ABSOLUTENESS FOR THE THEORY OF THE INNER MODEL CONSTRUCTED FROM FINITELY MANY COFINALITY QUANTIFIERS

UR YA'AR

ABSTRACT. We prove that the theory of the models constructible using finitely many cofinality quantifiers $-C^*_{\lambda_1,...,\lambda_n}$ and $C^*_{<\lambda_1,...,<\lambda_n}$ for $\lambda_1,...,\lambda_n$ regular cardinals – is set-forcing absolute under the assumption of class many Woodin cardinals, and is independent of the regular cardinals used. Towards this goal we prove some properties of the generic embedding induced from the stationary tower restricted to $< \mu$ -closed sets.

1. INTRODUCTION AND PRELIMINARIES

Following the general framework set by Kennedy, Magidor and Väänänen in [3] for inner models constructed from extended logics, our aim is to investigate $C^*_{\lambda_1,...,\lambda_n}$ and $C^*_{<\lambda_1,\ldots,<\lambda_n}$ – the models of sets constructible using the logics $\mathcal{L}(Q^{\mathrm{cf}}_{\lambda_1},\ldots,Q^{\mathrm{cf}}_{\lambda_n})$ and $\mathcal{L}(Q_{<\lambda_1}^{\mathrm{cf}},\ldots,Q_{<\lambda_n}^{\mathrm{cf}})$ respectively, where $Q_{\lambda}^{\mathrm{cf}}$ (resp. $Q_{<\lambda}^{\mathrm{cf}}$) is the quantifier asserting that an ordinal has cofinality λ (resp. $\langle \lambda \rangle$) and $\lambda_1 < \cdots < \lambda_n$ are regular cardinals. Kennedy, Magidor and Väänänen have proved that, assuming the existence of a proper class of Woodin cardinals, the theory of $C^*=C^*_\omega=C^*_{<\omega_1}$ is set-forcing absolute, and equals the theory of $C^*_{<\kappa}$ for any regular κ [3, theorem 5.18]. Our goal it to generalize this theorem to $C^*_{<\lambda_1,\ldots,<\lambda_n}$ for any regular uncountable $\lambda_1,\ldots,\lambda_n$, from which also the case of $C^*_{\lambda_1,\ldots,\lambda_n}$ can be deduced. To obtain this result, we use a variation of Woodin's stationary tower – the $< \mu$ -closed stationary tower, introduced by Foreman and Magidor in [2]. This tower has the property that it does not change the notion of "being of cofinality $< \mu$ ". We begin by stating and proving some facts regarding this tower, and then prove the main theorem. We end by showing that this method cannot be simply pushed to the case of infinitely many cofinalities, which leaves open the question of absoluteness for theory of this kind of models.

Our notation will mostly follow [2]. A set $S \subseteq \mathcal{P}(Y)$ is called *stationary in* Y if for every algebra $\mathfrak{A} = \langle Y, f_i \rangle_{i \in \omega}$ there is $Z \in S$ such that Z is a subalgebra of \mathfrak{A} . The collection of non-stationary subsets of Y is an ideal denoted by NS(Y). If S is stationary in Y then we denote the restriction of the non-stationary ideal to S by $NS(Y) \upharpoonright S$. We

²⁰²⁰ Mathematics Subject Classification. 03E45 (Primary) 03E47, 03E55, 03E57 (Secondary).

Key words and phrases. Inner models, Cofinality logic, Woodin cardinal, Stationary tower, Generic Absoluteness. I would like to thank my advisor, Prof. Menachem Magidor, for his guidance and support without which this work would not have been possible.

say that a set S is simply *stationary* if it is stationary in $\bigcup S$. If $\emptyset \neq X \subseteq Y$, then for $S \subseteq \mathcal{P}(Y)$ we define its projection to X by $S \downarrow X = \{Z \cap X \mid Z \in S\}$ and for $T \subseteq \mathcal{P}(X)$ we define its lift to Y by $T \uparrow Y = \{Z \in \mathcal{P}(Y) \mid Z \cap X \in T\}$.

A sequence $\mathcal{I} = \langle \mathcal{I}_{\beta} \mid \beta < \kappa \rangle$ such that each \mathcal{I}_{β} is an ideal on $\mathcal{P}(H(\beta))$ is called a tower of ideals if for every $\alpha < \beta$, if for any $S \subseteq \mathcal{P}(H(\beta)), S \notin \mathcal{I}_{\beta} \implies S \downarrow H(\alpha) \notin \mathcal{I}_{\alpha}$ and for any $T \subseteq \mathcal{P}(H(\alpha)), T \notin \mathcal{I}_{\alpha} \implies T \uparrow H(\beta) \notin \mathcal{I}_{\beta}$ (i.e. positive sets project/lift to positive sets). Any tower of ideals \mathcal{I} gives rise to a Boolean algebra $b(\mathcal{I})$ such that a generic $G \subseteq b(\mathcal{I})$ induces a generic embedding $j_G : V \to Ult(V,G) \subseteq V[G]$ (for more details see [4, 2]). The tower is called *precipitous* if every such generic embedding is well-founded. For a strongly inaccessible cardinal δ , the (full) stationary tower on δ is $\langle NS(H(\beta)) \mid \beta < \delta \rangle$, and a *restricted tower* is of the form $\langle NS(H(\beta)) \mid S_{\beta} \mid \beta < \delta \rangle$ where each S_{β} is stationary in $H(\beta)$. The Boolean algebra $b(\langle NS(H(\beta)) \mid \beta < \delta \rangle)$ of the full stationary tower on δ is forcing equivalent to as the poset

 $\mathbb{P}_{<\delta} = \{a \in V_{\delta} \mid a \text{ is stationary in } \cup a\}$

with $a \geq b$ iff $\cup a \subseteq \cup b$ and $\forall Z \in b \ Z \cup (\cup a) \in a$ (i.e. $b \downarrow (\cup a) \subseteq a$). Restricted stationary towers on δ usually correspond to subsets of $\mathbb{P}_{<\delta}$ with the same order. We will not distinguish the two notions.

