# Treewidth-aware Reductions of Normal ASP to SAT- Is Normal ASP Harder than SAT after All? 

Markus Hecher<br>TU Wien, Vienna, Austria<br>hecher@dbai.tuwien.ac.at

October 10, 2022


#### Abstract

Answer Set Programming (ASP) is a paradigm for modeling and solving problems for knowledge representation and reasoning. There are plenty of results dedicated to studying the hardness of (fragments of) ASP. So far, these studies resulted in characterizations in terms of computational complexity as well as in fine-grained insights presented in form of dichotomy-style results, lower bounds when translating to other formalisms like propositional satisfiability (SAT), and even detailed parameterized complexity landscapes. A generic parameter in parameterized complexity originating from graph theory is the so-called treewidth, which in a sense captures structural density of a program. Recently, there was an increase in the number of treewidth-based solvers related to SAT. While there are translations from (normal) ASP to SAT, no reduction that preserves treewidth or at least keeps track of the treewidth increase is known. In this paper we propose a novel reduction from normal ASP to SAT that is aware of the treewidth, and guarantees that a slight increase of treewidth is indeed sufficient. Further, we show a new result establishing that, when considering treewidth, already the fragment of normal ASP is slightly harder than SAT (under reasonable assumptions in computational complexity). This also confirms that our reduction probably cannot be significantly improved and that the slight increase of treewidth is unavoidable. Finally, we present an empirical study of our novel reduction from normal ASP to SAT, where we compare treewidth upper bounds that are obtained via known decomposition heuristics. Overall, our reduction works better with these heuristics than existing translations.


## 1 Introduction

Answer Set Programming (ASP) [20] is an active research area of knowledge representation and reasoning. ASP provides a declarative modeling language
and problem solving framework [54 for hard computational problems, which has been widely applied [7, 81, 82, 59, 86, 1]. There are very efficient ASP solvers [54, 3, 21] as well as several recent (language) extensions [21, 22, 24, 23]. In ASP, questions are encoded into rules and constraints that form a program (over atoms), whose solutions are called answer sets.

In terms of computational complexity, the consistency problem of deciding the existence of an answer set is well-studied, i.e., the problem is $\Sigma_{2}^{P}$ complete [35]. Some fragments of ASP have lower complexity though. A prominent example is the class of head-cycle-free (HCF) programs [9, which is a generalization of the class of normal programs and requires the absence of cycles in a dependency graph representation of the program. Deciding whether such a program has an answer set is NP-complete.

There is also a wide range of more fine-grained studies 88, for ASP, also in parameterized complexity [27, 80, 31, 52], where certain (combinations of) parameters [50, 71] are taken into account. In parameterized complexity, the "hardness" of a problem is classified according to the impact of a parameter for solving the problem. There, one often distinguishes the runtime dependency of the parameter, e.g., levels of exponentiality [76, [79] in the parameter, required for problem solving. Concretely, under the reasonable Exponential Time Hypothesis (ETH) [63], propositional satisfiability (SAT) is single exponential in the structural parameter treewidth, whereas evaluating Quantified Boolean formulas (QBFs) of quantifier depth two is [72] double exponential ${ }^{1}$ in the treewidth $k$.

For ASP there is growing research on treewidth [64, 44, 41, which even involves grounding [11, 14]. Algorithms of these works exploit structural restrictions (in form of treewidth) of a given program, and often run in polynomial time in the program size, while being exponential only in the treewidth. Intuitively, treewidth gives rise to a tree decomposition, which allows solving numerous NP-hard problems in parts (via dynamic programming) and indicates the maximum number of variables one has to investigate in such parts during evaluation. There were also dedicated competitions [28] and notable progresses in SAT 49, 25] and other areas [8.

Naturally, there are numerous reductions of ASP [26, 9, 74, 65, 4] and extensions thereof [19, 17] to SAT. These reductions have been investigated in the context of resulting formula size and number of auxiliary variables. However, structural dependency in form of, e.g., treewidth, has not been considered yet. Notably, there are existing reductions causing a sub-quadratic blow-up in the number of variables (auxiliary variables), which is unavoidable [73] if the answer sets should be preserved (bijectively). However, if one considers the structural dependency in form of treewidth, existing reductions could cause quadratic or even unbounded overhead in the treewidth.

On the contrary, we present a novel reduction for HCF programs that increases the treewidth $k$ at most sub-quadratically $(k \cdot \log (k))$. This is indeed interesting as there is a close connection [6] between resolution-width and treewidth,

[^0]resulting in efficient SAT solver runs on instances of small treewidth. As a result, our reduction could be of use for solving approaches based on SAT solvers [75, 65]. Then, we establish lower bounds, under ETH, for exploiting treewidth for consistency of normal programs. This renders normal ASP "harder" than SAT. At the same time we prove that one cannot significantly improve the reduction, i.e., avoid the sub-quadratic increase of treewidth.
Contributions. Concretely, we provide the following.

1. First, we present a novel reduction from HCF programs to SAT, which only requires linearly many auxiliary variables plus a number of auxiliary variables that is linear in the instance size and slightly superexponential in the treewidth of the SAT instance. This is achieved by guiding the whole reduction along a tree decomposition of the program. Thereby the reduction only increases the treewidth sub-quadratically, i.e., the treewidth of the resulting SAT formula is slightly larger than the treewidth of the given program.
2. Then, we develop and discuss a slightly different reduction from HCF programs to SAT, where we show a bijective correspondence between answer sets of the program and models of the propositional formula for a certain sub-class of programs. This reduction, while preserving bijectivity for certain programs, comes at the cost of a quadratic increase of treewidth.
3. We show that avoiding a sub-quadratic increase in the treewidth is very unlikely. Concretely, we establish that under the widely believed Exponential Time Hypothesis (ETH), one cannot decide ASP in time $2^{o(k \cdot \log (k))} \cdot n$, with treewidth $k$ and program size $n$. This is in contrast to the runtime for deciding SAT: $2^{\mathcal{O}(k)} \cdot n$ with treewidth $k$ and size $n$ of the formula. As a result, this establishes that the consistency of normal ASP programs is already harder than SAT using treewidth. Note that this is surprising as both problems are of similar hardness according to classical complexity (NP-complete).
4. Finally, we present an empirical study of the first contribution, where we compare treewidth upper bounds that are obtained via existing decomposition heuristics. Interestingly, compared with existing translations, in both acyclic and cyclic scenarios, our reduction overall works better with these heuristics than existing translations.

