

ON COUNTABLY PERFECTLY MEAGER AND COUNTABLY PERFECTLY NULL SETS

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ABSTRACT. We study a strengthening of the notion of a universally meager set and its dual counterpart that strengthens the notion of a universally null set.

We say that a subset A of a perfect Polish space X is countably perfectly meager (respectively, countably perfectly null) in X , if for every perfect Polish topology τ on X , giving the original Borel structure of X , A is covered by an F_σ -set F in X with the original Polish topology such that F is meager with respect to τ (respectively, for every finite, non-atomic, Borel measure μ on X , A is covered by an F_σ -set F in X with $\mu(F) = 0$).

We prove that if $2^{\aleph_0} \leq \aleph_2$, then there exists a universally meager set in $2^{\mathbb{N}}$ which is not countably perfectly meager in $2^{\mathbb{N}}$ (respectively, a universally null set in $2^{\mathbb{N}}$ which is not countably perfectly null in $2^{\mathbb{N}}$).

1. INTRODUCTION

We continue the study of countably perfectly meager sets undertaken by Pol and Zakrzewski [19]. We say (cf. [19]) that a subset A of a perfect Polish space X is *countably perfectly meager in X* ($A \in \mathbf{PM}_\sigma$), if for every sequence of perfect subsets $\{P_n : n \in \mathbb{N}\}$ of X , there exists an F_σ -set F in X such that $A \subseteq F$ and $F \cap P_n$ is meager in P_n for each n . Let us also recall that A is *universally meager* ($A \in \mathbf{UM}$), if for every Borel isomorphism f between X and any perfect Polish space Y the image of A under f is meager in Y (see [26], [27], [1], [2] and also [10], [11], [12], where this class was earlier studied by Grzegorek and denoted by $\overline{\mathbf{AFC}}$). By [2, Theorem 7] we have $\mathbf{PM}_\sigma \subseteq \mathbf{UM}$ and by [19, Theorem 1.1], this inclusion is consistently proper, namely it holds if there exists a universally meager set of cardinality 2^{\aleph_0} , in particular, if CH is true.

In this note we prove (see Theorem 2.2) that $\mathbf{PM}_\sigma \neq \mathbf{UM}$ follows also from the assumption that $2^{\aleph_0} = \aleph_2$. Whether it is consistent that $\mathbf{PM}_\sigma = \mathbf{UM}$ remains an open problem (it is consistent that $\mathbf{UM} \subsetneq$

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PM but also that $\mathbf{UM} = \mathbf{PM}$ (see [1]), where **PM** denotes the family of all perfectly meager subsets of X .

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If I is a σ -ideal of subsets of X , i.e., it is hereditary, closed under taking countable unions and contains all singletons, then by I^* we denote the σ -ideal on X generated by the closed subsets of X which belong to I (cf. [22]).

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If τ is a perfect Polish topology on X giving the original Borel structure of X , then by $\mathcal{M}(X, \tau)$ we denote the σ -ideal of meager sets with respect to τ . Let us note that $\mathcal{M}^*(X, \tau)$ consists of such $A \subseteq X$ that there exists an F_σ -set F in X (with the original Polish topology) with $A \subseteq F$ and $F \in \mathcal{M}(X, \tau)$. By [26, Theorem 2.1], A is universally meager in X if and only if A belongs to the intersection of all σ -ideals of the form $\mathcal{M}(X, \tau)$, whereas by [19, Proposition 4.6], A is countably perfectly meager in X if and only if A belongs to the intersection of all σ -ideals of the form $\mathcal{M}^*(X, \tau)$.

Universally meager sets may be seen as a category counterpart of *universally null* sets in X . Namely, if for a finite, non-atomic, Borel measure μ is on X (i.e., a countably additive measure $\mu : \mathbf{B}(X) \rightarrow [0, +\infty)$ defined on the σ -algebra $\mathbf{B}(X)$ of Borel subsets of X and vanishing on singletons of X), we denote by $\mathcal{N}(X, \mu)$ the σ -ideal of μ -null sets (i.e., sets of outer μ -measure zero), then the collection **UN** of universally null subsets of X is the intersection of all σ -ideals of the form $\mathcal{N}(X, \mu)$.

