# Omitting Types Theorem in hybrid-dynamic first-order logic with rigid symbols

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### Abstract

In the the present contribution, we prove an Omitting Types Theorem (OTT) for an arbitrary fragment of hybriddynamic first-order logic with rigid symbols (i.e. symbols with fixed interpretations across worlds) closed under *negation* and *retrieve*. The logical framework can be regarded as a parameter and it is instantiated by some well-known hybrid and/or dynamic logics from the literature. We develop a *forcing* technique and then we study a *forcing property* based on local satisfiability, which lead to a refined proof of the OTT. For uncountable signatures, the result requires compactness, while for countable signatures, compactness is not necessary. We apply the OTT to obtain upwards and downwards Löwenheim-Skolem theorems for our logic, as well as a completeness theorem for its *constructor-based* variant. The main result of this paper can easily be recast in the institutional model theory framework, giving it a higher level of generality.

Keywords: Institution, hybrid logic, dynamic logic, forcing, Omitting Types Theorem

# 1. Introduction

*Kripke semantics and hybrid-dynamic logics.* Modal logics are formalisms for describing and reasoning about multigraphs. These structures appear naturally in many areas of research. For example, in knowledge representation formalisms, role assertions describe relationships between individuals/objects grouped into classes determined by concepts. Linguistic information can be represented by multi-graphs. Other mathematical entities that can be viewed as multi-graphs are transition systems, derivation trees, semantic networks, etc. Therefore, it is useful to think of a Kripke structure in the following way:

- a frame consisting of a set of nodes together with a family of (typed) edge sets, and
- a mapping from the set of nodes to a class of local models that gives meaning to the nodes.

However, modal logics have no mechanisms for referring to the individual nodes in such structures, which is necessary when they are used as representation formalisms. Hybrid logics increase the expressive power of ordinary modal logics by adding an additional sort of symbols called *nominals* such that each nominal is true relative to exactly one point. The history of hybrid logics goes back to Arthur Prior's work [43]. Further developments can be found in works such as [1, 2, 3, 9]. The research on hybrid logics received an additional boost due to the recent interest in the logical foundations of the *reconfiguration paradigm*. Dynamic logics provide a powerful language for describing programs and reason about their correctness. Logics of programs have the roots in the work in the late 1960s of computer scientists interested in assigning meaning to programming languages and finding a rigorous standard for proofs about the programs. In the present contribution, we consider a logical system endowed with features from both hybrid and dynamic logics, which is built on top of many-sorted first-order logic with equality. Despite its complexity, it displays a certain simplicity due to its modular construction, which is a reminiscent of the hybridization of institutions from [39].

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*Applications of hybrid-dynamic logics.* The application domain of the work reported in this contribution refers to a broad range of reconfigurable systems whose states or configurations can be presented explicitly, based on some kind of context-independent data types, and for which we distinguish the computations performed at the local/configuration level from the dynamic evolution of the configurations. This suggests a two-layered approach to the design and analysis of reconfigurable systems, involving:

- *a local view*, which amounts to describing the structural properties of configurations, and
- *a global view*, which corresponds to a specialized language for specifying and reasoning about the way system configurations evolve.

Since configurations can be represented by local models and the dynamic evolution of configurations can be depicted by the accessibility relations of the Kripke structures, hybrid-dynamic logics and their fragments are acknowledged as suitable for describing and reasoning about systems with reconfigurable features. In addition, it is well-known (see e.g., [7]) that hybrid logics specialize to temporal logics [35], description logics [5] and feature logics [45]. Therefore, the area of applications of the present work is rather large and it involves knowledge representation, computational linguistics, artificial intelligence, biomedical informatics, semantic networks and ontologies. We recommend [7] for more information on this topic.

*Omitting Types Theorem (OTT).* In this paper we focus on obtaining an OTT for hybrid-dynamic first-order logic with rigid symbols and sufficiently expressive fragments. Observe that an OTT for the full logic would not necessarily have given us the property for its fragments. For this reason, we work within an arbitrary fragment of hybrid-dynamic first-order logic with rigid symbols, which can be viewed as a parameter. Thus the generality of our proofs is an important feature, since the parameter is instantiated by many concrete hybrid and/or dynamic logical systems which appear in the literature. We provide a version of OTT for countable languages without any restrictions and a version for uncountable languages provided that the fragment in question is compact. We show that compactness is necessary at least for one fragment of the underlying logic. This situation is similar to that described in a theorem by Lindström for first-order logic with only relational symbols [38]. The OTT for countable first-order languages is a result originally from [21]. The extension of the OTT to uncountable languages is from [10]. One of the best known applications of the OTT is a simple proof of the completeness of  $\omega$ -logic (a more complex proof without using the OTT can be found in [40]). In the present contribution, we develop this idea further to provide one important application of OTT to computer science, which is described briefly in the following paragraph.

Formal methods practitioners are often interested in properties that are true of a restricted class of models whose elements are reachable by some constructor operations [6, 31, 22]. For this reason, several algebraic specification languages incorporate features to express reachability and to deal with constructors like, for instance, Larch [34], CASL [4] or CITP [33, 29]. This situation is similar to the one in classical model theory, where the models of  $\omega$ -logic are reachable by the constructors *zero* and *successor*. In the present contribution, the completeness of  $\omega$ -logic is generalized by replacing the signature of arithmetics with an arbitrary vocabulary for which we distinguish a set of constructor operators. Then we apply OTT to obtained completeness of the logical system resulted from restricting the semantics of the underlying fragment of hybrid-dynamic first-order logic with rigid symbols to constructor-based Kripke structures.

*Institutions.* Our approach is rooted in institutional model theory [20], which provides a unifying setting for studying logical systems using category theory. The concept of institution formalizes the intuitive notion of logic, including syntax, semantics and the satisfaction relation between them. The theory of institutions is one major approach in universal logic which promotes the development of logical properties at the most general level of abstraction. However, to make the study available to a broader audience, the authors decided to present the results in a framework given by a concrete logical system, that is, hybrid-dynamic first-order logic with rigid symbols. It should be obvious, at least for the experts in institutions, that the main result, OTT, can be easily cast in a more general framework such as the one provided by the definition of stratified institution [18], similarly to the work reported in [27]. Therefore, the area of applications of our results covers a much broader range of hybrid-dynamic logics than the one mentioned in the present contribution.

*Forcing.* OTT is proved by applying a forcing technique, a method of constructing models based on consistency results. Forcing was invented by Paul Cohen [11, 12] in set theory to prove the independence of the continuum hypothesis from the other axioms of Zermelo-Fraenkel set theory. Robinson [44] developed an analogous forcing method in model theory. In institutional model theory, forcing was introduced in [32] to prove a Gödel Completeness Theorem. It was developed further for stratified institutions [27] to prove the completeness of a large class of hybrid logics. The present contribution extends the forcing introduced in [27] to cover logics with both hybrid and dynamic features and studies a forcing property based on local satisfiability to deliver an Omitting Types Theorem.

*Structure of the paper.* The article is arranged as follows: §2 reviews the framework of many-sorted first-order logic in the institutional setting. §3 introduces all the necessary preliminaries about hybrid dynamic first-order logic with rigid symbols, which expands the base system introduced in §2. §4 presents some necessary technical notions for the arguments that follow later, such as that of a reachable model and a language fragment. §5 develops the basics of the forcing technique in our present context. §6 presents a semantic forcing property that is instrumental in proving the main result of the paper. §7 contains the proof of the main result, an Omitting Types Theorem for both countable and uncountable signatures. §8 gives an application of the main result by establishing a completeness theorem for the constructor-based variant of the logic. §9 establishes Löwenheim-Skolem theorems (upwards and downwards) as consequences of the OTT. §10 shows that for a certain fragment of the logic in question compactness is a necessary condition for the OTT for uncountable signatures to hold.

# 2. Many-sorted first-order logic (FOL)

In this section, we recall the definition of first-order logic as presented in institutional model theory [20].

Signatures. Signatures are of the form (S, F, P), where *S* is a set of sorts,  $F = \{F_{ar \rightarrow s}\}_{(ar,s) \in S^* \times S}$  is a  $(S^* \times S \text{ -indexed})$  set of operation symbols, and  $P = \{P_{ar}\}_{ar \in S^*}$  is a  $(S^* \text{ -indexed})$  set of relation symbols. If  $ar = \varepsilon$  then an element of  $F_{ar \rightarrow s}$  is called a *constant symbol*. Generally, ar ranges over arities, which are understood here as strings of sorts; in other words an arity gives the number of arguments together with their sorts. We overload the notation and let *F* and *P* also denote  $\biguplus_{(ar,s)\in S^*\times S} F_{ar \rightarrow s}$  and  $\biguplus_{ar\in S^*} P_{ar}$ , respectively. Therefore, we may write  $\sigma \in F_{ar \rightarrow s}$  or  $(\sigma : ar \rightarrow s) \in F$ ; both have the same meaning, which is:  $\sigma$  is an operation symbol of type  $ar \rightarrow s$ . Throughout this paper, we let  $\Sigma, \Sigma'$  and  $\Sigma_i$  to range over first-order signatures of the form (S, F, P), (S', F', P') and  $(S_i, F_i, P_i)$ , respectively.

Signature morphisms. A number of usual tricks, such as adding constants, but also, importantly, quantification, are viewed as expansions of the signature, so moving between signatures is common. To make such transitions smooth, a notion of a signature morphism is introduced. A signature morphism  $\varphi: \Sigma \to \Sigma'$  is a triple  $\chi = (\chi^{st}, \chi^{op}, \chi^{rl})$  of maps: (a)  $\chi^{st}: S \to S'$ , (b)  $\chi^{op} = \{\chi^{op}_{ar \to s}: F_{ar \to s} \to F'_{\chi^{st}(ar) \to \chi^{st}(s)} | ar \in S^*, s \in S\}$ , and (c)  $\chi^{rl} = \{\chi^{rl}_{ar}: P_{ar} \to P'_{\chi^{st}(ar)} | ar \in S^*\}$ . When there is no danger of confusion, we may let  $\chi$  denote either of  $\chi^{st}, \chi^{op}_{ar \to s}, \chi^{rl}_{ar}$ .

Fact 1. First-order signature morphisms form a category Sig<sup>FOL</sup> under the componentwise composition as functions.

*Models.* Given a signature  $\Sigma$ , a  $\Sigma$ -model is a triple

$$\mathfrak{A} = (\{\mathfrak{A}_s\}_{s\in S}, \{\mathfrak{A}_\sigma\}_{(\mathsf{ar},s)\in S^*\times S, \sigma\in F_{\mathsf{ar}\to s}}, \{\mathfrak{A}_\pi\}_{\mathsf{ar}\in S^*, \pi\in P_{\mathsf{ar}}})$$

interpreting each sort *s* as a non-empty set  $\mathfrak{A}_s$ , each operation symbol  $\sigma \in F_{ar \to s}$  as a function  $\mathfrak{A}_{\sigma} \colon \mathfrak{A}_{ar} \to \mathfrak{A}_s$  (where  $\mathfrak{A}_{ar}$  stands for  $\mathfrak{A}_{s_1} \times \ldots \times \mathfrak{A}_{s_n}$  if  $ar = s_1 \ldots s_n$ ), and each relation symbol  $\pi \in P_{ar}$  as a relation  $\mathfrak{A}_{\pi} \subseteq \mathfrak{A}_{ar}$ . Morphisms between models are the usual  $\Sigma$ -homomorphisms, i.e., *S*-sorted functions that preserve the structure.

**Fact 2.** For any signature  $\Sigma$ , the  $\Sigma$ -homomorphisms form a category  $Mod^{FOL}(\Sigma)$  under the obvious composition as many-sorted functions.

For any signature morphism  $\chi \colon \Sigma \to \Sigma'$ , the reduct functor  $\Box_{\chi} \colon \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$  is defined as follows:

- The reduct 𝔄' ↾<sub>𝑋</sub> of a Σ'-model 𝔄' is a defined by (𝔄' ↾<sub>𝑋</sub>)<sub>𝔅</sub> = 𝔄'<sub>𝔅(𝔅)</sub> for each sort 𝔅 ∈ 𝔅, operation symbol 𝔅 ∈ 𝔅
   or relation symbol 𝔅 ∈ 𝔅. Note that, unlike the single-sorted case, the reduct functor modifies the universes of models. For the universe of 𝔄' ↾<sub>𝔅</sub> is {𝔄'<sub>𝔅(𝔅)</sub>}<sub>𝔅∈𝔅</sub>, which means that the sorts outside the image of 𝔅 are discarded. Otherwise, the notion of reduct is standard.
- 2. The reduct  $h' \upharpoonright_{\chi}$  of a homomorphism h' is defined by  $(h' \upharpoonright_{\chi})_s = h'_{\chi(s)}$  for all sorts  $s \in S$ .

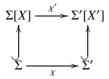
**Fact 3.** Mod<sup>FOL</sup> becomes a functor Sig<sup>FOL</sup>  $\rightarrow \mathbb{C}at^{op}$ , with Mod<sup>FOL</sup>( $\chi$ )(h') =  $h' \upharpoonright_{\chi}$  for each signature morphism  $\chi \colon \Sigma \rightarrow \Sigma'$  and each  $\Sigma'$ -homomorphism h'.

Sentences. We assume a countably infinite set of variable names  $\{v_i \mid i < \omega\}$ . A variable for a signature  $\Sigma$  is a triple  $\langle v_i, s, \Sigma \rangle$ , where  $v_i$  is a variable name, and s is a sort in  $\Sigma$ . Given a signature  $\Sigma$ , the S-sorted set of  $\Sigma$ -terms is denoted by  $T_{\Sigma}$ . The set Sen<sup>FOL</sup>( $\Sigma$ ) of sentences over  $\Sigma$  is given by the following grammar:

$$\gamma ::= t = t' \mid \pi(t_1, \dots, t_n) \mid \neg \gamma \mid \lor \Gamma \mid \exists X \cdot \gamma'$$

where (a) t = t' is an equation with  $t, t' \in T_{\Sigma,s}$  and  $s \in S$ , (b)  $\pi(t_1, \ldots, t_n)$  is a relational atom with  $\pi \in P_{s_1...s_n}$ ,  $t_i \in T_{\Sigma,s_i}$ and  $s_i \in S$ , (c)  $\Gamma$  is a finite set of  $\Sigma$ -sentences, (d) X is a finite set of variables for  $\Sigma$ , (e)  $\gamma'$  is a  $\Sigma[X]$ -sentence, where  $\Sigma[X] = (S, F[X], P)$ , and F[X] is the set of function symbols obtained by adding the variables in X as constants to F.

Sentence translations. Quantification comes with some subtle issues related to the translation of quantified sentences along signature morphisms that require a closer look. The translation of a variable  $\langle v_i, s, \Sigma \rangle$  along a signature morphism  $\chi: \Sigma \to \Sigma'$  is  $\langle v_i, \chi(s), \Sigma' \rangle$ . Therefore, any signature morphism  $\chi: \Sigma \to \Sigma'$  can be extended canonically to a function  $\chi: \text{Sen}^{FOL}(\Sigma) \to \text{Sen}^{FOL}(\Sigma')$  that translates sentences symbolwise.



Notice that  $\chi(\exists X \cdot \gamma) = \exists X' \cdot \chi'(\gamma)$ , where  $X' = \{\langle v_i, \chi(s), \Sigma' \rangle \mid \langle v_i, s, \Sigma \rangle \in X\}$  and  $\chi' \colon \Sigma[X] \to \Sigma'[X']$  is the extension of  $\chi$  that maps each variable  $\langle v_i, s, \Sigma \rangle \in X$  to  $\langle v_i, \chi(s), \Sigma' \rangle \in X'$  and such that the diagram of signature morphisms above is commutative.

**Fact 4.** Sen<sup>FOL</sup> is a functor Sig<sup>FOL</sup>  $\rightarrow$  Set.

For the sake of simplicity, we will identify a variable only by its name and sort provided that there is no danger of confusion. Using this convention, each inclusion  $\iota: \Sigma \hookrightarrow \Sigma'$  is canonically extended to an inclusion of sentences  $\iota: Sen^{FOL}(\Sigma) \hookrightarrow Sen^{FOL}(\Sigma')$ , which corresponds to the approach of classical model theory.

Satisfaction relation. Satisfaction is the usual first-order satisfaction and it is defined using the natural interpretations of ground terms t as elements  $\mathfrak{A}_t$  in models  $\mathfrak{A}$ . For example,  $\mathfrak{A} \models t_1 = t_2$  iff  $\mathfrak{A}_{t_1} = \mathfrak{A}_{t_2}$ .

*Non-void signatures.* A first-order signature  $\Sigma$  is called *non-void* if all sorts in  $\Sigma$  are inhabited by terms, that is  $T_{\Sigma,s} \neq \emptyset$  for all sorts s in  $\Sigma$ . If  $\Sigma$  is a *non-void* signature then the set of  $\Sigma$ -terms  $T_{\Sigma}$  can be regarded as a first-order model which interprets (a) any function symbol ( $\sigma$ : ar  $\rightarrow s$ )  $\in F$  as a function  $T_{\Sigma,\sigma}$ :  $T_{\Sigma,ar} \rightarrow T_{\Sigma,s}$  defined by  $T_{\Sigma,\sigma}(t) = \sigma(t)$  for all  $t \in T_{\Sigma,ar}$ , and (b) any relation symbol as the empty set.

*Notations.* For each first-order signature  $\Sigma$ , we denote by  $\perp$  the  $\Sigma$ -sentence  $\vee \emptyset$ . Obviously,  $\perp$  is not satisfiable and  $\chi(\perp) = \perp$  for all signature morphisms  $\chi: \Sigma \to \Sigma'$ . Let *T* and  $\Gamma$  be two theories over  $\Sigma$ .

- $\mathfrak{A} \models T$  if  $\mathfrak{A} \models \varphi$  for all  $\varphi \in T$ , where  $\mathfrak{A}$  is any first-order  $\Sigma$ -structure.
- $T \models \Gamma$  if for all first-order structures  $\mathfrak{A}$  over  $\Sigma$ , we have  $\mathfrak{A} \models T$  implies  $\mathfrak{A} \models \Gamma$ .
- $T \models \Gamma$  if  $T \models \Gamma$  and  $\Gamma \models T$ . In this case, we say that T and  $\Gamma$  are semantically equivalent.

## 3. Hybrid-dynamic first-order logic with rigid symbols (HDFOLR)

In this section, we present hybrid-dynamic first-order logic with rigid symbols, which is an extension of hybrid first-order logic with rigid symbols [27] with features of dynamic logics. Some preliminary attempts to the presentation of this logic framework can be found in [28].

*Signatures.* The signatures are of the form  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$ , where

- 1.  $\Sigma^n = (S^n, F^n, P^n)$  is a single-sorted first-order signature such that  $S^n = \{any\}$  is a singleton,  $F^n$  is a set of constants called *nominals*, and  $P^n$  is a set of binary relation symbols called *modalities*,
- 2.  $\Sigma = (S, F, P)$  is a many-sorted first-order signature such that S is a set of sorts, F is a  $(S^* \times S)$ -indexed set of function symbols, and P is a  $S^*$ -indexed set of relation symbols, and
- 3.  $\Sigma^{r} = (S^{r}, F^{r}, P^{r})$  is a many-sorted first-order subsignature of *rigid* symbols.

Throughout this paper, we let  $\Delta$  and  $\Delta_i$  range over HDFOLR signatures of the form  $(\Sigma^n, \Sigma^r \subseteq \Sigma)$  and  $(\Sigma_i^n, \Sigma_i^r \subseteq \Sigma_i)$ , respectively.

Signature morphisms. A signature morphism  $\chi: \Delta \to \Delta_1$  consists of a pair of first-order signature morphisms  $\chi^n: \Sigma^n \to \Sigma_1^n$  and  $\chi: \Sigma \to \Sigma_1$  such that  $\chi(\Sigma^r) \subseteq \Sigma_1^r$ .

**Fact 5.** HDFOLR signature morphisms form a category Sig<sup>HDFOLR</sup> under the component-wise composition as first-order signature morphisms.

Kripke structures. For every signature  $\Delta$ , the class of Kripke structures over  $\Delta$  consists of pairs (W, M), where

- 1. W is a first-order structure over  $\Sigma^n$ , called a frame, with the universe |W| consisting of a non-empty set of possible worlds, and
- 2.  $M: |W| \to |\mathsf{Mod}^{\mathsf{FOL}}(\Sigma)|$  is a mapping from the universe of W to the class of first-order  $\Sigma$ -structures such that the following sharing condition holds:  $M_{w_1} \upharpoonright_{\Sigma^r} = M_{w_2} \upharpoonright_{\Sigma^r}$  for all possible worlds  $w_1, w_2 \in |W|$ .

*Kripke homomorphisms.* A morphism  $h: (W, M) \to (W', M')$  is also a pair  $(W \xrightarrow{h} W', \{M_w \xrightarrow{h_w} M'_{h(w)}\}_{w \in |W|})$  consisting of first-order homomorphisms such that  $h_{w_1,s} = h_{w_2,s}$  for all possible worlds  $w_1, w_2 \in |W|$  and all rigid sorts  $s \in S^r$ .

**Fact 6.** For any signature  $\Delta$ , the  $\Delta$ -homomorphisms form a category Mod<sup>HDFOLR</sup>( $\Delta$ ) under the component-wise composition as first-order homomorphisms.

Every signature morphism  $\chi: \Delta \to \Delta'$  induces appropriate *reductions of models*, as follows: every  $\Delta'$ -model (W', M') is reduced to a  $\Delta$ -model  $(W', M') \upharpoonright_{\chi}$  that interprets every symbol x in  $\Delta$  as  $(W', M')_{\chi(x)}$ . When  $\chi$  is an inclusion, we usually denote  $(W', M') \upharpoonright_{\chi}$  by  $(W', M') \upharpoonright_{\Delta} -$  in this case, the model reduct simply forgets the interpretation of those symbols in  $\Delta'$  that do not belong to  $\Delta$ .

**Fact 7.** Mod<sup>HDFOLR</sup> becomes a functor Sig<sup>HDFOLR</sup>  $\rightarrow \mathbb{C}$ at<sup>*op*</sup>, with Mod<sup>HDFOLR</sup>( $\chi$ )(W, M) = (W, M)  $\upharpoonright_{\chi}$  for each signature morphism  $\chi: \Delta \rightarrow \Delta'$  and each Kripke structure (W, M) over  $\Delta'$ .