This is an extended version of a paper [60] at the 17th International Conference on Principles of Knowledge Representation and Reasoning (KR 2020). In addition to the conference version, this work contains additional examples and more detailed explanations. Further, we added the second contribution and worked out details in terms of bijective preservation of answer sets. We were also able to simplify the reduction of the third contribution and discuss relations to other works in detail. Further, we added an empirical study of treewidth, where we
compare treewidth upper bounds of our reduction and an exisiting translation via an efficient decomposer based on heuristics.
Related Work. For disjunctive ASP and extensions thereof, algorithms have been proposed 64, 83, 44 running in time linear in the instance size, but double exponential in the treewidth. Under ETH, one cannot significantly improve this runtime, using a result 72 for QBFs with quantifier depth two and a standard reduction 35 from this QBF fragment to disjunctive ASP. Unsurprisingly, SAT only requires single exponential runtime [85] in the treewidth. However, for normal and HCF programs only a slightly superexponential algorithm 41 for solving consistency is known so far. Still, the question whether the slightly superexponentiality can be avoided was left open. The proposed algorithm was used for counting answer sets involving projection [57], which is at least double exponential 45] in the treewidth.
There are also further studies on certain classes of programs and their relationships in the form of whether there exist certain reductions between these classes, thereby bijectively preserving the answer sets. These studies result in an expressive power hierarchy among program classes 65. Note that while existing results [73] and the expressive power hierarchy weakly indicate that normal ASP might be slightly harder than SAT, these results mostly deal with bijectively preserving all answer sets. However, our work considers the plain consistency problem, where ASP and SAT are both NP-complete.

## 2 Preliminaries

Before we discuss our reductions, we briefly recall some basics.
Answer Set Programming (ASP). We assume familiarity with propositional satisfiability (SAT) [13, 67, and follow standard definitions of propositional ASP 20, 66. Let $\ell, m, n$ be non-negative integers such that $\ell \leq m \leq n$, $a_{1}, \ldots, a_{n}$ be distinct propositional atoms. Moreover, we refer by literal to a propositional variable (atom) or the negation thereof. A program $\Pi$ is a set of rules of the form

$$
a_{1} \vee \cdots \vee a_{\ell} \leftarrow a_{\ell+1}, \ldots, a_{m}, \neg a_{m+1}, \ldots, \neg a_{n}
$$

For a rule $r$, we let $H_{r}:=\left\{a_{1}, \ldots, a_{\ell}\right\}, B_{r}^{+}:=\left\{a_{\ell+1}, \ldots, a_{m}\right\}$, and $B_{r}^{-}:=$ $\left\{a_{m+1}, \ldots, a_{n}\right\}$. We denote the sets of atoms occurring in a rule $r$ or in a program $\Pi$ by at $(r):=H_{r} \cup B_{r}^{+} \cup B_{r}^{-}$and at $(\Pi):=\bigcup_{r \in \Pi}$ at $(r)$. A rule $r$ is normal if $\left|H_{r}\right| \leq 1$ and unary if $\left|B_{r}^{+}\right| \leq 1$. Then, a program $\Pi$ is normal or unary if all its rules $r \in \Pi$ are normal or unary, respectively. The positive dependency digraph $D_{\Pi}$ of $\Pi$ is the directed graph defined on the set of atoms from $\bigcup_{r \in \Pi} H_{r} \cup B_{r}^{+}$, where for every rule $r \in \Pi$ two atoms $a \in B_{r}^{+}$and $b \in H_{r}$ are joined by an edge $(a, b)$. A head-cycle of $D_{\Pi}$ is an $\{a, b\}$-cycle ${ }^{2}$ for two distinct atoms $a, b \in H_{r}$ for some rule $r \in \Pi$. Program $\Pi$ is head-cycle-free

[^1]

Figure 1: Dependency graph $D_{\Pi}$ of $\Pi$ (cf., Example 1 .
$(H C F)$ if $D_{\Pi}$ contains no head-cycle 9 and we say $\Pi$ is tight if there is no directed cycle in $D_{\Pi}$. 38 .

An interpretation $I$ is a set of atoms. I satisfies a rule $r$ if $\left(H_{r} \cup B_{r}^{-}\right) \cap I \neq \emptyset$ or $B_{r}^{+} \backslash I \neq \emptyset . I$ is a model of $\Pi$ if it satisfies all rules of $\Pi$, in symbols $I \models \Pi$. For brevity, we view propositional formulas as sets of formulas (e.g., clauses) that need to be satisfied, and use the notion of interpretations, models, and satisfiability analogously. The Gelfond-Lifschitz (GL) reduct of $\Pi$ under $I$ is the program $\Pi^{I}$ obtained from $\Pi$ by first removing all rules $r$ with $B_{r}^{-} \cap I \neq \emptyset$ and then removing all $\neg z$ where $z \in B_{r}^{-}$from the remaining rules $r$ [58. $I$ is an answer set of a program $\Pi$ if $I$ is a minimal model of $\Pi^{I}$. The problem of deciding whether an ASP program has an answer set is called consistency, which is $\Sigma_{2}^{\mathrm{P}}$-complete [35]. If the input is restricted to normal programs, the complexity drops to NP-complete [12, 78. A head-cycle-free program $\Pi$ can be translated into a normal program in polynomial time [9. Further, the answer sets of a tight program can be represented by means of the models of a propositional formula, obtainable in linear time via, e.g., Clark's completion [26]. The following characterization of answer sets is often applied when considering normal programs [74]. For a given set $A \subseteq \operatorname{at}(\Pi)$ of atoms, a function $\varphi: A \rightarrow\{0, \ldots,|A|-1\}$ is an ordering over $A$. Let $I$ be a model of a normal program $\Pi$, and $\varphi$ be an ordering over $I$. We say a rule $r \in \Pi$ is suitable for proving $a \in I$ if (i) $a \in H_{r}$, (ii) $B_{r}^{+} \subseteq I$, (iii) $I \cap B_{r}^{-}=\emptyset$, as well as (iv) $\left.I \cap\left(H_{r} \backslash\{a\}\right]\right)=\emptyset$. An atom $a \in I$ is proven if there is a rule $r \in \Pi$ proving $a$, which is the case if $r$ is suitable for proving $a$ and $\varphi(b)<\varphi(a)$ for every $b \in B_{r}^{+}$. Then, $I$ is an answer set of $\Pi$ if (i) $I$ is a model of $\Pi$, and (ii) $I$ is proven, i.e., every $a \in I$ is proven. This characterization vacuously holds also for head-cycle-free programs 9, since in HCF programs, vaguely speaking, all but one atom of the head of any rule can be "shifted" to the negative body 30 .