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The following definition of a measure analogue of countably perfectly meager sets was suggested by Taras Banach. We say that A is *countably perfectly null in X* ($A \in \mathbf{PN}_\sigma$), if A belongs to the intersection of all σ -ideals of the form $\mathcal{N}^*(X, \mu)$. In other words, $A \in \mathbf{PN}_\sigma$ if for every finite, non-atomic, Borel measure μ on X , A is covered by an F_σ -set F in X with $\mu(F) = 0$. Let us note that if λ is the standard probability product measure on the Cantor space $2^\mathbb{N}$, then $\mathcal{N}^*(2^\mathbb{N}, \lambda)$ is a well-known σ -ideal which is usually denoted by \mathcal{E} (cf. [3]).

The name of the class \mathbf{PN}_σ is further justified by the following observation.

Proposition 1.1. *A set $A \subseteq X$ is countably perfectly null in X if and only if for every sequence of perfect subsets $\{P_n : n \in \mathbb{N}\}$ of X with associated probability non-atomic Borel measures μ_n on P_n , there exists an F_σ -set F in X such that $A \subseteq F$ and $\mu_n(F \cap P_n) = 0$ for each n .*

Proof. If $A \in \mathbf{PN}_\sigma$ and for each n we have a perfect set P_n together with the respective measure μ_n on P_n , then it is enough to cover A by

an F_σ -set F with $\mu(F) = 0$ for μ defined by

$$\mu(B) = \sum_n \frac{1}{2^n} \mu_n(B \cap P_n) \quad \text{for } B \in \mathbf{B}(X).$$

For the other direction, given a finite, non-atomic, Borel measure μ on X let us note that the regularity of μ (cf. [16, 17.C]) implies the existence of (pairwise disjoint) perfect sets $\{P_n : n \in \mathbb{N}\}$ of positive μ -measure such that $\mu(X \setminus \bigcup_n P_n) = 0$. Then it suffices to cover A by an F_σ -set F with $\mu(F \cap P_n) = 0$ for each n . \square

Clearly, we have $\mathbf{PN}_\sigma \subseteq \mathbf{UN}$. One easily observes that we also have $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma$.

Proposition 1.2. *Every countably perfectly null subset of X is countably perfectly meager.*

Proof. Let us assume that $A \in \mathbf{PN}_\sigma$ and let $\{P_n : n \in \mathbb{N}\}$ be a sequence of perfect subsets of X . For each n let μ_n be a Borel probability, non-atomic measure on P_n which assigns positive values to all non-empty, relatively open subsets of P_n (e.g., one may concentrate μ_n on a dense in P_n homeomorphic copy of the irrationals). Let F be an F_σ -set in X such that $A \subseteq F$ and $\mu_n(F \cap P_n) = 0$ for each n (cf. Proposition 1.1). Clearly, $F \cap P_n$ is meager in P_n for each n , so $A \in \mathbf{PM}_\sigma$. \square

The inclusion $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma$ is, at least consistently, proper. Indeed, if $A \subseteq 2^\mathbb{N}$ is a Sierpiński set with respect to the measure λ , then $A \in \mathbf{PM}_\sigma$ in $2^\mathbb{N}$ (cf. [19, Corollary 2.9 and Remark 2.11]) but A has positive outer measure λ .

An analogous argument shows the consistency of $\mathbf{PN}_\sigma \neq \mathbf{UN}$. Namely, if $A \subseteq 2^\mathbb{N}$ is a Luzin set in $2^\mathbb{N}$ (which exists e.g. under CH), then $A \in \mathbf{UN}$ but A is not even meager in $2^\mathbb{N}$.

In this note we prove (see Theorem 3.2) that the inequality $\mathbf{PN}_\sigma \neq \mathbf{UN}$ follows also from the assumptions that either there exists a universally null set in $2^\mathbb{N}$ of cardinality 2^{\aleph_0} (then we actually have that even $\mathbf{UN} \setminus \mathbf{PM}_\sigma \neq \emptyset$; cf. Proposition 1.2) or $2^{\aleph_0} = \aleph_2$. Whether it is consistent that $\mathbf{PN}_\sigma = \mathbf{UN}$, remains an open problem.

Section 2 is devoted to the proof of Theorem 2.2 stating that if $2^{\aleph_0} = \aleph_2$, then there is a universally meager set in $2^\mathbb{N}$ which is not countably perfectly meager in $2^\mathbb{N}$.