Actions. As in dynamic logic, HDFOLR supports structured actions obtained from atoms using sequential composition, union, and iteration. The set  $A^n$  of actions over  $\Sigma^n$  is defined in an inductive fashion, according to the grammar:

where  $\lambda \in P^n$  is a binary relation on nominals. Given a natural number m > 0, we denote by  $a^m$  the composition  $\mathfrak{a}_{\mathfrak{f}} \cdots \mathfrak{f} \mathfrak{a}$ (where the action  $\mathfrak{a}$  occurs m times). Actions are interpreted in Kripke structures as *accessibility relations* between possible worlds. This is done by extending the interpretation of binary modalities (from  $P^n$ ):  $W_{\mathfrak{a}_1\mathfrak{f}\mathfrak{a}_2} = W_{\mathfrak{a}_1} \mathfrak{g} W_{\mathfrak{a}_2}$ (diagrammatic composition of relations),  $W_{\mathfrak{a}_1 \cup \mathfrak{a}_2} = W_{\mathfrak{a}_1} \cup W_{\mathfrak{a}_2}$  (union), and  $W_{\mathfrak{a}^*} = (W_{\mathfrak{a}})^*$  (reflexive & transitive closure). *Hybrid terms.* For any signature  $\Delta$ , we make the following notational conventions:

- 1.  $S^{e} := S^{r} \cup \{any\}$  the extended set of rigid sorts, where any is the sort of nominals,
- 2.  $S^{f} \coloneqq S \setminus S^{r}$  the subset of flexible sorts,
- 3.  $F^{f} := F \setminus F^{r}$  the subset of flexible function symbols, where  $F \setminus F^{r} = \{F_{ar \to s} \setminus F^{r}_{ar \to s}\}_{(ar,s) \in S^{*} \times S}$ ,
- 4.  $P^{f} := P \setminus P^{r}$  the subset of flexible relation symbols, where  $P \setminus P^{r} = \{P_{ar} \setminus P^{r}_{ar}\}_{ar \in S^{*}}$ .

The *rigidification* of  $\Sigma$  with respect to  $F^{n}$  is the signature  $@\Sigma = (@S, @F, @P)$ , where

- 1.  $@S := \{@_k \ s \mid k \in F^n \text{ and } s \in S\},\$
- 2.  $@F := \{@_k \sigma : @_k ar \to @_k s \mid k \in F^n \text{ and } (\sigma : ar \to s) \in F\}, ^1 \text{ and }$
- 3.  $@P := \{@_k \pi : @_k \text{ ar } | k \in F^n \text{ and } (\pi : ar) \in P\}.$

It should be noted that  $@_k$  is used polymorphically. Here it is a device from metalanguage that creates new symbols out of existing ones. Later on  $@_k$  will also be used as a sentence-building operator. The context always decides which of these uses are intended. Since the rigid symbols have the same interpretation across the worlds, we define  $@_k x := x$  for all nominals  $k \in F^n$  and all symbols x in  $\Sigma^r$ . The set of *rigid*  $\Delta$ -*terms* is  $T_{@\Sigma}$ , while the set of *open*  $\Delta$ -*terms* is  $T_{\Sigma}$ . The set of *hybrid*  $\Delta$ -*terms* is  $T_{\overline{\Sigma}}$ , where  $\overline{\Sigma} = (\overline{S}, \overline{F}, \overline{P}), \overline{S} = S \cup @S^f, \overline{F} = F \cup @F^f$ , and  $\overline{P} = P \cup @P^f$ .

**Remark 8.** The set of hybrid terms include both open and rigid terms, that is,  $T_{\Sigma} \subseteq T_{\overline{\Sigma}}$  and  $T_{@\Sigma} \subseteq T_{\overline{\Sigma}}$ .

The interpretation of the hybrid terms into Kripke structures is defined as follows: for any  $\Delta$ -model (*W*, *M*), and any possible world  $w \in |W|$ ,

- 1.  $M_{w,\sigma(t)} = (M_{w,\sigma})(M_{w,t})$ , where  $(\sigma: ar \to s) \in F$ , and  $t \in T_{\overline{\Sigma}}$  ar, <sup>2</sup>
- 2.  $M_{w,(@_k \sigma)(t)} = (M_{w',\sigma})(M_{w,t})$ , where  $(@_k \sigma : @_k ar \to @_k s) \in @F^{f}, t \in T_{\overline{\Sigma},@_k ar}$  and  $w' = W_k$ .

Sentences. The simplest sentences defined over a signature  $\Delta$ , usually referred to as atomic, are given by:

$$o ::= k \mid t_1 = t_2 \mid \pi(t)$$

where (a)  $k, k' \in F^n$  are nominals, (b)  $t_i \in T_{\overline{\Sigma},s}$  are hybrid terms,  $s \in \overline{S}$  is a hybrid sort, (c)  $\pi \in \overline{P}_{ar}$ ,  $ar \in (\overline{S})^*$ and  $t \in T_{\overline{\Sigma},ar}$ . We refer to these sentences, in order, as *nominal sentences*, hybrid equations and hybrid relations, respectively. The set Sen<sup>HDFOLR</sup>( $\Delta$ ) of *full sentences* over  $\Delta$  are given by the following grammar:

$$\gamma \coloneqq \rho \mid @_k \gamma \mid \neg \gamma \mid \lor \Gamma \mid \downarrow z \cdot \gamma' \mid \exists X \cdot \gamma'' \mid \langle \mathfrak{a} \rangle \gamma$$

where (a)  $\rho$  is a nominal sentence or a hybrid equation or a hybrid relation, (b)  $k \in F^n$  is a nominal, (c)  $\mathfrak{a} \in A^n$  is an action, (d)  $\Gamma$  is a finite set of sentences over  $\Delta$ , (e) z is a nominal variable for  $\Delta$ , (f)  $\gamma'$  is a sentence over the signature  $\Delta[z]$  obtained by adding z as a new constant to  $F^n$ , (g) X is a set of variables for  $\Delta$  of sorts from the extended set  $S^e$  of rigid sorts, and (h)  $\gamma''$  is a sentence over the signature  $\Delta[X]$  obtained by adding the variables in X as new constants to  $F^n$  and  $F^r$ . Other than the first kind of sentences (*atoms*), we refer to the sentence-building operators, in order, as *retrieve*, *negation*, *disjunction*, *store*, *existential quantification* and *possibility*, respectively. Notice that *possibility* is parameterized by actions.

Sentence translations. Every signature morphism  $\chi \colon \Delta \to \Delta'$  induces translations of sentences, as follows: each  $\Delta$ -sentence  $\gamma$  is translated to a  $\Delta'$ -sentence  $\chi(\gamma)$  by replacing (in an inductive manner) the symbols in  $\Delta$  with symbols from  $\Delta'$  according to  $\chi$ .

**Fact 9.** Sen<sup>HDFOLR</sup> is a functor Sig<sup>HDFOLR</sup>  $\rightarrow$  Set.

 $<sup>{}^1@</sup>_k(s_1\ldots s_n) \coloneqq @_k s_1\ldots @_k s_n$  for all arities  $s_1\ldots s_n$ .

 $<sup>{}^{2}</sup>M_{w,(t_1,\ldots,t_2)} := M_{w,t_1},\ldots,M_{w,t_n}$  for all tuples of hybrid terms  $(t_1,\ldots,t_n)$ .

Local satisfaction relation. Given a  $\Delta$ -model (W, M) and a world  $w \in |W|$ , we define the satisfaction of  $\Delta$ -sentences at w by structural induction as follows:

- 1. For atomic sentences:
  - $(W, M) \models^{w} k$  iff  $W_k = w$  for all nominals k;
  - $(W, M) \models^w t_1 = t_2$  iff  $M_{w,t_1} = M_{w,t_2}$  for all hybrid equations  $t_1 = t_2$ ;
  - $(W, M) \models^{w} \pi(t)$  iff  $M_{w,t} \in M_{w,\pi}$  for all hybrid relations  $\pi(t)$ .
- 2. For full sentences:
  - $(W, M) \models^{w} @_k \gamma \text{ iff } (W, M) \models^{w'} \gamma, \text{ where } w' = W_k;$
  - $(W, M) \models^{w} \neg \gamma$  iff  $(W, M) \not\models^{w} \gamma$ ;
  - $(W, M) \models^{w} \lor \Gamma$  iff  $(W, M) \models^{w} \gamma$  for some  $\gamma \in \Gamma$ ;
  - (W, M) ⊨<sup>w</sup> ↓z · γ iff (W<sup>z←w</sup>, M) ⊨<sup>w</sup> γ,
     where (W<sup>z←w</sup>, M) is the unique Δ[z]-expansion of (W, M) that interprets the variable z as w; <sup>3</sup>
  - $(W, M) \models^{w} \exists X \cdot \gamma \text{ iff } (W', M') \models^{w} \gamma \text{ for some expansion } (W', M') \text{ of } (W, M) \text{ to the signature } \Delta[X];^{3}$
  - $(W, M) \models^{w} \langle \mathfrak{a} \rangle \gamma$  iff  $(W, M) \models^{w'} \gamma$  for some  $w' \in |W|$  such that  $(w, w') \in W_{\mathfrak{a}}$ .

The following *satisfaction condition* can be proved by induction on the structure of  $\Delta$ -sentences. The proof is essentially identical to those developed for several other variants of hybrid logic presented in the literature (see, e.g. [17]).

**Proposition 10** (Local satisfaction condition for signature morphisms). For every signature morphism  $\chi \colon \Delta \to \Delta'$ ,  $\Delta'$ -model (W', M'), possible world  $w' \in |W'|$ , and  $\Delta$ -sentence  $\gamma$ , we have  $(W', M') \models^{w} \chi(\gamma)$  iff  $(W', M') \upharpoonright_{\chi} \models^{w} \gamma$ .<sup>4</sup>

*Non-void signatures.* A signature  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  is called *non-void* if both  $\Sigma^n$  and  $\Sigma$  are non-void first-order signatures. Notice that for any non-void signature, the set of nominals is not empty, that is,  $F^n \neq \emptyset$ , and the set of hybrid terms of any sort is not empty, that is,  $T_{\overline{\Sigma}_s} \neq \emptyset$  for all sorts  $s \in S$ .

**Lemma 11.** If  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  is non-void then there exists an initial model of terms  $(W^{\Delta}, M^{\Delta})$  defined as follows: (1)  $W^{\Delta} = F^n$ , and (2)  $M^{\Delta}$ :  $F^n \to |\mathsf{Mod}^{\mathsf{FOL}}(\Sigma)|$ , where for all  $k \in F^n$ ,  $M_k^{\Delta}$  is a first-order structure such that

- (a)  $M_{k_s}^{\Delta} = T_{@\Sigma, @_k s}$  for all sorts  $s \in S$ ,
- (b)  $M_{k,\sigma}^{\Delta}$ :  $T_{@\Sigma,@_kar} \to T_{@\Sigma,@_ks}$  is defined by  $M_{k,\sigma}^{\Delta}(t) = (@_k \sigma)(t)$  for all function symbols  $(\sigma: ar \to s) \in F$  and all tuples of hybrid terms  $t \in T_{@\Sigma,@_kar}$ , and
- (c)  $M_{k,\pi}^{\Delta}$  is the empty set for all relation symbols  $(\pi: ar) \in P$ .

The proof of Lemma 11 is based on the unique interpretation of terms into models, and it is straightforward. We leave it as an exercise for the reader.

*Notations.* Take a signature  $\Delta$ , a Kripke structure  $(W, M) \in |\mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta)|$ , a sentence  $\varphi \in \mathsf{Sen}^{\mathsf{HDFOLR}}(\Delta)$ , and two theories  $T, \Gamma \subseteq \mathsf{Sen}^{\mathsf{HDFOLR}}(\Delta)$ .

- We say that (W, M) (globally) satisfies  $\varphi$ , in symbols,  $(W, M) \models \varphi$ , if  $(W, M) \models^{w} \varphi$  for all  $w \in |W|$ .
- We say that (W, M) satisfies  $\Gamma$ , in symbols,  $(W, M) \models \Gamma$ , if  $(W, M) \models \gamma$  for all  $\gamma \in \Gamma$ .

<sup>&</sup>lt;sup>3</sup>An expansion of (W, M) to  $\Delta[X]$  is a Kripke structure (W', M') over  $\Delta[X]$  that interprets all symbols in  $\Delta$  in the same way as (W, M). <sup>4</sup>By the definition of reducts, (W', M') and (W', M')<sub>V</sub> have the same possible worlds.

- We say that T (globally) satisfies  $\Gamma$ , in symbols,  $T \models \Gamma$ , if  $(V, N) \models T$  implies  $(V, N) \models \Gamma$  for all  $(V, N) \in |\mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta)|$ .<sup>5</sup>
- We say that T is semantically equivalent to  $\Gamma$ , in symbols,  $T \models \Gamma$ , if  $T \models \Gamma$  and  $\Gamma \models T$ .

#### **Lemma 12.** Let $\Delta$ be a signature.

- 1. For all sentences  $\varphi$  over  $\Delta$ , all nominal variables z for  $\Delta$ , all  $(W, M) \in |\mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta)|$  and all  $w \in |W|$ ,  $(W, M) \models^w \forall z \cdot @_z \varphi$  iff  $(W, M) \models \forall z \cdot @_z \varphi$  iff  $(W, M) \models \varphi$ .
- 2. For all sentences  $\varphi$  and  $\gamma$  over  $\Delta$ , all nominal variables z for  $\Delta$ , and all nominals k in  $\Delta$ , we have  $\varphi \models \forall z \cdot @_z \varphi \models @_k \forall z \cdot @_z \varphi$ , while  $\varphi \Rightarrow \gamma \models (\forall z \cdot @_z \varphi) \Rightarrow \gamma$  does not hold, in general.
- 3. For all theories T over  $\Delta$ , all sentences  $\varphi$  and  $\gamma$  over  $\Delta$  and all nominals k in  $\Delta$ ,  $T \models @_k(\varphi \Rightarrow \gamma) \text{ iff } T \cup \{@_k \varphi\} \models @_k \gamma.$
- 4. For all theories T over  $\Delta$ , all sentences  $\varphi$  over  $\Delta$  and all nominals k in  $\Delta$ ,  $T \models @_k \neg \varphi \text{ iff } T \cup \{@_k \varphi\} \models \bot.$
- 5. For all theories T over  $\Delta$ , all nominal variables x and z for  $\Delta$  and all sentences  $\psi$  over  $\Delta[x]$ ,  $T \cup \{\psi\}$  is satisfiable over  $\Delta[x]$  iff  $T \cup \{\exists x \cdot \forall z \cdot @_z \psi\}$  is satisfiable over  $\Delta$ .<sup>6</sup>

The proof of this lemma is straightforward and we leave it as an exercise for the interested reader. Informally, the key is that in the sentence  $\forall z \cdot @_z \varphi$  the quantifier  $\forall z$  binds the free variable z in  $@_z$ , so  $\forall z \cdot @_z \varphi$  means ' $\varphi$  holds at all worlds w'.

By using 'storing and retrieving' intuition it is easy to define complex properties. For example, temporal until operator U – with the following semantics:  $U(\varphi, \psi)$  is true at a state w if there is a future state w' where  $\varphi$  holds, such that  $\psi$  holds in all states between w and w' – can be defined as follows:

$$U(\varphi,\psi) \coloneqq \downarrow x \cdot \Diamond \downarrow y \cdot (\varphi \land @_x \Box(\Diamond y \Rightarrow \psi))$$

The idea is to name the current state x using  $\downarrow$ , and then by  $\Diamond$ , we identify a successor state, which we call y, where  $\varphi$  holds. Using @, the point of evaluation is changed to x, and then at all successors of x connected to y,  $\psi$  holds.

# 4. Logical concepts

In this section, we recall some concepts necessary to prove our results.

## 4.1. Substitutions

Let  $\Delta$  be a signature,  $C_1$  and  $C_2$  two sets of new constants for  $\Delta$  of sorts in  $S^e$ , the extended set of rigid sorts. A substitution  $\theta : C_1 \to C_2$  over  $\Delta$  is a mapping from  $C_1$  to  $|(W^{\Delta[C_2]}, M^{\Delta[C_2]})|$ , the carrier sets of the initial Kripke structure  $(W^{\Delta[C_2]}, M^{\Delta[C_2]})$  over  $\Delta[C_2]$  defined in Lemma 11.

**Proposition 13** (Local satisfaction condition for substitutions [26]). A substitution  $\theta$  :  $C_1 \rightarrow C_2$  over  $\Delta$  uniquely determines:

1. a sentence function  $\theta$ : Sen<sup>HDFOLR</sup>( $\Delta[C_1]$ )  $\rightarrow$  Sen<sup>HDFOLR</sup>( $\Delta[C_2]$ ), which preserves  $\Delta$  and maps each constant  $c \in C_1$  to a rigid term  $\theta(c)$  over  $\Delta[C_2]$ , and

<sup>&</sup>lt;sup>5</sup>Notice that the semantics of  $\varphi \models \gamma$  is different from the standard one, where  $\varphi \models \gamma$  is interpreted locally, that is,  $(V, N) \models^{w} \varphi$  implies  $(V, N) \models^{w} \gamma$  for all Kripke structures (V, N) and all possible worlds *w* in *V*.

<sup>&</sup>lt;sup>6</sup>If we take into consideration the third component of a variable, the correct statement is  $T \cup \{\psi\}$  is satisfiable over  $\Delta[x]$  iff  $T \cup \{\exists x \cdot \forall z \cdot @_z \iota(\psi)\}$  is satisfiable over  $\Delta$ , where  $\iota \colon \Delta[x] \hookrightarrow \Delta[x, z]$ .

2. a reduct functor  $\upharpoonright_{\theta} : \mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta[C_2]) \to \mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta[C_1])$ , which preserves the interpretation of  $\Delta$  and interprets each  $c \in C_1$  as  $\theta(c)$ ,

such that the following local satisfaction condition holds:

$$(W, M) \models^{w} \theta(\gamma) iff(W, M) \upharpoonright_{\theta} \models^{w} \gamma$$

for all  $\Delta[C_1]$ -sentences  $\gamma$ , all Kripke structures (W, M) over  $\Delta[C_2]$  and all possible worlds  $w \in |W|$ .

#### 4.2. Fragments

By restricting the signatures and/or the sentences of HDFOLR, one can obtain well-known hybrid logics studied in the literature.

**Definition 14 (Fragment).** A fragment  $\mathcal{L}$  of HDFOLR is obtained by restricting the syntax of HDFOLR, that is, Sig<sup> $\mathcal{L}$ </sup> is a subcategory of Sig<sup>HDFOLR</sup> and Sen<sup> $\mathcal{L}$ </sup>: Sig<sup> $\mathcal{L}$ </sup>  $\rightarrow$  Set is a subfunctor of Sen<sup>HDFOLR</sup>: Sig<sup>HDFOLR</sup>  $\rightarrow$  Set, such that

- 1. for any signature  $\Delta \in |Sig^{\mathcal{L}}|$ , any set *C* of new nominals and any set *D* of new rigid constants, we have  $\Delta \hookrightarrow \Delta[D, C] \in Sig^{\mathcal{L}}$ ,
- 2. for any substitution  $\theta$ :  $\langle C_1, D_1 \rangle \rightarrow \langle C_2, D_2 \rangle$  over a signature  $\Delta \in |Sig^{\mathcal{L}}|$  and any sentence  $\gamma \in Sen^{\mathcal{L}}(\Delta[C_1, D_1])$ , we have  $\theta(\gamma) \in Sen^{\mathcal{L}}(\Delta[C_2, D_2])$ , and
- 3.  $\mathcal{L}$  is closed under subsentence relation, that is,
  - if  $\langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\langle \mathfrak{a}_1 \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  and  $\langle \mathfrak{a}_2 \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$ ,
  - if  $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\langle \mathfrak{a}_1 \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  and  $\langle \mathfrak{a}_2 \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$ ,
  - if  $\langle \mathfrak{a}^* \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\langle \mathfrak{a}^n \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  for some  $n \in \omega$ ,
  - if  $\langle \mathfrak{a} \rangle \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$ ,
  - if  $\neg \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$ ,
  - if  $\forall \Gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  for all  $\gamma \in \Gamma$ ,
  - if  $@_k \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$ ,
  - if  $\downarrow z \cdot \gamma \in \text{Sen}^{\mathcal{L}}(\Delta)$  then  $\gamma \in \text{Sen}^{\mathcal{L}}(\Delta[z])$ , and
  - if  $\exists X \cdot \gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta)$  then  $\gamma \in \operatorname{Sen}^{\mathcal{L}}(\Delta[X])$ .

According to Definition 14, a fragment  $\mathcal{L}$  of HDFOLR has the same models as HDFOLR. By the closure under the subsentence relation, the sentences of  $\mathcal{L}$  are constructed from some atomic sentences by applying Boolean connectives, possibility over action relations, retrieve, store or existential quantifiers, if these sentence building operators are available in  $\mathcal{L}$ . It does not imply that  $\mathcal{L}$  is closed under any of these operators.

**Example 15 (Hybrid First-Order Logic with Rigid symbols (HFOLR) [27]).** This is the hybrid variant of HDFOLR obtained by discarding structured actions and allowing possibility over binary modalities. According to [27], HFOLR is compact.

**Example 16 (Hybrid-Dynamic Propositional Logic (HDPL)).** This is the dynamic variant of the most common form of multi-modal hybrid logic (e.g. [1]). HDPL is obtained from HDFOLR by restricting the signatures  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  such that the set of sorts in  $\Sigma$  is empty, and the set of sentences is given by the following grammar:

$$\gamma ::= \rho \mid k \mid @_k \gamma \mid \neg \gamma \mid \lor \Gamma \mid \langle \mathfrak{a} \rangle \gamma$$

where (a)  $\rho$  is a propositional symbol, (b)  $k \in F^n$  is a nominal, (c)  $a \in A^n$  is an action, and (d)  $\Gamma$  is a finite set of sentences over  $\Delta$ . Notice that if  $\Sigma = (S, F, P)$  and  $S = \emptyset$  then P contains only propositional symbols. HPL is the fragment of HDPL obtained by discarding structured actions.

**Example 17** (**Rigid First-Order Hybrid Logic** (RFOHL) [8]). This logic is obtained from HFOLR by restricting the signatures  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  such that (a)  $\Sigma^n$  has only one binary modality, (b)  $\Sigma$  is single-sorted, (c) there are no rigid function symbols except variables (regarded here as special constants), and (d) there are no rigid relation symbols.

All examples of logics given above are fragments of HDFOLR. In the following, we give an example of logic which is obtained from HDFOLR by some syntactic restrictions and it is not a fragment according to Definition 14.