Given a (perhaps restricted) tower on $\delta \mathbb{P}$ and a generic $G \in \mathbb{P}$, members of Ult(V, G)can be represented as equivalence classes $[f]_G$ for some $f : a \to V$ where $a \in G$. For every $x \in V j_G(x)$ is represented by the constant function c_x (on any $a \in G$), and every $\alpha < \delta$ is represented by the function $Z \mapsto otp(Z \cap \alpha)$ (cf. [4]).

Lemma 1. Let δ be a Woodin cardinal and $\rho > \delta$ strongly inaccessible. Let \mathbb{P} be a (perhaps restricted) precipitous stationary tower on δ , $G \subseteq \mathbb{P}$ generic and $j : V \to M$ the derived embedding. Then:

- (1) $j(\rho) = \rho$.
- (2) For every ordinal η the following are equivalent:

(a) $V \vDash \operatorname{cf}(\eta) < \rho$ (b) $V[G] \vDash \operatorname{cf}(\eta) < \rho$ (c) $M \vDash \operatorname{cf}(\eta) < \rho$.

Proof. For (1): If $\eta < \rho$ then every $x \in j(\eta)$ is of the form $[f]_G$ for $f: X \to \eta$ where $X \in V_{\delta}$, so there are at most ${}^{\delta}\eta < \rho$ many such functions, hence $j(\eta) < \rho$. Similarly every $x \in j(\rho)$ is of the form $[f]_G$ for $f: X \to \rho$ where $X \in V_{\delta}$, but now $|X| < \rho$ so f is bounded by some $\eta < \rho$, so $[f]_G < j(\eta)$. Hence $j(\rho) = \sup \{j(\eta) \mid \eta < \rho\} \le \rho$.

For (2): (c) \Rightarrow (b) since a cofinal sequence in M remains so in V[G].

 $(\mathbf{b}) \Rightarrow (\mathbf{a}) \text{ since } \rho > \delta = |\mathbb{P}| \text{ so by } \delta^+ \text{-c.c. cofinalities} \geq \rho \text{ are preserved from } V \text{ to } V[G].$

For (a) \Rightarrow (c) assume towards contradiction that $\operatorname{cf}^{V}(\eta) = \lambda < \rho$ while $\operatorname{cf}^{M}(\eta) \ge \rho$. Let $a \in \mathbb{P}$ such that $a \Vdash \operatorname{cf}^{M}(\eta) \ge \rho$ and $f_{\eta} : a \to V$ represent η (without loss of generalisation $f_{\eta}(X) \le \eta$ for every $X \in a$, since $\eta \le j(\eta)$). Let $\langle x_{\alpha} \mid \alpha < \lambda \rangle \in V$ be cofinal in η . For $X \in a$, let g(X) be the supremum of all values f(X) for some function $f \in V$ with domain $b \leq a$ which may represent some x_{α} . More precisely – for each $b \leq a$ choose some $f_{b,\alpha} : b \to V$ such that $b \Vdash [f_{b,\alpha}]_C = \check{x}_{\alpha}$, and let

$$g(X) = \sup \{ f_{b,\alpha}(Y) \mid b \le a, \alpha < \lambda, Y \in b \text{ s.t. } Y \cap (\cup a) = X \}$$

For every $\alpha < \lambda$ there are at most δ such functions, each with domain of size $< \delta$, so all-in-all there are at most $\lambda \cdot \delta$ possible values. Hence cf $(g(X)) \leq \lambda \cdot \delta < \rho$ (by strong inaccessibility) for all $X \in a$, so $a \Vdash cf([g]) < j(\rho) = \rho$. Since for every α $a \Vdash \check{x}_{\alpha} < [f_{\eta}]_{G}$, by the definition of g we get that $[g] \leq [f_{\eta}] = \eta$, and since cf^M $(\eta) \geq \rho$, $[g] < \eta$. But by the construction for every $b \leq a$ and $\alpha < \lambda$, $b \Vdash \check{x}_{\alpha} = [f_{b,\alpha}]_{G} \leq [g]_{G}$, so $a \Vdash \check{x}_{\alpha} \leq [g]_{G}$ for every α , so $\langle x_{\alpha} \mid \alpha < \lambda \rangle$ is bounded below η , contradicting the assumption that it is cofinal in η .

2. The $< \mu$ -closed stationary tower

In [2] we have the following theorem (theorem 1.3):

Theorem 2. Let δ Woodin, $\mu < \lambda < \delta$ regular, for every strong limit γ , let

 $S_{\gamma} = \{ Z \in \mathcal{P}_{\lambda} (H(\gamma)) \mid Z \cap \lambda \in \lambda \land Z \cap \gamma \text{ is } < \mu \text{ closed} \}$ NS $(\lambda, \gamma) = \text{non-stationary subsets of } \mathcal{P}_{\lambda} (H(\gamma))$ $\mathcal{I} = \{ \text{NS} (\lambda, \gamma) \upharpoonright S_{\gamma} \mid \gamma < \delta \}$

then \mathcal{I} is a tower of ideals and $b(\mathcal{I})$ is δ -presaturated.

Remark 3. $b(\mathcal{I})$ is forcing equivalent to $\mathbb{P}(\delta, \lambda, < \mu$ -closed), namely

$$\{a \in V_{\delta} \mid a \subseteq \mathcal{P}_{\lambda}(\cup a) \text{ is stationary s.t. } \forall Z \in a, Z \cap \mathbf{Ord} \text{ is } < \mu\text{-closed} \land Z \cap \lambda \in \lambda\}$$

with the relation defined above, $b \leq a$ iff $\cup a \subseteq \cup b$ and $b \downarrow (\cup a) \subseteq a$.