In Section 3 we give some examples of countably perfectly null sets and prove Theorem 3.2 which shows the inequality $\mathbf{PN}_\sigma \neq \mathbf{UN}$ under the assumption that either there exists a universally null set in $2^\mathbb{N}$ of cardinality 2^{\aleph_0} (then we actually have that even $\mathbf{UN} \setminus \mathbf{PM}_\sigma \neq \emptyset$, cf. Proposition 1.2) or $2^{\aleph_0} = \aleph_2$.

In Section 4 we collect some remarks and open problems.

2. UNIVERSALLY MEAGER NOT COUNTABLY PERFECTLY MEAGER SETS

Let us recall that the cardinal number \mathfrak{b} is the minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is unbounded in the ordering \leq^* of eventual domination. Following [24, Definition 2.8], by a \mathfrak{b} -scale (in $\mathbb{N}^{\mathbb{N}}$) we mean a subset $B = \{f_\alpha : \alpha < \mathfrak{b}\}$ of $\mathbb{N}^{\mathbb{N}}$ with the following properties:

- $f_\alpha : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing,
- $\alpha < \beta < \mathfrak{b}$ implies $f_\alpha <^* f_\beta$,
- for every $f \in \mathbb{N}^{\mathbb{N}}$ there is $\alpha < \mathfrak{b}$ with $f_\alpha \not\leq^* f$.

By identifying each f_α with the characteristic function of its range (or just its range, respectively), we obtain a homeomorphic copy A of B in $2^{\mathbb{N}}$ (respectively, in $\mathcal{P}(\mathbb{N})$ with the Cantor set topology) which we also call a \mathfrak{b} -scale in $2^{\mathbb{N}}$ (respectively, in $\mathcal{P}(\mathbb{N})$) (cf. [24]). It is well-known and easy to see that \mathfrak{b} -scales can be constructed in ZFC. They are also classical examples of sets which are both universally meager and universally null (cf. [18]).

Let us recall that given a subset A of a perfect Polish space X , by a γ -cover of A we mean a countable relatively open cover \mathcal{U} of A which is infinite and such that for each $x \in A$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite. We say that A satisfies *property* $S_1(\Gamma, \Gamma)$ if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of γ -covers of A we can select for each n a set $V_n \in \mathcal{U}_n$ such that $\{V_n : n \in \mathbb{N}\}$ is a γ -cover of A (cf. [14], [24]). It is well-known (and due to Hurewicz [13]) that property $S_1(\Gamma, \Gamma)$ implies the Hurewicz property (for a definition of the Hurewicz property see Section 3).

If $\mathfrak{b} = \omega_1$, then there exists a \mathfrak{b} -scale $A = \{a_\alpha : \alpha < \mathfrak{b}\}$ in $\mathcal{P}(\mathbb{N})$ with the additional property that $\alpha < \beta < \mathfrak{b}$ implies that $a_\beta \setminus a_\alpha$ is finite (see [24, page 8]) and by a theorem of Scheepers [23] (see also [5, Theorem 123]), if A is such a \mathfrak{b} -scale in $\mathcal{P}(\mathbb{N})$, then $A \cup [\mathbb{N}]^{<\aleph_0}$ has property $S_1(\Gamma, \Gamma)$. The following observation is an easy corollary of this result. Let us recall that if κ is an infinite cardinal, then a set $A \subseteq X$ is κ -concentrated on a set $Q \subseteq X$, if $|A \setminus U| < \kappa$ for each open set U in X containing Q .

Lemma 2.1. *Assume that $\mathfrak{b} = \omega_1$. Let $A = \{a_\alpha : \alpha < \mathfrak{b}\}$ be a \mathfrak{b} -scale in $\mathcal{P}(\mathbb{N})$ with the additional property that $\alpha < \beta < \mathfrak{b}$ implies that $a_\beta \setminus a_\alpha$ is finite.*

For each n let $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$ be an ascending (i.e., $U_k^n \subseteq U_{k+1}^n$) sequence of open sets in $\mathcal{P}(\mathbb{N})$ with $[\mathbb{N}]^{<\aleph_0} \subseteq \bigcup_k U_k^n$ but $[\mathbb{N}]^{<\aleph_0} \not\subseteq U_k^n$ for no k . Then we can select for each n a set $V_n = U_{k_n}^n$ such that $\{V_n : n \in \mathbb{N}\}$ is a γ -cover of $(A \cup [\mathbb{N}]^{<\aleph_0}) \setminus Y$ for a certain countable set $Y \subseteq A$.