**Example 18 (Hybrid First-Order Logic with user-defined Sharing** (HFOLS)). This logic has the same signatures and Kripke structure as HFOLR. The sentences are obtained from atoms constructed with open terms only, that is, if  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$ , all (ground) equations over  $\Delta$  are of the form  $t_1 = t_2$ , where  $t_1, t_2 \in T_{\Sigma}$ , and all (ground) relation over  $\Delta$  are of the form  $\pi(t)$ , where ( $\pi$  : ar)  $\in P$  and  $t \in T_{\Sigma,ar}$ . Variants of HFOLS have been used in works such as [39, 19, 17].

HFOLS is not a fragment of HDFOLR in the sense of Definition 14, as it is not closed under substitutions. Retrieve is applied only to sentences and not to function or relation symbols. However, according to [27], that is no loss of expressivity as HFOLS has the same expressive power as HFOLR.

**Lemma 19.** For each signature  $\Delta$  and each sentence  $\gamma \in \text{Sen}^{\text{HFOLR}}(\Delta)$  there exists a sentence  $\gamma' \in \text{Sen}^{\text{HFOLS}}(\Delta)$  such that  $(W, M) \models^{w} \gamma$  iff  $(W, M) \models^{w} \gamma'$  for all Kripke structures (W, M) over  $\Delta$  and all possible worlds in W.

*Proof.* By using [27, Lemma 2.20] which shows that for any atomic sentence in HFOLR there exists a sentence in HFOLS which is satisfied by the same class of Kripke structures.  $\Box$ 

The forcing technique and the Omitting Types Theorem are not applicable to HFOLS even if it has the same expressivity power as HFOLR. This is due to the absence of a proper support for the substitutions described in Section 4.1. By Lemma 19, the results can be borrowed from HFOLR to HFOLS. It is worth noting that HFOLS can be extended with features of dynamic logics such that the dynamic variant of HFOLS matches the expressivity of HDFOLR by the same arguments used in the proof of Lemma 19.

#### 4.3. Reachable models

In this section, we give a category-based description of the models which consist of elements that are denotations of terms. The concept of reachable model appeared in institutional model-theory in [42], and it has been used successfully in several abstract developments such as proof-theoretic results [32, 31, 24] as well as model-theoretic results [22, 23, 30, 25, 26, 13].

**Definition 20.** A Kripke structure (W, M) over a signature  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  is *reachable* if for each set of new nominals *C*, each set of new rigid constants *D*, and any expansion (W', M') of (W, M) to  $\Delta[C]$ , there exists a substitution  $\theta: \langle C, D \rangle \rightarrow \langle \emptyset, \emptyset \rangle$  over  $\Delta$  such that  $(W, M) \upharpoonright_{\theta} = (W', M')$ .

Proposition 21 (Reachable Kripke structures [26]). A Kripke structure is reachable iff

- 1. its set of states consists of denotations of nominals, and
- 2. its carrier sets for the rigid sorts consist of denotations of rigid terms.

By Proposition 21, a model (W, M) is reachable iff the unique homomorphism from the initial Kripke structure  $h: (W^{\Delta}, M^{\Delta}) \to (W, M)$  is surjective, that is,  $h: W^{\Delta} \to W$  is surjective and  $h_w: M_w^{\Delta} \to M_{h(w)}$  is surjective for all possible worlds  $w \in |W^{\Delta}|$ .

#### 4.4. Basic sentences

In this section, we recall an important property of certain simple sentences of hybrid logics, which play the role analogous to atomic sentences of first-order logic.

**Definition 22 (Basic set of sentences [14]).** A set of sentences *B* over a signature  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  is *basic* if there exists a Kripke structure  $(W^B, M^B)$  such that

 $(W, M) \models B$  iff there exists a homomorphism  $h : (W^B, M^B) \rightarrow (W, M)$ 

for all Kripke structures (W, M). We say that  $(W^B, M^B)$  is a *basic model* of *B*. If in addition the homomorphism *h* is unique then the set *B* is called *epi-basic*.

According to [14, 16], in first-order logic, any set of atomic sentences is basic. One important property of basic sentences is the preservation of their satisfaction along homomorphisms: given a set of basic sentences *B* and a homomorphism  $h: M \to N$ , if  $M \models B$  then  $N \models B$ . In hybrid logics, this property does not hold, in general. The following example is from [27].

**Example 23.** Consider the following HPL signature  $\Delta = (\Sigma^n, \text{Prop})$  such that  $F^n = \{k\}, P^n = \{\lambda : \text{ any any}\}$  and  $\text{Prop} = \{\rho\}$ . Let  $h: (W, M) \hookrightarrow (W', M')$  be the inclusion homomorphism defined by:

- 1.  $|W| = \{k\}, W_{\lambda} = \{(k, k)\}, \rho$  is true in  $M_k$ , and
- 2.  $|W'| = \{k, w\}, W'_{\lambda} = \{(k, k)\}, \rho$  is true in  $M'_{k}, \rho$  is not true in  $M'_{w}$ .

Example 23 points out a significant difference between ordinary logics and hybrid (or, more generally, modal) logics. Note that  $(W, M) \models^{\mathsf{HPL}} k$ ,  $(W, M) \models^{\mathsf{HPL}} \langle \lambda \rangle k$  and  $(W, M) \models^{\mathsf{HPL}} \rho$ . Since  $(W', M') \not\models^{w} k$ ,  $(W', M') \not\models^{w} \langle \lambda \rangle k$  and  $(W', M') \not\models^{w} \rho$  we have  $(W', M') \not\models^{\mathsf{HPL}} k$ ,  $(W', M') \not\models^{\mathsf{HPL}} \langle \lambda \rangle k$  and  $(W', M') \not\models^{\mathsf{HPL}} \rho$ . Thus, homomorphisms do not preserve satisfaction of atomic sentences. Hence, atomic sentences are not basic in HPL (the same example works for any modal logic). Note however that local satisfaction (satisfiaction at a world) is preserved, and in hybrid logic the retrieve operator (@) lifts local satisfaction to global. This motivates the next definition.

**Definition 24 (Locally basic set of sentences [27]).** A set of sentences  $\Gamma$  over a signature  $\Delta$  is *locally (epi-)basic* if  $@\Gamma := \{@_k \gamma \mid k \in F^n \text{ and } \gamma \in \Gamma\}$  is (epi-)basic.

Notice that  $@\Gamma$  is semantically equivalent to  $@@\Gamma$ . We denote by  $Sen_0^{HDFOLR}(\Delta)$  the set of all *extended atomic* sentences.

- 1. nominals  $k \in F^n$ ,
- 2. nominal relations  $\langle \lambda \rangle k$ , where  $\lambda \in P^n$  is a binary modality and  $k \in F^n$ ,
- 3. hybrid equations  $t_1 = t_2$ , where  $t_1, t_2 \in T_{\overline{\Sigma}}$ , and
- 4. hybrid relations  $\pi(t)$ , where  $\pi \in \overline{P}_{ar}$ ,  $t \in (T_{\overline{\Sigma}})_{ar}$  and  $ar \in (\overline{S})^*$ .

We denote by  $\operatorname{Sen}_{b}^{\mathsf{HDFOLR}}(\Delta)$  the set of all sentences obtained from an extended atomic sentence by applying retrieve (@) at most once.

**Proposition 25** (Locally basic set of sentences [26, 27]). Given a signature  $\Delta$ , every set of sentences  $B \subseteq \text{Sen}_{b}^{\text{HDFOLR}}(\Delta)$  is locally basic. Moreover, if  $\Delta$  is non-void, then B is locally epi-basic and its basic model ( $W^{B}, M^{B}$ ) is reachable.

**Definition 26 (Rigidification).** For any signature  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$ , the *rigidification function*  $at_{k-}: T_{\overline{\Sigma}} \to T_{@\Sigma}$ , where  $k \in F^n$ , is recursively defined by:

•  $\operatorname{at}_k \sigma(t) \coloneqq \begin{cases} (@_k \sigma)(\operatorname{at}_k t) & \text{if } (\sigma \colon \operatorname{ar} \to s) \in F^{\mathrm{f}}, \\ \sigma(\operatorname{at}_k t) & \text{if } (\sigma \colon \operatorname{ar} \to s) \in F^{\mathrm{r}} \cup @F^{\mathrm{f}}. \end{cases}$ 

Its extension  $at_k$  :: Sen<sup>HFOLR</sup>( $\Delta$ )  $\rightarrow$  Sen<sup>HFOLR</sup>( $\Delta$ ) is recursively defined by:

•  $\operatorname{at}_k k' \coloneqq @_k k'$ 

•  $\operatorname{at}_{k} \pi(t) := \mathfrak{G}_{k} \pi$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle \lambda \rangle(k')$ •  $\operatorname{at}_{k} \langle \lambda \rangle(k') := \mathfrak{G}_{k} \langle$ 

Any sentence semantically equivalent to a sentence in the image of  $at_k$  is called a *rigid sentence*.

The proof of the following lemma is straightforward and we leave it as an exercise for the readers.

**Lemma 27.** Any sentence  $@_k \gamma$  is semantically equivalent to  $at_k \gamma$ . Hence,  $@_k \gamma$  is rigid.

## 5. Forcing

Forcing is a method of constructing models satisfying some properties forced by some conditions. In this section, we generalize the forcing relation for hybrid logics defined in [27] to hybrid-dynamic first-order logic with rigid symbols. It is worth mentioning that the present developments can be cast in the framework of stratified institutions following the ideas presented in [27].

**Framework 1.** The results in this paper will be developed in a fragment  $\mathcal{L}$  of HDFOLR that is semantically closed under negation and retrieve.<sup>7</sup> We make the following notational conventions:

- We let  $\operatorname{Sen}_0^{\mathcal{L}}$  to denote the subfunctor of  $\operatorname{Sen}^{\mathcal{L}}$  which maps each signature  $\Delta$  to the set of extended atomic sentences of  $\mathcal{L}$  over the signature  $\Delta$ . This means that  $\operatorname{Sen}_0^{\mathcal{L}}(\Delta) = \operatorname{Sen}^{\mathsf{HDFOLR}}(\Delta) \cap \operatorname{Sen}_0^{\mathsf{HDFOLR}}(\Delta)$  for all signatures  $\Delta$ .
- We let Sen<sup>L</sup><sub>b</sub> to denote the subfunctor of Sen<sup>L</sup> which maps each signature Δ to the set of basic sentences of L over the signature Δ. This means that Sen<sup>L</sup><sub>b</sub>(Δ) = Sen<sup>HDFOLR</sup>(Δ) ∩ Sen<sup>HDFOLR</sup><sub>b</sub>(Δ) for all signatures Δ.

Since  $\mathcal{L}$  is the logic in which we develop our results, we drop the superscript  $\mathcal{L}$  from the notations  $\operatorname{Sen}_{0}^{\mathcal{L}}$ ,  $\operatorname{Sen}_{0}^{\mathcal{L}}$  and  $\operatorname{Sen}_{h}^{\mathcal{L}}$  if there is no danger of confusion.

Examples of fragments can be found in Section 4.2.

**Definition 28 (Forcing property).** Given a signature  $\Delta$ , a forcing property over  $\Delta$  is a triple  $\mathbb{P} = \langle P, \leq, f \rangle$  such that:

1.  $(P, \leq)$  is a partially ordered set with a least element 0.

The elements of *p* are traditionally called *conditions*.

- 2.  $f: P \to \mathcal{P}(\mathsf{Sen}_b(\Delta))$  is a function,
- 3. if  $p \le q$  then  $f(p) \subseteq f(q)$ , and
- 4. if  $f(p) \models @_k \gamma$  then  $@_k \gamma \in f(q)$  for some  $q \ge p$ ,

where  $p \in P$ ,  $q \in P$ ,  $k \in F^{n}$  and  $\gamma \in \text{Sen}_{0}(\Delta)$ .

As for ordinary first-order logics, a forcing property generates a forcing relation on the set of all sentences.

**Definition 29 (Forcing relation).** Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be a forcing property over  $\Delta$ .

The family of relations  $\Vdash = \{ \Vdash^k \}_{k \in F^n}$ , where  $\Vdash^k \subseteq P \times \text{Sen}(\Delta)$ , is inductively defined as follows:

 $<sup>{}^{7}\</sup>mathcal{L}$  is semantically closed under negation whenever for all  $\mathcal{L}$ -sentences  $\gamma$  there exists another  $\mathcal{L}$ -sentence  $\varphi$  such that we have:  $(W, M) \models^{w} \varphi$  iff  $(W, M) \not\models^{w} \gamma$  for all Kripke structures (W, M) and all possible worlds  $w \in |W|$ . When there is no danger of confusion, we denote  $\varphi$  by  $\neg \gamma$ . Similarly, one can define the semantic closer of  $\mathcal{L}$  under any sentence building operator.

- 1. For  $\gamma$  extended atomic:  $p \Vdash^k \gamma$  if  $@_k \gamma \in f(p)$ .
- 2. For  $\langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''$ :  $p \Vdash^k \langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''$  if  $p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$  and  $p \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$  for some  $k' \in F^n$ .
- 3. For  $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ :  $p \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$  if  $p \Vdash^k \langle \mathfrak{a}_1 \rangle k''$  or  $p \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ .
- 4. *For*  $\langle \mathfrak{a}^* \rangle k''$ :  $p \Vdash^k \langle \mathfrak{a}^* \rangle k''$  if  $p \Vdash^k \langle \mathfrak{a}^n \rangle k''$  for some  $n \in \mathbb{N}$ .
- 5. For  $\langle \mathfrak{a} \rangle \gamma$  with  $\gamma \notin F^{\mathsf{n}}$ :  $p \Vdash^{k} \langle \mathfrak{a} \rangle \gamma$  if  $p \Vdash^{k} \langle \mathfrak{a} \rangle k'$  and  $p \Vdash^{k'} \gamma$  for some nominal  $k' \in F^{\mathsf{n}}$ .
- 6. For  $\neg \gamma: p \Vdash^k \neg \gamma$  if there is no  $q \ge p$  such that  $q \Vdash^k \gamma$ .
- 7. *For*  $\forall \Gamma$ :  $p \Vdash^k \forall \Gamma$  if  $p \Vdash^k \gamma$  for some  $\gamma \in \Gamma$ .
- 8. For  $@_{k'} \gamma$ :  $p \Vdash^k @_{k'} \gamma$  if  $p \Vdash^{k'} \gamma$ .
- 9. For  $\downarrow z \cdot \gamma$ :  $p \Vdash^k \downarrow z \cdot \gamma$  if  $p \Vdash^k \gamma(z \leftarrow k)$ .
- 10. For  $\exists X \cdot \gamma$ :  $p \Vdash^k \exists X \cdot \gamma$  if  $p \Vdash^k \theta(\gamma)$  for some substitution  $\theta$ :  $X \to \emptyset$  over  $\Delta$ .

The forcing relation defined in the present contribution consists of the forcing relation defined in [27] plus the items 2—4 of Definition 29. The notation  $p \Vdash^k \gamma$  is read *p* forces  $\gamma$  at *k*.

**Remark 30.** Notice that Definition 29 does not rely on the fact that  $\mathcal{L}$  is closed under disjunction or quantifiers. For example, the last item from Definition 29 should be interpreted as follows: if  $\exists X \cdot \gamma$  is a sentence in  $\mathcal{L}$  and  $p \Vdash^k \theta(\gamma)$  for some substitution  $\theta : X \to \emptyset$  over  $\Delta$  then  $p \Vdash^k \exists X \cdot \gamma$ .

In regard to the satisfaction relation, one may consider a global forcing relation:  $p \Vdash \gamma$  iff  $p \Vdash^k \gamma$  for all nominals *k*. This remark establishes a connection between the results in the present contribution and the results in [32] and [23], where there exists only a global forcing relation.

**Lemma 31.** Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be a forcing property as in Definition 28. We have:

- 1.  $p \Vdash^k \neg \neg \gamma$  iff for each  $q \ge p$  there is  $r \ge q$  such that  $r \Vdash^k \gamma$ .
- 2. If  $q \ge p$  and  $p \Vdash^k \gamma$  then  $q \Vdash^k \gamma$ .
- *3. If*  $p \Vdash^k \gamma$  *then*  $p \Vdash^k \neg \neg \gamma$ *.*
- 4. We cannot have both  $p \Vdash^k \gamma$  and  $p \Vdash^k \neg \gamma$ .

*Proof.* Notice that the statements 1 and 3 are well-defined as  $\mathcal{L}$  is semantically closed under negation.

- *p* ||-<sup>k</sup> ¬¬γ iff for each *q* ≥ *p* we have *q* ||-<sup>k</sup> ¬γ iff for each *q* ≥ *p* there is *r* ≥ *q* such that *r* ||-<sup>k</sup> γ.
- 2. By induction on the structure of sentences:

[For  $\gamma$  extended atomic] The conclusion follows easily from  $f(p) \subseteq f(q)$ .

- [ For  $\langle \mathfrak{a}_1 \mathfrak{g} \mathfrak{a}_2 \rangle k''$ ]  $p \Vdash^k \langle \mathfrak{a}_1 \mathfrak{g} \mathfrak{a}_2 \rangle k''$  iff  $p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$  and  $p \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$  for some  $k' \in F^n$ . By the induction hypothesis,  $q \Vdash^k \langle \mathfrak{a}_1 \rangle k'$  and  $q \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ . Hence,  $q \Vdash^k \langle \mathfrak{a}_1 \mathfrak{g} \mathfrak{a}_2 \rangle k''$ .
- [ For  $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ ]  $p \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$  iff  $p \Vdash^k \langle \mathfrak{a}_1 \rangle k''$  or  $p \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ . By the induction hypothesis,  $q \Vdash^k \langle \mathfrak{a}_1 \rangle k''$  or  $q \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ . Hence,  $q \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ .
- [For  $\langle \mathfrak{a}^* \rangle k''$ ]  $p \Vdash^k \langle \mathfrak{a}^* \rangle k''$  iff there exists  $n \in \mathbb{N}$  such that  $p \Vdash^k \mathfrak{a}^n \langle k'' \rangle$ . By the induction hypothesis,  $q \Vdash^k \langle \mathfrak{a}^n \rangle k''$ . Hence,  $q \Vdash^k \langle \mathfrak{a}^* \rangle k''$ .

- [For  $\langle \mathfrak{a} \rangle \gamma$  with  $\gamma \notin F^n$ ]  $p \Vdash^k \langle \mathfrak{a} \rangle \gamma$  iff  $p \Vdash^k \langle \mathfrak{a} \rangle k'$  and  $p \Vdash^{k'} \gamma$ . By the induction hypothesis,  $q \Vdash^k \langle \mathfrak{a} \rangle k'$  and  $q \Vdash^{k'} \gamma$ . Hence,  $q \Vdash^k \langle \mathfrak{a} \rangle \gamma$ .
- [For  $@_{k'}\gamma$ ] We have  $p \Vdash^k @_{k'}\gamma$  iff  $p \Vdash^{k'}\gamma$ . By induction hypothesis,  $q \Vdash^{k'}\gamma$ . Hence,  $q \Vdash^k @_{k'}\gamma$ .
- [For  $\neg \gamma$ ] We have  $p \Vdash^k \neg \gamma$ . This means  $r \nvDash^k \gamma$  for all  $r \ge p$ . In particular,  $r \nvDash^k \gamma$  for all  $r \ge q$ . Hence,  $q \Vdash^k \neg \gamma$ .
- [For  $\lor \Gamma$ ]  $p \Vdash^k \gamma$  for some  $\gamma \in \Gamma$ . By induction hypothesis,  $q \Vdash^k \gamma$  which implies  $q \Vdash^k \lor \Gamma$ .
- [For  $\downarrow z \cdot \gamma$ ] We have  $p \Vdash^k \downarrow z \cdot \gamma$  iff  $p \Vdash^k \gamma(z \leftarrow k)$ . By the induction hypothesis,  $q \Vdash^k \gamma(z \leftarrow k)$ , which implies  $q \Vdash^k \downarrow z \cdot \gamma$ .
- [For  $\exists X \cdot \gamma$ ] Since  $p \Vdash^k \exists X \cdot \gamma$  then  $p \Vdash^k \theta(\gamma)$  for some substitution  $\theta: X \to \emptyset$  over  $\Delta$ . By the induction hypothesis,  $q \Vdash^k \theta(\gamma)$ . Hence,  $q \Vdash^k \exists X \cdot \gamma$ .
- 3. It follows from 1 and 2.
- 4. By the reflexivity of  $(P, \leq)$ .

Lemma 31 is a generalization of [27, Lemma 4.4] from hybrid logics to hybrid dynamic logics. In the present contribution, since the proof of the second statement is by induction, we need to consider possibility over structured actions.

**Definition 32** (Generic set [27]). Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be a forcing property over a signature  $\Delta$ .

A subset  $G \subseteq P$  is generic if it has the following properties:

- 1.  $r \in G$  if  $r \leq p$  and  $p \in G$ ;
- 2. there exists  $r \in G$  such that  $r \ge p$  and  $r \ge q$ , for all  $p, q \in G$ ;
- 3. there exists  $r \in G$  such that  $r \Vdash^k \gamma$  or  $r \Vdash^k \neg \gamma$ , for all  $\Delta$ -sentences  $@_k \gamma$ .

We write  $G \Vdash^k \gamma$  whenever  $p \Vdash^k \gamma$  for some  $p \in G$ .

Note that G in Definition 32 is well-defined, as  $\mathcal{L}$  is semantically closed under negation. The following lemma ensures the existence of generic sets. The result is based on the assumption that signatures consist of a countable number of symbols.

**Lemma 33** (Existence of generic sets [27]). Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be a forcing property over a signature  $\Delta$ . If Sen( $\Delta$ ) is countable then every *p* belongs to a generic set.

For the semantic forcing property defined in the next section it is possible to construct generic sets even if the underlying signature consists of an uncountable number of symbols. Notice that the definition of forcing relation and the definition of generic set are based on syntactic compounds. The following definition gives the semantics/meaning to these concepts.

**Definition 34 (Generic model [27]).** Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be a forcing property over a signature  $\Delta$ .

- (W, M) is a model for a generic set  $G \subseteq P$  when  $(W, M) \models @_k \gamma$  iff  $G \Vdash^k \gamma$ , for all  $\Delta$ -sentences  $@_k \gamma$ .
- (W, M) is a model for  $p \in P$  if there is a generic set  $G \subseteq P$  such that  $p \in G$  and (W, M) is a model for G.

The models (W, M) from Definition 34 are called, traditionally, *generic models*. The following result ensures the existence of generic models.

**Theorem 35** (Generic Model Theorem). Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be a forcing property over  $\Delta$ . Then each generic set G of  $\mathbb{P}$  has a generic Kripke structure (W, M). If in addition  $\Delta$  is non-void, (W, M) is reachable.