Theorem 4. If $G \subseteq b(\mathcal{I})$ is generic then the induced generic embedding $j : V \to M$ satisfies:

- (1) *M* is well-founded and ${}^{<\delta}M \cap V[G] \subseteq M$.
- (2) crit $(j) = \lambda$ and $j(\lambda) = \delta$.
- (3) For any θ ,
 - (a) If $V \models cf(\theta) < \lambda$ then $cf^{V}(\theta) = cf^{V[G]}(\theta) = cf^{M}(\theta)$;
 - (b) If $V \vDash \operatorname{cf}(\theta) \ge \mu$ then $V[G], M \vDash \operatorname{cf}(\theta) \ge \mu$;
 - (c) $V \vDash \operatorname{cf}(\theta) < \delta$ iff $V[G] \vDash \operatorname{cf}(\theta) < \delta$ iff $M \vDash \operatorname{cf}(\theta) < \delta$.

To prove this theorem we introduce some notions defined in [2] and prove variations of two lemmas.

Definition 5. Fix regular cardinals μ , λ and $\theta \gg \lambda$, and let $\mathfrak{A} = \langle H(\theta), \in, \triangle, \mu, f_i \rangle_{i \in \omega}$ be a Skolemized algebra on $H(\theta)$ where the f_i s are closed under composition and \triangle is a well-order. Define by recursion a sequence $\langle (\mathfrak{A}_i, F_i, G_i, F_i^*, G_i^*) | i \leq \mu \rangle$ of functions and expansions of \mathfrak{A} as follows. $\mathfrak{A}_0 = \mathfrak{A}, F_0^* = G_0^* = \emptyset$. For every *i*, we will define $\mathfrak{A}_i = \langle \mathfrak{A}, F_i^*, G_i^* \rangle$ and

$$F_i: \theta \times H(\theta) \to \theta \qquad F_i(\xi, x) = \begin{cases} \sup \left(\mathrm{Sk}^{\mathfrak{A}_i}(x \cap \theta) \right) & \xi = 0\\ \sup \left(\mathrm{Sk}^{\mathfrak{A}_i}(x \cap \xi) \right) & \xi \neq 0 \end{cases}$$
$$G_i: i \times H(\theta) \to H(\theta) \qquad G_i(j, Z) = (G_j \upharpoonright Z, F_i \upharpoonright Z).$$

At stage i > 0 we define

$$\begin{split} F_i^* : i \times \theta \times H(\theta) &\to \theta \qquad F_i^*(j,\xi,x) = F_j(\xi,x) \\ G_i^* : i \times i \times H(\theta) &\to H(\theta) \qquad G_i^*(j,k,x) = \begin{cases} G_j(k,x) & k < j \\ \varnothing & k \geq j \end{cases} \end{split}$$

Denote $\bar{F} = F^*_{\mu}, \bar{G} = G^*_{\mu}, \bar{\mathfrak{A}} = \mathfrak{A}_{\mu} = \langle \mathfrak{A}, \bar{F}, \bar{G} \rangle.$

Lemma 6 (Variation of [2, Lemma 1.6]). Let $\gamma_0 < \gamma_1$ be strong limit cardinals of cofinality $\geq \mu$. Let \mathfrak{A} be a Skolemized algebra on $H(\gamma_1)$ expanding $\langle H(\gamma_1), \in, \Delta, \{\gamma_0\}, \{\mu\} \rangle$ where Δ is a well order on $H(\gamma_1)$. Then for every $x \subseteq H(\gamma_0)$ which is $\langle \mu$ -closed, if $\bar{x} \subseteq H(\gamma_1)$ is the subalgebra of $\bar{\mathfrak{A}}$ generated by x and $\bar{x} \cap H(\gamma_0) = x$ then $\bar{x} \cap \gamma_1$ is $\langle \mu$ -closed in γ_1 .

Remark. The lemma in [2] is stated for $\leq \mu$ -closed sets rather than for $< \mu$ -closed sets.

Proof. Assume \bar{x} is not $\langle \mu$ -closed and let $\xi \in \bar{x}$ be least such that $\bar{x} \cap \xi$ is not $\langle \mu$ closed (or γ_1 if there is no such ξ). By assumption $\xi > \gamma_0$. So there is some sequence $\langle a_\beta \mid \beta < \nu \rangle \subseteq \bar{x} \cap \xi$ for some $\nu < \mu$ such that $\sup \langle a_\beta \mid \beta < \nu \rangle \notin \bar{x} \cap \xi$. Note that by minimality, $\langle a_\beta \mid \beta < \nu \rangle$ is cofinal in $\bar{x} \cap \xi$. We first claim that ξ is regular (or γ_1). Otherwise, if $\xi \in \bar{x}$ and is singular, since $\bar{x} \prec \langle H(\gamma_1), \epsilon, \Delta \rangle \bar{x}$ knows that ξ is singular, so there is some $\xi' \in \bar{x}, \xi' < \xi$, and a sequence $\langle \xi_\alpha \mid \alpha < \xi' \rangle \in \bar{x}$ cofinal in $\bar{x} \cap \xi$. So both $\langle \xi_\alpha \mid \alpha < \xi' \rangle \langle a_\beta \mid \beta < \nu \rangle$ are cofinal in $\bar{x} \cap \xi$ so by letting α_β be the first $\alpha < \xi'$ such that $\xi_\alpha > a_\beta$ we get a sequence $\langle \alpha_\beta \mid \beta < \nu \rangle$ cofinal in $\bar{x} \cap \xi'$ by contradiction to the minimality of ξ .

