatorname{tp}(h_1^{a_2} h_2 / M^{\operatorname{ext}}) = u * u * v = u * v.$$

So u \* v is an idempotent.

We now show that  $\mathcal{I}_A^M * \mathcal{J}_H^M$  is minimal. It suffices to show that  $\mathcal{I}_A^M * \mathcal{J}_H^M \subseteq S_B(M^{\mathrm{ext}}) * r * s$  for any  $r \in \mathcal{I}_A^M$  and  $s \in \mathcal{J}_H^M$ . Let  $\alpha \in \mathcal{I}_A^M$  and  $\beta \in \mathcal{J}_H^M$ , then there are  $a \in A(N^*)$  and  $h \in H(N^*)$  such that  $\alpha = \operatorname{tp}(a/M^{\mathrm{ext}}) * r$  and  $\beta = \operatorname{tp}(h/M^{\mathrm{ext}}) * s$ . Let  $a' \in A(N^*)$  realize  $r|(M^{\mathrm{ext}}, a, h_0)$  and  $h_0 \in H(N^*)$  such that  $h_0^{a'}/H^0 = h/H^0$ . Let  $h' \models s|N^*$ . Then

$$\operatorname{tp}\left(ah_{0}/M^{\operatorname{ext}}\right) * r * s = \operatorname{tp}\left(ah_{0}a'h'/M^{\operatorname{ext}}\right) = \operatorname{tp}\left(aa'h_{0}^{a'}h'/M^{\operatorname{ext}}\right),$$

and, since  $\operatorname{tp}(h_0^{a'}h'/N^*)$  is the heir of  $\operatorname{tp}(h_0^{a'}h'/M^{\operatorname{ext}})$  by Fact 1.3.6, it is easy to see that

$$\operatorname{tp}\left(aa'h_0^{a'}h'/M^{\operatorname{ext}}\right) = \operatorname{tp}\left(aa'/M^{\operatorname{ext}}\right) * \operatorname{tp}\left(h_0^{a'}h'/M^{\operatorname{ext}}\right)$$

$$= \operatorname{tp}\left(a/M^{\operatorname{ext}}\right) * \operatorname{tp}\left(a'/M^{\operatorname{ext}}\right) * \operatorname{tp}\left(h_0^{a'}h'/M^{\operatorname{ext}}\right)$$

$$= \operatorname{tp}\left(a/M^{\operatorname{ext}}\right) * \operatorname{tp}\left(a'/M^{\operatorname{ext}}\right) * \operatorname{tp}\left(hh'/M^{\operatorname{ext}}\right)$$

$$= \alpha * \beta.$$

This completes the proof.

**Lemma 2.1.3.** Let  $\mathcal{I}_A^M$  and  $\mathcal{J}_H^M$  be minimal subflows of  $S_A(M^{\mathrm{ext}})$  and  $S_H(M^{\mathrm{ext}})$  respectively. Let  $u \in \mathcal{I}_A^M$  and  $v \in \mathcal{J}_H^M$  be idempotents. Then  $u * \mathcal{I}_A^M * \mathcal{J}_H^M$  is the Ellis group in  $S_B(M^{\mathrm{ext}})$  generated by u \* v.

*Proof.* By Lemma 2.1.2,  $\mathcal{I}_A^M * \mathcal{J}_H^M \subseteq S_B(M^{\mathrm{ext}})$  is a miniaml subflow and u \* v is an idempotent. The Ellis group containing u \* v is  $u * v * S_B(M^{\mathrm{ext}}) * u * v$ . We first show that  $u * v * S_B(M^{\mathrm{ext}}) * u * v \subseteq u * \mathcal{I}_A^M * \mathcal{J}_H^M$ . Let  $p \in S_B(M^{\mathrm{ext}})$ .

Take  $a_1, a_2, a_3 \in A$  and  $h_1, h_2, h_3 \in H$  such that

$$a_1 \models u, h_1 \models v | (M^{\text{ext}}, a_1), \ a_2 h_2 \models p | (M^{\text{ext}}, a_1, h_1),$$
  
 $a_3 \models u | (M^{\text{ext}}, a_1, h_1, a_2, h_2), \text{ and}$   
 $h_3 \models v | (M^{\text{ext}}, a_1, h_1, a_2, h_2, a_3).$ 

Then  $u * v * p * u * v = \text{tp}(a_1h_1a_2h_2a_3h_3/M^{\text{ext}})$ . Now

$$a_1h_1a_2h_2a_3h_3 = a_1a_2h_1^{a_2}h_2a_3h_3 = a_1a_2a_3(h_1^{a_2}h_2)^{a_3}h_3.$$

Let  $N^* \prec N^{**}$  be extensions of  $M^{\text{ext}}$  such that  $N^*$  is  $|M|^+$ -saturated and  $N^{**}$  is  $|N^*|^+$ -saturated. Without lose of generality, we may assume that  $a_1 \in A(N^{**})$  realizes the coheir of u over  $N^*$  and  $h_3$  realizes the heir of v over  $N^{**}$ , and  $h_1, a_2, h_2, a_3 \in B(N^*)$ . Since  $\operatorname{tp}(h_3/\mathbb{M}^*)$  is a definable fgeneric type and  $(h_1^{a_2}h_2) \in H(N^*)$ ,  $\operatorname{tp}((h_1^{a_2}h_2)h_3/N^{**})$  is the unique heir of  $\operatorname{tp}((h_1^{a_2}h_2)h_3/M^{\operatorname{ext}})$ . Similarly,  $\operatorname{tp}(a_1/\mathbb{M}^*)$  is a finitely satisfiable generic type, so  $\operatorname{tp}(a_1a_2a_3/N^*)$  is the unique coheir of  $\operatorname{tp}(a_1a_2a_3/M^{\operatorname{ext}})$ . We conclude that

$$tp(a_1 a_2 a_3 (h_1^{a_2} h_2)^{a_3} h_3 / M^{\text{ext}}) = tp(a_1 a_2 a_3 / M^{\text{ext}}) * tp((h_1^{a_2} h_2)^{a_3} h_3 / M^{\text{ext}})$$

Clearly,

$$\operatorname{tp}(a_1 a_2 a_3 / M^{\operatorname{ext}}) = \operatorname{tp}(a_1 / M^{\operatorname{ext}}) * \operatorname{tp}(a_2 a_3 / M^{\operatorname{ext}}) \in u * S_A(M^{\operatorname{ext}}),$$

and

$$\operatorname{tp}((h_1^{a_2}h_2)^{a_3}h_3/M^{\operatorname{ext}}) = \operatorname{tp}((h_1^{a_2}h_2)^{a_3}/M^{\operatorname{ext}}) * \operatorname{tp}(h_3/M^{\operatorname{ext}}) \in S_H(M^{\operatorname{ext}}) * v.$$