*Proof.* Let *G* be a generic set. We define  $T = \{ @_k \gamma \in Sen(\Delta) \mid G \Vdash^k \gamma \}$  and  $B = T \cap Sen_b(\Delta)$ . By Proposition 25, *B* is basic, and there exists a basic model  $(W^B, M^B)$  for *B* that is reachable. We show that  $(W^B, M^B) \models @_k \gamma$  iff  $G \Vdash^k \gamma$ , for all  $\Delta$ -sentences  $@_k \gamma$ .

[For  $\gamma$  extended atomic] Assume that  $(W^B, M^B) \models @_k \gamma$ .

| 1  | <i>B</i> and $\{@_k \gamma\}$ are basic  | by Proposition 25   |
|----|--|---|
| 2  | there exists an arrow $(W^{@_k\gamma}, M^{[@_k\gamma]}) \to (W^B, M^B)$  | since $\{@_k \gamma\}$ is basic and $(W^B, M^B) \models @_k \gamma$ |
| 3  | $B \models @_k \gamma$   | since both B and $\{@_k \gamma\}$ are basic                         |
| 4  | there exists $B_f \subseteq B$ finite such that $B_f \models @_k \gamma$   | since $HDFOLR_b$ is compact   |
| 5  | $B_f = \{ @_{k_1} \gamma_1, \dots, @_{k_n} \gamma_n \}$ for some $\gamma_i \in Sen_0(\Delta)$ and some $k_i \in F^n$ | by the definition of B  |
| 6  | for all $i \in \{1,, n\}$ , there exists $p_i \in G$ such that $p_i \Vdash^{k_i} \gamma_i$                           | by the definition of B  |
| 7  | there exists $p \in G$ such that $p \ge p_i$ for all $i \in \{1,, n\}$   | since $G$ is generic  |
| 8  | $B_f \subseteq f(p)$   | since $B_f \subseteq Sen_b(\Delta)$                                 |
| 9  | $q \Vdash^k \gamma$ or $q \Vdash^k \neg \gamma$ for some $q \in G$   | since $G$ is generic  |
| 10 | suppose towards a contradiction that $q \Vdash^k \neg \gamma$  |   |
|    |  |   |
|    | 10.1 $r \ge p$ and $r \ge q$ for some $r \in G$  | since $G$ is generic  |
|    | 10.2 $r \Vdash^k \neg \gamma$  | by Lemma 31 (2), since $r \ge q$ and $q \Vdash^k \neg \gamma$       |
|    | $10.3 \qquad B_f \subseteq f(r)$   | since $B_f \subseteq f(p)$ and $r \ge p$                            |
|    | 10.4 there exists $s \ge r$ such that $@_k \gamma \in f(s)$  | since $B_f \models @_k \gamma$ , we have $f(r) \models @_k \gamma$  |
|    | 10.5 $s \Vdash^k \gamma$   | by Definition 29  |
|    | 10.6 $s \Vdash^k \neg \gamma$  | by Lemma 31 (2)   |
|    | 10.7 contradiction   | by Lemma 31 (4)   |
|    |  |   |
| 11 | $q \Vdash^k \gamma$  | by 9 and 10   |
| 12 | $G \Vdash^k \gamma$  | since $q \in G$   |
|    |  |   |

If  $G \Vdash^k \gamma$  then by the definition of *B*, we have  $@_k \gamma \in B$ , which implies  $B \models @_k \gamma$ ; hence,  $(W^B, M^B) \models @_k \gamma$ .

[For  $\langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''$ ] Assume that  $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''$ .

| 1  | $(W_k^B, W_{k''}^B) \in W_{(\mathfrak{a}_1\mathfrak{a}_2)}^B$   | by definition   |
|----|---|---|
| 2  | $(W_k^B, w) \in W_{\mathfrak{a}_1}^B$ and $(w, W_{k''}^B) \in W_{\mathfrak{a}_2}^B$ for some $w \in  W^B $                                  | since $\mathfrak{a}_1 \mathfrak{z}_2$ is the composition of the relations $\mathfrak{a}_1$ and $\mathfrak{a}_2$ |
| 3  | $w = W_{k'}^B$ for some nominal $k' \in F^n$  | since $(W^B, M^B)$ is reachable   |
| 4  | $(W_k^B, W_{k'}^B) \in W_{\mathfrak{a}_1}^B$ and $(W_{k'}^B, W_{k''}^B) \in W_{\mathfrak{a}_2}^B$   | by 2 and 3  |
| 5  | $G \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $G \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$                                       | by the induction hypothesis   |
| 6  | $p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ for some $p \in G$ and $q \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ for some $q \in G$ | by Definition 32  |
| 7  | $r \ge p$ and $r \ge q$ for some $r \in G$  | since G is generic  |
| 8  | $r \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $r \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$                                       | by Lemma 31 (2) applied to 6 and 7  |
| 9  | $r \Vdash^k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k''$   | by Definition 29  |
| 10 | $G \Vdash^k \langle \mathfrak{a}_1 \  aight angle \mathfrak{a}_2  angle k''$  | by Definition 32  |
| As | ssume that $G \Vdash^k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k''$ .  |   |
| 1  | $p \Vdash^k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k'' \text{ for some } p \in G$                                   |   |
| 2  | $p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $p \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ for some $k' \in F^n$                 | by Definition 29  |
| 3  | $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \rangle k' \text{ and } (W^B, M^B) \models @_{k'} \langle \mathfrak{a}_2 \rangle k''$        | by the induction hypothesis   |
| 4  | $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k''$   | by the semantics of $\mathfrak{a}_1$ § $\mathfrak{a}_2$   |

[ For  $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ ] The following are equivalent:

 $G \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ 1  $p \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$  for some  $p \in G$ 2 by Definition 29  $p \Vdash^k \langle \mathfrak{a}_1 \rangle k'' \text{ or } p \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ 3 by Definition 29 4  $G \Vdash^k \langle \mathfrak{a}_1 \rangle k''$  or  $G \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ by Definition 29  $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \rangle k'' \text{ or } (W^B, M^B) \models @_k \langle \mathfrak{a}_2 \rangle k''$ 5 by the induction hypothesis 6  $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ by the semantics of  $\mathfrak{a}_1 \cup \mathfrak{a}_2$ 

[ *For*  $\langle \mathfrak{a}^* \rangle k''$  ] The following are equivalent:

1 $(W^B, M^B) \models @_k \langle a^* \rangle k''$ 2 $(W^B, M^B) \models @_k \langle a^n \rangle k'' \text{ for some } n \in \mathbb{N}$ by the semantics of  $a^*$ 3 $G \Vdash^k \langle a^n \rangle k'' \text{ for some } n \in \mathbb{N}$ by the induction hypothesis

4  $G \Vdash^k \langle \mathfrak{a}^* \rangle k''$ 

[ For  $\langle \mathfrak{a} \rangle \gamma$  with  $\gamma \notin F^n$  ] The following are equivalent:

| 1 | $(W^B, M^B) \models @_k \langle \mathfrak{a} \rangle \gamma$   |  |
|---|--|--|
| 2 | $(W^B, M^B) \models^{w_1} \gamma$ for some $w_1 \in  W^B $ such that $(W^B_k, w_1) \in W^B_a$            | by the definition of $\models$                     |
| 3 | $(W^B, M^B) \models @_k \langle \mathfrak{a} \rangle k_1 \text{ and } (W^B, M^B) \models @_{k_1} \gamma$ | by Proposition 21, since $(W^B, M^B)$ is reachable |
|   | for some $k_1 \in F^n$ such that $W_{k_1}^B = w_1$   |  |
| 4 | $G \Vdash^k \langle \mathfrak{a} \rangle k_1$ and $G \Vdash^{k_1} \gamma$ for some $k_1 \in F^n$         | by the induction hypothesis                        |
| 5 | $G \Vdash^k \langle \mathfrak{a}  angle \gamma$  | since $G$ is generic                               |

by Definition 29

[*For*  $\neg \gamma$ ] The following are equivalent:

| 1 | $(W^B, M^B) \models @_k \neg \gamma$                |                               |
|---|---|-------------------------------|
| 2 | $(W^B, M^B) \not\models @_k \gamma$                 | by the semantics of negation  |
| 3 | $G ut{\hspace{-0.1em}\not=\hspace{-0.1em}}^k\gamma$ | by the induction hypothesis   |
| 4 | $p \not\Vdash^k \gamma$ for all $p \in G$           | by the definition of $\Vdash$ |
| 5 | $p \Vdash^k \neg \gamma$ for some $p \in G$         | since $G$ is generic          |
| 6 | $G\Vdash^k \neg \gamma$                             |                               |

[*For*  $\lor \Gamma$ ] The following are equivalent:

| 1 | $(W^B, M^B) \models @_k \lor \Gamma$                                |                                 |
|---|---|---------------------------------|
| 2 | $(W^B, M^B) \models @_k \gamma \text{ for some } \gamma \in \Gamma$ | by the semantics of disjunction |
| 3 | $G \Vdash^k \gamma$ for some $\gamma \in \Gamma$                    | by the induction hypothesis     |
| 4 | $G \Vdash^k \lor \Gamma$  | by the definition of $\Vdash$   |

[For  $\exists X \cdot \gamma$ ] Let  $w = W_k^B$ . The following are equivalent:

 $\begin{array}{ll} 1 & (W^B, M^B) \models @_k \exists X \cdot \gamma \\ 2 & (W', M') \models^w \gamma \text{ for some expansion } (W', M') \text{ of } (W^B, M^B) \text{ to } \Delta[X] & \text{ by the definition of } \models \\ 3 & (W^B, M^B) \models^w \theta(\gamma) \text{ for some substitution } \theta \colon X \to \emptyset \text{ over } \Delta \text{ such that} & \text{ since } (W^B, M^B) \text{ is reachable} \\ (W^B, M^B) \upharpoonright_{\theta} = (W', M') & & \\ 4 & G \Vdash^k \theta(\gamma) \text{ for some substitution } \theta \colon X \to \emptyset \text{ over } \Delta & \text{ by the induction hypothesis} \\ 5 & G \Vdash^k \exists X \cdot \gamma & & \text{ by the definition of } \Vdash \\ \end{array}$ 

[For  $\downarrow z \cdot \gamma$ ] This case is straightforward since  $@_k \downarrow z \cdot \gamma$  is semantically equivalent to  $@_k \gamma(z \leftarrow k)$ .

[For  $@_{k'} \gamma$ ] This case is straightforward since  $@_k @_{k'} \gamma$  is semantically equivalent to  $@_{k'} \gamma$ .

Theorem 35 is a generalization of Generic Model Theorem for hybrid logics from [27]. The new cases from the present contribution correspond to structured actions, which include second, third and fourth cases.

# 6. Semantic forcing property

We study a semantic forcing property, which will be used to prove the Omitting Types Theorem for a fragment  $\mathcal{L}$  of HDFOLR semantically closed under negation and retrieve.

Framework 2. In this section, we arbitrarily fix

- 1. a signature  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  of  $\mathcal{L}$ ,
- 2. a class  $\mathcal{K}$  of Kripke structures over the signature  $\Delta$ , and
- 3. a sorted set  $C = \{C_s\}_{s \in S^e}$  of new rigid constants for  $\Delta$  such that  $card(C_s) = \alpha$  for all sorts  $s \in S^e$ , where  $S^e = S^r \cup \{any\}$  is the extended set of rigid sorts and any is the sort of nominals.

We let  $\alpha$  denote the power of  $\Delta$ .

If the set of sorts in  $\Sigma$  is empty then *C* consists only of nominals.

**Definition 36.** The semantic forcing property  $\mathbb{P} = (P, \leq, f)$  over the signature  $\Delta[C]$  relative to the class of Kripke structures  $\mathcal{K}$  is defined as follows:

- 1.  $P = \{p \subseteq \text{Sen}(\Delta[C]) \mid \text{card}(p) < \alpha \text{ and } (W, M) \models p \text{ for some } (W, M) \in |\text{Mod}(\Delta[C])| \text{ s.t. } (W, M) \upharpoonright_{\Delta} \in \mathcal{K}\},\$
- 2.  $\leq$  is the inclusion relation, and
- 3.  $f(p) = p \cap \operatorname{Sen}_b(\Delta[C])$  for all  $p \in P$ .

**Lemma 37.**  $\mathbb{P} = \langle P, \leq, f \rangle$  described in Definition 36 is a forcing property.

*Proof.* All conditions enumerated in Definition 28 obviously hold except the last one. Assume that  $f(p) \models @_k \gamma$ , where  $p \in P$  and  $@_k \gamma \in Sen_b(\Delta)$ . Since  $f(p) \subseteq p$ , we have  $p \models @_k \gamma$ . By Definition 36,  $(W, M) \models p$  for some  $(W, M) \in |Mod(\Delta[C])|$  such that  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ . Since  $(W, M) \models p$  and  $p \models @_k \gamma$ ,  $(W, M) \models p \cup \{@_k \gamma\}$ . Hence,  $q \coloneqq p \cup \{@_k \gamma\} \in P$  and  $p \leq q$ 

**Proposition 38.**  $\mathbb{P} = \langle P, \leq, f \rangle$  described in Definition 36 has the following properties:

- P1) If  $p \in P$  and  $@_k \langle \mathfrak{a}_1 \rangle \mathfrak{s}_2 \rangle k'' \in p$  then  $p \cup \{@_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k''\} \in P$  for some nominal  $k' \in C_{anv}$ .
- *P2)* If  $p \in P$  and  $@_k \langle \mathfrak{a} \rangle \gamma \in p$  with  $\gamma \notin F^n \cup C_{any}$  then  $p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\} \in P$  for some nominal  $k' \in C_{any}$ .
- *P3)* If  $p \in P$  and  $@_k \lor \Gamma \in p$  then  $p \cup \{@_k \gamma\} \in P$  for some  $\gamma \in \Gamma$ .
- P4) If  $p \in P$  and  $@_k \exists X \cdot \gamma \in p$  then there exists an injective mapping  $f: X \to C$  such that  $p \cup \{@_k \chi(\gamma)\} \in P$ , where  $\chi: \Delta[C, X] \to \Delta[C]$  is the unique extension of f to a signature morphism which preserves  $\Delta[C]$ .

*Proof.* Let  $p \in P$  be a condition. By the definition of  $\mathbb{P}$ , we have that  $p \subseteq \text{Sen}(\Delta[C'])$  for some  $C' \subset C$  with  $\operatorname{card}(C'_s) < \alpha$  for all  $s \in S^e$ .

P1) Assume that  $(a_1 \circ a_2)k'' \in p$ . Since card $(C_{any}) = \alpha$  and card $(C'_{any}) < \alpha$ , there exists  $k' \in C_{any} \setminus C'_{any}$ . We show that  $p \cup \{(a_k \land a_1)k', (a_k)k'' \in P\}$ :

by the definition of  $\mathbb{P}$ 

by the satisfaction condition

since  $(W', M') \models @_k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k''$ 

- 1  $(W, M) \models p$  for some model (W, M) over  $\Delta[C]$  with  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$
- 2  $(W', M') \coloneqq (W, M) \upharpoonright_{\Delta[C']} \models p$

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- 3  $(W'_k, w) \in W'_{\mathfrak{g}_1}$  and  $(w, W'_{k''}) \in W'_{\mathfrak{g}_2}$  for some  $w \in |W'|$
- 4  $(W'', M') \models @_k \langle a_1 \rangle k'$  and  $(W'', M') \models @_{k'} \langle a_2 \rangle k''$ , where (W'', M') is the unique expansion of (W', M') to  $\Delta[C', k']$  interpreting k' as w
- 5  $(V,N) \models p \cup \{@_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k''\}$ , where (V,N) is any expansion of by the satisfaction condition, since (W'', M') to  $\Delta[C]$   $(W'', M') \models p \cup \{@_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k''\}$

- P2) Assume that  $@_k \langle \mathfrak{a} \rangle \gamma \in p$  with  $\gamma \notin F^n \cup C_{any}$ . Since  $card(C_{any}) = \alpha$  and  $card(C'_{any}) < \alpha$ , there exists  $k' \in C_{any} \setminus C'_{any}$ . We show that  $p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\} \in P$ :
  - 1  $(W, M) \models p$  for some model (W, M) over  $\Delta[C]$  with  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$
  - 2  $(W', M') \coloneqq (W, M) \upharpoonright_{\Delta[C']} \models p$
  - 3  $(W'_k, w) \in W'_a$  and  $(W', M') \models^w \gamma$  for some  $w \in |W'|$
  - 4  $(W'', M') \models @_k \langle \mathfrak{a} \rangle k'$  and  $(W'', M') \models @_{k'} \gamma$ , where (W'', M') is the unique expansion of (W', M') to  $\Delta[C, k']$  interpreting k' as w
  - 5  $(V,N) \models p \cup \{@_k \langle a \rangle k', @_{k'} \gamma\}$ , where (V,N) is any expansion of (W'', M') to  $\Delta[C]$

$$6 \qquad p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\} \in P$$

by the definition of  $\mathbb{P}$ by the satisfaction condition since  $(W', M') \models @_k \langle \mathfrak{a} \rangle \gamma$ by semantics

by the satisfaction condition, since  $(W'', M') \models p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\}$ since  $(V, N) \upharpoonright_{\Delta} = (W', M') \upharpoonright_{\Delta} \in \mathcal{K}$ 

- P3) Assume that  $@_k \lor \Gamma \in p$ . There exists a Kripke structure (W, M) over  $\Delta[C]$  such that  $(W, M) \models p$  and  $(W, M) \upharpoonright \Delta \in \mathcal{K}$ . Since  $(W, M) \models @_k \lor \Gamma$ , we have  $(W, M) \models @_k \gamma$  for some  $\gamma \in \Gamma$ . Since  $(W, M) \models p$ ,  $(W, M) \models @_k \gamma$  and  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ , we obtain  $p \cup \{@_k \gamma\} \in P$ .
- P4) Assume that  $@_k \exists X \cdot \gamma \in p$ . Since  $card(C'_s) < \alpha$  and  $card(C_s) = \alpha$  for all sorts  $s \in S^e$ , by the finiteness of X, there exists an injective mapping  $f: X \to C \setminus C'$ . Let  $C'' := C' \cup f(X)$ . Let  $\chi' : \Delta[C', X] \to \Delta[C'']$  be the unique extension of f to a signature morphism which preserves  $\Delta[C']$ . Let  $\chi: \Delta[C, X] \to \Delta[C]$  be the unique extension of f to a signature morphism which preserves  $\Delta[C]$ . Let  $\iota: \Delta[C''] \hookrightarrow \Delta[C]$  and  $\iota': \Delta[C', X] \to \Delta[C, X]$  be inclusions. Since  $\chi$  and  $\chi'$  agree on X and they preserve the rest of the symbols, we have  $\chi' \ \mathfrak{gl} \iota = \iota' \ \mathfrak{gl} \chi$ .

We show that  $p \cup \{@_k \chi(\gamma)\} \in P$ :

 $(W, M) \models p$  for some Kripke structure (W, M) over the 1 by the definition of  $\ensuremath{\mathbb{P}}$ signature  $\Delta[C]$  such that  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ 2  $(W', M') \coloneqq (W, M) \upharpoonright_{\Delta[C']} \models p$ by the satisfaction condition  $(V', N') \models^{w} \gamma$  for some expansion (V', N') of (W', M') to 3 since  $@_k \exists X \cdot \gamma \in p$  and  $(W', M') \models p$ the signature  $\Delta[C', X]$ , where  $w = W'_k = V'_k$ let (V'', N'') be the unique  $\chi'$ -expansion of (V', N')4 (V'', N'') exists, as  $\chi'$  is a bijection 5 let (V, N) be any expansion of (V'', N'') to  $\Delta[C]$ 6  $(V,N)\upharpoonright_{\chi}\upharpoonright_{\iota'} = (V,N)\upharpoonright_{\iota}\upharpoonright_{\chi'} = (V'',N'')\upharpoonright_{\chi'} = (V',N')$ from 4 and 5, since  $\iota' \Im \chi = \chi' \Im \iota$ 7 (V, N)  $\upharpoonright_{\chi} \models^{w} \gamma$ by the local satisfaction condition, since  $(V, N) \upharpoonright_{\chi} \upharpoonright_{\iota'} = (V', N') \models^{w} \gamma$ 8  $(V, N) \models^{w} \chi(\gamma)$ by the local satisfaction condition  $(V, N) \models @_k \chi(\gamma)$ 9 since  $w = V'_k = (V \upharpoonright_{\chi} \upharpoonright_{\iota'})_k = V_k$ 10  $(V, N) \models p$ by the satisfaction condition, since  $(V, N) \upharpoonright_{\Delta[C']} = (W', M') \models p$ 11  $(V, N) \upharpoonright_{\Delta} \in \mathcal{K}$ since  $(V, N) \upharpoonright_{\Delta[C']} = (W', M')$  and  $(W', M') \upharpoonright_{\Delta} \in \mathcal{K}$ 12  $p \cup \{@_k \chi(\gamma)\} \in P$ from 9-11

Proposition 38 sets the basis for the following important result concerning semantic forcing properties, which says that all sentences of a given condition are forced eventually by some condition greater or equal than the initial one.

**Theorem 39** (Semantic Forcing Theorem). Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be the semantic forcing property described in Definition 36. For all  $\Delta[C]$ -sentences  $@_k \gamma$  and conditions  $p \in P$  we have:

 $q \Vdash^k \gamma$  for some  $q \ge p$  iff  $p \cup \{@_k \gamma\} \in P$ .

*Proof.* We proceed by induction on the structure of  $\gamma$ .