Example 1. Consider the given program $\Pi:=\{\overbrace{a \vee b \leftarrow}^{r_{1}} ; \overbrace{c \vee e \leftarrow d}^{r_{2}} ; \overbrace{d \leftarrow b, \neg e}^{r_{3}}$; $\overbrace{e \leftarrow b, \neg d}^{r_{4}} ; \overbrace{b \leftarrow e, \neg d}^{r_{5}} ; \overbrace{d \leftarrow \neg b}^{r_{6}}\}$. Observe that $\Pi$ is not tight, since the dependency graph $D_{\Pi}$ of Figure 1 contains the cycle $b, d, e$. However, the program $\Pi$ is head-cycle-free since there is neither an $\{a, b\}$-cycle, nor a $\{c, e\}$-cycle in $D_{\Pi}$. Therefore, rule $r_{1}$ allows shifting [30] and actually corresponds to the two rules $a \leftarrow \neg b$ and $b \leftarrow \neg a$. Analogously, rule $r_{2}$ can be seen as the rules $c \leftarrow d$, $\neg e$ and $e \leftarrow d, \neg c$. Then, $I:=\{b, c, d\}$ is an answer set of $\Pi$, since $I \models \Pi$, and we can prove with ordering $\varphi:=\{b \mapsto 0, d \mapsto 1, c \mapsto 2\}$ atom $b$ by rule $r_{1}$, atom $d$ by rule $r_{3}$, and atom $c$ by rule $r_{2}$. Further answer sets of $\Pi$ are $\{b, e\},\{a, c, d\}$, and $\{a, d, e\}$.

The characterization above already fails for simple programs that are not $H C F$. Consider for example program $\Pi^{\prime}:=\{a \vee b \leftarrow ; a \leftarrow b ; b \leftarrow a\}$, which
has only one answer set $I^{\prime}=\{a, b\}$. However, $I^{\prime}$ cannot be proven. If the first rule $a \vee b \leftarrow$ shall prove $a$, we require $b \notin I^{\prime}$ (and vice versa). Then, the remaining two rules of $\Pi^{\prime}$ can only prove either $a$ or $b$, but fail to prove $I^{\prime}$, since both rules proving $I^{\prime}$ (together) prohibit every ordering due to the cyclic dependency.

Tree Decompositions (TDs). We assume familiarity with graph terminology, cf., [29]. A tree decomposition (TD) [84] of a given graph $G=(V, E)$ is a pair $\mathcal{T}=(T, \chi)$ where $T$ is a tree rooted at $\operatorname{root}(T)$ and $\chi$ assigns to each node $t$ of $T$ a set $\chi(t) \subseteq V$, called bag, such that (i) $V=\bigcup_{t \text { of } T} \chi(t)$, (ii) $E \subseteq\{\{u, v\} \mid t$ in $T,\{u, v\} \subseteq \chi(t)\}$, and (iii) for each $r, s, t$ of $T$, such that $s$ lies on the path from $r$ to $t$, we have $\chi(r) \cap \chi(t) \subseteq \chi(s)$. For every node $t$ of $T$, we denote by $\operatorname{chldr}(t)$ the set of child nodes of $t$ in $T$. The bags $\chi \leq t$ below $t$ consists of the union of all bags of nodes below $t$ in $T$, including $t$. We let $\operatorname{width}(\mathcal{T}):=\max _{t}$ of $T|\chi(t)|-1$. The treewidth $\operatorname{tw}(G)$ of $G$ is the minimum $\operatorname{width}(\mathcal{T})$ over all TDs $\mathcal{T}$ of $G$. For a graph $G$, one can compute a TD of $G$, whose width is at most $5 \cdot t w(G)$, in single exponential time [16] in the treewidth (5approximation). The so-called pathwidth of $G$ refers to the minimum width $(\mathcal{T})$ over all TDs $\mathcal{T}$ of $G$, whose trees are just paths. For a node $t$ of $T$, we say that type $(t)$ is leaf if $t$ has no children and $\chi(t)=\emptyset$; join if $t$ has children $t^{\prime}$ and $t^{\prime \prime}$ with $t^{\prime} \neq t^{\prime \prime}$ and $\chi(t)=\chi\left(t^{\prime}\right)=\chi\left(t^{\prime \prime}\right)$; int ("introduce") if $t$ has a single child $t^{\prime}$, $\chi\left(t^{\prime}\right) \subseteq \chi(t)$ and $|\chi(t)|=\left|\chi\left(t^{\prime}\right)\right|+1$; forget if $t$ has a single child $t^{\prime}, \chi\left(t^{\prime}\right) \supseteq \chi(t)$ and $\left|\chi\left(t^{\prime}\right)\right|=|\chi(t)|+1$. If for every node $t$ of $T$, type $(t) \in\{$ leaf, join, int, forget $\}$, the TD is called nice. A TD can be turned into a nice TD [68] [Lem. 13.1.3] without increasing the width in linear time.

Example 2. Figure 2 illustrates a graph $G$ and a $T D \mathcal{T}$ of $G$ of width 2, which is also the treewidth of $G$, since $G$ contains a complete graph on vertices e,b,d. In general, if a graph contains a complete graph among $k+1$ vertices, the treewith of the graph is at least $k$, cf., [68].

Dynamic Programming on TDs. Solvers based on dynamic programming ( $D P$ ) evaluate a given input instance $\mathcal{I}$ in parts along a TD of a graph representation $G$ of the instance. Thereby, for each node $t$ of the TD, intermediate results are stored in a table $\tau_{t}$. This is achieved by running a table algorithm, which is designed for $G$, and stores in $\tau_{t}$ results of problem parts of $\mathcal{I}$, thereby considering tables $\tau_{t^{\prime}}$ for child nodes $t^{\prime}$ of $t$. DP works for many problems as follows.

1. Construct a graph representation $G$ of $\mathcal{I}$.
2. Compute a $\operatorname{TD} \mathcal{T}=(T, \chi)$ of $G$, which is obtainable via heuristics, e.g., [2].
3. Traverse the nodes of $T$ in post-order (bottom-up tree traversal of $T$ ). At every node $t$ of $T$ during post-order traversal, execute a table algorithm that takes as input a bag $\chi(t)$, a certain bag instance $\mathcal{I}_{t}$ depending on the problem, as well as previously computed child tables of $t$. Then, the results of this execution is stored in table $\tau_{t}$.


Figure 2: Graph $G$ (left) and a tree decomposition $\mathcal{T}$ of $G$ (right).
4. Finally, interpret table $\tau_{n}$ for the root node $n$ of $T$ in order to output the solution to the problem for instance $\mathcal{I}$.

In order to use TDs for ASP, we need dedicated graph representations of programs [64. The primal grapt ${ }^{3} \mathcal{G}_{\Pi}$ of program $\Pi$ has the atoms of $\Pi$ as vertices and an edge $\{a, b\}$ if there exists a rule $r \in \Pi$ and $a, b \in$ at $(r)$. Let $\mathcal{T}=(T, \chi)$ be a TD of primal graph $\mathcal{G}_{\Pi}$ of a program $\Pi$, and let $t$ be a node of $T$. The bag program $\Pi_{t}$ contains rules entirely covered by the bag $\chi(t)$. Formally, $\Pi_{t}:=\{r \mid r \in \Pi$, at $(r) \subseteq \chi(t)\}$.