Second, let $\langle a_{\beta} \mid \beta < \nu \rangle \subseteq \bar{x} \cap \xi$ for some $\nu < \mu$ such that $\sup \langle a_{\beta} \mid \beta < \nu \rangle \notin \bar{x} \cap \xi$. For every $\beta < \nu$ there are an $\bar{\mathfrak{A}}$ -term τ_{β} and parameters $\vec{\alpha}_{\beta} \in (x \cap \gamma_0)^{<\omega}$ such that $a_{\beta} = \tau_{\beta}(\vec{\alpha}_{\beta})$. By the construction of $\bar{\mathfrak{A}}$, for every $\beta < \nu$ there is some $\rho_{\beta} \in \mu \cap \bar{x}$ such that $a_{\beta} = \tau_{\beta}(\vec{\alpha}_{\beta}) < \bar{F}(\rho_{\beta}, \xi, \gamma_0)$ (or $< \bar{F}(\rho_{\beta}, 0, \gamma_0)$ if $\xi = \gamma_1$). By assumption $\mu \cap \bar{x} = \mu \cap x$, and since $\nu < \mu$ and x is $< \mu$ closed, there is $\rho \in x$ such that for every $\beta < \nu$, $a_{\beta} < \bar{F}(\rho, \xi, \gamma_0)$ (or $< \bar{F}(\rho, 0, \gamma_0)$). But since $\xi > \gamma_0 > \mu$ and is regular, $\bar{F}(\rho, \xi, \gamma_0) < \xi$ (or $\bar{F}(\rho, 0, \gamma_0) < \gamma_1$) in contradiction to the minimality of ξ .

Lemma 7 (Variation of [2, Lemma 1.7]). Let θ be a regular cardinal > μ . Let \mathfrak{A} be a Skolemized algebra on $H(\theta)$ expanding $\langle H(\theta), \in, \Delta \rangle$. Let $\langle x_{\alpha} \mid \alpha < \eta \rangle$ be a continuous increasing sequence of $< \mu$ -closed substructures of $\overline{\mathfrak{A}}$ of cardinality $\geq \mu$ and let $\langle \gamma_{\alpha} \mid \alpha < \eta \rangle \subseteq x_0 \setminus \mu$ be an increasing sequence of cardinals closed in $\gamma = \sup \langle \gamma_{\alpha} \mid \alpha < \eta \rangle$. Suppose that for every $\alpha < \eta$,

(1) $x_{\alpha+1}$ is a $\gamma_{\alpha+1}$ -end-extension of x_{α} i.e $x_{\alpha+1} \cap \sup(x_{\alpha} \cap \gamma_{\alpha+1}) = x_{\alpha} \cap \gamma_{\alpha+1}$,

(2) x_{α} is the substructure of \mathfrak{A} generated by $x_{\alpha} \cap \gamma_{\alpha}$.

Then $z = \bigcup_{\alpha < n} x_{\alpha}$ is $< \mu$ -closed.

Proof. Assume towards contradiction that z is not $< \mu$ closed and let $\langle \beta_{\alpha} \mid \alpha < \nu \rangle$ witness this. Note that there is no γ such that x_{γ} contains unboundedly many β_{α} s, since x_{γ} is $< \mu$ closed, so we can find a strictly increasing subsequence $\langle \beta_{\alpha(\gamma)} \mid \gamma < \eta \rangle$, hence cf $(\nu) = cf(\eta)$. So without loss of generalization we just assume that in fact $\nu = \eta$, and is regular and $< \mu$. Note that this implies $\gamma \in x_0$ as x_0 is $< \mu$ -closed.

We claim that there must be some $\xi \in z$ such that $z \cap \xi$ is not $< \mu$ closed. If $i \in z \cap \theta$, then there is some $\alpha < \eta$, an $\overline{\mathfrak{A}}$ term τ and $\overline{\alpha} \in x_{\alpha} \cap \gamma_{\alpha}$ such that $i = \tau(\overline{\alpha}) \in x_{\alpha}$. This means that for some $\rho \in x_{\alpha} \cap \mu$, $i < \overline{F}(\rho, 0, \gamma)$. But since $\rho < \gamma_0 < \gamma_{\alpha}$ and x_{α} is in particular a γ_0 -end extension of $x_0, \rho \in x_0$. So in fact $i < \sup(x_0 \cap \theta)$. So we've shown that $\sup(z \cap \theta) = \sup(x_0 \cap \theta)$. x_0 is $< \mu$ closed so cf $(\sup(x_0 \cap \theta)) \ge \mu$, so every sequence in $z \cap \theta$ of cofinality $< \mu$ must be bounded by some $\xi \in z \cap \theta$, and in particular any sequence witnessing that $z \cap \theta$ is not $< \mu$ closed, will be bounded by some $\xi \in z$.

Let ξ be the minimal ordinal in z such that $z \cap \xi$ is not $< \mu$ closed. Note that $z \prec \langle H(\theta), \in, \Delta \rangle$ so as in the previous lemma ξ must be regular. In $z \cap \xi$ we have a sequence $\langle \beta_{\alpha} \mid \alpha < \eta \rangle$ such that it's supremum is not in $z \cap \xi$. By minimality this sequence is cofinal in $z \cap \xi$. But if $\xi \in x_{\alpha}$, then as before for every $i \in z \cap \xi$ there is $\rho \in x_{\alpha}$ such that $i < F(\rho, \xi, \gamma)$, so $\sup(z \cap \xi) = \sup(x_{\alpha} \cap \xi)$ and since x_{α} is $< \mu$ -closed this is of cofinality $\geq \mu$, by contradiction.

We are now ready to prove our theorem.

Proof of theorem 4. (1) This is a standard consequence of presaturation, see e.g. [1, section 9].

(2) As we noted earlier every ordinal $\gamma < \delta$ is represented by the function $Z \mapsto$ otp $(Z \cap \gamma)$. By concentrating on stationary sets of subsets of size $< \lambda$, otp $(Z \cap \gamma) < \lambda$, so $\gamma < j(\lambda)$ for every $\gamma < \delta$, hence $\delta \leq j(\lambda)$, so in particular crit $(j) \leq \lambda$.

Let $\eta < \lambda$. Let $b \in \mathbb{P}$. There is $a \in \mathbb{P}$, $a \leq b$, such that $\eta \in \cup a$. The set of $Z \in \mathcal{P}_{\lambda}(\cup a)$ such that $\eta \in Z$ is club, so we may assume $\forall Z \in a, \eta \in Z$. By the definition of \mathbb{P} , $Z \cap \lambda \in \lambda$ so in fact $\eta \in Z$ implies $Z \cap \eta = \eta$. So the set of d such that $\forall Z \in d Z \cap \eta = \eta$ is dense in \mathbb{P} . Since η is represented by $\operatorname{otp}(Z \cap \eta)$, every such d forces $\eta = j(\eta)$, so by genericity, this is the case. Thus λ must be the critical point.