By Fact 1.3.3(vi), we have  $u*S_A(M^{\text{ext}}) = u*\mathcal{I}_A^M$ . Clearly,  $S_H(M^{\text{ext}})*v = \mathcal{J}_H^M$ . So  $u*v*S_B(M^{\text{ext}})*u*v \subseteq u*\mathcal{I}_A^M*\mathcal{J}_H^M$ . On the other side, if  $u_1, u_2 \in u*\mathcal{I}_A^M$  and  $v_1, v_2 \in \mathcal{J}_H^M$  such that  $u_1*v_1/B^{00} = u_2*v_2/B^{00}$ , then we conclude that  $u_1/A^{00} = u_2/A^{00}$  and  $v_1/H^0 = v_2/H^0$ . Since  $\mathcal{J}_H^M$  is an Ellis group by Lemma 1.3.7, we have  $u_1 = u_2$  and  $v_1 = v_2$ , which implies that  $p \mapsto p/B^{00}$  is a bijection from  $u*\mathcal{I}_A^M*\mathcal{J}_H^M$  to  $B/B^{00}$ . So  $u*\mathcal{I}_A^M*\mathcal{J}_H^M = u*v*S_B(M^{\text{ext}})*u*v$  is the Ellis group generated by u\*v by Fact 1.3.2(iv).

The above Lemma shows that  $E_A^M * \mathcal{J}_H^M$  is an Ellis group in  $S_B(M^{\mathrm{ext}})$  when  $E_A^M$  is a Ellis group of  $S_A(M^{\mathrm{ext}})$  and  $\mathcal{J}_H^M$  a minimal subflow (or Ellis group) of  $S_H(M^{\mathrm{ext}})$ .

From now on, we use notation  $E_A^M$  to denote a Ellis group in  $S_A(M^{\mathrm{ext}})$  and  $E_B^M = E_A^M * \mathcal{J}_H^M$ .

**Lemma 2.1.4.** Let  $q \in E_A^M$  and  $p \in \mathcal{J}_H^M$ , then  $E_B^M = q * S_B(M^{\mathrm{ext}}) * p$ .

*Proof.* Clearly,  $E_A^M = q * S_A(M^{\text{ext}})$  and  $\mathcal{J}_H^M = S_H(M^{\text{ext}}) * p$ . So  $E_A^M * \mathcal{J}_H^M \subseteq q * S_B(M^{\text{ext}}) * p$ .

Conversely, let  $r \in S_B(M^{\text{ext}})$ . Let  $N^* \succ M^{\text{ext}}$  such that  $N^*$  is  $|M|^+$ -saturated. Take  $a, a_0 \in A(N^*)$  and  $h_0 \in H(N^*)$  such that  $a \models$  the coheir of q over  $(M^{\text{ext}}, a_0, h_0)$  and  $a_0h_0 \models r$ . Let  $h \models p|N^*$  Then  $\operatorname{tp}(h/N^*)$  is f-generic and definable. So the transition  $\operatorname{tp}(h_0h/N^*)$  is definable over  $M^{\text{ext}}$ , and thus an heir of  $\operatorname{tp}(h_0h/M^{\text{ext}})$ . We see that

$$q * r * p = \operatorname{tp}(aa_0h_0h/M^{\operatorname{ext}}) = \operatorname{tp}(aa_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0h/M^{\operatorname{ext}}) \subseteq E_A^M * \mathcal{J}_H^M.$$

This completes the proof.

Corollary 2.1.5. Let  $E_A^M$  and  $\mathcal{J}_H^M$  be Ellis groups in  $S_A(M^{\mathrm{ext}})$  and  $S_H(M^{\mathrm{ext}})$  respectively. Then for any  $r_1, r_2, r^* \in E_A^M$  and  $s_1, s_2, s^* \in \mathcal{J}_H^M$  such that  $r_1 \vdash a_1/A^{00}$ ,  $r_2 \vdash a_2/A^{00}$ ,  $s_1 \vdash h_1/H^0$ ,  $s_2 \vdash h_2/H^0$ ,  $r^* \vdash a_1a_2/A^{00}$ , and  $s^* \vdash h_1^{a_2}h_2/H^0$ . Then

$$(r_1 * s_1) * (r_2 * s_2) = r^* * s^*.$$

*Proof.* It is easy to see from Lemma 2.1.3 and Fact 1.3.2 that  $(r_1 * s_1) * (r_2 * s_2) \vdash a_1 a_2 h_1^{a_2} h_2 / B^{00}$ . On the other side, if  $r^* \vdash a_1 a_2 / A^{00}$  and  $s^* \vdash h_1^{a_2} h_2 / H^0$ , then  $r^* * s^* \vdash a_1 a_2 h_1^{a_2} h_2 / B^{00}$ . So  $(r_1 * s_1) * (r_2 * s_2) = r^* * s^*$  as required.  $\square$ 

### 2.2 Groups with compact-dfg decomposition

We assume in this section that  $G = G(\mathbb{M})$  is a group definable in  $\mathbb{M}$ , with parameters from  $M_0 = \mathbb{Q}_p$ , and G = CH is a decomposition of G, where H is a  $\mathbb{Q}_p$ -definable subgroup of G with dfg, and C a  $\mathbb{Q}_p$ -definable subset of G such that  $C(M_0)$  is definably compact, and open in  $G(M_0)$ .

Since C is open in G, the infinitesimals of C over  $M_0$ , which is the intersection of all  $\mathbb{Q}_p$ -definable open subsets of C, denoted by  $\mu_C$ , coincides with  $\mu_G$ , the infinitesimals of G over  $M_0$ . By the continuity of the group operation, we see that  $\mu_G^g = \mu_G$  for all  $g \in G(M_0)$ . Let  $V_G = G(M_0)\mu_G$ , then it is the subgroup of G consisting of all elements have its standard part in  $G(M_0)$ . It is easy to see that  $V_G \leq N_G(\mu_G) = N_G(\mu_C)$ .

For any  $N \succ M_0$ , we use  $G^0(N)$  to denote  $G^0 \cap G(N)$ . By  $V_G(N)$  we mean set  $G(M_0)\mu_G(N)$ , which is the subgroup of G(N) consisting of all elements have its standard part in  $G(M_0)$ .

Let Y be an N-definable subset of G. By Y(N)/H we mean the set  $\{g/H(N)| g \in Y(N)\}$ . Let X = G/H, we write Def(X(N)) for the boolean algebra of all sets of the form  $\{Y(N)/H| Y \in Def(G(N))\}$ , and  $S_X(N)$  is the space of all ultrafilters of Def(X(N)), similarly for  $Def^{ext}(X)$  and  $S_X(N^{ext})$ .