Example 3. Recall program $\Pi$ from Example 1. Observe that graph $G$ of Figure 2 is the primal graph of $\Pi$. Further, we have $\Pi_{t_{1}}=\left\{r_{2}\right\}, \Pi_{t_{2}}=\left\{r_{1}\right\}$, and $\Pi_{t_{3}}=\left\{r_{3}, r_{4}, r_{5}, r_{6}\right\}$. Note that in general a rule might appear in several bag programs.

Now, the missing ingredient for solving problems via dynamic programming along a given TD, is a suitable table algorithm. Such table algorithms have been already presented for SAT [85] and ASP [64, 44, 41]. We only briefly sketch the ideas of a table algorithm using the primal graph that computes models of a given program $\Pi$. Each table $\tau_{t}$ consists of rows storing interpretations over atoms in the bag $\chi(t)$. Then, the table $\tau_{t}$ for a leaf node $t$ consists of the empty interpretation. For a node $t$ with introduced variable $a \in \chi(t)$, we store in $\tau_{t}$ interpretations of the child table, but for each such interpretation we decide whether $a$ is in the interpretation or not, and ensure that $\Pi_{t}$ is satisfied. When an atom $b$ is forgotten in a forget node $t$, we store interpretations of the child table, but projected to $\chi(t)$. By the properties of a TD, it is then guaranteed that all rules containing $b$ have been processed so far. For a join node $t$, we store in $\tau_{t}$ interpretations that are in both child tables of $t$.

## 3 Treewidth-Aware Reductions to SAT

Having the basic concept of dynamic programming in mind, we use this idea to design a reduction of an HCF program $\Pi$ to a SAT formula $F$, which is treewidth-aware. The reduction is inspired by ideas of a DP algorithm for consistency of HCF programs 41] and the idea of level mappings 65]. Intuitively, global orderings can cause already huge blow-up in the treewidth, e.g., reductions, where all atoms are ordered at once, often cause long rules, whose number of atoms vastly exceeds treewidth. This is indeed not surprising, since

[^2]

Figure 3: High-level illustration of the treewidth-awareness of our reduction from ASP to SAT. We assume a given program $\Pi$ and a tree decomposition $\mathcal{T}=(T, \chi)$ of $G_{\Pi}$. Then, our reduction is constructed for each node $t$ of $T$ and it immediately gives rise to a tree decomposition $\mathcal{T}^{\prime}=\left(T, \chi^{\prime}\right)$ of the resulting SAT formula. Thereby, each resulting bag $\chi^{\prime}(t)$ functionally depends on $\chi(t)$ (as well as on child or parent bags).
for a given program with $n$ many atoms, we have that global level mappings use $\mathcal{O}(\log (n))$ additional auxiliary atoms. Consequently, already one rule that actually utilizes these mappings causes a treewidth blow-up factor $\mathcal{O}(\log (n))$, which obviously is unbounded by the treewidth of the given program.

As a result, we apply these orderings only locally within the bags of a TD. Note that while this approach might look similar to existing techniques that are applied on a component-by-component basis [65, 53, 18], the approach is slightly different, as different components of the positive dependency graph $D_{\Pi}$ might be spread among different bags of a TD and a bag might only contain parts of components. If components are required to be spread across any TD of primal graph $\mathcal{G}_{\Pi}$ whose width coincides with the treewidth, only parts of cycles of $D_{\Pi}$ can be analyzed by a table algorithm in a bag. This is the reason why already the consistency problem of ASP remains slightly harder than the decision problem SAT (under ETH). Consequently, instead of global orderings or orderings per component, we use local orderings, which only order within bags. However, these orderings need to be "synchronized" between bags and come at the price of loosing bijective preservation of answer sets in general. The reason for that will be discussed in Section B

More concretely, our reduction is guided by a $\mathrm{TD} \mathcal{T}=(T, \chi)$ of primal graph $\mathcal{G}_{\Pi}$ and uses core ideas of dynamic programming along TD $\mathcal{T}$ to ensure only a slight increase in treewidth of the resulting SAT formula. This increase of treewidth is by construction, since our reduction essentially yields a tree decomposition of the primal graph of the resulting SAT formula, which functionally depends on $\mathcal{T}$, as sketched in Figure 3. Intuitively, thereby the aforementioned reduction takes care to keep the increase of width local, i.e., the increase of width happens within the bags of $\mathcal{T}$. Concretely, if width $(\mathcal{T})$ is bounded by some value $\mathcal{O}(k)$, the treewidth of the resulting formula $F$ is at most $\mathcal{O}(k \cdot \log (k))$.

For encoding orderings along a TD, we need the following notation. Let us
consider a $\operatorname{TD} \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$, and a node $t$ of $T$. We refer to an ordering over $\chi(t)$ by $t$-local ordering.

Definition 1. $A \mathcal{T}$-local ordering is a set containing one $t$-local ordering $\varphi_{t}$ for every $t$ of $T$ such that there is an interpretation $I$ with (1) satisfiability: $I \models \Pi_{t}$ for every node $t$ of $T$, (2) provability: for every $a \in I$, there is a node $t$ of $T$ and a rule $r \in \Pi_{t}$ proving a, and (3) compatibility: for every nodes $t, t^{\prime}$ of $T$ and every $a, b \in \chi(t) \cap \chi\left(t^{\prime}\right)$, whenever $\varphi_{t}(a)<\varphi_{t}(b)$ then $\varphi_{t^{\prime}}(a)<\varphi_{t^{\prime}}(b)$.