Proof. The set A being a \mathfrak{b} -scale in $\mathcal{P}(\mathbb{N})$, is \mathfrak{b} -concentrated on $[\mathbb{N}]^{<\aleph_0}$ (see [24, Lemma 2.10]). Consequently, since $\mathfrak{b} = \omega_1$, there is $\xi < \omega_1$ such that if we let $A' = \{a_\alpha : \xi < \alpha < \mathfrak{b}\}$, then for each n we have

$A' \cup [\mathbb{N}]^{<\aleph_0} \subseteq \bigcup_k U_k^n$ and by the properties of the sequence $\{U_k^n : k \in \mathbb{N}\}$, $\{(A' \cup [\mathbb{N}]^{<\aleph_0}) \cap U_k^n : k \in \mathbb{N}\}$ is a γ -cover of $A' \cup [\mathbb{N}]^{<\aleph_0}$. Since at the same time A' is still a \mathfrak{b} -scale in $\mathcal{P}(\mathbb{N})$ with the additional property above, Scheepers's theorem gives the desired conclusion. \square

Let us recall that $\text{non}(\mathcal{M})$ is the smallest cardinality of a non-meager subset of $2^{\mathbb{N}}$. It is well-known that if τ is a perfect Polish topology on a Polish space X , then $\text{non}(\mathcal{M})$ is the smallest cardinality of a subset of X not in $\mathcal{M}(X, \tau)$. We denote by \mathbb{Q} the copy of the rationals in $2^{\mathbb{N}}$ consisting of all eventually zero binary sequences.

\mathfrak{B}

Now we are ready to prove the main result of this section (cf. the proof of [25, Theorem 4]).

Theorem 2.2. *If $2^{\aleph_0} \leq \aleph_2$, then there is a universally meager set in $2^{\mathbb{N}}$ which is not countably perfectly meager in $2^{\mathbb{N}}$.*

Proof. If $2^{\aleph_0} = \aleph_1$, then the result follows from [19, Theorem 1.1], so from now on let us assume that $2^{\aleph_0} = \aleph_2$.

We shall split the argument into three cases.

Case (A): $\text{non}(\mathcal{M}) = \aleph_2$.

Then, by a result of Grzegorek (see [11, Theorem 1]), there exists a universally meager set in $2^{\mathbb{N}}$ of cardinality $\aleph_2 = 2^{\aleph_0}$ and the result follows from [19, Theorem 1.1].

Case (B): $\mathfrak{b} = \aleph_2$.

This case is already covered by the previous one, since it is well-known that $\mathfrak{b} \leq \text{non}(\mathcal{M})$.

Case (C): $\text{non}(\mathcal{M}) = \mathfrak{b} = \aleph_1$.

Let C and D be disjoint copies of the Cantor set in $2^{\mathbb{N}}$ such that

- (1) the operation $+$ of addition is a homeomorphism between $C \times D$ and $C + D$ (cf. [20]).

Let us fix a homeomorphism $h : 2^{\mathbb{N}} \rightarrow C$.

Let $A = \{a_\alpha : \alpha < \mathfrak{b}\}$ be a \mathfrak{b} -scale in $\mathcal{P}(\mathbb{N})$ with the additional property that $\alpha < \beta < \mathfrak{b}$ implies that $a_\beta \setminus a_\alpha$ is finite (cf. the paragraph preceding Lemma 2.1) and let us identify it with its homeomorphic copy in $2^{\mathbb{N}}$.

Let $X = A \cup \mathbb{Q}$ and $\tilde{X} = h(X)$. Since X is universally meager, so is \tilde{X} .

Let us fix a set $M \subseteq D$ of cardinality $\text{non}(\mathcal{M}) = \aleph_1$ such that

- (2) M is relatively non-meager in D .