Assume towards contradiction that $j(\lambda) > \delta$. This means that δ is represented by a function f with domain $a \in V_{\delta}$ such that $[f] < [c_{\lambda}]$, so we can assume $f : a \to \lambda$. We

can also assume $\cup a = V_{\eta}$ for some strongly inaccessible $\eta < \delta$. Since every $\gamma < \delta$ is represented by the function $g_{\gamma} : Z \mapsto \operatorname{otp}(Z \cap \gamma)$, and $[f] > [g_{\gamma}]$, we know that for every $\gamma < \delta$ there is some $a_{\gamma} \leq a$ such that for every $Z \in a_{\gamma}$, $f(Z \cap V_{\eta}) > \operatorname{otp}(Z \cap \gamma)$. We want to get a contradiction by finding some $a' \in \mathbb{P}$, $a' \leq a$ such that for some γ and every $Z \in a'$, $f(Z \cap V_{\eta}) \leq \operatorname{otp}(Z \cap \gamma)$.

Let $\rho \in (\eta, \delta)$ be a measurable cardinal (exists from Woodinness) and fix a measure U on ρ . Now we use lemmas 6 and 7: Choose some strongly inaccessible $\theta \in (\rho, \delta)$ and let $\overline{\mathfrak{A}}$ be a Skolemized expansion of $\langle H(\theta), \in, \Delta, U, \{\rho\}, \{\mu\} \rangle$. Let $Z \prec \overline{\mathfrak{A}}$ with $|Z| < \lambda$, $Z \cap \lambda \in \lambda$ and consider $A_Z = \bigcap (U \cap Z)$. Since $|Z| < \lambda$, $A_Z \in U$, although Z doesn't know this fact, and in fact $A_Z \cap Z = \emptyset$ since Z satisfies that U is non-principal, i.e. for every $\zeta \in Z$, $Z \models \exists A \in U \ (\zeta \notin A)$ so $\zeta \notin A_Z$. Now fix some $\zeta \in A_Z$ and some function G_Z from $Z \cap \lambda$ onto Z, and let \overline{Z} be the substructure of $\overline{\langle \mathfrak{A}, \{\zeta\}, G_Z \rangle}$ generated by Z. Let $\rho_Z = \sup(Z \cap \rho)$.

Claim 8. $\overline{Z} \cap \rho_Z = Z \cap \rho_Z$.

Proof. Let t be some Skolem term, $p \in Z^{<\omega}$ such that $\tau = t(p, \zeta) \in \overline{Z} \cap \rho_Z$. Consider the function $\xi \mapsto t(p, \xi)$ for $\xi < \rho$. We have two cases

- *Case* 1. The function is constant on some $A \in U$. So $H(\theta) \models \exists A \in U \forall \xi, \xi' \in A(t(p,\xi) = t(p,\xi'))$. $Z \prec H(\theta)$ so there is such $A \in Z$. But then by the definition of $A_Z, \zeta \in A$, so for some (any) $\xi \in A \cap Z, t(p,\xi) = t(p,\zeta) = \tau$ so $\tau \in Z$.
- *Case* 2. There is $\sigma < \rho$ such that $\{\xi < \rho \mid t(p,\xi) \le \sigma\} \in U$. Then the function is regressive on the interval $(\sigma, \rho) \in U$ so by normality it is constant on some set in U, so we are back to the previous case.
- *Case* 3. For every $\sigma < \rho$ we'd get $\{\xi < \rho \mid t(p,\xi) > \sigma\} \in U$. So this holds also in Z. In particular, if $\sigma = \min(Z \smallsetminus \tau)$, we have $\{\xi < \rho \mid t(p,\xi) > \sigma\} \in U \cap Z$ so $\zeta \in \{\xi < \rho \mid t(p,\xi) > \sigma\}$, i.e. $t(\rho,\zeta) = \tau > \sigma$, by contradiction to the choice of σ .

So the first case must hold, and we have $\tau \in Z$ as required.

Let $Z_0 = Z$, and for every $\alpha \leq f(Z \cap V_\eta)$, if Z_α is defined and is, by induction, $< \mu$ -closed, denote $\rho_\alpha = Z_\alpha \cap \rho$, $Z'_\alpha = Z_\alpha \cap H(\rho_\alpha)$, $A_{Z_\alpha} = \bigcap (U \cap Z_\alpha)$, choose a function $G_{Z_\alpha} : Z_\alpha \cap \lambda \twoheadrightarrow Z_\alpha$, fix some $\zeta_\alpha \in A_{Z_\alpha}$ of cardinality $> \rho_\alpha$, and let $Z_{\alpha+1}$ be the substructure of $\overline{\langle \mathfrak{A}, \{\zeta_\alpha\}, G_{Z_\alpha} \rangle}$ generated by Z'_α . By the claim we have $Z_{\alpha+1} \cap \rho_\alpha = Z_\alpha \cap \rho_\alpha$, in particular $Z_{\alpha+1} \cap H(\rho_\alpha) = Z_\alpha \cap H(\rho_\alpha)$, so by lemma 6 $Z_{\alpha+1}$ is $< \mu$ closed. Note that by including the predicate G_{Z_α} we have $Z_\alpha \subseteq Z_{\alpha+1}$. At limit stages we use lemma 7 to get that $Z_\alpha = \bigcup_{\beta < \alpha} Z_\beta$ is $< \mu$ closed. Note that at every stage we have that $Z_\alpha \cap V_\eta = Z \cap V_\eta$. So we get, after $f(Z \cap V_\eta)$ stages, a set $Z_* = Z_{f(Z \cap V_\eta)}, Z_* \prec \overline{\mathfrak{A}}$, which is $< \mu$ closed, $Z_* \cap V_\eta = Z \cap V_\eta \in a$, and $\operatorname{otp}(Z_* \cap \rho) \ge f(Z_* \cap V_\eta) = f(Z \cap V_\eta)$.