For an ordering $\varphi$, we use the canonical t-local ordering $\hat{\varphi}_{t}$ for each $t$ of $T$ as follows. Intuitively, atoms $a \in \chi(t)$ with smallest ordering position $\varphi(a)$ among all atoms in $\chi(t)$ get $\hat{\varphi}_{t}(a)=0$, second-smallest get value 1 , and so on. Formally, we define $\hat{\varphi}_{t}(a):=\operatorname{ord}_{t}(a, \varphi)-1$ for each $a \in \chi(t)$, where $\operatorname{ord}_{t}(a, \varphi)$ is the ordinal number (rank) of $a$ according to smallest ordering position $\varphi(a)$ among $\chi(t)$.
Example 4. Consider program $\Pi$, answer set $I=\{b, c, d\}$, and ordering $\varphi=$ $\{b \mapsto 0, d \mapsto 1, c \mapsto 2\}$ of Example 1. Ordering $\varphi$ can easily be extended to ordering $\varphi^{\prime}:=\{a \mapsto 0, e \mapsto 0, b \mapsto 0, d \mapsto 1, c \mapsto 2\}$ over at( $\Pi$ ). Then, using $T D \mathcal{T}$ of $\mathcal{G}_{\Pi}$, we can construct $\mathcal{T}$-local ordering $\mathcal{M}:=\left\{\hat{\varphi}_{t_{1}}, \hat{\varphi}_{t_{2}}, \hat{\varphi}_{t_{3}}\right\}$ of $\varphi^{\prime}$, where $\hat{\varphi}_{t_{1}}=\{e \mapsto 0, d \mapsto 1, c \mapsto 2\}, \hat{\varphi}_{t_{2}}=\{a \mapsto 0, b \mapsto 0\}$, and $\hat{\varphi}_{t_{3}}=\{e \mapsto 0, b \mapsto 0, d \mapsto 1\}$. Consider a TD $\mathcal{T}^{\prime}$ of $\mathcal{G}_{\Pi}$, which is similar to $\mathcal{T}$, but $t_{1}$ has a child node $t^{\prime}$, whose bag is $\{c, e\}$. Then, $\mathcal{M} \cup\left\{\hat{\varphi}_{t^{\prime}}\right\}$ with $\hat{\varphi}_{t^{\prime}}=\{e \mapsto 0, c \mapsto 1\}$ is a $\mathcal{T}^{\prime}$-local ordering.

In our reduction, we use the following propositional variables. For each atom $x \in$ at( $\Pi$ ), we use $x$ also as propositional variable. For each atom $x \in \chi(t)$ of each node $t$ of $T$, we use $\lceil\log (|\chi(t)|)\rceil$ many variables of the form $b_{x_{t}}^{i}$ forming the $i$-th bit of the $t$-local ordering position (in binary) of $x$. By the shortcut notation $\llbracket x \rrbracket_{t, j}$, we refer to the conjunction of literals over bits $b_{x_{t}}^{i}$ for $1 \leq i \leq$ $\lceil\log (|\chi(t)|)\rceil$ according to the representation of the number $j$ in binary. For atoms $x, x^{\prime} \in \chi(t)$ of node $t$ of $T$, we use the following notation to indicate that atom $x$ is ordered before atom $x^{\prime}$ :

$$
x \prec_{t} x^{\prime}:=\bigvee_{1 \leq i \leq\lceil\log (|\chi(t)|)\rangle}\left(b_{x_{t}^{\prime}}^{i} \wedge \neg b_{x_{t}}^{i} \wedge \bigwedge_{i<j \leq\lceil\log (|x(t)|)\rceil}\left(b_{x_{t}}^{j} \longrightarrow b_{x_{t}^{x_{t}}}^{j}\right)\right) .
$$

Example 5. Consider Example 4 and the $\mathcal{T}$-local ordering $\mathcal{M}=\left\{\varphi_{t_{1}}, \varphi_{t_{2}}, \varphi_{t_{3}}\right\}$. One could encode ordering position $\varphi_{t_{1}}(e)=0$ using two bit variables $b_{e_{t_{1}}}^{1}, b_{e_{t_{1}}}^{2}$ and forcing it to false. This results in formula $\llbracket e \rrbracket_{t_{1}, 0}=\neg b_{e_{t_{1}}}^{1} \wedge \neg b_{e_{t_{1}}}^{0}$. Then, we formulate $\varphi_{t_{1}}(d)=1$ by $\llbracket d \rrbracket_{t_{1}, 1}=\neg b_{d_{t_{1}}}^{1} \wedge b_{d_{t_{1}}}^{0}$, and $\varphi_{t_{1}}(c)=2$ by $\llbracket c \rrbracket_{t_{1}, 2}=$ $b_{c_{t_{1}}}^{1} \wedge \neg b_{c_{t_{1}}}^{0}$. For the whole resulting formula, $\left(e \prec_{t_{1}} d\right)$, $\left(d \prec_{t_{1}} c\right)$ as well as ( $e \prec_{t_{1}} c$ ) hold.

### 3.1 TD-guided Reduction to SAT

For solving consistency, we require to construct the following Formulas (1)(6) below for each TD node $t$ of $T$ having child nodes $\operatorname{chldr}(t)=\left\{t_{1}, \ldots, t_{\ell}\right\}$.

Thereby, these formulas aim at constructing $\mathcal{T}$-local orderings along the TD $\mathcal{T}$, where Formulas (1) ensure satisfiability, Formulas (2) take care of compatibility along the TD, and Formulas (6) enforce provability within a node, which is then guided along the TD by Formulas (3)-(5).
Concretely, Formulas (1) ensure that the variables of the constructed SAT formula $F$ are such that all (bag) rules are satisfied. Then, whenever in node $t$ an atom $x$ has a smaller ordering position than an atom $y$ (using $\prec_{t}$ ), this must hold also for the parent node of $t$ and vice versa, cf., Formulas (2). Formulas (3) guarantee, for nodes $t$ removing bag atom $x$, i.e., $x \in \chi(t) \backslash \chi\left(t^{\prime}\right)$, that $x$ is proven if $x$ is set to true. Similarly, this is required for atoms $x \in \chi(n)$ that are in the root node $n=\operatorname{root}(T)$ and therefore never forgotten, cf., Formulas (4). At the same time we ensure by Formulas (5) that an atom $x$ is proven up to node $t$ if and only if it is proven up to some child node of $t$ or freshly proven in node $t$. Finally, Formulas (6) make sure that an atom $x$ is freshly proven in node $t$ if and only if there is at least one rule $r \in \Pi_{t}$ proving $x$.

$$
\begin{align*}
& \bigvee_{b \in B_{r}^{+}} \neg b \vee \bigvee_{a \in B_{r}^{-} \cup H_{r}} a \quad \text { for each } r \in \Pi_{t}  \tag{1}\\
& \left(x \prec_{t^{\prime}} y\right) \longleftrightarrow\left(x \prec_{t} y\right) \quad \begin{array}{l}
\text { for } \left.\operatorname{each} t^{\prime} \in \operatorname{chldr}(t) \text { and } x, y \in \chi(t) \cap \chi\left(t^{\prime}\right){ }_{(2)}\right) \\
\text { with } x \neq y
\end{array} \\
& x \longrightarrow p_{<t^{\prime}}^{x} \quad \text { for each } t^{\prime} \in \operatorname{chldr}(t) \text { and } x \in \chi\left(t^{\prime}\right) \backslash \chi(t) \\
& x \longrightarrow p_{<n}^{x} \quad \text { for each } x \in \chi(n) \text { with } n=\operatorname{root}(T)  \tag{4}\\
& p_{<t}^{x} \longleftrightarrow p_{t}^{x} \vee\left(\underset{t^{\prime} \in \operatorname{chldr}(t), x \in \chi\left(t^{\prime}\right)}{ } p_{<t^{\prime}}^{x}\right) \quad \text { for each } x \in \chi(t)  \tag{5}\\
& p_{t}^{x} \longleftrightarrow \bigvee_{r \in \Pi_{t}, x \in H_{r}}\left(\bigwedge_{b \in B_{r}^{+}} b \wedge x \wedge \quad \text { for each } x \in \chi(t)\right.  \tag{6}\\
& \left.\left(b \prec_{t} x\right) \wedge \bigwedge \neg a\right) \\
& a \in B_{r}^{-} \cup\left(H_{r} \backslash\{x\}\right)
\end{align*}
$$

Example 6. Recall program $\Pi$ from Example 1, and $T D \mathcal{T}$ of $\mathcal{G}_{\Pi}$ given in Figure 2. We briefly show Formula $F$ for node $t_{3}$.
Formulas Formula F


Next, we show that the reduction is indeed aware of the treewidth and that the treewidth is slightly increased.