[For  $\gamma$  extended atomic] Assume that there is  $q \ge p$  such that  $q \Vdash^k \gamma$ . We show that  $p \cup \{@_k \gamma\} \in P$ : by Definition 29 1  $@_k \gamma \in q$ 2  $p \cup \{@_k \gamma\} \le q$ since  $q \ge p$  $(W, M) \models q$  for some Kripke structure (W, M) over 3 since  $q \in P$ the signature  $\Delta[C]$  such that  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$  $p \cup \{@_k \gamma\} \in P$ since  $(W, M) \models p \cup \{@_k \gamma\}$  and  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ 4 Assume that  $p \cup \{@_k \gamma\} \in P$ . Let  $q = p \cup \{@_k \gamma\}$ . By Definition 29,  $q \Vdash^k \gamma$ . [*For*  $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ ] The following are equivalent:  $q \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$  for some  $q \ge p$ 1 2  $q \Vdash^k \langle \mathfrak{a}_1 \rangle k''$  or  $q \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ by Definition 29 3  $p \cup \{@_k \langle \mathfrak{a}_1 \rangle k''\} \in P \text{ or } p \cup \{@_k \langle \mathfrak{a}_2 \rangle k''\} \in P$ by the induction hypothesis 4  $(W, M) \models p \cup \{@_k \langle \mathfrak{a}_1 \rangle k''\}$  or  $(W, M) \models p \cup \{@_k \langle \mathfrak{a}_2 \rangle k''\}$  for some by Definition 36 Kripke structure (W, M) over  $\Delta[C]$  such that  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$  $(W, M) \models p \cup \{ @_k \langle a_1 \cup a_2 \rangle k'' \}$  for some Kripke structure (W, M)5 by the semantics of  $\mathfrak{a}_1 \cup \mathfrak{a}_2$ over  $\Delta[C]$  such that  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ 6  $p \cup \{@_k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''\} \in P$ since  $(W, M) \models p \cup \{ @_k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k'' \}$  and  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ [For  $\langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''$ ] Assume that  $q \Vdash^k \langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''$  for some  $q \ge p$ . We show that  $p \cup \{\langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''\} \in P$ :  $q \Vdash^k \langle \mathfrak{a}_1 \rangle k'$  and  $q \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$  for some  $k' \in F^n \cup C_{anv}$ 1 by Definition 29  $q \cup \{@_k \langle \mathfrak{a}_1 \rangle k'\} \in P$ 2 by the induction hypothesis, since  $q \le q$  $q \cup \{@_k \langle \mathfrak{a}_1 \rangle k'\} \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ by Lemma 31 (2), since  $q \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$  and  $q \leq q \cup \{ @_k \langle \mathfrak{a}_1 \rangle k' \}$ 3 4  $p \cup \{@_k \langle \mathfrak{a}_1 \rangle k'\} \cup \{@_{k'} \langle \mathfrak{a}_2 \rangle k''\} \in P$ by the induction hypothesis, since  $p \cup \{\langle \mathfrak{a}_1 \rangle k'\} \le q \cup \{\langle \mathfrak{a}_1 \rangle k'\}$ 5  $p \cup \{@_k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k''\} \in P$ by the definition of  $\mathbb{P}$ Assume that  $p \cup \{@_k \langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''\} \in P$ . We show that  $q \Vdash^k \langle \mathfrak{a}_1 \ \mathfrak{g} \ \mathfrak{a}_2 \rangle k''$  for some  $q \ge p$ :  $p \cup \{ @_k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k'', @_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k'' \} \in P \text{ for some } k \in C_{any}$ 1 by Proposition 38 (P1) let  $r := p \cup \{ @_k \langle \mathfrak{a}_1 \ \mathfrak{s} \ \mathfrak{a}_2 \rangle k'', @_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k'' \}$ 2  $s \Vdash^k \langle \mathfrak{a}_1 \rangle k'$  for some  $s \ge r$ 3 by the induction hypothesis, since  $r \cup \{ @_k \langle \mathfrak{a}_1 \rangle k' \} = r \in P$  $q \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$  for some  $q \ge s$ 4 by the induction hypothesis, since  $s \cup \{ @_{k'} \langle \mathfrak{a}_2 \rangle k'' \} = s \in P$  $q \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ 5 by Lemma 31 (2), since  $s \Vdash^k \langle \mathfrak{a}_1 \rangle k'$  and  $q \ge s$  $q \Vdash^k \langle \mathfrak{a}_1 \mathfrak{s} \mathfrak{a}_2 \rangle k''$ 6 from 4 and 5

[*For*  $\langle \mathfrak{a}^* \rangle k''$ ] The following are equivalent:

| 1 | $q \Vdash^k \langle \mathfrak{a}^* \rangle k''$ for some $q \ge p$                             |                             |
|---|--|-----------------------------|
| 2 | $q \Vdash^k \langle \mathfrak{a}^n \rangle k''$ for some $q \ge p$ and $n \in \mathbb{N}$      | by Definition 29            |
| 3 | $p \cup \{ @_k \langle \mathfrak{a}^n \rangle k'' \} \in P \text{ for some } n \in \mathbb{N}$ | by the induction hypothesis |

 $p \cup \{@_k \langle \mathfrak{a}^* \rangle k''\} \in P$ 4

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by the semantics of \mathfrak{a}^* and the definition of \mathbb P
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[For  $\langle \mathfrak{a} \rangle \gamma$  with  $\gamma \notin F^{\mathsf{n}} \cup C_{\mathsf{any}}$ ] Assume that  $q \Vdash^k \langle \mathfrak{a} \rangle \gamma$  for some  $q \ge p$ . We show that  $p \cup \{ @_k \langle \mathfrak{a} \rangle \gamma \} \in P$ :

| 1  | $q \Vdash^k \langle \mathfrak{a} \rangle k'$ and $q \Vdash^{k'} \gamma$ for some nominal $k'$  | from $q \Vdash^k \langle \mathfrak{a} \rangle \gamma$ , by Definition 29                                       |
|----|--|--|
| 2  | $q \cup \{@_{k'} \gamma\} \in P$   | from $q \leq q$ and $q \Vdash^{k'} \gamma$ , by the induction hypothesis                                       |
| 3  | $@_k \langle \mathfrak{a} \rangle k' \in q$  | from $q \Vdash^k \langle \mathfrak{a} \rangle k'$ , by Definition 29   |
| 4  | $(W, M) \models q \cup \{ @_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma \}$ for some Kripke structure $(W, M)$ over $\Delta[C]$ such that $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ | since $q \cup \{@_{k'} \gamma\} \in P$ and $@_k \langle \mathfrak{a} \rangle k' \in q$                         |
| 5  | $(W, M) \models q \cup \{@_k \langle a \rangle \gamma\}$ for some Kripke structure $(W, M)$<br>over $\Delta[C]$ such that $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$                      | since $\{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\} \models @_k \langle \mathfrak{a} \rangle \gamma$ |
| 6  | $q \cup \{@_k \langle \mathfrak{a} \rangle \gamma\} \in P$   | by Definition 36   |
| 7  | $p \cup \{ @_k \langle \mathfrak{a} \rangle \gamma \} \in P$   | since $p \subseteq q$  |
| As | sume that $p \cup \{@_k \langle \mathfrak{a} \rangle \gamma\} \in P$ . We show that $q \Vdash^k \langle \mathfrak{a} \rangle \gamma$ for   | some $q \ge p$ :   |
| 1  | $(p \cup \{@_k \langle \mathfrak{a} \rangle \gamma\}) \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\} \in P \text{ for some nominal } k' \in F$                                      | $^{n} \cup C_{any}$ by Proposition 38 (P2)   |
| 2  | $q \Vdash^{k'} \gamma \text{ for some } q \ge p \cup \{ @_k \langle \mathfrak{a} \rangle \gamma, @_k \langle \mathfrak{a} \rangle k' \}$   | by the induction hypothesis  |
| 3  | $q \Vdash^k \langle \mathfrak{a}  angle k'$  | since $@_k \langle \mathfrak{a} \rangle k' \in f(q)$   |
| 4  | $q \Vdash^k \langle \mathfrak{a}  angle \gamma$  | by 3 and 2   |

[*For*  $\neg \gamma$ ] By the induction hypothesis, for each  $q \in P$  we have

(S1)  $r \Vdash^k \gamma$  for some  $r \ge q$  iff  $q \cup \{@_k \gamma\} \in P$ , which is equivalent to

- (S2)  $r \nvDash^k \gamma$  for all  $r \ge q$  iff  $q \cup \{@_k \gamma\} \notin P$ , which is equivalent to
- (S3)  $q \Vdash^k \neg \gamma$  iff  $q \cup \{@_k \gamma\} \notin P$ .

Assume that  $q \Vdash^k \neg \gamma$  for some  $q \ge p$ . We show that  $p \cup \{@_k \neg \gamma\} \in P$ :

 $q \cup \{@_k \gamma\} \notin P$ 1 by statement S3  $(W, M) \models q$  for some Kripke structure (W, M) over  $\Delta[C]$  such that 2 by Definition 36, since  $q \in P$  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ 3  $(W, M) \not\models @_k \gamma$ since  $q \cup \{@_k \gamma\} \notin P$ 4  $(W, M) \models @_k \neg \gamma$ by the semantics of  $\neg$ 5  $q \cup \{@_k \neg \gamma\} \in P$ since  $(W, M) \models q \cup \{@_k \neg \gamma\}$  and  $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ 6  $p \cup \{@_k \neg \gamma\} \in P$ since  $p \cup \{@_k \neg \gamma\} \subseteq q \cup \{@_k \neg \gamma\}$ 

Assume that  $p \cup \{@_k \neg \gamma\} \in P$ . We show that  $q \Vdash^k \neg \gamma$  for some  $q \ge p$ :

1 let  $q = p \cup \{@_k \neg \gamma\}$ 2  $q \cup \{@_k \gamma\} \notin P$  since  $@_k \neg \gamma \in q$ 3  $q \Vdash^k \neg \gamma$  by statement S3

[For  $\lor \Gamma$ ] Assume that there exists  $q \ge p$  such that  $q \Vdash^k \lor \Gamma$ . We show that  $p \cup \{@_k \lor \Gamma\} \in P$ :

| 1  | $q \Vdash^k \gamma$ for some $\gamma \in \Gamma$   | by Definition 29                   |
|----|--|------------------------------------|
| 2  | $p \cup \{@_k \gamma\} \in P$  | by the induction hypothesis        |
| 3  | $(W, M) \models p \cup \{@_k \gamma\}$ for some Kripke structure<br>$(W, M)$ over $\Delta[C]$ such that $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$      | by Definition 36                   |
| 4  | $(W, M) \models p \cup \{@_k \lor \Gamma\}$ for some Kripke structure<br>$(W, M)$ over $\Delta[C]$ such that $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ | by the semantics of $\vee$         |
| 5  | $p \cup \{@_k \lor \Gamma\} \in P$   | by Definition 36                   |
| As | sume that $p \cup \{@_k \lor \Gamma\} \in P$ . We show that $q \Vdash^k$   | $\vee \Gamma$ for some $q \ge p$ : |
| 1  | $(p \cup \{@_k \lor \Gamma\}) \cup \{@_k \gamma\} \in P \text{ for some } \gamma \in \Gamma$   | by Proposition 38 (P3)             |
| 2  | $q \Vdash^k \gamma$ for some $q \ge p \cup \{@_k \lor \Gamma\}$  | by the induction hypothesis        |

| 3 $q \Vdash^k \lor \Gamma$ for some $q \ge p$ by Defi | inition 29 |
|---|------------|
|---|------------|

[For  $\exists X \cdot \gamma$ ] Assume that  $q \Vdash^k \exists X \cdot \gamma$  for some  $q \ge p$ . We show that  $p \cup \{@_k \exists X \cdot \gamma\} \in P$ :

| 1 | $q \Vdash^k \theta(\gamma)$ for some substitution $\theta : X \to \emptyset$   | by Definition 29            |
|---|--|-----------------------------|
| 2 | $p \cup \{ @_k \theta(\gamma) \} \in P$  | by the induction hypothesis |
| 3 | $(W, M) \models p \cup \{@_k \theta(\gamma)\}$ for some Kripke structure $(W, M)$ over $\Delta[C]$ such that $(W, M) \upharpoonright_{\Delta} \in \mathcal{K}$ | by Definition 36            |
| 4 | $(W, M) \models p \cup \{@_k \exists X \cdot \gamma\} \text{ and } (W, M) \upharpoonright_{\Delta} \in \mathcal{K}$  | by semantics                |
| 5 | $p \cup \{@_k \exists X \cdot \gamma\} \in P$  | by Definition 36            |

We assume that  $p \cup \{@_k \exists X \cdot \gamma\} \in P$ . We show that  $q \Vdash^k \exists X \cdot \gamma$  for some  $q \ge p$ :

| 1 | $(p \cup \{@_k \exists X \cdot \gamma\}) \cup \{@_k \chi(\gamma)\} \in P$                        | by Proposition 38 (P4)      |
|---|--|-----------------------------|
|   | for some signature morphism $\chi \colon \Delta[C, X] \to \Delta[C]$ which preserves $\Delta[C]$ |                             |
| 2 | $q \Vdash^k \chi(\gamma)$ for some $q \ge p \cup \{@_k \exists X \cdot \gamma\}$                 | by the induction hypothesis |
| 3 | $q \Vdash^k \exists X \cdot \gamma \text{ for some } q \geq p$                                   | by Definition 29            |

[For  $\downarrow z \cdot \gamma$ ] This case is straightforward, as  $@_k \downarrow z \cdot \gamma$  is semantically equivalent to  $@_k \gamma(z \leftarrow k)$ .

[For  $@_{k'} \gamma$ ] This case is straightforward, as  $@_k @_{k'} \gamma$  is semantically equivalent to  $@_{k'} \gamma$ .

The following result is a corollary of Theorem 39. It shows that each generic set of a given semantic forcing property has a reachable model that satisfies all its conditions.

**Corollary 40.** Let  $\mathbb{P} = \langle P, \leq, f \rangle$  be the semantic forcing property described in Definition 36. *Then for each generic set G we have:* 

- C1)  $G \Vdash^k \gamma$  for all conditions  $p \in G$ , sentences  $\gamma \in p$  and nominals  $k \in F^n \cup C_{any}$ .
- C2) There exists a generic structure  $(W^G, M^G)$  for G which is reachable and satisfies each condition  $p \in G$ .

## Proof.

C1) Suppose towards a contradiction that  $G \nvDash^k \gamma$  for some  $p \in G, \gamma \in p$  and nominal  $k \in F^n \cup C_{any}$ . Then:

| 1 | $q \Vdash^k \neg \gamma$ for some $q \in G$ | since G is generic   |
|---|---|--|
| 2 | $r \ge p$ and $r \ge q$ for some $r \in G$  | since $G$ is generic   |
| 3 | $\gamma \in r$                              | since $\gamma \in p$ and $r \ge p$                                     |
| 4 | $r \cup \{@_k \gamma\} \in P$               | since $r \models @_k \gamma$   |
| 5 | $s \Vdash^k \gamma$ for some $s \ge r$      | by Theorem 39  |
| 6 | $s \Vdash^k \neg \gamma$                    | by Lemma 31 (2) since $s \ge q$ and $q \Vdash^k \neg \gamma$           |
| 7 | contradiction                               | by Lemma 31 (4) since $s \Vdash^k \gamma$ and $s \Vdash^k \neg \gamma$ |

It follows that  $G \Vdash^k \gamma$  for all  $p \in G, \gamma \in p$  and nominals *k*.

C2) By Theorem 35, there exists a generic model  $(W^G, M^G)$  for G which is reachable. Let  $p \in G$ ,  $\gamma \in p$  and  $w \in |W^G|$ . Since  $(W^G, M^G)$  is reachable, w is the denotation of some nominal  $k \in F^n \cup C_{any}$ . By the first part of the proof,  $G \Vdash^k \gamma$ . Since  $(W^G, M^G)$  is a model for  $G, (W^G, M^G) \models @_k \gamma$ . Hence,  $(W^G, M^G) \models^w \gamma$ . As w was arbitrary, we have  $(W^G, M^G) \models \gamma$ .

# 7. Omitting Types Theorem

Let  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma^n)$  be a countable signature. Let  $C = \{C_s\}_{s \in S^e}$  be a finite set of new constants of extended rigid sorts. We say that a Kripke structure (W, M) over  $\Delta$  realizes a set  $\Gamma$  of sentences over  $\Delta[C]$  iff there exists an expansion

(V, N) of (W, M) to  $\Delta[C]$  such that  $(V, N) \models \Gamma$ . We say that (W, M) omits  $\Gamma$  if (W, M) does not realize  $\Gamma$ . We say that a satisfiable set T of sentences over  $\Delta$  locally realizes  $\Gamma$  if there exists a finite set p of sentences over  $\Delta[C]$  such that  $T \cup p$  is satisfiable, and  $T \cup p \models \Gamma$ . In the following we generalize these definitions to signatures of any power.

**Definition 41 (Omitting Types semantically).** Assume a signature  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma^n)$ , and let  $\alpha$  be the power of  $\Delta$ . Let  $X = \{X_s\}_{s \in S^e}$  be a sorted set of variables for  $\Delta$  such that  $\operatorname{card}(X_s) < \omega$  for all sorts  $s \in S^e$ .

- A Kripke structure (W, M) over  $\Delta$  realizes a type  $\Gamma \subseteq \text{Sen}(\Delta[X])$  if there exists an expansion (V, N) of (W, M) to  $\Delta[X]$  such that  $(V, N) \models \Gamma$ .
- A Kripke structure (W, M) over  $\Delta$  omits a set  $\Gamma$  of  $\Delta[X]$ -sentences if (W, M) does not realize  $\Gamma$ .

Classically,  $\Gamma$  from Definition 41 is called a type with free variables *X*.

**Definition 42 (Omitting Types syntactically).** Let  $\Delta$  be a signature, and let  $\alpha$  be the power of  $\Delta$ . Let  $X = \{X_s\}_{s \in S^e}$  be a sorted set of variables for  $\Delta$  such that  $X_s$  is finite for all sorts  $s \in S^e$ . A theory  $T \subseteq \text{Sen}(\Delta) \alpha$ -realizes a type  $\Gamma \subseteq \text{Sen}(\Delta[X])$  if there exist

- a sorted set  $C = \{C_s\}_{s \in S^e}$  of new constants for  $\Delta$  with  $card(C_s) < \alpha$  for all  $s \in S^e$ ,
- a substitution  $\theta : X \to C$ , and
- a set of sentences p over  $\Delta[C]$  with  $card(p) < \alpha$ ,

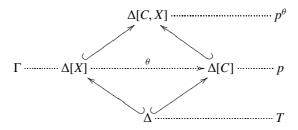
such that  $T \cup p$  is satisfiable and  $T \cup p \models \theta(\Gamma)$ . We say that T  $\alpha$ -omits  $\Gamma$  if T does not  $\alpha$ -realize  $\Gamma$ .

Notice that the power of any signature is at least  $\omega$ . If  $\alpha = \omega$ , we say that *T* locally omits  $\Gamma$  instead of *T*  $\alpha$ -omits  $\Gamma$ . Definition 42 is similar to the definition of locally omitting types for first-order logic without equality from [37]. Our results are applicable to fragments  $\mathcal{L}$  without equality. We give a couple of equivalent descriptions of the omitting types property which can be found in the literature.

## Lemma 43.

- *L1)* Assume that  $\mathcal{L}$  is semantically closed under equality. <sup>8</sup> Then  $T \alpha$ -realizes  $\Gamma$  as described in Definition 42 iff there exist (a) a sorted set  $C = \{C_s\}_{s \in S^e}$  of new constants for  $\Delta[X]$  with  $\operatorname{card}(C_s) < \alpha$  for all  $s \in S^e$ , and (b) a set of sentences p over  $\Delta[C, X]$  with  $\operatorname{card}(p) < \alpha$ , such that  $T \cup p$  is satisfiable and  $T \cup p \models \Gamma$ .
- *L2)* Assume that  $\mathcal{L}$  is semantically closed under equality, conjunction and quantifiers. Then T locally realizes  $\Gamma$  iff there exists a finite set of  $\Delta[X]$ -sentences p such that  $T \cup p$  is satisfiable and  $T \cup p \models \Gamma$ .
- L3) Assume that  $\mathcal{L}$  is compact and semantically closed under equality, conjunction and quantifiers. Then  $T \alpha$ -realizes  $\Gamma$  iff there exists a set of  $\Delta[X]$ -sentences p with  $card(p) < \alpha$  such that  $T \cup p$  is satisfiable and  $T \cup p \models \Gamma$ .

Proof. The backward implication is straightforward for all cases. Therefore, we will focus on the forward implication.



Let  $\theta: X \to C$  be a substitution with  $\operatorname{card}(C_s) < \alpha$  for all  $s \in S^e$ , and let p be a set of sentences over  $\Delta[C]$  with  $\operatorname{card}(p) < \alpha$  such that  $T \cup p$  is satisfiable and  $T \cup p \models \theta(\Gamma)$ . Without loss of generality, we assume that  $X \cap C = \emptyset$ . Since in all three cases  $\mathcal{L}$  is semantically closed under equality, there exists a set of sentences  $p^\theta$  over  $\Delta[C, X]$  semantically equivalent with  $\{x = \theta(x) \mid x \in X\}^9$ . Since  $T \cup p$  is satisfiable,  $T \cup p \cup p^\theta$  is satisfiable too. Now we consider three

<sup>&</sup>lt;sup>8</sup>  $\mathcal{L}$  is semantically closed under equality whenever (a) for any nominal k there exists an  $\mathcal{L}$ -sentence  $\varphi$  such that  $(W, M) \models^{w} \varphi$  iff  $w = W_{k}$  for all Kripke structures (W, M) and all possible worlds  $w \in |W|$ , and (b) for any open terms  $t_{1}, t_{2} \in T_{\overline{\Sigma}}$  there exists an  $\mathcal{L}$ -sentence  $\varphi$  such that  $(W, M) \models^{w} \varphi$  iff  $M_{w,t_{1}} = M_{w,t_{2}}$  for all Kripke structures (W, M) and all possible worlds  $w \in |W|$ .

<sup>&</sup>lt;sup>9</sup>Here = is a shorthand from the metalanguage. In particular, for nominals  $x = \theta(x)$  means that  $(@_x \theta(x))$  for all  $x \in X_{any}$ .

cases.