To conclude, what we have shown is that the set of all $Z \prec H(\theta)$ which are $\langle \mu$ closed, $|Z| < \lambda$, $Z \cap (\bigcup a) \in a$ and $\operatorname{otp}(Z \cap \rho) \ge f(Z \cap (\cup a))$ is stationary, which is a contradiction. So $j(\lambda) = \delta$.

For (3), first note that since ${}^{<\delta}M \cap V[G] \subseteq M$, if $\operatorname{cf}^{V[G]}(\theta) < \delta$ or $\operatorname{cf}^{M}(\theta) < \delta$ then M and V[G] agree on this cofinality.

(a) If $\theta < \lambda$ is regular in V, then since $\lambda = \operatorname{crit}(j)$, $\theta = j(\theta)$ remains regular in M, and is $< \delta$, so it is also regular in V[G]. So regular cardinals $< \lambda$ from V are preserved in M and V[G], hence all cofinalities $< \lambda$ are preserved.

(b) Assume V[G], $M \models cf(\theta) < \mu$ ($\mu < \delta$ so if one satisfies this, so does the other). The size of the forcing is $\delta > \mu$ and δ remains regular (since $j(\lambda) = \delta$ is regular in M, and if it were singular in V[G], then by closure of M under $< \delta$ -sequences it would have been singular in M as well) so we must have $V \models cf(\theta) < \delta$. By (a) if $\mu \le cf^V(\theta) < \lambda$ then it is preserved, so we may assume (towards contradiction) that $\lambda \le cf^V(\theta) < \delta$, and we can also assume $cf^V(\theta) = \theta$. Let $a \Vdash M \models cf(\theta) = \beta < \mu$. This means that there is a function f with domain a such that $a \Vdash "[f]$ is a sequence of length β cofinal in θ ." $\beta < crit(j)$ and θ is represented by $otp(\cdot \cap \theta)$ so without loss of generalization, for every $Z \in a f(Z)$ is a sequence of length β cofinal in $otp(Z \cap \theta)$. But $Z \cap \theta$ is $< \mu$ -closed and $\beta < \mu \le \theta$ so it cannot have a cofinal sequence of length β , a contradiction.

(c) $V \models cf(\alpha) < \delta$ and $M \models cf(\alpha) < \delta$ both imply $V[G] \models cf(\alpha) < \delta$ since a cofinal sequence from V or M will remain so in V[G]. $V[G] \models cf(\alpha) < \delta$ implies $M \models cf(\alpha) < \delta$ since ${}^{<\delta}M \cap V[G] \subseteq M$. Assume $V[G] \models cf(\alpha) < \delta$. $|\mathbb{P}| = \delta$, so it preserves cofinalities $> \delta$, so we must have $V \models cf(\alpha) \leq \delta$. If $V \models cf(\alpha) = \delta$ while $V[G] \models cf(\alpha) < \delta$, it means that δ is not regular in V[G] – in V there is a δ sequence cofinal in α , and in V[G] there is also a shorter sequence cofinal in α , which can be pulled back to a cofinal sequence in δ . But since ${}^{<\delta}M \cap V[G] \subseteq M$ this means that δ is not regular in M as well, contradicting $j(\lambda) = \delta$ for λ regular and j elementary. So we must have $V \models cf(\alpha) < \delta$.

3. The main theorem

Theorem 9. Let $\lambda_1 < \cdots < \lambda_n < \delta_1 < \cdots < \delta_n$ such that the λ_i s are regular uncountable cardinals and the δ_i s are Woodin cardinals. Then $C^*_{<\lambda_1,\ldots,<\lambda_n} \equiv C^*_{<\delta_1,\ldots,<\delta_n}$.

Proof. Let $M_0 = V$ and take some δ_0 regular $\lambda_n < \delta_0 < \delta_1$. We define by induction generic elementary embeddings $j_i : M_{i-1} \to M_i$ for i = 1, ..., n. Given $j_1, ..., j_{i-1}$, set $\overline{j}_{i-1} = j_{i-1} \circ \cdots \circ j_1$ (if i = 1 $\overline{j}_0 = \text{Id}$) and $\overline{\lambda}_l = \overline{j}_{i-1} (\lambda_l)$ for $l \leq i$ and we inductively require that

- (1) crit $(j_i) = \overline{\lambda}_i$ and $j_i(\overline{\lambda}_i) = \delta_i$;
- (2) For k > i, $j_i(\delta_k) = \delta_k$;
- (3) For every l = 1, ..., n and every $\alpha, V \vDash cf(\alpha) < \delta_l \iff M_i \vDash cf(\alpha) < \delta_l;$

(4) $M_i \vDash \delta_{i+1}, \ldots, \delta_n$ are Woodin.

Let $\mathbb{P}_i = \mathbb{P}\left(\delta_i, \bar{\lambda}_i, < \delta_{i-1}^+ \text{-closed}\right)^{M_{i-1}}$ (the stationary tower on δ_i consisting of stationary sets with $< \delta_{i-1}^+ \text{-closed}$ elements of size $\bar{\lambda}_i$, as computed in M_{i-1}), $G_i \subseteq \mathbb{P}_i$ an M_{i-1} -generic filter and $j_i : M_{i-1} \to M_i$ the associated embedding, i.e we use the previous theorem with $V = M_{i-1}, \mu = \delta_{i-1}^+$ and $\lambda = \bar{\lambda}_i$. By elementarity of $\bar{j}_{i-1}, \bar{\lambda}_i$ is a regular uncountable cardinal in M_{i-1} so (1) follows from theorem 4.2, (2) follows from lemma 1.1 and (4) is a standard fact regarding stationary towers.