We now consider quotient space X = G/H, which admits a quotient topology. Let  $\pi$  be the projection from G to X, then it is easy to see that  $\pi$  could be naturally extended to a onto homomorphism from  $S_G(M^{\text{ext}})$  to  $S_X(M^{\text{ext}})$ .

**Lemma 2.2.1.** Let  $g, h \in V_G$  such that  $\operatorname{tp}((g/H)/M_0) = \operatorname{tp}((h/H)/M_0)$ , then there is  $\epsilon_1, \epsilon_2 \in \mu_G$  such that  $\epsilon_1 gH = hH$  and  $g\epsilon_2 H = hH$ .

*Proof.* For any  $g, h \in V_G$ , we have  $\mu_G g = g \mu_G$  and  $\mu_G h = h \mu_G$ . So it sufficies to show that  $g \mu_G H \cap h \mu_G H \neq \emptyset$ .

If  $g\mu_G H \cap h\mu_G H = \emptyset$ , then by compactness there is a  $M_0$ -definable open subgroup D of C such that  $D \supseteq \mu_G$  and  $gDH \cap hDH = \emptyset$ . Since  $\mu_G \subseteq D$ , we see that  $gD = \operatorname{st}(g)D$  and  $hD = \operatorname{st}(h)D$ , and thus  $g/H \in \operatorname{st}(g)D/H$  and  $h/H \notin \operatorname{st}(g)D/H$ . We conclude that  $\operatorname{tp}((g/H)/M_0) \neq \operatorname{tp}((h/H)/M_0)$ . A contradiction.

**Lemma 2.2.2.** Let  $p \in S_G(M^{\text{ext}})$  be  $H^0(M)$ -invariant. Then for any  $k \in K$  and  $h \in H$  such that  $kh \models p$ ,  $H^0(M)^k \subseteq \mu_G H$ .

*Proof.* Let  $h_0 \in H^0(M)$ . Since p is  $H^0(M)$ -invariant, we see that

$$\operatorname{tp}(h_0 k h / M^{\operatorname{ext}}) = \operatorname{tp}(k h / M^{\operatorname{ext}}),$$

and hence  $\operatorname{tp}((h_0k/H)/M_0) = \operatorname{tp}((k/H)/M_0)$ . By Lemma 2.2.1, we have  $h_0kH = k\epsilon H$  for some  $\epsilon \in \mu_G$ . So  $h_0^k \in \mu_G H$  for all  $h_0 \in H^0(M)$  as required.

# 2.3 Minimal subflows and Ellis groups of groups admitting Iwasawa decomposition

In this section we assume that L is the language of the rings,  $M_0 = (\mathbb{Q}_p, +, \times, 0, 1)$  is the standard model of pCF, M will denote an elementary extension of  $M_0$ , and  $L_M^{\text{ext}}$  the associated language of  $M^{\text{ext}}$ . We assume that the  $L_M^{\text{ext}}$ -structure  $\mathbb{M}^*$  is a monster model of  $\text{Th}(M^{\text{ext}})$ , and  $\mathbb{M} = \mathbb{M}^* \upharpoonright L$  is the reduction of  $\mathbb{M}^*$  on L. Clearly,  $\mathbb{M} \succ M_0$  is a monster model of pCF.

We now consider the case that  $G = G(\mathbb{M})$  is a linear algebraic group over  $\mathbb{Q}_p$  admitting a Iwasawa decomposition G = KB, where B is a borel subgroup of G, definable over  $\mathbb{Q}_p$ , and K is a  $\mathbb{Q}_p$ -definable open subgroup of G such that  $K(\mathbb{Q}_p)$  is compact.

We can decompose B into a semi-product  $B = T \rtimes B_u$ , where  $B_u(\mathbb{Q}_p)$  is the maximal unipotent subgroup of  $B(\mathbb{Q}_p)$  and  $T(\mathbb{Q}_p)$  a torus. a basic fact is that  $N_G(B_u) = N_G(B) = B$  (see [31]). Moreover, T is an almost direct product of  $T_{spl}$  and  $T_{an}$ , where  $T_{spl}(\mathbb{Q}_p)$  is  $\mathbb{Q}_p$ -split, thus is isomorphic to  $\mathbb{G}_m^k$ for some  $k \in \mathbb{N}$ , and  $T_{an}(\mathbb{Q}_p)$  is anisotropic, which is compact [26, 33].

For simplity, We assume that  $B = A \rtimes H$  where  $A = T_{an}$  and  $H = S \rtimes B_u$ . By [34], H has dfg and  $H^{00} = H^0 = S^0 \rtimes B_u$ . Let C = KA, it is easy to see that G = CH is a compact-dfg decomposition. By [22], both K and A have fsg,  $K^0 = K^{00} = \mu_G$ , and  $A^0 = A^{00} = \mu_G \cap A$ . By Lemma 2.1.1,  $B^{00} = A^0 \rtimes H^0 = B^0$ , and  $B/B^0 \cong T/T^0$  is commutative.

**Lemma 2.3.1.** Let  $p \in S_G(M^{\text{ext}})$  be  $H^0(M)$ -invariant. Suppose that  $g \models p$ , then  $g = \epsilon b$  for some  $\epsilon \in \mu_G$ ,  $b \in B$  with  $\epsilon, b \in \text{dcl}(M, g)$ .

Proof. Let g = kb' for some  $k \in K$  and  $b' \in B$ . Since pCF has definable Skolem functions (see [32]), we may assume  $k, b' \in \operatorname{dcl}(M, g)$ . By Lemma 2.2.2, we have that  $H^0(M)^k \subseteq \mu_G H$ . Note that  $B_u(M_0) \subseteq H^0(M)$ . So in particular, we have  $B_u(M_0)^k \subseteq \mu_G H \cap V_G$ . Take a standard part map, we have

$$\operatorname{st}(B_u(M_0)^k) = B_u(M_0)^{\operatorname{st}(k)} \subseteq \operatorname{st}(\mu_G H \cap V_G) = H(M_0) \subseteq B(M_0).$$

Thus  $\operatorname{st}(k) \in N_G(B_u(M_0)) = B(M_0)$ . Let  $\epsilon \in \mu_G$  such that  $k = \epsilon \cdot \operatorname{st}(k)$ , then  $g = \epsilon \cdot \operatorname{st}(k)b'$ . Let  $b = \operatorname{st}(k)b'$ , then  $g = \epsilon b \in \mu_G B$  as required. Clearly  $\epsilon, b \in \operatorname{dcl}(M, g)$ .