Since $|\tilde{X}| = \aleph_1$, we can fix a surjection $m : \tilde{X} \rightarrow M$ onto \tilde{X} and let $H = \{(x, m(x)) : x \in \tilde{X}\} \subseteq C \times D$ be the graph of m . Let us note that since \tilde{X} is the injective continuous image of H under the projection onto the first axis and \tilde{X} is universally meager, so is H .

Finally, let $Z = \{x+m(x) : x \in \tilde{X}\}$. Clearly, Z is universally meager as the image of H under the homeomorphism $+$ between $C \times D$ and $C + D$ (cf. (1)).

We shall show that

(3) Z is not a \mathbf{PM}_σ -set in $2^\mathbb{N}$

and this will end the proof of the theorem.

To that end, let $\tilde{\mathbb{Q}} = h(\mathbb{Q}) = \{q_n : n \in \mathbb{N}\}$ and let us suppose, towards a contradiction, that there are closed sets F_n in $2^\mathbb{N}$ such that $Z \subseteq \bigcup_n F_n$ and F_n is relatively nowhere dense in $q_k + D$ or equivalently, $(q_k + F_n) \cap D$ is relatively nowhere dense in D for each n and k .

Let $\{I_n : n \in \mathbb{N}\}$ be an enumeration with infinitely many repetitions of the elements of a countable basis \mathcal{B} of D .

Let us fix an arbitrary i and let $F = F_i$.

As the set F is compact, for each n we can define by induction on k an ascending sequence $\{U_k^n : k \in \mathbb{N}\}$ of open sets in C with $\{q_i : i < k\} \subseteq U_k^n \cap \tilde{\mathbb{Q}} \neq \tilde{\mathbb{Q}}$ for every k together with a sequence $\{D_k^n : k \in \mathbb{N}\}$ of non-empty, relatively clopen sets in D such that

(4) $D_{k+1}^n \subseteq D_k^n \subseteq I_n$ and $cl_D((U_k^n + F) \cap D) \cap D_k^n = \emptyset$ for every k .

Now, since \tilde{X} and $\tilde{\mathbb{Q}}$ are the respective images of X and \mathbb{Q} under the homeomorphism h , and $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$ is an ascending sequence of open sets in C with $\tilde{\mathbb{Q}} \subseteq \bigcup_k U_k^n$ but $\tilde{\mathbb{Q}} \subseteq U_k^n$ for no k , Lemma 2.1 enables us to select for each n a set $V_n = U_{k_n}^n$ such that

(5) $\{V_n : n \in \mathbb{N}\}$ is a γ -cover of $\tilde{X} \setminus Y$ for a certain countable set $Y \subseteq \tilde{X}$.

We will show that

(6) $((\tilde{X} \setminus Y) + F) \cap D$ is meager in D .

To see this, for each m let $K_m = \bigcap_{n \geq m} cl_D((V_n + F) \cap D)$ and let us note that K_m is a closed relatively nowhere dense subset of D . Indeed, any open set from \mathcal{B} is of the form I_n for some $n \geq m$ and $I_n \not\subseteq K_m$ by (4).

Moreover, we have $((\tilde{X} \setminus Y) + F) \cap D \subseteq \bigcup_m K_m$. Indeed, if $c \in \tilde{X} \setminus Y$, then there is m such that $c \in V_n$ for every $n \geq m$ (cf. (5)). Consequently, $(c + F) \cap D \subseteq \bigcap_{n \geq m} ((V_n + F) \cap D) \subseteq K_m$, completing the proof of (6).

Let us summarize: for each i we have found a countable set $Y_i \subseteq \tilde{X}$ such that $((\tilde{X} \setminus Y_i) + F_i) \cap D$ is meager in D .

Consequently, letting $\tilde{Y} = \bigcup_i Y_i$ we get a countable subset of C such that $((\tilde{X} \setminus \tilde{Y}) + \bigcup_n F_n) \cap D$ is meager in D .

But since $Z \subseteq \bigcup_n F_n$, we conclude that

(7) $((\tilde{X} \setminus \tilde{Y}) + Z) \cap D$ is meager in D .