Theorem 1 (Treewidth-awareness). The reduction from an HCF program $\Pi$ and a nice $T D \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$ to $S A T$ formula $F$ consisting of Formulas (1)(6) ensures that if $k$ is the width of $\mathcal{T}$, then the treewidth of $\mathcal{G}_{F}$ is at most $\mathcal{O}(k$. $\log (k))$.

Proof. We construct a $\mathrm{TD} \mathcal{T}^{\prime}=\left(T, \chi^{\prime}\right)$ of $\mathcal{G}_{F}$ to show that the width of $\mathcal{T}^{\prime}$ increases from $k$ to $\mathcal{O}(k \cdot \log (k))$. To this end, let $t$ be a node of $T$ with $\operatorname{chldr}(t)=$ $\left\langle t_{1}, \ldots, t_{\ell}\right\rangle$ and let $\hat{t}$ be the parent of $t$ (if exists). We define $B(t, x):=\left\{b_{x_{t}}^{j} \mid\right.$ $x \in \chi(t), 1 \leq j \leq\lceil\log (|\chi(t)|)\rceil\}$. We inductively define $\chi^{\prime}(t):=\chi(t) \cup$ $\left(\bigcup_{x \in \chi(t)} B(t, x) \cup B(\hat{t}, x)\right) \cup\left\{p_{<t^{\prime}}^{y}, p_{t}^{x} \mid t^{\prime} \in\left\{t, t_{1}, \ldots, t_{\ell}\right\}, x \in \chi(t), y \in \chi\left(t^{\prime}\right)\right\}$. Observe that indeed $\mathcal{T}^{\prime}$ is a TD of $\mathcal{G}_{F}$. Further, $\left|\chi^{\prime}(t)\right| \leq k+k \cdot\lceil\log (k)\rceil \cdot 2+$ $k \cdot(\ell+2)$. Thus, the width of $\mathrm{TD} \mathcal{T}^{\prime}$ is in $\mathcal{O}(k \cdot \log (k))$, since for nice TDs we have $\ell=2$.

Note that the result above can be also lifted to non-nice TDs, as long as the number of child nodes is limited by $\mathcal{O}(\log (k))$.

Later we will see the lower bound for consistency of normal ASP, which indicates that one cannot expect to significantly improve this increase of treewidth. Next, we present consequences for auxiliary variables and runtime.

Corollary 1 (Runtime). The reduction from an HCF program $\Pi$ and a nice $T D \mathcal{T}$ of $\mathcal{G}_{\Pi}$ to $S A T$ formula $F$ consisting of Formulas (1)- (6) uses at most $\mathcal{O}(k$. $\log (k) \cdot h)$ many variables and runs in time $\mathcal{O}\left(k^{2} \cdot \log (k) \cdot h+|\Pi|\right)$, where $k$ and $h$ are the width and the number of nodes of $\mathcal{T}$, respectively.

Proof. The result follows from Theorem 1. Linear time in the size of $\Pi$ can be obtained by slightly modifying Formulas (1) and (6) such that each rule $r \in \Pi$ is used in only one node $t$, where $r \in \Pi_{t^{\prime}}$, but $r \notin \Pi_{t}$, for some $t^{\prime} \in \operatorname{chldr}(t)$. Runtime $\mathcal{O}\left(k^{2} \log (k)\right)$ in $k$ is due to the definition of $\left(x \prec_{t} y\right)$ as used in Formulas (2) and (6).

Note that a nice TD of $\mathcal{G}_{\Pi}$ of width $k=t w\left(\mathcal{G}_{\Pi}\right)$, having only $h=\mathcal{O}(|\operatorname{at}(\Pi)|)$ many nodes 68][Lem. 13.1.2] always exists. Further, since $k \cdot \log (k)$ might be much smaller than $\log (|\operatorname{at}(\Pi)|)$, for some programs this reduction could be a benefit compared to global or component-based orderings used in tools like lp2sat [65, 53, 18]. An empirical study addressing this is given in Section 5 .
Correctness of the Reduction. Now, we discuss the correctness of our reduction, which establishes that $\mathcal{T}$-local orderings encoded by Formulas (1)(6) follow ideas of the characterization of answer sets for HCF programs.

Theorem 2 (Correctness). The reduction from an $H C F$ program $\Pi$ and $a$ $T D \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$ to $S A T$ formula $F$ consisting of Formulas (1)-(6) is correct. Precisely, for each answer set of $\Pi$ there is a model of $F$ and vice versa.

Proof. The proof is given in $A$.

The statement above can be strengthened to derive the following corollary, which concludes that the reduction above only weakly preserves answer sets (and not bijectively).

Corollary 2 (Preservation of Answer Sets). The reduction from an HCF program $\Pi$ and a $T D \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$ to $S A T$ formula $F$ consisting of Formulas (1)-(6) preserves answer sets with respect to at(П). Concretely, for each answer set $M$ of $\Pi$ there is exactly one model of $F$ that when restricted to the variables in at( $\Pi)$, coincides with $M$. Conversely, for each model of $F$ there is exactly one answer set of $\Pi$.

However, in general we have that for an answer set of $\Pi$, there might be several models of the propositional formula obtained by the reduction above. Overall, one can strenghten the reduction in order to remove some models, which is presented in $B$.

In the next subsection, we present another reduction from HCF ASP to SAT that bijectively preserves all the answer sets at least for uniquely provable programs, at the cost of a higher increase of the treewidth from $k$ to $k^{2}$. The difference to the reduction above is that the increase from $k$ to $k^{2}$ explicitly allows us to verify whether the (up to $k^{2}$ many) relations per tree decomposition node in terms of provability are applicable when proving an answer set.