Now it is easy to see that

Corollary 2.3.2. Let  $\mathcal{I}_{B}^{M}$  be a minimal subflow of  $S_{B}(\mathbb{M}^{\mathrm{ext}})$ ,  $b \models p_{1} \in \mathcal{I}_{B}^{M}$ , and  $g \in G(M)$ . Then there exist  $\epsilon \in \mu_{G} \cap \operatorname{dcl}(M, b, g)$  and  $b' \in B \cap \operatorname{dcl}(M, b, g)$  such that  $bg = \epsilon b' \in \mu_{G}B$ .

*Proof.* Since  $\operatorname{tp}(h_0g/M^{\operatorname{ext}})$  is  $B^0(M)$ -invariant, thus is  $H^0(M)$ -invariant.  $\square$ 

**Lemma 2.3.3.** Let  $g \in G$  and  $\epsilon \in \mu_G$  such that  $\operatorname{tp}(g/M_0, \epsilon)$  is finitely satisfiable in  $M_0$ , then  $\epsilon^g \in \mu_G$ . Particularly, if  $N \succ M_0$  and  $g \in G$  such that tp (g/N) is finitely satisfiable in  $M_0$ , then  $(\mu_G(N))^g \leq \mu_G$ .

*Proof.* Suppose that g and  $\epsilon$  satisfy the condition, and  $\epsilon^g \notin \mu_G$ , then there exists an  $M_0$ -definable open neighbrhood U around  $\mathrm{id}_G$  satisfies  $\epsilon^g \notin U$ . Thus the formula  $(\epsilon^x \notin U)$  is in tp  $(g/M_0, \epsilon)$ . So by the type is finitely satisfiable in  $M_0$ , we know that there exists  $g_0 \in G(M_0)$  such that  $\epsilon^{g_0} \notin U$ , hence is not in  $\mu_G$ . A contradiction.

**Lemma 2.3.4.** Let  $\epsilon_0, \epsilon \in \mu_G$  and  $b_0, b \in B$  such that  $\epsilon_0 b_0 = \epsilon b$ , then  $b_0, b$ are in the same coset of  $B^0$ .

*Proof.*  $bb_0^{-1} = \epsilon^{-1}\epsilon_0 \in \mu_G \cap B = \mu_B$ . Since each  $\emptyset$ -definable subgroup A of B with finite index is open, we have  $\mu_B \models A$ , so  $\mu_B \leq B^0$ , which means  $b_0$  and b are in the same coset of  $B^0$ .

We will freely use the above fact. Note that for any  $b \in B$ , and any finite-index subgroup A of B defiable over  $M_0$ , there is  $b_0 \in B(M_0)$  such that  $bA = b_0 A$ . So the coset  $bB^0$  is defined by a partial type over  $M_0$  for any  $b \in B$ .

Let  $\mathcal{I}_A^M \subseteq S_A(M^{\mathrm{ext}})$  and  $\mathcal{J}_H^M \subseteq S_H(M^{\mathrm{ext}})$  be minimal subflows with  $u \in \mathcal{I}_A^M$  and  $v \in \mathcal{J}_H^M$  are idempotents. Then by Lemma 2.1.2,  $\mathcal{I}_B^M = \mathcal{I}_A^M * \mathcal{J}_H^M$  is a minimal subflow of  $S_B(M^{\mathrm{ext}})$  and  $p_0 = u * v \vdash B^0$  is an idempotent. By Lemma 2.1.3, the Ellis group  $E_B^M = p_0 * \mathcal{I}_B^M$  of  $S_B(M^{\mathrm{ext}})$  equals to  $u * \mathcal{I}_A^M * \mathcal{J}_H^M$ . By Fact 1.3.2(iv),  $\tau : p \mapsto p/B^0$  is an isomorphic from  $E_B^M$  to  $B/B^0$ . Suppose that  $\delta \in B$ , consider the map  $l_\delta : E_B^M \to E_B^M$  defined by  $p \mapsto p_\delta * p$ , where  $p_\delta \in E_B^M$  such that  $p_\delta/B^0 = \delta/B^0$ . Then  $l_\delta$  is a bijection from  $E_B^M$  to itself and  $l_\delta = l_\delta$  iff  $\delta_1/B^0 = \delta_2/B^0$ . Besides, for any  $\delta_1, \delta_2 \in B$ , we have

itself, and  $l_{\delta_1} = l_{\delta_2}$  iff  $\delta_1/B^0 = \delta_2/B^0$ . Besides, for any  $\delta_1, \delta_2 \in B$ , we have  $l_{\delta_1} \circ l_{\delta_2} = l_{\delta_1 \delta_2}$ .

By Lemma 2.3.1, we see that  $p * q \vdash \mu_G B$  for each  $p \in \mathcal{I}_B^M$  and  $q \in$  $S_K(M^{\text{ext}})$ . We now assume that  $u \in \mathcal{I}_A^M$  and  $v \in \mathcal{J}_H^M$  are an idempotents,  $E_B^M = E_A^M * \mathcal{J}_H^M$  is the Ellis group of  $S_B(M^{\text{ext}})$ , and  $p_0 = u * v \in E_B^M$  is the idempotent.

**Lemma 2.3.5.** If  $\delta = ah$  with  $a \in A$  and  $h \in H$ . Let  $q \in E_A^M$  such that  $q/A^0 = a/A^0$ ,  $p \in S_H(M^{\text{ext}})$  such that p is finitely satisfiable in  $M_0$  and  $p/N^0 = h/H^0$ . Then

$$l_{\delta}(p_0) = q * p * p_0.$$

Proof. Assume that  $p_0 = u * v$  with u and v be idempotents of  $E_A^M$  and  $\mathcal{J}_H^M$  respectively. Let  $N^*$  be an  $|M|^+$ -sarurated extension of  $M^{\mathrm{ext}}$ ,  $a_1 \in A(N^*)$  and  $h_1 \in H(N^*)$  such that  $a_1 \models u$  and  $h_1 \models v \mid (M^{\mathrm{ext}}, a_1)$ . Let  $N^{**}$  be an  $|N^*|^+$ -sarurated extension of  $N^*$ . Take  $a_0 \in A$  realizing the generic extension of q over  $N^{**}$ . Take  $h_0 \in H(N^{**})$  realizing p such that  $\mathrm{tp}(h_0/N^*)$  is finitely satisfiable in  $M_0$ . Then

$$q * p * p_0 = \operatorname{tp}(a_0 h_0 a_1 h_1 / M^{\operatorname{ext}}) = \operatorname{tp}(a_0 a_1^{h_0} h_0 h_1 / M^{\operatorname{ext}}).$$