L1) As  $p \cup p^{\theta}$  is a set of sentences over  $\Delta[C, X]$ , we show that  $T \cup p \cup p^{\theta} \models \Gamma$ :

- 1 let  $(W, M) \in |\mathsf{Mod}(\Delta[C, X])|$  such that  $(W, M) \models T \cup p \cup p^{\theta}$
- 2  $(W, M) \upharpoonright_{\Delta[C]} \upharpoonright_{\theta} = (W, M) \upharpoonright_{\Delta[X]}$
- $3 \qquad (W, M) \upharpoonright_{\Delta[C]} \models T \cup p$
- 4  $(W, M) \upharpoonright_{\Delta[C]} \models \theta(\Gamma)$
- 5  $(W, M) \upharpoonright_{\Delta[X]} = (W, M) \upharpoonright_{\Delta[C]} \upharpoonright_{\theta} \models \Gamma$
- $6 \qquad (W, M) \models \Gamma$
- 7  $T \cup p \cup p^{\theta} \models \Gamma$

since  $(W, M) \models p^{\theta}$ by the satisfaction condition, since  $(W, M) \models T \cup p$ 

since  $T \cup p \models \theta(\Gamma)$  and  $(W, M) \upharpoonright_{\Delta[C]} \models T \cup p$ by the satisfaction condition for substitutions by the satisfaction condition

by its definition, p' is in one-to-one correspondence with  $\{p^{\gamma} \mid \gamma \in \Gamma\}$ 

since (W, M) was arbitrarily chosen

L2) If  $\alpha = \omega$ , we show  $T \cup \{\varphi\} \models \Gamma$  for a single sentence  $\varphi$  over  $\Delta[X]$ :

| 1 | the sets C, p and $p^{\theta}$ are finite   | since their cardinals are strictly less than $\omega$                        |
|---|---|--|
| 2 | there exists a $\Delta[X]$ -sentence $\varphi$ semantically equivalent with $\exists C \cdot \forall z \cdot @_z \land (p \cup p^{\theta})$ | since $\mathcal{L}$ is semantically closed under conjunction and quantifiers |
| 3 | $T \cup \{\varphi\}$ is satisfiable over $\Delta[X]$  | since $T \cup p \cup p^{\theta}$ is satisfiable over $\Delta[C, X]$          |
| 4 | $T\cup\{\varphi\}\models\Gamma$   | since $T \cup p \cup p^{\theta} \models \Gamma$                              |

L3) If  $\mathcal{L}$  is compact, we show that  $T \cup p' \models \Gamma$  for a set p' of sentences over  $\Delta[X]$  with card $(p') < \alpha$ :

- 1 for each  $\gamma \in \Gamma$  there exists  $p^{\gamma} \subseteq p \cup p^{\theta}$  finite such that  $T \cup p^{\gamma} \models \gamma$  by compactness, since  $T \cup p \cup p^{\theta} \models \gamma$  for all  $\gamma \in \Gamma$
- 2 let  $C^{\gamma}$  be all constants from *C* which occur in  $p^{\gamma}$  for all  $\gamma \in \Gamma$
- there exists a set p' of  $\Delta[X]$ -sentences semantically equivalent with 3 since  $\mathcal{L}$  is semantically closed under conjunction,  $\{\exists C^{\gamma} \cdot \forall z \cdot @_z \, \wedge \, p^{\gamma} \mid \gamma \in \Gamma\}$ retrieve and quantifiers 4  $T \cup p'$  is satisfiable over  $\Delta[C]$ since  $T \cup p \cup p^{\theta}$  is satisfiable over  $\Delta[C, X]$  $T \cup p' \models \Gamma$ since  $T \cup p^{\gamma} \models \gamma$  for all  $\gamma \in \Gamma$ 5  $\operatorname{card}(\mathcal{P}_{\omega}(p \cup p^{\theta})) < \alpha$ 6 since  $card(p) < \alpha$  and  $card(p^{\theta}) < \alpha$ 7  $\operatorname{card}(\{p^{\gamma} \mid \gamma \in \Gamma\}) < \alpha$ since  $\{p^{\gamma} \mid \gamma \in \Gamma\} \subseteq \mathcal{P}_{\omega}(p \cup p^{\theta})$

8  $\operatorname{card}(p') < \alpha$ 

The following result is needed for proving the Omitting Types Theorem.

**Lemma 44.** Assume that  $T \alpha$ -omits  $\Gamma$  as described in Definition 42. Then for any substitution  $\theta : X \to C$  over  $\Delta$  such that  $\operatorname{card}(C_s) < \alpha$  for all  $s \in S^e$ , and any set of  $\Delta[C]$ -sentences p such that  $\operatorname{card}(p) < \alpha$  and  $T \cup p$  is satisfiable, there exists  $\gamma \in \Gamma$  such that  $T \cup p \cup \{@_z \neg \theta(\gamma)\}$  is satisfiable, where z is a nominal variable for  $\Delta[C]$ .

*Proof.* Let  $C = \{C_s\}_{s \in S^e}$  be a set of new constants for  $\Delta$  with  $\operatorname{card}(C_s) < \alpha$  for all  $s \in S^e$ . Let  $\theta : X \to C$  be a substitution over  $\Delta$ . Let p be a set of  $\Delta[C]$ -sentences such that  $\operatorname{card}(p) < \alpha$  and  $T \cup p$  satisfiable. Since  $T \alpha$ -omits  $\Gamma$ , we have  $T \cup p \not\models \theta(\Gamma)$ . There exists a Kripke structure (W, M) over  $\Delta[C]$  such that  $(W, M) \models T \cup p$  and  $(W, M) \not\models \theta(\Gamma)$ . It follows that  $(W, M) \models^w \neg \theta(\gamma)$  for some possible world  $w \in |W|$  and some sentence  $\gamma \in \Gamma$ . Let z be a new nominal for  $\Delta[C]$ , and let  $(W^{z \leftarrow w}, M)$  be the unique expansion of (W, M) to  $\Delta[z, C]$  which interprets z as w. Since  $(W, M) \models^w \neg \theta(\gamma)$ , we get  $(W^{z \leftarrow w}, M) \models @_z \neg \theta(\gamma)$ . Hence,  $T \cup p \cup \{@_z \neg \theta(\gamma)\}$  is satisfiable.

**Definition 45 (Omitting Types Property).** We say that  $\mathcal{L}$  has  $\alpha$ -Omitting Types Property ( $\alpha$ -OTP), where  $\alpha$  is an infinite cardinal, whenever

- for all signatures  $\Delta$  of power at most  $\alpha$ ,
- all satisfiable theories  $T \subseteq \text{Sen}(\Delta)$ , and
- all families of types  $\{\Gamma^i \subseteq \text{Sen}(\Delta[X^i]) \mid i < \alpha\},\$

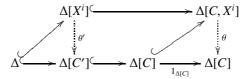
where  $X^i = \{X^i_s\}_{s \in S^e}$  is a set of variables for  $\Delta$  with  $card(X^i_s) < \omega$  for all  $s \in S^e$ ,

such that  $T \alpha$ -omits  $\Gamma^i$  for all  $i < \alpha$ , there exists a Kripke structure over  $\Delta$  which satisfies T and omits  $\Gamma^i$  for all  $i < \alpha$ . If for all signatures  $\Delta \in |Sig^{\mathcal{L}}|$  and all cardinals  $\alpha$  equal or greater than the power of  $\Delta$ ,  $\mathcal{L}$  has  $\alpha$ -OTP then  $\mathcal{L}$  has OTP.

All the ingredients for proving Omitting Types Theorem are in place.

**Theorem 46** (Extended Omitting Types Theorem). Let  $\alpha$  be an infinite cardinal. Assume that  $\mathcal{L}$  is semantically closed under retrieve and negation, and if  $\alpha > \omega$  assume that  $\mathcal{L}$  is compact. Then  $\mathcal{L}$  has  $\alpha$ -OTP.

*Proof.* Assume that  $T \alpha$ -omits  $\Gamma^i$  as described in Definition 42. Let  $C = \{C_s\}_{s \in S^e}$  be a sorted set of new constants for  $\Delta$  such that  $\operatorname{card}(C_s) = \alpha$  for all  $s \in S^e$ . Let  $\mathbb{P} = (P, \leq, f)$  be the semantic forcing property described in Definition 36 with  $\mathcal{K} = |\operatorname{Mod}(\Delta, T)|$ . The proof is performed in four steps.



- S1) We show that for any condition  $p \in P$ , any index  $i < \alpha$ , and any substitution  $\theta : X^i \to \emptyset$  over  $\Delta[C]$ , there exist a sentence  $\gamma \in \Gamma^i$  and a nominal  $c \in C$  such that  $p \cup \{@_c \neg \theta(\gamma)\} \in P$ :
  - 1 let  $C^p$  be the set of all constants from C which occur in p
  - 2 there exists  $c \in C_{any} \setminus (\theta(X_{any}^i) \cup C_{any}^p)$

since  $\operatorname{card}(\theta(X_{\operatorname{any}}^i)) < \omega$ ,  $\operatorname{card}(C_{\operatorname{any}}^p) < \alpha$  and  $\operatorname{card}(C_{\operatorname{any}}) = \alpha$ 

- 3 let  $C' \coloneqq \theta(X^i) \cup C^p \cup \{c\}$
- 4 let  $\theta' : X^i \to C'$  be the substitution over  $\Delta$  defined by  $\theta(x) = \theta'(x)$  for all  $x \in X^i$
- 5  $T \cup p \cup \{@_c \neg \theta'(\gamma)\}$  is satisfiable for some  $\gamma \in \Gamma^i$
- $6 \qquad p \cup \{@_c \neg \theta(\gamma)\} \in P$

by Lemma 44, since  $T \alpha$ -omits  $\Gamma^i$ since  $@_c \neg \theta(\gamma) = @_c \neg \theta'(\gamma)$ , we have  $(W, M) \models T \cup p \cup \{@_c \neg \theta(\gamma)\}$  for some  $(W, M) \in |\mathsf{Mod}(\Delta[C])|$ 

S2) The cardinality of the set  $S^i$  of all substitutions  $\theta : X^i \to \emptyset$  over  $\Delta[C]$  is equal or less than  $\alpha$ . It follows that the cardinality of  $S := \bigcup_{i < \alpha} S^i$  is equal or less than  $\alpha$ . Let  $\{\theta^j : X^{i_j} \to \emptyset \in S \mid j < \alpha\}$  be an enumeration of S. Let  $\{\widehat{a}_{k_j} \varphi_j \in \text{Sen}(\Delta[C]) \mid j < \alpha\}$  be an enumeration of the  $\Delta[C]$ -sentences with retrieve as the top operator. We define an increasing chain of conditions  $p_0 \le p_1 \le \ldots$  by induction on ordinals:

 $[j=0] p_0 \coloneqq \emptyset.$ 

- $[j \Rightarrow j+1]$  If  $p_j \Vdash^{k_j} \neg \varphi_j$  then let  $q \coloneqq p_j$  else let  $q \ge p_j$  be a condition such that  $q \Vdash^{k_j} \varphi_j$ . By the first part of the proof, there exist  $\gamma \in \Gamma^{i_j}$  and  $c \in C$  such that  $q \cup \{ @_c \neg \theta^j(\gamma) \} \in P$ . Let  $p_{j+1} \coloneqq q \cup \{ @_c \neg \theta^j(\gamma) \}$ .
- $[\beta < \alpha \text{ limit ordinal}] p_{\beta} := \bigcup_{j < \beta} p_j$ . Since  $card(p_j) < \alpha$  for all  $j < \beta$  and  $\beta < \alpha$ , we have  $card(p_{\beta}) < \alpha$ . Since  $p_j \in P$  for all  $j < \beta$ , the set  $T \cup p_j$  is satisfiable for all  $j < \beta$ . By compactness<sup>10</sup>,  $(\bigcup_{j < \beta} p_j) \cup T$  is satisfiable too. Hence,  $p_{\beta} \in P$ .

The set  $G = \{q \in P \mid q \leq p_{j+1} \text{ for some } j < \alpha\}$  is generic. Let  $k \in F^n \cup C_{any}$  and  $\psi \in T$ . Suppose towards a contradiction that  $q \Vdash^k \neg \psi$  for some  $q \in G$  then:

| 1 $q \cup \{@_k \psi\} \in P$ since $\psi \in T$ and $q$  | $q \cup T$ is satisfiable |
|---|---------------------------|
| 2 $r \Vdash^k \psi$ for some $r \ge q$ by Theorem 39      |                           |
| 3 $r \Vdash^k \neg \psi$ since $q \Vdash^k \neg \psi$ and | nd $r \ge q$              |
| 4 contradiction by Lemma 31 (4)                           | ) from 2 and 3            |

Since *G* is generic,  $q \Vdash^k \psi$  for some  $q \in G$ .

<sup>&</sup>lt;sup>10</sup>If there exists a limit ordinal  $\beta < \alpha$  then  $\alpha$  is not countable, so we assume  $\mathcal{L}$  is compact.

S3) By Theorem 35, there exists a generic Kripke structure (W, M) for *G* that is reachable. Let  $(V, N) := (W, M) \upharpoonright_{\Delta}$ . We show that  $(V, N) \models T$ :

by the satisfaction condition, since  $(W, M) \upharpoonright_{\Delta} = (V, N)$ 

generic for G

since  $\gamma \in \Gamma^i$ 

since  $@_c \neg \theta^j(\gamma) \in p_{j+1}$ 

 $(W', M') \upharpoonright_{\Delta[X^i]} = (V', N')$ 

by the local satisfaction condition for  $\theta^{j}$ 

by the local satisfaction condition, since

since (V', N') is an arbitrary expansion of (V, N)

1 let  $w \in |W|$  and  $\psi \in T$ 

| 2 | $W_k = w$ for some $k \in F^n \cup C_{any}$ | since $(W, M)$ is reachable                                |
|---|---|--|
| 3 | $G\Vdash^k\psi$                             | by the second part of the proof                            |
| 4 | $(W, M) \models @_k \psi$                   | since $(W, M)$ is generic for $G$                          |
| 5 | $(W, M) \models^w \psi$                     | by the semantics of $@$ , since $W_k = w$                  |
| 6 | $(W, M) \models T$                          | since $w \in  W $ and $\psi \in T$ were arbitrarily chosen |

7  $(V, N) \models T$ 

S4) We show that (V, N) omits  $\Gamma^i$  for all  $i < \alpha$ :

- 1let (V', N') be an arbitrary expansion of (V, N) to  $\Delta[X^i]$ 2there exists an expansion (W', M') of (W, M) to  $\Delta[C, X^i]$  such that<br/> $(W', M') \upharpoonright_{\Delta[X^i]} = (V', N')$ by interpreting  $X^i$  as (V', N') interprets  $X^i$ 3there exists  $\theta^j : X^i \to \emptyset \in S$  such that  $(W, M) \upharpoonright_{\theta^j} = (W', M')$ since (W, M) is reachable4there exist  $c \in C$  and  $\gamma \in \Gamma^i$  such that  $@_c \neg \theta^j(\gamma) \in p_{j+1}$ by the construction of the chain  $p_0 \le p_1 \le \dots$ 5 $(W, M) \models p_{j+1}$ by Corollary 40, since  $p_{j+1} \in G$  and (W, M) is
- 6  $(W, M) \models^{w} \neg \theta^{j}(\gamma)$ , where  $w = W_{c}$
- 7  $(W', M') \models^w \neg \gamma$
- 8  $(V', N') \models^w \neg \gamma$
- 9  $(V', N') \not\models \Gamma^i$
- 10 (V, N) omits  $\Gamma^i$

We conclude that (V, N) is a Kripke structure over  $\Delta$  which satisfies T and omits  $\Gamma^i$  for all  $i < \alpha$ .

Any fragment  $\mathcal{L}$  of HDFOLR free of the Kleene operator is compact. If, in addition,  $\mathcal{L}$  is semantically closed under negation and retrieve,  $\mathcal{L}$  is an instance of Theorem 46. In particular, any fragment presented in Examples 15 — 18 can be an instance of  $\mathcal{L}$  from Theorem 46. Omitting Types Theorem is obtained from Theorem 46 by restricting the signatures  $\Delta$  to countable ones. By Lemma 43 (L2), Omitting Types Theorem is a corollary of Extended Omitting Types Theorem.

Notice that the forcing technique developed in the present contribution is not applicable to HFOLS as this logic lacks support for the substitutions described in Section 4.1. However, by Lemma 19, OTP can be borrowed from HFOLR to HFOLS.

**Theorem 47.** HFOLS has  $\alpha$ -OTP for all infinite cardinals  $\alpha$ .

*Proof.* Recall that for all HFOLS signatures  $\Delta$ , we have:

- $\operatorname{Sen}^{\operatorname{HFOLS}}(\Delta) \subseteq \operatorname{Sen}^{\operatorname{HFOLR}}(\Delta)$ , and
- by Lemma 19, for every sentence γ ∈ Sen<sup>HFOLR</sup>(Δ) there exists a sentence γ' ∈ Sen<sup>HFOLS</sup>(Δ) which is satisfied by the same class of Kripke structures as γ.

Assume that  $T \alpha$ -omits  $\Gamma^i$  as described in Definition 42. By the remarks above,  $T \alpha$ -omits  $\Gamma^i$  in HFOLR for all  $i < \alpha$ . By Theorem 46, there exists a Kripke structure (W, M) over  $\Delta$ , which satisfies T and omits  $\Gamma^i$  for all  $i < \alpha$ .

It is worth noting that in general the Omitting Types Property cannot be borrowed from a given logic to its restrictions. If T omits  $\Gamma^i$  in a restriction then T might not omit  $\Gamma^i$  in the full underlying logic. This is the reason for developing Theorem 46 in an arbitrary fragment  $\mathcal{L}$  of HDFOLR.

## 8. Constructor-based completeness

Constructor-based completeness is a modern approach to the well-known  $\omega$ -completeness, which has applications in formal methods. We make the result independent of the arithmetic signature by working over an arbitrary vocabulary where we distinguish a set of constructors which determines a class of Kripke structures reachable by constructors. Throughout this section we assume that the fragment  $\mathcal{L}$  is semantically closed under equality, negation, retrieve, disjunction and quantifiers. An example of such fragment  $\mathcal{L}$  is HDFOLR or HDPL.

#### 8.1. Semantic restrictions

Given a theory *T* over a vocabulary  $\Delta$ , not all Kripke structures are of interest. In many cases, formal methods practitioners are interested in the properties of a class Kripke structures that are reachable by a set of constructor operators. Let  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  be a signature and  $\Sigma^c \subseteq \Sigma^r$  a subset of constructor operators. The constructors create a partition of the set of rigid sorts  $S^r$ . A *constrained* sort is a rigid sort  $s \in S^r$  that has a constructor, that is, there exists a constructor  $\sigma: w \to s$  in  $\Sigma^c$ . A rigid sort that is not constrained it is called *loose*. We denote by  $S^c$  the set of all constrained sorts, and by  $S^1$  the set of all loose sorts. Let  $Y = \{Y_s\}_{s \in S^1}$  be a set of loose variables such that  $Y_s$  is countably infinite for all  $s \in S^1$ . A constructor-based Kripke structure is a Kripke structure (*W*, *M*) such that

- for all possible worlds  $w \in |W|$  there exists a nominal  $k \in F^n$  such that  $w = W_k$ , and
- for all rigid sorts  $s \in S^r$ , all possible worlds  $w \in |W|$ , and all elements  $m \in M_{w,s}$  there exist an expansion (W, N) of (W, M) to  $\Delta[Y]$ , and a rigid term  $t \in T_{@\Sigma^c}(Y)$  such that  $m = N_{w,t}$ .

**Example 48.** Let  $\Delta$  be the signature defined as follows:

- $\Sigma^n = (F^n, P^n)$  such that  $F^n$  consists of all natural numbers, and  $P^n$  has one element  $\lambda$ .
- $\Sigma = (S, F, P), S = \{Elt, List\}, F = \{empty : \rightarrow List, cons : Elt List \rightarrow List, delete : List \rightarrow List\}$  and  $P = \emptyset$ .
- $S^{r} = S$  and  $F^{r} = \{empty : \rightarrow List, cons : Elt List \rightarrow List\}.$
- The set of constructors  $F^{c}$  is  $F^{r}$ .

This means that *List* is constrained while *Elt* is loose.

We define a theory T over  $\Delta$ , which deletes n elements from a list in each possible world n:

- $\{@_n \langle \lambda \rangle n + 1 \mid n \ge 0\} \cup \{\neg @_n m \mid n \ne m\},\$
- $\{\forall N \cdot (@_N \ delete)(empty) = empty, \forall L \cdot (@_0 \ delete)(L) = L\}$ , and
- { $\forall E, L \cdot (@_{n+1} delete) cons(E, L) = (@_n delete)(L) | n > 0$ }.

A constructor-based Kripke structure which satisfies *T* has the set of possible worlds isomorphic with  $\omega$ . Let (W, M) be the constructor-based Kripke structure such that (a)  $|W| = \omega$  and  $W_{\lambda}$  is <, the natural order on  $\omega$ , and (b) for all possible worlds  $n \in \omega$ , the first-order-structure  $M_n$  interprets *Elt* as an arbitrary set, and *List* as the set of lists with elements from  $M_{n,Elt}$ , while the function  $M_{n,delete}: M_{n,List} \to M_{n,List}$  deletes the first *n* elements from the list given as argument. Obviously, (W, M) satisfies *T*.

By enhancing the syntax with a subset of rigid constructor operators and by restricting the semantics to constructorbased Kripke structures, we obtain a new logic  $\mathcal{L}^{c}$  from  $\mathcal{L}$ . Note that restricting the semantics also changes the relation  $\models$ , applied to theories:  $T \models \varphi$  now means that all restricted models of T are models of  $\varphi$ , so there may be non-restricted models of T which are not models of  $\varphi$ .

#### 8.2. Entailment systems

Given a system of proof rules for  $\mathcal{L}$  which is sound and complete, the goal is to add some new proof rules such that the resulting proof system is sound and complete for  $\mathcal{L}^{c}$ .

**Definition 49 (Entailment relation).** An *entailment relation* for  $\mathcal{L}$  is a family of binary relations between sets of sentences indexed by signatures  $\vdash \{ \vdash_{\Delta} \}_{\Delta \in |\mathsf{Sig}^{\mathcal{L}}|}$  with the following properties:

$$(Monotonicity)\frac{\Phi_{1} \subseteq \Phi_{2}}{\Phi_{2} \vdash \Phi_{1}} \qquad (Transitivity)\frac{\Phi_{1} \vdash \Phi_{2} \quad \Phi_{2} \vdash \Phi_{3}}{\Phi_{1} \vdash \Phi_{3}}$$
$$(Union)\frac{\Phi_{1} \vdash \varphi_{2} \text{ for all } \varphi_{2} \in \Phi_{2}}{\Phi_{1} \vdash \Phi_{2}} \qquad (Translation)\frac{\Phi_{1} \vdash_{\Sigma} \Phi_{2}}{\chi(\Phi_{1}) \vdash \chi(\Phi_{2})} \text{ where } \chi \colon \Delta \to \Delta'$$

The entailment relation is sound (complete) if  $\vdash \subseteq \models$  ( $\models \subseteq \vdash$ ). Examples of sound and complete entailment relations for HFOLR and HPL can be found in [27].