### 3.2 Bijective and Treewidth-Aware Reduction to SAT

This section deals with a different approach that is inspired by an early attempt [74, combined with the ideas of Clark's completion [26] that is guided along a tree decomposition. To this end, consider an HCF program $\Pi$ and a $\mathrm{TD} \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$. Then, instead of orderings, we use fresh auxiliary variables of the form $(x \prec y)$ (and $(y \prec x)$ ) to indicate that an atom $x$ has to precede an other atom $y$ (and vice versa). This is done locally for each tree decomposition bag and consequently results in a quadratic number of additional auxiliary variables per bag and causes a quadratic increase of the treewidth in the worst-case. Further, we also require auxiliary variables $p_{t}^{x}, p_{t}^{x \prec y}$, and $p_{t}^{y \prec x}$ for a node $t$ of $T$ and $x, y \in \chi(t)$ in order to indicate whether there is a rule proving atom $x$, auxiliary variable $(x \prec y)$, and auxiliary variable $(y \prec x)$, respectively. Finally, this approach also uses auxiliary variable $p_{t, r}^{x}$ to indicate that an atom $x \in \chi(t)$ is proven by a rule $r \in \Pi_{t}$ in a node $t$ of $T$.

The reduction constructs a formula $F^{\prime}$ that consists of Formulas (1), (3)-(5) as well as Formulas (7)-(15), as given below, for each node $t$ of $T$. As before, Formulas (1) ensure that each rule $r \in \Pi$ is satisfied. Then, Formulas (7) guarantee that we have provability $p_{t, r}^{x}$ for an atom $x \in \chi(t)$ using a rule $r \in \Pi_{t}$, if $r$ proves $x$ with any ordering, i.e., if $x$ does not precede any atom of positive body $B_{r}^{+}$. Formulas (8) make sure that if we have $p_{t, r}^{x}$, indeed all positive body atoms $b \in B_{r}^{+}$precede $x$, i.e., whenever a rule is suitable for proving an atom, it actually has to be applied (greedy application). We also ensure by Formulas (9) that the precedence is transitive. Formulas 10 take care of not allowing cycles over $\prec$, i.e., we cannot have both $(x \prec y)$ and $(y \prec x)$ at the same time.

$$
\begin{align*}
& \bigvee_{b \in B_{r}^{+}} \neg b \vee \bigvee_{a \in B_{r}^{-} \cup H_{r}} a \quad \text { for each } r \in \Pi_{t}  \tag{11}\\
& p_{t, r}^{x} \longleftrightarrow\left(\bigwedge_{b \in B_{r}^{+}} b \wedge x \wedge \neg(x \prec b) \quad \text { for each } r \in \Pi_{t} \text { and } x \in H_{r}\right.  \tag{7}\\
& a \in B_{r}^{-} \cup\left(H_{r} \backslash\{x\}\right) \\
& p_{t, r}^{x} \longrightarrow(b \prec x)  \tag{8}\\
& (x \prec y) \wedge(y \prec z) \longrightarrow(x \prec z) \quad \begin{array}{l}
\text { for each } \left.x, y, z \in \underset{\sim}{x \neq y, x \neq} \begin{array}{l}
\text { and }(t) \\
y \neq z
\end{array}\right) \text { with }
\end{array}  \tag{9}\\
& \neg(x \prec y) \vee \neg(y \prec x) \quad \text { for each } x, y \in \chi(t) \text { with } x \neq y  \tag{10}\\
& p_{t}^{x} \longleftrightarrow \bigvee_{r \in \Pi_{t}, x \in H_{r}} p_{t, r}^{x} \quad \text { for each } x \in \chi(t)  \tag{11}\\
& p_{t}^{y \prec x} \longleftrightarrow \bigvee \quad p_{t, r}^{x} \quad \vee \quad \text { for each } x, y \in \chi(t) \text { with } x \neq y \\
& r \in \Pi_{t}, x \in H_{r}, y \in B_{r}^{+} \\
& \bigvee[(y \prec z) \wedge(z \prec x)]  \tag{12}\\
& z \in \chi(t), z \neq x, z \neq y \\
& x \longrightarrow p_{<t^{\prime}}^{x} \\
& \text { for each } t^{\prime} \in \operatorname{chldr}(t), x \in \chi\left(t^{\prime}\right) \backslash \chi(t)  \tag{3}\\
& x \longrightarrow p_{<n}^{x} \quad \text { for each } x \in \chi(n) \text { with } n=\operatorname{root}(T)  \tag{4}\\
& p_{<t}^{x} \longleftrightarrow p_{t}^{x} \vee\left(\bigvee p_{<t^{\prime}}^{x}\right) \quad \text { for each } x \in \chi(t)  \tag{5}\\
& (x \prec y) \longrightarrow p_{<t^{\prime}}^{x \prec y}
\end{align*}
$$

$$
\begin{align*}
& (x \prec y) \longrightarrow p_{<n}^{x \prec y}  \tag{14}\\
& p_{<t}^{x \prec y} \longleftrightarrow p_{t}^{x \prec y} \vee\left(\underset{t^{\prime} \in \operatorname{chldr}(t), x, y \in \chi\left(t^{\prime}\right)}{ } p_{<\prime^{\prime}}^{x \prec y}\right) \quad \text { for each } x, y \in \chi(t) \text { with } x \neq y  \tag{15}\\
& \text { for each } r \in \Pi_{t}, x \in H_{r} \text {, and } b \in B_{r}^{+}
\end{align*}
$$

$$
\begin{aligned}
& \left.\begin{array}{l}
\text { for each } \\
n=\operatorname{root}(T) \\
y
\end{array}\right) \in \chi(n) \text { with } x \neq y \text { and }
\end{aligned}
$$

We define $p_{t}^{x}$ by Formulas (11), similarly to Formulas (6), but here we are able to use auxiliary variables $p_{t, r}^{x}$ for every $r \in \Pi_{t}$. Further, we also define provability $p_{t}^{y \prec x}$ for auxiliary variables $(y \prec x)$ in node $t$, which is the case if we either derive $(y \prec x)$ due to Formulas $(8)$, where we also need $p_{t, r}^{x}$, or we obtain $(y \prec x)$ due to Formulas (9), where we require $(y \prec z)$ and ( $z \prec x$ ) for some $z \in \chi(t)$. For ensuring provability for an atom $x \in \chi(t)$ and guiding it along the TD, we use Formulas (3), (4), and (5) as before. Analogously, we require Formulas (13), (14), and (15) for guiding provability of an auxiliary variable $(x \prec y)$ along the TD.
Example 7. Recall program $\Pi$ from Example $1, T D \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$ given in Figure 2, as well as Formulas (1) and (3)-(5) from Example 6. Next, we briefly show Formulas (7)-(15) for node $t_{3}$ of $T$.