Since  $\operatorname{tp}(h_0/N^*)$  is finitely satisfiable in  $M_0$ , we see that

$$a_1^{h_0} \in (\mu_G(N^*))^{h_0} \cap A(N^{**}) \le \mu_G \cap A(N^{**}) = A^0(N^{**})$$

by Lemma 2.3.3. Now  $\operatorname{tp}(a_0/N^{**})$  is a generic type and thus  $A^0(N^{**})$ -invariant under the right action. So  $\operatorname{tp}(a_0/N^{**}) = \operatorname{tp}(a_0a_1^{h_0}/N^{**})$  is finitely satisfiable in  $M_0$ . We conclude that

$$tp(a_0 a_1^{h_0} h_0 h_1 / M^{\text{ext}}) = tp(a_0 a_1^{h_0} / M^{\text{ext}}) * tp(h_0 h_1 / M^{\text{ext}})$$
$$= tp(a_0 / M^{\text{ext}}) * tp(h_0 h_1 / M^{\text{ext}}),$$

which is in  $E_B^M$  By Lemma 2.1.3. Since

$$\operatorname{tp}(a_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0 h_1/M^{\operatorname{ext}}) \vdash a_0 h_0/B^0 = ah/B^0 = \delta/B = l_\delta(p_0)/B,$$

we have

$$q * p * p_0 = \text{tp}(a_0/M^{\text{ext}}) * \text{tp}(h_0 h_1/M^{\text{ext}}) = l_{\delta}(p_0).$$

**Lemma 2.3.6.** If  $p = l_{\delta}(p_0)$  and  $p' = l_{\delta'}(p_0)$ , then  $p * p' = l_{\delta\delta'}(p_0)$ .

*Proof.* Since  $p \vdash \delta/B^0$  and  $p \vdash \delta'/B^0$ , we have  $p * p' \vdash \delta\delta'/B^0$ . On the other side  $l_{\delta\delta'}(p_0) \vdash \delta\delta'/B^0$ , which implies that  $p * p' = l_{\delta\delta'}(p_0)$  by Fact 1.3.2(iv).  $\square$ 

**Lemma 2.3.7.** Let  $q \in S_G(M^{\text{ext}})$  such that  $p_0 * q \vdash \mu_G \delta_0 B^0$ , let  $p_1 = l_{\delta}(p_0)$ , where  $\delta_0, \delta \in B$ . Then  $p_1 * q \vdash \mu_G \delta_0 B^0$ .

*Proof.* Let  $N^{**} \succ N^* \succ M$ , where  $N^*$  is  $|M|^+$ -saturated and  $N^{**}$  is  $|N^*|^+$ -saturated. Let  $\delta = ah$  with  $a \in A$  and  $h \in H$ .

Let and  $a_0 \in A$  such that  $\operatorname{tp}(a_0/M^{\operatorname{ext}}) \in E_A^{\operatorname{ext}}$  and  $\operatorname{tp}(a_0/N^{**}) \vdash a/A^0$  is finitely satisfiable in  $M_0$ . As  $h/H^0$  is a partial type over  $M_0$ , there is  $h_0 \in H(N^{**})$  such that  $\operatorname{tp}(h_0/N^*) \vdash h/H^0$  is finitely satisfiable in  $M_0$ . Without loss of generality, we may assume that  $\delta, \delta_0 \in B^(N^*)$ .

By Lemma 2.3.5,

$$l_{\delta}(p_0) = \text{tp}(a_0/M^{\text{ext}}) * \text{tp}(h_0/M^{\text{ext}}) * p_0.$$

Let  $b \in B(N^*)$  and  $g \in G(N^*)$  such that  $b \models p_0$  and  $g \models q \mid (M^{\text{ext}}, b)$ , then there are  $c \in \mu_G(N^*)$  and  $b' \in \delta_0 B^0(N^*)$  such that bg = cb'. Now

$$l_{\delta}(p_0) * q = \operatorname{tp}(a_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0/M^{\operatorname{ext}}) * p_0 * q = \operatorname{tp}(a_0h_0cb'/M^{\operatorname{ext}}).$$

By Lemma 2.3.3,  $c^{h_0} \in \mu_G(N^{**})$  and thus

$$a_0h_0cb' = a_0c^{h_0}h_0b' \in a_0\mu_Gh_0\delta_0B^0 = \mu_Ga_0h_0\delta_0B^0 = \mu_G\delta\delta_0B^0.$$

This completes the proof.

Take a generic type  $q_M \in S_K(M^{\text{ext}})$  such that  $q_M \vdash K^0$ . We fix some  $\delta_0 \in B$  such that  $p_0 * q_M \vdash K^0 \delta_0 B^0$ . By Lemma 2.3.7 we have:

Corollary 2.3.8. Write  $p_M = l_{\delta_0^{-1}}(p_0)$ . Then  $l_{\delta}(p_M) * q_M \vdash \mu_G B^0 \delta$  for each  $\delta \in B$ . Particularly,  $p_M * q_M \vdash \mu_G B^0 = K^0 B^0$ 

We now prove that  $q_M * p_M$  is an idempotent in a minimal subflow. We will denote the space of generic types in  $S_K(M^{\text{ext}})$  by  $\mathcal{I}_K^M$ .

**Lemma 2.3.9.** Suppose  $q_1 \in \mathcal{I}_K^M$ ,  $p, p' \in E_B^M$ . If  $p = l_\delta(p_M)$  for some  $\delta \in B$ . Then we have  $q_1 * p * q_M * p' = q_1 * l_\delta(p')$ .

*Proof.* By Corollary 2.3.8,  $p * q_M \vdash \mu_G \delta B^0$ . Without loss of generality, we main assume that

$$q_1 * p * q_M * p' = q_1 * \operatorname{tp}(\epsilon \delta / M^{\operatorname{ext}}) * p'$$

for some  $\epsilon \in \mu_G$ . It is easy to see that  $q_1 * \operatorname{tp}(\epsilon \delta/M^{\operatorname{ext}}) = q_1 * \operatorname{tp}(\delta/M^{\operatorname{ext}})$  since  $q_1$  is generic.

since  $q_1$  is generic. Let  $u \in E_A^M$ , then by Lemma 2.1.4,  $u * S_B(M^{\text{ext}}) * p' \subseteq E_B^M$ . We assume that  $u \vdash A^0 \subseteq \mu_G$  is an idempotent. So  $q_1 * u = q_1$ . We have

$$q_1 * p * q_M * p' = q_1 * \operatorname{tp}(\delta/M^{\operatorname{ext}}) * p' = q_1 * (u * \operatorname{tp}(\delta/M^{\operatorname{ext}}) * p').$$

Assume that  $p' \vdash \delta' B^0$ , then we have  $u * \operatorname{tp}(\delta/M^{\operatorname{ext}}) * p' \vdash \mu_G \delta \delta' B^0$ . Since both  $u * \operatorname{tp}(\delta/M^{\operatorname{ext}}) * p'$  and  $l_{\delta}(p')$  are in  $E_B^M$ , and

$$l_{\delta}(p')/B^0 = (\delta \delta')/B^0 = (u * \operatorname{tp}(\delta/M^{\operatorname{ext}}) * p')/B^0,$$

we conclude that  $l_{\delta}(p') = u * \operatorname{tp}(\delta/M^{\operatorname{ext}}) * p'$  by Fact 1.3.2. Thus

$$q_1 * p * q_M * p' = q_1 * (u * \operatorname{tp}(\delta/M^{\operatorname{ext}}) * p') = q_1 * l_{\delta}(p')$$

as required.