**Definition 50 (Constructor-based entailment relation).** Let  $\vdash$  be an entailment relation for  $\mathcal{L}$ . The entailment relation  $\vdash^{c}$  for  $\mathcal{L}^{c}$  is the least entailment relation closed under the following proof rules:

$$(R0) \frac{\Phi \vdash \varphi}{\Phi \vdash^{c} \varphi} \quad (R1) \frac{\Phi \vdash^{c} @_{k_{1}} \varphi(k_{2}) \text{ for all } k_{1}, k_{2} \in F^{n}}{\Phi \vdash^{c} \forall x \cdot \varphi(x)} \quad (R2) \frac{\Phi \vdash^{c} @_{k} \forall Y_{t} \cdot \psi(t) \text{ for all } k \in F^{n} \text{ and } t \in T_{\Sigma^{c}}(Y)}{\Phi \vdash^{c} \forall y \cdot \psi(y)}$$
where  $Y_{t}$  is the set of variables occurring in  $t$ 

According to [15], the entailment relation  $\vdash^{c}$  is well defined. We say that a theory *T* in  $\mathcal{L}$  is *semantically closed under* (*R1*) if  $T \models @_{k_1} \varphi(k_2)$  for all  $k_1, k_2 \in F^n$  implies  $T \models \forall x \cdot \varphi(x)$ . Similarly, we define the closure under (R2), that is,  $T \models @_k \forall Y_t \cdot \psi(t)$  for all  $k \in F^n$  and  $t \in T_{@\Sigma}(Y)$  implies  $T \models \forall y \cdot \psi(y)$ . It is not difficult to check that  $\vdash^{c}$  is sound for  $\mathcal{L}^{c}$  provided that  $\vdash$  is sound for  $\mathcal{L}$ . Completeness is much more difficult to establish in general, but it can be done with the help of the OTP.

**Theorem 51** (Constructor-based completeness). *The entailment relation*  $\vdash^{c}$  *is complete for*  $\mathcal{L}^{c}$  *if*  $\vdash$  *is complete for*  $\mathcal{L}$  *and*  $\mathcal{L}$  *has OTP.* 

*Proof.* Let  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  be a signature and *T* a theory over  $\Delta$  in  $\mathcal{L}$ . Let  $\Sigma^c \subseteq \Sigma^r$  be a set of constructors, and *Y* a set of loose variables. We perform the proof in two steps.

- (S1) We show that if *T* is satisfiable in  $\mathcal{L}$  and semantically closed under (R1) and (R2) then *T* is satisfiable in  $\mathcal{L}^c$ . Let  $\Gamma^n := \{\neg @_k x \mid k \in F^n\}$  be a type in one nominal variable *x*, and let  $\Gamma^r := \{\forall Y_t \cdot \neg t = y \mid t \in T_{\Sigma^c}(Y)\}$  be a type in one constrained variable *y*. Any Kripke structure over  $\Delta$  which omits  $\Gamma^n$  and  $\Gamma^r$  is reachable by the constructors in  $\Sigma^c$ . Firstly, we show that *T* locally omits  $\Gamma^n$ :
  - 1 let  $\rho(x)$  be a  $\Delta[x]$ -sentence such that  $T \cup \{\rho(x)\}$  is satisfiable

| $T \cup \{\exists x \cdot \forall z \cdot @_z \rho(x)\}$ is satisfiable  | by semantics since $T \cup \{\rho(x)\}$ is satisfiable   |
|--|--|
| $T \not\models \forall x \cdot \neg \forall z \cdot @_z \rho(x)$   | since $(W, M) \models T \cup \{\forall z \cdot @_z \rho(x)\}$ for some Kripke structure $(W, M)$ over $\Delta[x]$  |
| $T \not\models @_{k_1} \neg \forall z \cdot @_z \rho(k_2)$ for some nominals $k_1, k_2 \in F^n$  | since $T$ is semantically closed under (R1)  |
| $T \cup \{ @_{k_1} \forall z \cdot @_z \rho(k_2) \}$ is satisfiable  | by semantics of negation and retrieve  |
| $T \cup \{ @_{k_1} \forall z \cdot @_z \rho(x) \} \cup \{ @_{k_2} x \}$ is satisfiable   | by the semantics of nominals   |
| $T \cup \{\rho(x)\} \cup \{@_{k_2} x\}$ is satisfiable   | by Lemma 12  |
| T locally omits $\Gamma^n$   | since $\rho(x)$ was arbitrarily chosen   |
| econdly, we show that T locally omits $\Gamma^r$ :   |  |
| let $\rho(y)$ be a $\Delta[y]$ -sentence such that $T \cup \{\rho(y)\}$ is satisfiable   |  |
| $T \cup \{\exists y \cdot \forall z \cdot @_z \rho(y)\}$ is satisfiable  | by Lemma 12  |
| $T \not\models \forall y \cdot \neg \forall z \cdot @_z \rho(y)$   | since $(W, M) \models T \cup \{\forall z \cdot @_z \rho(y)\}$ for some Kripke structure $(W, M)$ over $\Delta[y]$  |
| $T \not\models @_k \forall Y_t \cdot \neg \forall z \cdot @_z \rho(t) \text{ for some } k \in F^n \text{ and } t \in T_{@\Sigma^c}(Y)$ | since $T$ is semantically closed under (R2)  |
| $T \not\models @_k \neg \forall z \cdot @_z \rho(t) \text{ over } \Delta[Y_t]$   | by semantics of quantifiers  |
| $T \cup \{ @_k \forall z \cdot @_z \rho(t) \}$ is satisfiable over $\Delta[Y_t]$   | by semantics of negation and retrieve  |
| $T \cup \{\rho(t)\}$ is satisfiable over $\Delta[Y_t]$   | by Lemma 12  |
|  | $T \not\models \forall x \cdot \neg \forall z \cdot @_{z} \rho(x)$ $T \not\models @_{k_{1}} \neg \forall z \cdot @_{z} \rho(k_{2}) \text{ for some nominals } k_{1}, k_{2} \in F^{n}$ $T \cup \{@_{k_{1}} \forall z \cdot @_{z} \rho(k_{2})\} \text{ is satisfiable}$ $T \cup \{@_{k_{1}} \forall z \cdot @_{z} \rho(x)\} \cup \{@_{k_{2}} x\} \text{ is satisfiable}$ $T \cup \{\rho(x)\} \cup \{@_{k_{2}} x\} \text{ is satisfiable}$ $T \text{ locally omits } \Gamma^{n}$ econdly, we show that T locally omits $\Gamma^{T}$ :<br>$let \rho(y) \text{ be a } \Delta[y]\text{-sentence such that } T \cup \{\rho(y)\} \text{ is satisfiable}$ $T \cup \{\exists y \cdot \forall z \cdot @_{z} \rho(y)\} \text{ is satisfiable}$ $T \not\models \forall y \cdot \neg \forall z \cdot @_{z} \rho(y)$ $T \not\models @_{k} \forall Y_{t} \cdot \neg \forall z \cdot @_{z} \rho(t) \text{ for some } k \in F^{n} \text{ and } t \in T_{@\Sigma^{c}}(Y)$ $T \not\models @_{k} \neg \forall z \cdot @_{z} \rho(t) \text{ is satisfiable over } \Delta[Y_{t}]$ |

| 8 | $T \cup \{\rho(y)\} \cup \{\exists Y_t \cdot t = y\}$ is satisfiable | since $(W, M) \models T \cup \{\rho(t)\}$ for some Kripke structure $(W, M)$<br>over $\Delta[Y_t]$ |
|---|--|--|
| 9 | T locally omits $\Gamma^{r}$   | since $\rho(y)$ was arbitrarily chosen   |

By Theorem 46, there exists a Kripke structure (W, M) which satisfies T and omits  $\Gamma^n$  and  $\Gamma^r$ . By the definition of  $\Gamma^n$  and  $\Gamma^r$ , (W, M) is a constructor-based Kripke structure.

(S2) Next we assume T is consistent in  $\mathcal{L}^{c}$  and show that T is satisfiable in  $\mathcal{L}^{c}$ . Let  $T' := \{\varphi \in \text{Sen}(\Delta) \mid T \vdash^{c} \varphi\}$ . We have that T is consistent in  $\mathcal{L}^{c}$  iff T' is consistent in  $\mathcal{L}$ :

For the forward implication, suppose towards a contradiction that T' is not consistent in  $\mathcal{L}$ , that is,  $T' \vdash \bot$ ; By (R0),  $T' \vdash^{c} \bot$ ; by (*Union*),  $T \vdash^{c} T'$ ; by (*Transitivity*),  $T \vdash^{c} \bot$ , which is a contradiction with the consistency of T in  $\mathcal{L}^{c}$ .

For the backward implication, suppose towards a contradiction that  $T \vdash^{c} \bot$ ; we have  $\bot \in T'$ , and by *(Monotonicity)*,  $T' \vdash \bot$ , which is a contradiction with the consistency of T' in  $\mathcal{L}$ .

Assume that *T* is consistent in  $\mathcal{L}^{c}$ . It follows that *T'* is consistent in  $\mathcal{L}$ . By the completeness of  $\vdash$  in  $\mathcal{L}$ , *T'* is satisfiable in  $\mathcal{L}$ . By the completeness of  $\vdash$  in  $\mathcal{L}$ , *T'* is semantically closed under (R1) and (R2). By the first part of the proof, *T'* is satisfiable in  $\mathcal{L}^{c}$ . Since  $T \subseteq T'$ , *T* is satisfiable in  $\mathcal{L}^{c}$ .

# 9. Omitting types and Löwenheim-Skolem Theorems

Downwards and Upwards Löwenheim-Skolem Theorems are consequences of the Omitting Types Theorem. Throughout this section we assume that the fragment  $\mathcal{L}$  is semantically closed under equality, retrieve, negation, disjunction, possibility over binary modalities, and quantifiers. An example of such fragment  $\mathcal{L}$  is HDFOLR or HDPL, in which case  $\mathcal{L}$  has  $\omega$ -OTP. For cardinals greater than  $\omega$ , we need to drop the Kleene operator \* in order to have compactness and be able to apply our OTP (we will show in the next section that compactness is necessary at least for certain strong fragments of  $\mathcal{L}$ ). Some of the arguments in this and the next section are modelled after the technique used by Lindström [38] for first-order logic without equality.

**Theorem 52** (Downwards Löwenheim-Skolem Theorem). Assume that  $\mathcal{L}$  has  $\alpha$ -OTP. Let T be a satisfiable theory over a signature  $\Delta$  of power at most  $\alpha$ . Then T has a Kripke structure (W, M) such that  $card(W) \leq \alpha$  and  $card(M_{w,s}) \leq \alpha$  for all rigid sorts  $s \in S^r$ .

*Proof.* Let  $C = \{C_s\}_{s \in S^e}$  be a sorted set of new constants for  $\Delta$  such that  $card(C_s) = \alpha$  for all sorts  $s \in S^e$ . Let  $\Gamma^s := \{c \neq x \mid c \in C_s\}$  be a type<sup>11</sup> in one variable x of sort  $s \in S^e$ . We show that  $T \alpha$ -omits  $\Gamma^s$ :

1 let p be a set of sentences over  $\Delta[C, x]$  such that  $card(p) < \alpha$  and  $T \cup p$  is satisfiable

| 2 | $p \subseteq \Delta[C', x]$ for some $C' \subseteq C$ such that $card(C'_s) < \alpha$ | since $card(p) < \alpha$   |
|---|---|--|
| 3 | there exists $c \in C_s \setminus C'_s$   | since $\operatorname{card}(C_s) = \alpha$ and $\operatorname{card}(C'_s) < \alpha$ |
| 4 | $T \cup p \cup \{x = c\}$ is satisfiable  | since $T \cup p$ is satisfiable and <i>c</i> does not occur in $T \cup p$          |
| 5 | T $\alpha$ -omits $\Gamma^s$  | since $p$ was arbitrarily chosen   |

Since  $\mathcal{L}$  has  $\alpha$ -OTP, there exists a Kripke structure (W, M) over  $\Delta[C]$  which satisfies T and omits  $\Gamma^s$  for all  $s \in S^e$ .

**Theorem 53** (Upwards Löwenheim-Skolem Theorem). Assume that  $\mathcal{L}$  has  $\alpha$ -OTP, where  $\alpha$  is a regular cardinal. Let T be a satisfiable theory over a signature  $\Delta$  of power at most  $\alpha$ . For each model (W, M) of T there exists another model (V, N) of T such that  $card((V, N)_s) \geq \alpha$  for all sorts  $s \in S^e$ .

In fact, if  $\Delta'$  is obtained from  $\Delta$  by adding a rigid binary relation  $\leq$  on each sort  $s \in S^{e}$  interpreted by (W, M) as infinite then there exists an expansion (V', N') of (V, N) to  $\Delta'$  such that  $\langle (W, M)_s, (W, M)_{\leq} \rangle$  is a linear ordering of

<sup>&</sup>lt;sup>11</sup>Notice that for nominals,  $c \neq x$  means  $\neg @_c x$ . Compare Lemma 43 for a similar use.

## cofinality $\alpha$ for all sorts $s \in S^{e}$ .

*Proof.* Let  $\Omega \subseteq S^e$  be the set of all sorts interpreted by (W, M) as infinite. Let  $C = \{C_s\}_{s \in \Omega}$  be a set of new rigid constants such that  $C_s = \{c_i \mid i < \alpha\}$  for all  $s \in \Omega$ . Let T' be the theory over  $\Delta'[C]$  obtained from T by adding:

 $\{ \le \text{ is a linear order on } s \text{ without the greatest element} \} \cup \{c_i \le c_j \mid i < j < \alpha\} \text{ for each sort } s \in \Omega$ 

The definition of T' relies on the semantic closure of  $\mathcal{L}$  under the relevant sentence building operators. For example, for nominals,  $c_i \leq c_j$  means  $@_{c_i} \langle \leq \rangle c_j$ . There exists an expansion (W', M') of (W, M) to the signature  $\Delta'[C]$  such that  $(W', M') \models T'$ . For each sort  $s \in \Omega$  we define the following type in one variable x of sort s:

$$\Gamma^s := \{c_i \le x \mid i < \alpha\}$$

We show that  $T' \alpha$ -omits  $\Gamma^s$ :

let  $p \subseteq \text{Sen}(\Delta'[C, x])$  with  $\text{card}(p) < \alpha$  such that  $T' \cup p$  is satisfiable 2  $(V, N) \models T' \cup p$  for some Kripke structure (V, N) over  $\Delta'[C, x]$ since  $T' \cup p$  is satisfiable 3  $p \subseteq \text{Sen}(\Delta'[C^{\beta}, x])$  for some  $\beta < \alpha$ , where  $C^{\beta}$  is obtained from C by since  $\alpha$  is regular restricting the constants of sort s to  $C_s^{\beta} := \{c_i \in C_s \mid i < \beta\}$  $(V^{\beta}, N^{\beta}) \models T \cup p$ , where  $(V^{\beta}, N^{\beta}) := (V, N) \upharpoonright_{\Lambda' \cap C^{\beta}} \mathbb{R}^{1}$ 4 since  $(V, N) \models T' \cup p$  and  $T \subseteq T'$ 5 there exists  $w > max\{(V, N)_x, (V, N)_{c_B}\}$ since  $\langle (V, N)_s, (V, N)_s \rangle$  is a linear order without the greatest element  $w > (V, N)_{c_i}$  for all  $i < \beta$ 6 since  $w > (V, N)_{c_{\beta}}$  and  $(V, N)_{c_{\beta}} \ge (V, N)_{c_{i}}$  for all  $i < \beta$  $(V', N') \models T' \cup p$ , where (V', N') is the unique expansion of  $(V^{\beta}, N^{\beta})$ since  $(V^{\beta}, N^{\beta}) \models T \cup p$  and w is greater than the 7 interpretation of  $c_{\beta}$  in (V, N)to the signature  $\Delta'[C, x]$  such that  $(V', N')_{c_i} = w$  for all  $i \ge \beta$ 8  $(V', N') \not\models c_i \leq x \text{ for all } i \geq \beta$ since  $(V, N)_x < w$  and  $w = (V', N')_{c_i}$  for all  $i \ge \beta$ T' α-omits Γ 9 from 7 and 8, since p was arbitrarily chosen

By Theorem 46, there exists a model (V', N') which satisfies T' and omits  $\Gamma^s$  for all  $s \in \Omega$ . It follows that  $\langle (V', N')_s, (V', N')_{\leq} \rangle$  is a linear ordering of cofinality<sup>12</sup>  $\alpha$  for all sorts  $s \in \Omega$ . Let  $(V, N) := (V', N') \upharpoonright_{\Delta}$ , and notice that (V, N) satisfies T and its carrier sets corresponding to the sorts in  $\Omega$  have cardinalities greater than or equal to  $\alpha$ .

### 10. Omitting types and compactness

In this section, we show that at least at some occasions, compactness is a necessary condition for proving the Omitting Types Theorem for uncountable signatures. We work within a fragment  $\mathcal{L}$  with the following properties:

- P1)  $\mathcal{L}$  is semantically closed under (a) possibility applied to nominal sentences, (b) retrieve, (c) negation, (d) disjunction, and (e) quantifiers.
- P2) Signatures have only one rigid sort and all function symbols (except variables) are flexible.

Notice that  $\mathcal{L}$  is semantically closed under possibility, as  $\langle \lambda \rangle \varphi \models \exists x \cdot \langle \lambda \rangle x \land @_x \varphi$ .

# 10.1. Global substitutions

We begin by defining a notion of substitution which we then use to derive compactness for infinite models from  $\alpha$ -OTP using a technique originally developed by Lindström for first-order logic with only relational symbols [38]. Consider a signature  $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$  with only one rigid sort and no rigid function symbols, that is,  $S^n = \{s_1\}$ ,  $S^r = S = \{s_2\}$  and  $F^r = \emptyset$ . We define another signature  $\Delta_+ = (\Sigma^n, \Sigma^r \subseteq \Sigma_+)$  as follows:

1.  $\Sigma_{\pm}^{n}$  consists of only one sort, let us say,  $s_{0}$ , and  $\Sigma_{\pm}^{r}$  consists of two sorts  $s_{1}$  and  $s_{2}$ .

<sup>&</sup>lt;sup>12</sup>To be more precise, we can select a strictly increasing subsequence  $(c_{ij} : ij < \alpha)$  which is unbounded. This sequence is order-isomorphic with an ordinal  $\gamma$ , and since  $\alpha$  is regular we have  $\gamma = \alpha$ . In particular card $(C_s) \ge \alpha$  for each  $s \in \Omega$ .

2.  $\Sigma_+$  is obtained from  $\Sigma^n$  by adding the following sets of flexible symbols:

(a) 
$$\{\sigma_+ : s_1 \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \mid \sigma : \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \in F\}$$
 and  
(b)  $\{\pi_+ : s_1 \underbrace{s_2 \dots s_2}_{m-times} \mid \pi : \underbrace{s_2 \dots s_2}_{m-times} \in P\}.$ 

The signature  $\Delta_+$  provides a local environment for encoding the Kripke structures over  $\Delta$ . The following set of sentences over  $\Delta_+$  ensures that the interpretation of the rigid relation symbols in  $\Delta$  is 'locally rigid' in  $\Delta_+$ .

$$\Gamma := \{ \forall x_1, x_2, y_1, \dots, y_m \cdot \pi_+(x_1, y_1, \dots, y_m) \Leftrightarrow \pi_+(x_2, y_1, \dots, y_m) \mid \pi : \underbrace{s_2 \dots s_2}_{m-times} \in P^r \}$$

Let z be a distinguished nominal variable for  $\Delta_+$ . We define a substitution (\_)<sup>+</sup>:  $\Delta \rightarrow (\Delta_+[z], \Gamma)$ , that is,

- 1. a sentence function  $(\_)^+$ : Sen $(\Delta) \rightarrow$  Sen $(\Delta_+[\mathbf{z}], \Gamma)$  and
- 2. a reduct functor  $(\_)^-$ : Mod $(\Delta_+[\mathbf{z}], \Gamma) \to Mod(\Delta)$ ,

such that the following global satisfaction condition holds:

$$(W^{z \leftarrow w}, M) \models \gamma^+ \text{ iff } (W^{z \leftarrow w}, M)^- \models \gamma$$

for all Kripke structures  $(W, M) \in |Mod(\Delta_+, \Gamma)|$ , all possible worlds  $w \in |W|$  and all sentences  $\gamma \in Sen(\Delta)$ .

*Mapping on models.* Notice that a model in  $|\mathsf{Mod}(\Delta_+, \Gamma)|$  can be regarded as a collection of Kripke structures over the signature  $\Delta$ . Once z is assigned to a node, the functor (\_)<sup>-</sup> extracts the Kripke structure corresponding to the node denoted by z. Concretely, the functor (\_)<sup>-</sup> :  $\mathsf{Mod}(\Delta_+[z], \Gamma) \to \mathsf{Mod}(\Delta)$  maps each Kripke structure  $(W^{z \leftarrow w}, M) \in$  $|\mathsf{Mod}(\Delta_+[z], \Gamma)|$  to  $(W^-_w, M^-_w) \in |\mathsf{Mod}(\Delta)|$ , where

1. 
$$W_w^- \coloneqq M_w \upharpoonright_{\Sigma^n}, \mathbb{1}^3$$

- 2. the mapping  $M_w^-: M_{w,s_1} \to |\mathsf{Mod}(\Sigma)|$  is defined as follows:
  - For all  $v \in M_{w,s_1}$ , the carrier set  $(M_w^-)_{v,s_2}$  is  $M_{w,s_2}$ .
  - For all  $v \in M_{w,s_1}$  and all  $\sigma : \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \in F$ , the function  $(M_w^-)_{v,\sigma} : \underbrace{M_{w,s_2} \times \dots \times M_{w,s_2}}_{m-times} \to M_{w,s_2}$  is defined by  $(M_w^-)_{v,\sigma}(a_1, \dots, a_m) := M_{w,\sigma_+}(v, a_1, \dots, a_m)$  for all  $a_1, \dots, a_m \in M_{w,s_2}$ .
  - For all  $v \in M_{w,s_1}$  and all  $\pi : \underbrace{s_2 \dots s_2}_{m-times} \in P$ , the relation  $(M_w^-)_{v,\pi} \subseteq \underbrace{M_{w,s_2} \times \dots \times M_{w,s_2}}_{m-times}$  is defined by  $(M_w^-)_{v,\pi} := \{(a_1, \dots, a_m) \mid (v, a_1, \dots, a_m) \in M_{w,\pi_+}\}.$

Since  $(W, M) \models \Gamma$ , the Kripke structure  $(W_w^-, M_w^-)$  interprets all rigid symbols in  $P^r$  uniformly across the worlds, which means it is well-defined.