For (3), first for l < i by the induction hypothesis we have for every α , $V \models cf(\alpha) < \delta_l \iff M_{i-1} \models cf(\alpha) < \delta_l$. By the assumptions,

$$\delta_l = j_l(\bar{\lambda}_l) = j_l \circ j_{l-1} \circ \cdots \circ j_1(\lambda_l) = \bar{j}_l(\lambda_l) < \bar{j}_l(\lambda_i) \le \bar{j}_{i-1}(\lambda_i) = \bar{\lambda}_i.$$

By theorem 4.3(a), all cofinalities $\langle \overline{\lambda}_i | are preserved from M_{i-1} | to M_i$, so in particular we have $V \models cf(\alpha) \langle \delta_l \Rightarrow M_i \models cf(\alpha) \langle \delta_l$. If $M_i \models cf(\alpha) \langle \delta_l$, then in particular $M_i \models cf(\alpha) \langle \delta_{i-1}^+ \rangle$, so by theorem 4.3(b) also $M_{i-1} \models cf(\alpha) \langle \delta_{i-1}^+ \rangle \leq \overline{\lambda}_i$, so again by 4.3(a) this cofinality is preserved from M_{i-1} to M_i hence $M_{i-1} \models cf(\alpha) \langle \delta_l$ and by the induction hypothesis we get $V \models cf(\alpha) \langle \delta_l$. The case l > i is an application of lemma 1.2 to the induction hypothesis. For l = i, by the induction hypothesis we have $V \models$ $cf(\alpha) \langle \delta_i \iff M_{i-1} \models cf(\alpha) \langle \delta_i$, and by theorem 4.3(c) we get $M_{i-1} \models cf(\alpha) \langle \delta_i$ $\iff M_i \models cf(\alpha) \langle \delta_i$.

Now we look at $j = j_n \circ \cdots \circ j_1 : V \to M_n$. By the construction we get that for every $i = 1, \ldots, n$,

$$j(\lambda_i) = j_n \circ \cdots \circ j_{i+1} \circ j_i \circ \cdots \circ j_1(\lambda_i)$$
$$= j_n \circ \cdots \circ j_{i+1} \circ j_i(\bar{\lambda}_i)$$
$$= j_n \circ \cdots \circ j_{i+1}(\delta_i) = \delta_i$$

so if we restrict j to $\left(C^*_{<\lambda_1,\ldots,<\lambda_n}\right)^V$ we get an elementary embedding

$$\bar{j}: \left(C^*_{<\lambda_1,\dots,<\lambda_n}\right)^V \to \left(C^*_{< j(\lambda_1),\dots,< j(\lambda_n)}\right)^{M_n} = \left(C^*_{<\delta_1,\dots,<\delta_n}\right)^{M_n}$$

From the last step in the induction we obtain that for every i = 1, ..., n and ordinal α , $V \models \operatorname{cf}(\alpha) < \delta_l \iff M_n \models \operatorname{cf}(\alpha) < \delta_l$, and since both V and M_n are contained in $V[G_1, ..., G_n]$ – the least model containing V and $G_1, ..., G_n$ – in this model we get that the construction of $\left(C^*_{<\delta_1,...,<\delta_n}\right)^{M_n}$ and $\left(C^*_{<\delta_1,...,<\delta_n}\right)^V$ will yield the same results at each step, so indeed

$$\left(C^*_{<\lambda_1,\ldots,<\lambda_n}\right)^V \equiv \left(C^*_{<\delta_1,\ldots,<\delta_n}\right)^{M_n} = \left(C^*_{<\delta_1,\ldots,<\delta_n}\right)^V.$$

Corollary 10. If there is a proper class of Woodin cardinals, and $\lambda_1 < \cdots < \lambda_n$ are regular uncountable cardinals, then the theory of $C^*_{<\lambda_1,\ldots,<\lambda_n}$ is set-forcing absolute and does not

depend on $\lambda_1, \ldots, \lambda_n$, in the following sense: if \mathbb{P} is a set-forcing, $G \subseteq \mathbb{P}$ generic and $\lambda'_1 < \cdots < \lambda'_n$ are in V[G] regular cardinals, then $\left(C^*_{<\lambda_1,\ldots,<\lambda_n}\right)^V \equiv \left(C^*_{<\lambda'_1,\ldots,<\lambda'_n}\right)^{V[G]}$. Proof. Let \mathbb{P} be some forcing notion, and $\delta_n > \cdots > \delta_1 > \max(|\mathbb{P}|, \lambda_n, \lambda'_n)$ Woodin

cardinals. Note that after forcing with a generic $G \subseteq \mathbb{P}$, they remain Woodin. So we can apply the theorem both in V and V[G] to obtain

$$\left(C^*_{<\lambda_1,\ldots,<\lambda_n}\right)^V \equiv \left(C^*_{<\delta_1,\ldots,<\delta_n}\right)^V$$

$$\left(C^*_{<\lambda'_1,\ldots,\lambda'_n}\right)^{V[G]} \equiv \left(C^*_{<\delta_1,\ldots,<\delta_n}\right)^{V[G]}$$

but being of cofinality $< \delta_i$ is not affected by a forcing of size $< \delta_1$, so $\left(C^*_{<\delta_1,\ldots,<\delta_n}\right)^V = \left(C^*_{<\delta_1,\ldots,<\delta_n}\right)^{V[G]}$, and we get $\left(C^*_{<\lambda_1,\ldots,<\lambda_n}\right)^V \equiv \left(C^*_{<\lambda'_1,\ldots,<\lambda'_n}\right)^{V[G]}$. *Remark* 11. (1) Note that cf $(\alpha) = \omega \iff cf(\alpha) < \omega_1$ so for $\lambda_1 = \omega_1$ we get the

Remark 11. (1) Note that $cf(\alpha) = \omega \iff cf(\alpha) < \omega_1$ so for $\lambda_1 = \omega_1$ we get the original result for C^*_{ω} .