By Lemma 2.3.9, we see immediately that

Corollary 2.3.10.  $q_M * p_M * q_M * p_M = q_M * p_M$ . Namely,  $q_M * p_M$  is an idempotent

We now show that the subflow  $S_G(M^{\text{ext}}) * q_M * p_M$  generated by  $q_M * p_M$  is minimal. We use KA to denote the definable set  $\{ka | k \in K, a \in A\}$ .

**Lemma 2.3.11.** Suppose that  $p \in \mathcal{J}_{H}^{M}$ . Then

$$S_G(M^{\mathrm{ext}}) * p \subseteq S_{KA}(M^{\mathrm{ext}}) * \mathcal{J}_H^M$$
.

*Proof.* Let  $N^*$  be an  $|M|^+$ -saturated extension of  $M^{\text{ext}}$ . Let  $s \in S_G(M^{\text{ext}})$ . Take  $k_0 \in K(N^*)$ ,  $b_0 \in B(N^*)$  and  $b \in B$  such that  $k_0b_0 \models s$  and  $h \models p|N^*$ . Then:

$$s * p = \operatorname{tp}\left(k_0 b_0 h / M^{\operatorname{ext}}\right).$$

Suppose that  $b_0 = a_0 h_0$  with  $a_0 \in A(N^*)$  and  $h_0 \in H(N^*)$ , then s \* p is tp  $(k_0 a_0 h_0 h/M^{\text{ext}})$ .

By Fact 1.3.6, the heir  $p|N^*$  is an f-generic type in  $S_B(N^*)$  and any  $B(N^*)$ -translate of  $p|N^*$  is f-generic and definable over M, so there is  $q^* \in \mathcal{J}_H^M$  such that tp  $(h_0h/N^*) = q^*|N^*$ . Therefore we have:

$$s * p = \operatorname{tp} (k_0 a_0 h_0 h / M^{\operatorname{ext}})$$
  
=  $\operatorname{tp} (k_0 a_0 / M^{\operatorname{ext}}) * \operatorname{tp} (h_0 h / M^{\operatorname{ext}})$   
 $\in S_{KA}(M^{\operatorname{ext}}) * S_H(M^{\operatorname{ext}}).$ 

Therefore  $S_G(M^{\text{ext}}) * p \subseteq S_{KA}(M^{\text{ext}}) * \mathcal{J}_H^M$ .

Proposition 2.3.12. Suppose  $s \in S_G(M^{\text{ext}})$ , then

$$q_M * p_M \in S_G(M^{\text{ext}}) * s * q_M * p_M = \text{cl}(G(M) \cdot (s * q_M * p_M)).$$

Consequently,  $S_G(M^{\text{ext}}) * q_M * p_M$  is a minimal subflow.

*Proof.* By the previous lemma, we may assume that  $s * q_M * p_M = q * p_1$  where  $p_1 = \mathcal{J}_H^M$  and  $q \in S_{KA}(M^{\mathrm{ext}})$ . Let  $N^*$  be an  $|M|^+$  saturated extension of  $M^{\mathrm{ext}}$ . Let  $k_0 \in K(N^*)$  and  $a_0 \in A(N^*)$  such that  $k_0 a_0 \models q$ . Let  $u \in E_A^M$  be an idempotent. Then  $u * \mathrm{tp}(a_0/M^{\mathrm{ext}}) * p_1 \in E_A^M * \mathcal{J}_H^M = E_B^M$ .

If  $\operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1 \vdash \delta/B^0$ , then by Lemma 2.3.9 we see that

$$q_M * l_{\delta^{-1}}(p_M) * q_M * (u * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1) = q_M * p_M.$$

Note that  $q_M * u = q_M$  since  $u \vdash A^0 \subseteq K^0 = \mu_G$ . So we have

$$q_M * p_M = q_M * l_{\delta^{-1}}(p_M) * q_M * (u * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1)$$
  
=  $q_M * l_{\delta^{-1}}(p_M) * q_M * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1.$ 

By Lemma 1.3.4(iii), there is  $r \in q_M * \mathcal{I}_K^M$  such that  $q_M = r * \operatorname{tp}(k_0/M^{\operatorname{ext}})$ . So we have

$$q_M * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1 = r * \operatorname{tp}(k_0/M^{\operatorname{ext}}) * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1.$$

Take  $k \in K(N^*)$  realizing the unique generic (or coheir) extension of r over  $(M^{\text{ext}}, k_0, h_0)$  and  $h \in H$  realizing the heir of  $p_1$  over  $N^*$ . Then  $\operatorname{tp}(h_0 h/N^*)$  the unique heir of some  $p \in \mathcal{J}_H^M$ . We see that

$$r * \operatorname{tp}(k_0/M^{\operatorname{ext}}) * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1$$

$$= (\operatorname{tp}(k/M^{\operatorname{ext}}) * \operatorname{tp}(k_0/M^{\operatorname{ext}})) * (\operatorname{tp}(h_0/M^{\operatorname{ext}}) * \operatorname{tp}(h/M^{\operatorname{ext}}))$$

$$= \operatorname{tp}(kk_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0h/M^{\operatorname{ext}}) = \operatorname{tp}(kk_0h_0h/M^{\operatorname{ext}}).$$

On the other side,

$$r * q * p_1 = r * \operatorname{tp}(k_0 a_0 / M^{\text{ext}}) * p_1 = \operatorname{tp}(k k_0 h_0 h / M^{\text{ext}}).$$

We conclude that

$$q_M * p_M = q_M * l_{\delta^{-1}}(p_M) * r * \operatorname{tp}(k_0/M^{\operatorname{ext}}) * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * p_1$$
  
=  $q_M * l_{\delta^{-1}}(p_M) * r * q * p_1$   
=  $q_M * l_{\delta^{-1}}(p_M) * r * s * q_M * p_M$ ,

which is in  $S_G(M^{\text{ext}}) * s * q_M * p_M$  as required.