On the other hand, $M \setminus m(\tilde{Y}) \subseteq (\tilde{X} \setminus \tilde{Y}) + Z$. Indeed, if $m \in M \setminus m(\tilde{Y})$, then $m = m(x)$ for some $x \in \tilde{X} \setminus \tilde{Y}$ and then $m = (x + (x + m(x))) \in x + Z$. This implies that $((\tilde{X} \setminus \tilde{Y}) + Z) \cap D$ is not meager in D (cf. (2)) contradicting (7) and thus completing the proof of (3). \square

Let us note that under CH we have $\text{non}(\mathcal{M}) = \mathfrak{b} = \aleph_1$ and Case (C) of the proof above establishes the consistency of $\mathbf{PM}_\sigma \neq \mathbf{UM}$ in the way which avoids the use of [19, Theorem 1.1].

3. COUNTABLY PERFECTLY NULL SETS

Let us recall that given a perfect Polish space X a set $A \subseteq X$ has *the Hurewicz property*, if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of A , there are finite subfamilies $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $A \subseteq \bigcup_n \bigcap_{m \geq n} (\bigcup \mathcal{F}_m)$. If A is a zero-dimensional subspace of X , then by a result of Hurewicz (cf. [13] and [21]) this is equivalent to the statement that every continuous image of A in $\mathbb{N}^{\mathbb{N}}$ is bounded in the ordering \leq^* of eventual domination.

The smallest cardinality of a subset of $2^{\mathbb{N}}$ which is nonmeasurable with respect to the standard probability product measure λ on $2^{\mathbb{N}}$ is denoted by $\text{non}(\mathcal{N})$. It is well-known that if μ is a non-zero, finite, non-atomic, Borel measure on X , then $\text{non}(\mathcal{N})$ is the smallest cardinality of a subset of X not in $\mathcal{N}(X, \mu)$.

Let us also recall that by \mathbb{Q} we denote the copy of the rationals in $2^{\mathbb{N}}$ consisting of all eventually zero binary sequences.

\mathfrak{B}

The following result provides examples of universally null countably perfectly meager sets which are countably perfectly null as well.

Proposition 3.1. *The following collections of sets are countably perfectly null in the respective perfect Polish spaces:*

- (1) *universally null sets with the Hurewicz property in any perfect Polish space X ,*
- (2) *any sets of cardinality less than $\min(\text{non}(\mathcal{N}), \mathfrak{b})$ in any perfect Polish space X ,*
- (3) *γ -sets in any perfect Polish space X ,*
- (4) *\mathfrak{b} -scales in $2^{\mathbb{N}}$,*
- (5) *Hausdorff (ω_1, ω_1^*) -gaps in $\mathcal{P}(\mathbb{N})$.*

Proof. (1) Let $A \subseteq X$ be a universally null set with the Hurewicz property and let μ be a non-zero, finite, non-atomic Borel measure on X . Since $A \in \mathbf{UN}$, there is a G_δ -set G in X such that $A \subseteq G$ and $\mu(G) = 0$. Now, since A has the Hurewicz property, there is an F_σ set F in X such that $A \subseteq F \subseteq G$ (cf. [14, Theorem 5.7]). Consequently, $\mu(F) = 0$ which shows that $A \in \mathbf{PN}_\sigma$.

\mathfrak{B}

Statements (2) – (4) can be derived from (1) as follows.

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(2) Sets of cardinality less than $\text{non}(\mathcal{N})$ are universally null and sets of cardinality less than \mathfrak{b} have the Hurewicz property.

(3) γ -sets are universally null (as they actually have Rothberger's property C''' , cf. [8]) and they have the Hurewicz property, by [7, Theorem 2].

(4) Let us assume that A is a \mathfrak{b} -scale in $2^{\mathbb{N}}$. Let $B = A \cup \mathbb{Q}$. Then B is a universally null set with the Hurewicz property (see e.g., [19, Example 4.1 and Remark 4.2]), so $B \in \mathbf{PN}_{\sigma}$ in $2^{\mathbb{N}}$. Consequently, $A \in \mathbf{PN}_{\sigma}$ in $2^{\mathbb{N}}$.