(2) For any regular λ, cf (α) = λ ⇔ (cf (α) < λ⁺ ∧ ¬cf (α) < λ). For any regular λ₁ < λ₂, denote by C^{*}_[λ1,λ2] the model constructed with the logic obtained from first order logic by adding the quantifier Q^{cf}_[λ1,λ2](α) ⇔ cf (α) ∈ [λ₁, λ₂). We can write C^{*}_λ as C^{*}_{[λ,λ⁺)}. Now, if λ is regular uncountable, the proof of the main theorem using λ, λ⁺ and Woodins δ₁ < δ₂, also gives us a generic elementary embedding

$$\overline{j}: \left(C^*_{[\lambda,\lambda^+)}\right)^V \to \left(C^*_{[\delta_1,\delta_2)}\right)^V$$

so the proof of the corollary can be applied to obtain the absoluteness of $Th(C^*_{\lambda})$. (3) In general, we get absoluteness of $Th(C^*_{\lambda_1,\dots,\lambda_n})$ for any regular cardinals λ_i .

4. Open questions

A natural question at this stage would be whether our results are true also for infinitely many cofinality quantifiers. However, the naïve approach of taking a direct limit of the above construction and trying to prove similar results does not work, as cofinalities are not preserved to the limit stage. Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals and $\langle \delta_n \mid n < \omega \rangle$ an increasing sequence of Woodin cardinals such that $\delta_0 > \lambda_* := \sup \langle \lambda_n \mid n < \omega \rangle$. For every $i < \omega$ define by induction generic elementary embeddings $j_{i,i+1} : M_i \to M_{i+1} \subseteq V[G_1, \ldots, G_{i+1}]$ $(M_0 = V)$ as in the proof of the main theorem (setting $j_{0,0} = \operatorname{Id}, j_{0,n+1} = j_{n,n+1} \circ j_{0,n}, G_{i+1}$ is M_i -generic for the poset $\mathbb{P}\left(\delta_i, j_{0,i}(\lambda_i), < \delta^+_{i-1}\text{-closed}\right)^{M_i}$. We get that $j_{0,n+1}(\lambda_n) = \delta_n$, for every $m \neq n+1$ $j_{0,m}(\delta_n) = \delta_n$, and for every $k < \omega$ and $\eta \in \operatorname{Ord}, V \models \operatorname{cf}(\eta) < \delta_k \iff M_n \models \operatorname{cf}(\eta) <$ δ_k . We can now take the direct limit and get embeddings $j_{n,\omega} : M_n \to M_\omega$ which are defined in the finite support iteration of the forcings which we denote by $V[\mathcal{G}]$. So we get that for every $n j_{0,\omega}(\lambda_n) = \delta_n$ and for every $m > n + 1 j_{m,\omega}(\delta_n) = \delta_n$. Denote for every $n \leq \omega \ \theta_n := j_{0,n}((\lambda_*^+)^V)$, and consider θ_{ω} . First note that

$$j_{0,\omega}(\lambda_*) = \sup j_{0,\omega}\left(\langle \lambda_i \mid i < \omega \rangle\right) = \sup \langle j_{0,\omega}\left(\lambda_i\right) \mid i < \omega \rangle = \sup \langle \delta_i \mid i < \omega \rangle$$

so if we denote this by δ_* we get by elementarity that $\theta_{\omega} = j_{0,\omega}(\lambda_*^+) = (\delta_*^+)^{M_{\omega}}$. So in particular $M_{\omega} \models \operatorname{cf}(\theta_{\omega}) = \theta_{\omega} > \delta_*$, or equivalently (since $\delta_* = \sup \langle \delta_n | n < \omega \rangle$) for every $n \ M_{\omega} \models \operatorname{cf}(\theta_{\omega}) > \delta_n$. If this were to hold also in V, then it would hold also in $V[\mathcal{G}]$, since it is a forcing extension with a forcing of size at most δ_* . But in fact $V[\mathcal{G}] \models \operatorname{cf}(\theta_{\omega}) = \omega: \theta_{\omega} = j_{n,\omega}(\theta_n)$ for every n, and by definition of the direct limit, every $\eta < \theta_{\omega}$ is of the form $j_{n,\omega}(\bar{\eta})$ for some $n < \omega$ and $\bar{\eta} < \theta_n$, so the sequence $\langle \sup j_{n,\omega} [\theta_n] | n < \omega \rangle$ is cofinal in θ_{ω} . We claim that this sequence is strictly increasing. For every n,

$$\theta_n = j_{0,n}(\lambda_*^+) < j_{0,n}(\delta_n) = \delta_n.$$

 M_{n+1} is closed under $\langle \delta_n$ sequences in $M_n[G_n]$, so $j_{n,n+1}[\theta_n] \in M_{n+1}$ and is of ordertype $\theta_n \langle \delta_n$, while $\theta_{n+1} = j_{n,n+1}(\theta_n) \rangle j_{n,n+1}(\lambda_n) = \delta_n$, so $j_{n,n+1}[\theta_n]$ is bounded below θ_{n+1} so also $\sup j_{n,\omega}[\theta_n]$ is below $\sup j_{n+1,\omega}[\theta_{n+1}]$. Hence in $V[\mathcal{G}]$ $\operatorname{cf}(\theta_{\omega}) = \omega$.

So to conclude, the question of absoluteness of the theory of the model constructed with infinitely many cofinality quantifers remains open.

References

- 1. Matthew Foreman, *Ideals and generic elementary embeddings*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), Springer Netherlands, Dordrecht, 2010, pp. 885–1147.
- 2. Matthew Foreman and Menachem Magidor, *Large cardinals and definable counterexamples to the continuum hypothesis*, Annals of Pure and Applied Logic **76** (1995), no. 1, 47–97.
- Juliette Kennedy, Menachem Magidor, and Jouko Väänänen, Inner models from extended logics: Part 1, arXiv preprint (2020).
- 4. Paul B. Larson, *The stationary tower: Notes on a course by w. hugh woodin*, University lecture series, American Mathematical Society, 2004.

EINSTEIN INSTITUTE OF MATHEMATICS Hebrew University of Jerusalem Edmond J. Safra Campus, Givat Ram Jerusalem 91904, ISRAEL

Email address: ur.yaar@mail.huji.ac.il