Let  $q_M$ ,  $p_M$  be as in above, and  $\mathcal{M} = S_G(M^{\text{ext}}) * q_M * p_M$  be the minimal subflow generated by  $q_M * p_M$ , we now compute the Ellis group E of  $S_G(M^{\text{ext}})$ , which is of the form  $q_M * p_M * \mathcal{M}$ .

**Lemma 2.3.13.** Assume again that  $E_B^M$  is the Ellis group of  $S_B(M^{\text{ext}})$  generated by  $p_M$ . Then

$$q_M * p_M * \mathcal{M} = q_M * E_B^M$$

*Proof.* Let  $p_1 \in E_B^M$ . By Lemma 2.3.9,

$$q_M * p_1 = q_M * p_M * q_M * p_1$$

$$= q_M * p_M * q_M * p_M * q_M * p_1$$

$$= q_M * p_M * (q_M * p_1 * q_M * p_M) \in q_M * p_M * \mathcal{M}.$$

Thus, we have  $q_M * E_B^M \subseteq q_M * p_M * \mathcal{M}$ .

Now we show that  $q_M * p_M * \mathcal{M} \subseteq q_M * p_M * \mathcal{M}$ . By lemma 2.3.11,  $\mathcal{M}$  is a subset of  $S_{KA}(M^{\text{ext}}) * \mathcal{J}_H^M$ , so it suffices to show  $q_M * p_M * S_{KA}(M^{\text{ext}}) * \mathcal{J}_H^M \subseteq q_M * E_B^M$ .

Let  $q \in S_{KA}(M^{\text{ext}})$  and  $p_1 \in \mathcal{J}_H^M$ . By Lemma 2.3.1,  $p_M * q \vdash \mu_G B$ . Let  $N^* \succ M^{\text{ext}}$  be  $|M|^+$ -saturated. Assume that  $p_M * q = \text{tp}(\epsilon a_0 h_0/M^{\text{ext}})$  for  $\epsilon \in \mu_G(N^*) = K^0(N^*)$ ,  $a_0 \in A(N^*)$ , and  $h_0 \in H(N^*)$ .

Let  $k \in K(N^*)$  realize the coheir of  $q_M$  over  $dcl(M^{ext}, \epsilon, a_0, h_0)$  and  $h \models p_1|N^*$ . Then a similar argument as in Proposition 2.3.12 shows that

$$q_M * p_M * q * p_1 = \operatorname{tp}(k\epsilon a_0 h_0 h/M^{\operatorname{ext}})$$

$$= \operatorname{tp}(k\epsilon/M^{\operatorname{ext}}) * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0 h/M^{\operatorname{ext}})$$

$$= q_M * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0 h/M^{\operatorname{ext}}).$$

Let  $u \in E_A^M$  be the idempotent. Then  $q_M * u = q_M$ , so we have

$$q_M * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0 h/M^{\operatorname{ext}}) = q_M * u * \operatorname{tp}(a_0/M^{\operatorname{ext}}) * \operatorname{tp}(h_0 h/M^{\operatorname{ext}})$$
$$\in q_M * E_A^M * \mathcal{J}_H^M = q_M * E_B^M.$$

This completes the proof.

**Theorem 1.** The Ellis group of  $S_G(M^{\text{ext}})$  is isomorphic to Ellis group of  $S_B(M^{\text{ext}})$ . Namely,  $q_M * E_B^M \cong E_B^M$ .

*Proof.* Recall that  $p_M \vdash B^0 \delta_0^{-1}$ . Let  $r: E_B^M \to q_M * E_B^M$  be the map defined by  $p \mapsto q_M * l_{\delta_0^{-1}}(p)$ . Because  $l_{\delta_0^{-1}}$  is a bijection, we know r is an onto map.

Suppose  $p, p' \in E_B^M$  and assume  $p = l_{\delta}(p_M), p' = l_{\delta'}(p_M)$ , then by lemma 2.3.9:

$$r(p) * r(p') = q_M * l_{\delta_0^{-1}\delta}(p_M) * q_M * l_{\delta_0^{-1}\delta'}(p_M) = q_M * l_{\delta_0^{-2}\delta\delta'}(p_M).$$

Now  $l_{\delta_0^{-2}\delta\delta'}(p_M) \vdash \delta_0^{-3}\delta\delta' B^0$  and  $p * p' \vdash \delta_0^{-2}\delta\delta' B^0$ , so

$$l_{\delta_0^{-2}\delta\delta'}(p_M) = l_{\delta_0^{-1}}(p * p'),$$

and which implies that

$$r(p) * r(p') = q_M * l_{\delta_0^{-2} \delta \delta'}(p_M) = q_M * l_{\delta_0^{-1}}(p * p') = r(p * p').$$

So r is a group homomorphism.

Now it remains to show that r is injective. For  $p = l_{\delta}(p_M)$ , we have  $r(p) = q_M * l_{\delta_0^{-1}\delta}(p_M) \vdash \mu_G B^0 \delta_0^{-2} \delta$ . If  $r(p) = q_M * p_M \vdash \mu_G B^0 \delta_0^{-1}$ , then there are  $\epsilon \in \mu_G$  and  $b_1, b_2 \in B^0$  such that  $\epsilon b_1 \delta_0^{-2} \delta = b_1 \delta_0^{-1}$ . We see that  $\epsilon \in \mu_G \cap B \subseteq B^0$ , and conclude immediately that  $\delta/B^0 = \delta_0/B^0$ , thus  $p \vdash B^0$  is the idempotent. So  $\ker(r) = \{ \mathrm{id}_{E_R^M} \}$ , i.e. r is an isompophism.  $\square$ 

Thus finally we have our main theorem:

**Theorem 2.** Suppose that G is a linear algebraic group over  $\mathbb{Q}_p$  admits a Iwasawa decomposition KB, with K open and definably compact over  $\mathbb{Q}_p$  and B a borel subgroup deinable over  $\mathbb{Q}_p$ . Then the Ellis group of  $S_G(M^{\text{ext}})$  algebraically isomorphic to  $B/B^0$ .