**Fact 54.** The functor  $(\_)^-$ : Mod $(\Delta_+[z], \Gamma) \rightarrow Mod(\Delta)$  can be extended to  $(\_)^-$ : Mod $(\Delta_+[z, X], \Gamma) \rightarrow Mod(\Delta[X])$ , where  $X = \{X_s\}_{s \in S^e}$  is a set of variables for  $\Delta$ , such that the interpretation of all variables in X is preserved, that is,  $(W^{z \leftarrow w}, M)_x = (W^{z \leftarrow w}, M)_x^-$  for all  $x \in X$ .

<sup>&</sup>lt;sup>13</sup>Notice that  $M_w \in |\mathsf{Mod}(\Sigma_+)|$  and  $M_w \upharpoonright_{\Sigma^n}$  is well-defined since  $\Sigma^n \subseteq \Sigma_+$ .

*Mapping on sentences.* We define a mapping on sentences  $(\_)^+$ : Sen $(\Delta[X]) \rightarrow$  Sen $(\Delta_+[z, X])$  in three steps, where  $X = \{X_s\}_{s \in S^e}$  is any set of variables for  $\Delta$ .

- S1) We define a mapping from the rigid terms over  $\Delta[X]$  to the rigid terms over  $\Delta_+[z, X]$  by structural induction:
  - $x^+ := x$ , where x is any variable of rigid sort from X, and
  - $(@_k \sigma)(t_1, \ldots, t_m)^+ := (@_z \sigma_+)(@_z k, t_1^+, \ldots, t_m^+)$ , where  $k \in F^n \cup X_{any}$ , and  $t_i$  are rigid terms over  $\Delta_+[z, X]$ .

Notice that  $(_)^+$  is well-defined on rigid terms, as the set of rigid function symbols is empty.

**Lemma 55.** For all Kripke structures  $(W, M) \in |Mod(\Delta_+[X], \Gamma)|$ , all possible worlds  $w \in |W|$ , and all rigid terms t over  $\Delta[X]$ ,

$$(W^{\mathsf{Z}\leftarrow \mathsf{W}}, M)_{t^+} = (W^{\mathsf{Z}\leftarrow \mathsf{W}}, M)_t^-.$$
(1)

•  $(@_k \lor \Phi)^+ := \lor_{\varphi \in \Phi} (@_k \varphi)^+$ 

Proof. By structural induction on terms:

 $[ x \in \{X_s\}_{s \in S^r} ] \text{ Obviously, } (W^{z \leftarrow w}, M)_x = (W_w^-, M_w^-)_x.$   $[ @_k \sigma(t_1, \dots, t_m) ] \text{ Let } v \coloneqq (W^{z \leftarrow w}, M)_{@_z k} = M_{w,k}, \text{ and we have:}$   $(W^{z \leftarrow w}, M)_{(@_z \sigma_+)(@_z k, t_1^+, \dots, t_m^+)} = M_{w,\sigma_+}(v, (W^{z \leftarrow w}, M)_{t_1^+}, \dots, (W^{z \leftarrow w}, M)_{t_m^+}). \text{ By the induction hypothesis,}$   $M_{w,\sigma_+}(v, (W^{z \leftarrow w}, M)_{t_1^+}, \dots, (W^{z \leftarrow w}, M)_{t_m^+}) = (M_w^-)_{v,\sigma}((W_w^-, M_w^-)_{t_1}, \dots, (W_w^-, M_w^-)_{t_m}) =$   $(W_w^-, M_w^-)_{(@_k \sigma)(t_1, \dots, t_m)}.$ 

Since  $F^{r} = \emptyset$ , the cases considered above cover all possibilities.

S2) We define the mapping (\_)<sup>+</sup> on rigid sentences of the form  $@_k \varphi \in \text{Sen}(\Delta[X])$  such that every rigid sentence will be mapped to a rigid sentence  $(@_k \varphi)^+ \in \text{Sen}(\Delta_+[z, X])$ , which means that

$$(W^{z \leftarrow w}, M) \models (@_k \varphi)^+ \text{ iff } (W^{z \leftarrow w}, M) \models^w (@_k \varphi)^+$$

for all Kripke structures  $(W, M) \in |Mod(\Delta_+[X], \Gamma)|$  and all possible worlds  $w \in |W|$ . We proceed by structural induction.

- $(@_k k')^+ := @_z (k = k')$ •  $(@_k \langle \lambda \rangle k')^+ := @_z \lambda(k, k')$ •  $(@_k \exists X' \cdot \varphi)^+ := \exists X' \cdot (@_k \varphi)^+$
- $(@_k (t_1 = t_2))^+ := (\mathsf{at}_k t_1)^+ = (\mathsf{at}_k t_2)^+$ •  $(@_k @_{k'} \varphi)^+ := (@_{k'} \varphi)^+$
- $(@_k \pi(t_1, ..., t_m))^+ := (@_z \pi_+)(@_z k, (at_k t_1)^+, ..., (at_k t_m)^+)$ •  $(@_k \downarrow x \cdot \varphi)^+ := (@_k \varphi(x \leftarrow k))^+$

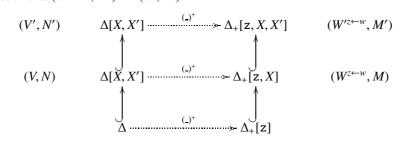
**Lemma 56** (Rigid satisfaction condition). For all sentences  $\varphi \in \text{Sen}(\Delta[X])$ , all nominals  $k \in F^n \cup X_{\text{any}}$ , all *Kripke structures*  $(W, M) \in |\text{Mod}(\Delta_+[X], \Gamma)|$  and all possible worlds  $w \in |W|$ ,

$$(W^{\mathsf{Z} \leftarrow \mathsf{W}}, M) \models (@_k \varphi)^+ iff (W^{\mathsf{Z} \leftarrow \mathsf{W}}, M)^- \models @_k \varphi.$$
<sup>(2)</sup>

*Proof.* Let  $v := M_{w,k}$ . We proceed by induction on the structure of  $\varphi$ :

- $\begin{bmatrix} k' \in F^{\mathsf{n}} \cup X_{\mathsf{any}} \end{bmatrix} (W^{\mathsf{z} \leftarrow w}, M) \models (@_k k')^+ \text{ iff } (W^{\mathsf{z} \leftarrow w}, M) \models @_{\mathsf{z}} (k = k') \text{ iff } (W^{\mathsf{z} \leftarrow w}, M) \models^w k = k' \text{ iff } M_{w,k} = M_{w,k'} \text{ iff } (M_w^-)_k = (M_w^-)_{k'} \text{ iff } (W_w^-, M_w^-) \models @_k k'.$
- $\begin{bmatrix} \langle \lambda \rangle k' \end{bmatrix} (W^{z \leftarrow w}, M) \models (@_k \langle \lambda \rangle k')^+ \text{ iff } (W^{z \leftarrow w}, M) \models @_z \lambda(k, k') \text{ iff } (W^{z \leftarrow w}, M) \models^w \lambda(k, k') \text{ iff } (M_k, M_{k'}) \in M_{w,\lambda} \text{ iff } (W^-_w, M^-_w) \models @_k \langle \lambda \rangle k'.$
- $\begin{bmatrix} t_1 = t_2 \end{bmatrix} (W^{z \leftarrow w}, M) \models (@_k (t_1 = t_2))^+ \text{ iff } (W^{z \leftarrow w}, M) \models (at_k t_1)^+ = (at_k t_2)^+ \text{ iff } (W^{z \leftarrow w}, M)_{(at_k t_1)^+} = (W^{z \leftarrow w}, M)_{(at_k t_2)^+} \text{ iff } (W^-_w, M^-_w)_{at_k t_1} = (W^-_w, M^-_w)_{at_k t_2} \text{ iff } (W^-_w, M^-_w)_{@_k t_1} = (W^-_w, M^-_w)_{@_k t_2} (W^-_w, M^-_w) \models @_k (t_1 = t_2).$

- $\begin{bmatrix} \pi(t_1, \dots, t_m) \end{bmatrix} (W^{z \leftarrow w}, M) \models (@_k \pi(t_1, \dots, t_m))^+ \text{ iff } (W^{z \leftarrow w}, M) \models @_z \pi_+ (@_z k, (\mathsf{at}_k t_1)^+, \dots, (\mathsf{at}_k t_m)^+) \text{ iff } (v, (W^{z \leftarrow w}, M)_{(\mathsf{at}_k t_1)^+}, \dots, (W^{z \leftarrow w}, M)_{(\mathsf{at}_k t_m)^+}) \in M_{w, \pi_+} \text{ iff } ((W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{\mathsf{at}_k t_m}) \in (M^-_w)_{v, \pi} \text{ iff } ((W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{\mathsf{at}_k t_m}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{\mathsf{at}_k t_m}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{\mathsf{at}_k t_m}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{\mathsf{at}_k t_m}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{\mathsf{at}_k t_1}, \dots, (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi}) \in (M^-_w)_{w, \pi} \text{ iff } (W^-_w, M^-_w)_{w, \pi} \text$
- $\begin{bmatrix} \neg \varphi \end{bmatrix} (W^{z \leftarrow w}, M) \models (@_k \neg \varphi)^+ \text{ iff } (W^{z \leftarrow w}, M) \models \neg (@_k \varphi)^+ \text{ iff } (W^{z \leftarrow w}, M) \models^w \neg (@_k \varphi)^+ \text{ iff } (W^{z \leftarrow w}, M) \not\models^w (@_k \varphi)^+ \text{ iff } (W^{z \leftarrow w}, M) \not\models (@_k \varphi)^+ \text{ iff } (W^-_w, M^-_w) \not\models @_k \varphi \text{ iff } (W^-_w, M^-_w) \models @_k \neg \varphi.$
- $[ \lor \Phi ] (W^{z \leftarrow w}, M) \models (@_k \lor \Phi)^+ \text{ iff } (W^{z \leftarrow w}, M) \models \lor_{\varphi \in \Phi} (@_k \varphi)^+ \text{ iff } (W^{z \leftarrow w}, M) \models \lor_{\varphi \in \Phi} (@_k \varphi)^+ \text{ iff } (W^{z \leftarrow w}, M) \models w \lor_{\varphi \in \Phi} (@_k \varphi)^+ \text{ for some } \varphi \in \Phi \text{ iff } (W_w^-, M_w^-) \models @_k \lor_{\varphi \in \Phi} \varphi.$
- $[\exists X' \cdot \varphi]$  Let  $(V, N) := (W_w, M_w)$ . Since  $(\_)^-$  preserves the interpretation of variables, we have:
  - (a) for any expansion (W', M') of (W, M) to  $\Delta_+[X, X']$ ,  $(W'^{z \leftarrow w}, M')^-$  is an expansion of (V, N) to  $\Delta[X, X']$ ,
  - (b) for any expansion (V', N') of (V, N) to  $\Delta[X, X']$ , there exists an expansion (W', M') of (W, M) to  $\Delta_+[X, X']$  such that  $(W'^{z \leftarrow w}, M')^- = (V', N')$ .



Based on the remark above, the following are equivalent:

 $\begin{array}{ll} 2 & (W^{z \leftarrow w}, M) \models \exists X' \cdot (@_k \varphi)^+ & \text{by the definition of } (\_)^+ \\ 3 & (W'^{z \leftarrow w}, M') \models (@_k \varphi)^+ \text{ for some expansion } (W', M') \text{ of } (W, M) \text{ to } \Delta_+[X, X'] & \text{since } (@_k \varphi)^+ \text{ is rigid} \\ 4 & (V', N') \models @_k \varphi \text{ for some expansion } (V', N') \text{ of } (V, N) \text{ to } \Delta[X, X'] & \text{by the induction hypothesis} \\ 5 & (V, N) \models @_k \exists X' \cdot \varphi & \text{since } @_k \varphi^+ \text{ is rigid} \end{array}$ 

[ $@_{k'}\varphi$ ] This case is straightforward, since  $@_k @_{k'}\varphi \models @_{k'}\varphi$ .

 $[\downarrow x \cdot \varphi]$  This case is straightforward, since  $@_k \downarrow x \cdot \varphi \models @_k \varphi[x \leftarrow k]$ .

S3) The function  $(\_)^+$ : Sen $(\Delta[X]) \to$  Sen $(\Delta_+[z, X])$  is defined by  $\varphi^+ = \forall \mathbf{x} \cdot (@_{\mathbf{x}} \varphi)^+$  for all  $\varphi \in$  Sen $(\Delta[X])$ , where **x** is a distinguished nominal variable for  $\Delta[X]$ .

**Proposition 57** (Global satisfaction condition). For all sentences  $\varphi \in \text{Sen}(\Delta[X])$ , all Kripke structures  $(W, M) \in |\text{Mod}(\Delta_+[X])|$ , and all possible worlds  $w \in |W|$ ,

$$(W^{\mathsf{z}\leftarrow w}, M) \models \varphi^+ iff (W^{\mathsf{z}\leftarrow w}, M)^- \models \varphi.$$
(3)

*Proof.* Let  $(V, N) := (W_w^-, M_w^-)$ .

 $(W^{\mathsf{z}\leftarrow w}, M) \models (@_k \exists X' \cdot \varphi)^+$ 

1

$$(V^{\mathbf{x}\leftarrow\nu}, N) \qquad \Delta[\mathbf{x}, X] \xrightarrow{(\Box)^+} \Delta_+[\mathbf{z}, \mathbf{x}, X] \qquad (W^{\mathbf{z}\leftarrow\nu}, M^{\mathbf{x}\leftarrow\nu})$$

$$(V, N) \qquad \Delta[X] \xrightarrow{(\Box)^+} \Delta_+[\mathbf{z}, X] \qquad (W^{\mathbf{z}\leftarrow\omega}, M)$$

$$(U, N) \qquad \Delta[X] \xrightarrow{(\Box)^+} \Delta_+[\mathbf{z}]$$

The following are equivalent:

- 1  $(W^{\mathsf{z}\leftarrow w}, M) \models \varphi^+$
- 2  $(W^{\mathsf{z}\leftarrow w}, M) \models \forall \mathsf{x} \cdot (@_{\mathsf{x}} \varphi)^+$
- 3  $(W^{z \leftarrow w}, M^{x \leftarrow v}) \models (@_x \varphi)^+$  for any expansion  $(W^{z \leftarrow w}, M^{x \leftarrow v})$  of  $(W^{z \leftarrow w}, M)$  to  $\Delta_+[z, x, X]$
- 4  $(V^{\mathbf{x}\leftarrow\nu}, N) \models @_{\mathbf{x}} \varphi$  for any expansion  $(V^{\mathbf{x}\leftarrow\nu}, N)$  of (V, N) to  $\Delta[\mathbf{x}, X]$
- 5  $(V, N) \models \forall \mathbf{x} \cdot @_{\mathbf{x}} \varphi$
- 6  $(V, N) \models \varphi$

since  $(W^{Z \leftarrow w}, M) \models \forall \mathbf{x} \cdot (@_{\mathbf{x}} \varphi)^+$ by Lemma 56, since  $(W^{Z \leftarrow w}, M^{\mathbf{x} \leftarrow v})^- = (V^{\mathbf{x} \leftarrow v}, N)$ by semantics by semantics

by the definition of  $(\_)^+$ 

## 10.2. Inf-compactness

We say that  $\mathcal{L}$  is *inf-compact* if each set of sentences  $\Gamma$  has an infinite model whenever each finite subset  $\Gamma_f \subseteq \Gamma$  has an infinite model. We say that L is  $\alpha$ -*inf-compact*, where  $\alpha$  is an infinite cardinal, if each set of sentences  $\Gamma$  of cardinality  $\alpha$  has an infinite model whenever each finite subset  $\Gamma_f \subseteq \Gamma$  has an infinite model. We show that inf-compactness is a consequence of omitting type property.

**Theorem 58.** If  $\mathcal{L}$  has  $\alpha$ -OTP, where  $\alpha$  is a regular cardinal then  $\mathcal{L}$  is  $\beta$ -inf-compact for all cardinals  $\beta < \alpha$ .

*Proof.* Let  $\Delta$  be a signature of power at most  $\alpha$ . By induction, it suffices to prove that each sequence  $\Phi_{\beta} = \{\varphi_i \in \text{Sen}(\Delta) \mid i < \beta\}$  has an infinite model whenever each subsequence  $\Phi_j := \{\varphi_i \mid i < j\}$  has an infinite model for all  $j < \beta$ . Let  $\{(W^i, M^i) \in |\text{Mod}(\Delta)| \mid 0 < i < \beta\}$  be a sequence of Kripke structures over  $\Delta$  such that

- the carrier sets of  $(W^i, M^i)$  are infinite for all indexes j with  $0 < j < \beta$ , and
- $(W^j, M^j) \models \Phi_j$  for all indexes j with  $0 < j < \beta$ .

By Löwenheim-Skolem properties, we can assume that all carrier sets of  $(W^i, M^i)$  are of cardinality  $\alpha$ . By renaming the elements, we assume furthermore that  $|W^i| = |W^j|$  and  $M^i_{w,s_2} = M^j_{w,s_2}$  for all  $i < j < \beta$  and all possible worlds  $w \in |W^i|$ . We define the following Kripke structure  $(W^+, M^+)$  over  $\Delta_+$ :

- $|W^+| = \{w_i \mid 0 < i < \beta\}$ , where  $\{w_i \mid 0 < i < \beta\}$  is a sequence of pairwise distinct possible worlds. The carrier sets of  $(W^+, M^+)$  for the sorts  $s_1$  and  $s_2$  are the carrier sets of  $(W^i, M^i)$  for the sorts  $s_1$  and  $s_2$ , where  $0 < i < \beta$ .
- For all  $k \in F^n$  and all  $0 < i < \beta$ , we define  $M^+_{w,k} \coloneqq W^i_k$ .
- For all  $\sigma: \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \in F$  and all  $0 < i < \beta$ , the function  $M^+_{w_i,\sigma^+}: M^+_{w_i,s_1} \times \underbrace{M^+_{w_i,s_2} \times \dots \times M^+_{w_i,s_2}}_{m-times} \to M^+_{w_i,s_2}$  is defined by  $M^+_{w_i,\sigma_+}(a, b_1, \dots, b_m) = M^i_{a,\sigma}(b_1, \dots, b_n)$  for all  $(a, b_1, \dots, b_m) \in M^+_{w_i,s_1} \times \underbrace{M^+_{w_i,s_2} \times \dots \times M^+_{w_i,s_2}}_{m-times}$ .
- For all  $\pi : \underbrace{s_2 \dots s_2}_{m-times} \in P$ , we define  $M^+_{w_i,\pi} \coloneqq \{(a, b_1, \dots, b_m) \mid (b_1, \dots, b_m) \in M^i_{a,\pi}\}.$

By the definition of  $(W^+, M^+)$ , we have

$$((W^+)^{z \leftarrow w_i}, M^+)^- = (W^i, M^i) \text{ for all } i < \beta.$$

$$\tag{4}$$

Let  $\Delta_{\bullet}$  be the signature obtained from  $\Delta_{+}$  by adding a set of new nominals  $C = \{k_i \mid 0 < i < \beta\}$  and a new binary modality  $\leq$ . Let  $(W^{\bullet}, M^{\bullet})$  be the expansion of  $(W^+, M^+)$  to  $\Delta_{\bullet}$  such that

- (a)  $W_{k_i}^{\bullet} = w_i$  for all ordinals *i* with  $0 < i < \beta$ , and
- (b)  $(w_i, w_j) \in W^{\bullet}_{\leq}$  iff i < j.

Let  $T = \Gamma \cup \{ \forall z \cdot @_{k_i} \langle \langle \rangle z \Rightarrow \varphi_i^+(z) \mid i < \beta \}$ . We show that  $(W^{\bullet}, M^{\bullet}) \models T$ :

| 2. Let the an ardian much that $0 < t < 0$  |                             |
|---|-----------------------------|
| 2 let <i>i</i> be an ordinal such that $0 < i < \beta$  |                             |
| 3 let $w_j \in  W^{\bullet} $ such that $(w_i, w_j) \in W^{\bullet}_{<}$ , meaning that $((W^{\bullet})^{z \leftarrow w_j}, M^{\bullet}) \models @_{k_i} \langle < \rangle_Z$ |                             |
| 4 $i < j$ by the definition of $(W^{\bullet}, M^{\bullet})$ , since $(w_i, w_j)$  | $) \in W^{ullet}_{<}$       |
| 5 $((W^+)^{z \leftarrow w_j}, M^+) \models \Phi_j^+$ by Proposition 57 and statement 4, since (W  | $(V^j, M^j) \models \Phi_j$ |
| 6 $((W^{\bullet})^{z \leftarrow w_j}, M^{\bullet}) \models \Phi_j^+$ by the satisfaction condition  |                             |
| 7 $((W^{\bullet})^{\mathbf{z}\leftarrow w_j}, M^{\bullet}) \models \varphi_i^+$ since $\varphi_i \in \Phi_j$  |                             |
| 8 $(W^{\bullet}, M^{\bullet}) \models \forall z \cdot @_{k_i} \langle \langle \rangle z \Rightarrow \varphi_i^+$ from 3 and 7   |                             |
| 9 $(W^{\bullet}, M^{\bullet}) \models T$ from 1 and 8   |                             |

By Theorem 53, there exists a model  $(V^{\bullet}, N^{\bullet})$  of T such that  $(V^{\bullet}, V^{\bullet}_{\leq})$  is of confinality  $\alpha$ . We define  $v_i \coloneqq V^{\bullet}_{k_i}$  for all  $i < \beta$ . By cofinality, there exists  $v \in V^{\bullet}_{s_0}$  such that  $(v_i, v) \in V^{\bullet}_{<}$  for all  $i < \beta$ . It follows that  $((V^{\bullet})^{z \leftarrow v}, N^{\bullet}) \models \varphi^+_i$  for all  $i < \beta$ . Let  $(V^+, N^+) \coloneqq (V^{\bullet}, N^{\bullet}) \upharpoonright_{\Delta_+}$ . By the satisfaction condition,  $((V^+)^{z \leftarrow v}, N^+) \models \varphi^+_i$  for all  $i < \beta$ . By Proposition 57,  $((V^+)^{z \leftarrow v}, N^+)^- \models \varphi_i$  for all  $i < \beta$ .

## 11. Conclusion

In this paper we established an omitting types theorem for first-order hybrid dynamic logic and sufficiently expressive fragments. For countable signatures, the result followed without needing compactness whereas for uncountable signatures we had to restrict our attention to compact fragments of the logic. It turns out that the latter restriction is actually necessary for some of these fragments, as compactness is a consequence of OTT for uncountable signatures. We also provided two applications of the OTT: (1) Löwenheim-Skolem theorems and (2) a completeness theorem for the constructor-based version of first-order hybrid dynamic logic. In future work we intend to explore other interesting consequences of OTT in this setting, particularly the Robinson Joint Consistency theorem.

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