(5). This may actually be established by a classical argument showing that the Hausdorff gap is universally null, which we sketch here for the sake of completeness. Following the proof of [15, Lemma 20.5], let $\langle \langle a_{\alpha} : \alpha < \omega_1 \rangle, \langle b_{\alpha} : \alpha < \omega_1 \rangle \rangle$ be a Hausdorff gap, $F_{\alpha} = \{c \in \mathcal{P}(\mathbb{N}) : a_{\alpha} \subseteq^* c \subseteq^* b_{\alpha}\}$ for $\alpha < \omega_1$ and let μ be a non-zero, finite, non-atomic Borel measure on $\mathcal{P}(\mathbb{N})$. Then F_{α} 's are F_{σ} -sets in $\mathcal{P}(\mathbb{N})$ and for a sufficiently large ξ we have $\mu(F_{\xi}) = 0$ (see [15, the proof of Lemma 20.5]). Letting

$$F = F_{\xi} \cup \{a_{\alpha} : \alpha < \xi\} \cup \{b_{\alpha} : \alpha < \xi\},$$

we get an F_{σ} -set with $\{a_{\alpha} : \alpha < \omega_1\} \cup \{b_{\alpha} : \alpha < \omega_1\} \subseteq F$ and $\mu(F) = 0$ which shows that $\{a_{\alpha} : \alpha < \omega_1\} \cup \{b_{\alpha} : \alpha < \omega_1\} \in \mathbf{PN}_{\sigma}$ in $\mathcal{P}(\mathbb{N})$. □

The main result of this section is a measure counterpart of [19, Theorem 1.1] and Theorem 2.2.

Theorem 3.2. *If either*

(a) *there exists a universally null set in $2^{\mathbb{N}}$ of cardinality 2^{\aleph_0}*

or

(b) $2^{\aleph_0} \leq \aleph_2$,

then there is a universally null set in $2^{\mathbb{N}}$ which is not countably perfectly null in $2^{\mathbb{N}}$.

Proof. (a) Let T be a universally null set in $2^{\mathbb{N}}$ of cardinality 2^{\aleph_0} .

By Proposition 1.2, it suffices to show that there is also one which is not countably perfectly meager.

Let us recall that by [19, Theorem 1.1], there exist a set $H \subseteq T \times 2^{\mathbb{N}}$ intersecting each vertical section $\{t\} \times 2^{\mathbb{N}}$, $t \in T$, in a singleton and a homeomorphic copy E of H in $2^{\mathbb{N}}$ which is not a \mathbf{PM}_{σ} -set in $2^{\mathbb{N}}$. Now, since T is universally null, so is E as a preimage of T under a continuous injective function.

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(b) If $2^{\aleph_0} = \aleph_1$, then any Luzin set in $2^{\mathbb{N}}$ provides an example of a non-meager, universally null set.

From now on let us assume that $2^{\aleph_0} = \aleph_2$.

Following closely the scheme of proof of the Theorem 2.2, we split the argument into three cases.

Case (A): $\text{non}(\mathcal{N}) = \aleph_2$.

Then, by a theorem of Grzegorek (see [9]), there exists a universally null set in $2^{\mathbb{N}}$ of cardinality $\aleph_2 = 2^{\aleph_0}$ and the result follows from part (a).

Case (B): $\mathfrak{b} = \aleph_2$.

In this case any \mathfrak{b} -scale in $2^{\mathbb{N}}$ is a universally null set of cardinality $\mathfrak{b} = 2^{\aleph_0}$ and the result again follows from part (a).

Case (C): $\text{non}(\mathcal{N}) = \mathfrak{b} = \aleph_1$.

As in the proof of Theorem 2.2, we fix copies C, D of the Cantor set in $2^{\mathbb{N}}$ such that

- (1) the operation $+$ of addition is a homeomorphism between $C \times D$ and $C + D$ (cf. [20]),

a homeomorphism $h : 2^{\mathbb{N}} \rightarrow C$, a \mathfrak{b} -scale X in $2^{\mathbb{N}}$ and we let $\tilde{X} = h(X)$. Since X is universally null, so is \tilde{X} .

We also fix a homeomorphism $g : 2^{\mathbb{N}} \rightarrow D$ and we define a Borel measure μ on $2^{\mathbb{N}}$ by letting

$$\mu(B) = \lambda(g^{-1}(B \cap D)), \quad \text{for } B \in \mathbf{B}(2^{\mathbb{N}}).$$

Then we fix a set $M \subseteq D$ of cardinality $\text{non}(\mathcal{N}) = \aleph_1$ with

- (2) $\mu^*(M) > 0$,

we let $m : \tilde{X} \rightarrow M$ be a surjection onto M and we put $H = \{(x, m(x)) : x \in \tilde{X}\}$. Since \tilde{X} is the injective continuous image of H under the projection onto the first axis and \tilde{X} is universally null, so is H .