**Example.** We now consider the general linear group G(x) = GL(n, x). Then  $G(\mathbb{Q}_p)$  has the Iwasawa decomposition  $G(\mathbb{Q}_p) = K(\mathbb{Q}_p)B(\mathbb{Q}_p)$ , where  $K(\mathbb{Q}_p) = GL(n, \mathbb{Z}_p)$  is a maxiaml open compact subgroup, and  $B(\mathbb{Q}_p)$  is the subgroup consisting of all upper triangular matrices. Since  $B = D \rtimes B_u$ , where D is the subgroup of diagonal matrices and  $B_u$  is the subgroup of strictly upper triangular matrices, we see that  $B/B^0 \cong D/D^0 \cong (\mathbb{G}_m/\mathbb{G}_m^0)^n$ . Now  $\mathbb{G}_m/\mathbb{G}_m^0$  is isomorphic to  $(\hat{\mathbb{Z}} \times \mathbb{Z}_p^*)$ , with  $\hat{\mathbb{Z}} = \lim_{\longleftarrow} \mathbb{Z}/n$  and  $\mathbb{Z}_p^* = \{x \in \mathbb{Z}_p | \nu(x) = 0\}$  by Remark 2.5 in [23]. We finally conclude that the Ellis group corresponding to GL(n,x) is isomorphic to  $(\hat{\mathbb{Z}} \times \mathbb{Z}_p^*)^n$ , independent of the models.

Similarly, for  $G(x) = \mathrm{SL}(n,x)$ , we have that the corresponding Ellis group is isomorphic to  $(\hat{\mathbb{Z}} \times \mathbb{Z}_p^*)^{n-1}$ .

Acknowledgement: The research is supported by The National Social Science Fund of China(Grant No.20CZX050).

# References

- [1] A. Pillay, Topological dynamics and definable groups, The Journal of Symbolic Logic, 78 (2013), p. 657–666.
- [2] A. PILLAY AND N. YAO, On minimal flows, definably amenable groups, and o-minimality,, Adv. Math., 90 (2016), p. 83–502.
- [3] J. Auslander, *Flows and minimal sets*, North-Holland Mathematics Studies, Elsevier Science Publishers, Amsterdam, 1988, pp. 1–34.
- [4] L. Belair, *Panorama of p-adic model theory*, Ann. Sci. Math. Quebec, 36 (2012).
- [5] A. CHERNIKOV, A. PILLAY, AND P. SIMON, External definability and groups in nip theories, Journal of the London Mathematical Society, 90 (2014), pp. 213–240.
- [6] A. Chernikov and P. Simon, Definably amenable nip groups, J. Amer. Math. Soc., 31 (2018), pp. 609–641.

- [7] F. Delon, Définissabilité avec paramètres extérieurs dans  $\mathbb{Q}_p$  et  $\mathbb{R}$ , Proceedings of the American Mathematical Society, 106 (1989), pp. 193–198.
- [8] R. Ellis, Lectures on Topological Dynamics, Benjamin, 1969.
- [9] J. GISMATULLIN, D. PENAZZI, AND A. PILLAY, Some model theory of  $SL(2, \mathbb{R})$ , Fundamenta Mathematicae, 229 (2015), pp. 117–128.
- [10] E. HRUSHOVSKI, Y. PETERZIL, AND A. PILLAY, *Groups, measures, and the nip*, Journal of the American Mathematical Society, 21 (2008), pp. 563–596.
- [11] E. HRUSHOVSKI AND A. PILLAY, Groups definable in local and pseudofinite fields, Israel J. Math., 85 (1994), pp. 203–262.
- [12] G. Jagiella, Definable topological dynamics and real lie groups, Mathematical Logic Quarterly, 61 (2015), p. 45–55.
- [13] G. Jagiella, The ellis group conjecture and variants of definable amenability, Journal of Symbolic Logic, 83 (2018), p. 1376–1390.
- [14] G. Jagiella, Topological dynamics and nip fields, arXiv, (2020).
- [15] T. Kirk, Definable topological dynamics of  $SL(2, \mathbb{C}((t)))$ , arXiv, (2019).
- [16] K. Krupiński, *Definable topological dynamics*, The Journal of Symbolic Logic, 82 (2017), p. 1080–1105.
- [17] A. Macintyre, On definable subsets of p-adic fields, Journal of Symbolic Logic, 41 (1976), p. 605–610.
- [18] D. Marker, *Model Theory: An introduction*, Graduate Texts in Mathematics 217, Springer-Verlag New York Inc, 2002.
- [19] L. Newelski, Topological dynamics of definable group actions, The Journal of Symbolic Logic, 74 (2009), pp. 50–72.
- [20] —, Model theoretic aspects of the ellis semigroup, Israel Journal of Mathematics, 190 (2012), p. 477–507.
- [21] —, Topological dynamics of stable groups, The Journal of Symbolic Logic, 79 (2014), p. 1199–1223.
- [22] A. Onshuus and A. Pillay, Definable groups and compact p-adic lie groups, Journal of the London Mathematical Society, 78 (2008), pp. 233–247.

- [23] D. Penazzi, A. Pillay, and N. Yao, Some model theory and topological dynamics of p-adic algebraic groups, Fundamenta Mathematicae, 247 (2019), pp. 191–216.
- [24] A. PILLAY, On fields definable in  $\mathbb{Q}$ , Archive for Mathematical Logic, 29 (1989), pp. 1–7.
- [25] A. PILLAY AND N. YAO, Definable f-generic groups over p-adic numbers, arXiv, 1911.01833 (2019).
- [26] V. Platonov and A. Rapinchuk, Algebraic Groups and Number Theory, Academic Press Inc, 1994.
- [27] B. Poizat, A course in Model Theory An Introduction to Contemporary Mathematical Logic, Universitext, Springer-Verlag New York Inc, 2000.
- [28] S. Shelah, Minimal bounded index subgroup for dependent theories, Proceedings of the American Mathematical Society, 136 (2008), p. 1087–1091.
- [29] —, Dependent first order theories, continued, ISRAEL JOURNAL OF MATHEMATICS, 173 (2009), pp. 1–60.
- [30] P. Simon, A Guide to NIP Theories, Lecture Notes in Logic, Cambridge University Press, Cambridge, 2015.
- [31] T. A. Springer, *Parabolic subgroups, Borel subgroups, solvable groups*, Birkhäuser Basel, Basel, 2 ed., 1998, pp. 98–113.
- [32] L. VAN DEN DRIES, Algebraic theories with definable skolem functions, Jornal of Symbolic Logic, 49 (1984), pp. 625–629.
- [33] S. Wang, On anisotropic solvable linear algebraic groups, Proceedings of the American Mathematical Society, 84 (1982), pp. 11–15.
- [34] N. YAO, Definable topological dynamics for trigonalizable algebraic groups over Q, Mathematical Logic Quarterly, 65 (2019), p. 376–386.
- [35] N. YAO AND D. LONG, Topological dynamics for groups definable in real closed field, Annals of Pure and Applied Logic, 166 (2015), pp. 261 273.