Finally, let $Z = \{x + m(x) : x \in \tilde{X}\}$. Clearly, Z is universally null as the image of H under the homeomorphism $+$ between $C \times D$ and $C + D$ (cf. (1)).

We shall show that on the other hand

- (3) Z is not a \mathbf{PN}_{σ} -set in $2^{\mathbb{N}}$,

thus completing the proof of the theorem.

To that end, let $\tilde{\mathbb{Q}} = h(\mathbb{Q}) = \{q_n : n \in \mathbb{N}\}$ and let us suppose, towards a contradiction, that there are closed μ -null sets F_n in $2^{\mathbb{N}}$ such that $Z \subseteq \bigcup_n F_n$ and $\mu(q_k + F_n) = 0$ for each n and k (cf. Proposition 1.1.)

Let us fix an arbitrary $\varepsilon > 0$.

For each n , F_n being compact and μ -null, there is an open set U_n in C such that $\tilde{\mathbb{Q}} \subseteq U_n$ and

- (4) $\mu(U_n + F_n) < \frac{\varepsilon}{2^{n+1}}$.

Now, X being a \mathfrak{b} -scale in $2^{\mathbb{N}}$, is \mathfrak{b} -concentrated on \mathbb{Q} (see [24, Lemma 2.10]). Consequently, \tilde{X} is \mathfrak{b} -concentrated on $\tilde{\mathbb{Q}}$ which, taking into account that $\mathfrak{b} = \aleph_1$, implies that for each n there is a countable set

$Y_n \subseteq \tilde{X}$ such that $\tilde{X} \setminus Y_n \subseteq U_n$. It follows (cf. (4)) that $\mu^*((\tilde{X} \setminus Y_n) + F_n) < \frac{\varepsilon}{2^{n+1}}$ which implies that, letting $F = \bigcup_n F_n$ and $\tilde{Y} = \bigcup_n Y_n$, we have $\mu^*((\tilde{X} \setminus \tilde{Y}) + F) < \varepsilon$. But since $Z \subseteq F$ and the choice of ε was arbitrary, we conclude that

$$(5) \quad \mu^*((\tilde{X} \setminus \tilde{Y}) + Z) = 0.$$

On the other hand, exactly as in the proof of Theorem 2.2, we have $M \setminus m(\tilde{Y}) \subseteq (\tilde{X} \setminus \tilde{Y}) + Z$ which, \tilde{Y} being countable, implies that $\mu^*((\tilde{X} \setminus \tilde{Y}) + Z) > 0$ (cf. (2)), contradicting (5) and thus completing the proof of (3). □

4. REMARKS AND OPEN PROBLEMS

The results of Sections 2 and 3 motivate the following questions. The first two are directly related to Theorems 2.2 and 3.2, respectively.

β

Problem 1. Is $\mathbf{PM}_\sigma = \mathbf{UM}$ consistent?

β **Problem 2.** Is $\mathbf{PN}_\sigma = \mathbf{UN}$ consistent?

β

Let us note that we consistently have $\mathbf{PM}_\sigma \subseteq \mathbf{UN}$ since in the model obtained by adding \aleph_2 Cohen reals to a model of GCH we have $\mathbf{UM} \subseteq \mathbf{UN}$ (see Corazza [6, Theorem 0.6(b)] and Miller [17]; by a theorem of Bartoszyński and Shelah, cf [4, Theorem 3], it is consistently true that even all perfectly meager sets are universally null). By the fact that $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma$ (see Proposition 1.2), the dual statement that $\mathbf{PN}_\sigma \subseteq \mathbf{UM}$ is just true but the following question remains open.

β

Problem 3. Is $\mathbf{PN}_\sigma = \mathbf{PM}_\sigma$ consistent?

β

Finally, in view of the inclusion $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma \cap \mathbf{UN}$ one may ask

β

Problem 4. Is $\mathbf{PN}_\sigma = \mathbf{PM}_\sigma \cap \mathbf{UN}$ true/consistent?

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