Formulas Formula $F^{\prime}$


Note that for practical implementations there is, of course, potential for optimizations. To demonstrate this, the table above does not contain every useless instance of Formulas (9). As an example, $(b \prec e) \wedge(e \prec d) \rightarrow(b \prec d)$ is not needed since the only way to prove $(e \prec d)$ is via the first instance of Formulas (9) in the table, which requires $(e \prec b)$. However, having both $(b \prec e)$ and $(e \prec b)$ is not possible anyway due to Formulas (10).

Treewidth-Awareness and Runtime. Next, we discuss consequences of the reduction consisting of Formulas (1), (3)-(5) as well as Formulas (7)-(15) for each node $t$ of $T$. Thereby, we show results for treewidth-awareness and runtime.

Theorem 3 (Treewidth-awareness). The reduction from an HCF program $\Pi$ and a nice $T D \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$ to SAT formula $F^{\prime}$ consisting of Formulas (1), (3)-(5) as well as Formulas (7)-(15) ensure that if $k$ is the width of $\mathcal{T}$, then the treewidth of $\mathcal{G}_{F^{\prime}}$ is at most $\mathcal{O}\left(k^{2}\right)$.

Proof. We construct a $\mathrm{TD} \mathcal{T}^{\prime}=\left(T, \chi^{\prime}\right)$ of $\mathcal{G}_{F^{\prime}}$ to show that the width of $\mathcal{T}^{\prime}$ increases from $k$ to $\mathcal{O}\left(k^{2}\right)$. To this end, let $t$ be a node of $T$ with $\operatorname{chldr}(t)=$ $\left\langle t_{1}, \ldots, t_{\ell}\right\rangle$ and let $\hat{t}$ be the parent node of $t$ (if it exists). Note that the number $\left|\Pi_{t}\right|$ of rules might be larger than $\mathcal{O}\left(k^{2}\right)$. However, one can easily modify TD $\mathcal{T}$ by adding intermediate nodes $t^{1}, \ldots, t^{o}$ between $t$ and $\hat{t}$, where for each node $t^{i}$ with $1 \leq i \leq o$, we have $\chi\left(t_{i}\right)=\chi(t)$. Then, instead of the actual bag program $\Pi_{t^{i}}=\Pi_{t}$ we only apply a small subset $\Pi_{t^{i}}^{\prime} \subseteq \Pi_{t}$ of $t^{i}$ such that $\bigcup_{1 \leq i \leq o} \Pi_{t^{i}}^{\prime}=\Pi_{t}$. So, intuitively instead of applying $\Pi_{t}$ in one node $t$, this allows us to partition $\Pi_{t}$ and apply the parts separately by using intermediate nodes. Therefore, in the following we assume for the ease of notation and without loss of generality that $\left|\Pi_{t}\right|$ is bounded by a constant $c$, i.e., we sloppily refer to $\Pi_{t}^{\prime}$ by $\Pi_{t}$. We inductively define $\chi^{\prime}(t):=\chi(t) \cup\{(x \prec y) \mid$ $\{x, y\} \subseteq \chi(t), x \neq y\} \cup\left\{p_{<t^{\prime}}^{y}, p_{t}^{x} \mid t^{\prime} \in\left\{t, t_{1}, \ldots, t_{\ell}\right\}, x \in \chi(t), y \in \chi\left(t^{\prime}\right)\right\} \cup\left\{p_{t, r}^{x} \mid\right.$
$\left.x \in \chi(t), r \in \Pi_{t}\right\} \cup\left\{p_{<t^{\prime}}^{y \prec y^{\prime}}, p_{t}^{x \prec x^{\prime}} \mid t^{\prime} \in\left\{t, t_{1}, \ldots, t_{\ell}\right\},\left\{x, x^{\prime}\right\} \subseteq \chi(t),\left\{y, y^{\prime}\right\} \subseteq\right.$ $\left.\chi\left(t^{\prime}\right), x \neq x^{\prime}, y \neq y^{\prime}\right\}$. Observe that indeed $\mathcal{T}^{\prime}$ is a TD of $\mathcal{G}_{F^{\prime}}$. Further, $\left|\chi^{\prime}(t)\right| \leq k+k^{2}+2 k \cdot(\ell+1)+c k+2 k^{2} \cdot(\ell+1)$. Thus, since $\ell \leq 2$ for nice TDs, the width of $\mathcal{T}^{\prime}$ is in $\mathcal{O}\left(k^{2}\right)$.

This increase of treewidth is also reflected in the runtime of the reduction.
Corollary 3 (Runtime). The reduction from an HCF program $\Pi$ and a nice $T D \mathcal{T}$ of $\mathcal{G}_{\Pi}$ to SAT formula $F^{\prime}$ consisting of Formulas (1), (3)-(5) as well as Formulas (7)-15) uses at most $\mathcal{O}\left(k^{2} \cdot h+|\Pi|\right)$ many variables and runs in time $\mathcal{O}\left(k^{3} \cdot h+|\Pi|\right)$, where $k$ and $h$ are the width and the number of nodes of $\mathcal{T}$, respectively.

Proof. The result follows from Theorem [3. Linear time in the size of $\Pi$ can be obtained by slightly modifying Formulas (1), (7), (8), (11), and (12) such that each rule $r \in \Pi$ is used in only one node $t$, where $r \in \Pi_{t^{\prime}}$, but $r \notin \Pi_{t}$, for some $t^{\prime} \in \operatorname{chldr}(t)$. The cubic runtime in $k$ is due to transitivity by Formulas 9 ) and (12).

For detailed discussions on correctness and further consequences, we refer to C

Compared to the reduction of Section 3.1, the worst-case increase of treewidth from $k$ to $k^{2}$ explicitly allows us to verify whether every single $\prec$-relation per tree decomposition node is indeed applicable when proving an answer set. However, this reduction does not bijectively preserve answer sets for HCF programs in general, and we believe that a different approach is needed in order to design a reduction that is both treewidth-aware and bijective at the same time.

Next, we show that indeed already deciding the consistency of a normal (uniquely provable) program is expected to be slightly harder than deciding the satisfiability of a propositional formula.

## 4 Why ASP Consistency is Harder than SAT

This section concerns the hardness of ASP consistency when considering treewidth. The high-level reason for ASP being harder than SAT when assuming bounded treewidth, lies in the issue that a TD, while capturing the structural dependencies of a program, might force an evaluation that is completely different from the orderings proving answer sets. Consequently, during dynamic programming for ASP, one needs to store in each table $\tau_{t}$ for each node $t$ during post-order traversal, in addition to an interpretation (candidate answer set), also an ordering among the atoms in those interpretations. We show that under reasonable assumptions in complexity theory, this worst-case cannot be avoided. Then, the resulting runtime consequences cause ASP to be slightly harder than SAT, where in contrast to ASP storing a table $\tau_{t}$ of only assignments for each node $t$ suffices.


Figure 4: An instance $I=(G, P)$ (left) of the Disjoint Paths Problem and a TD of $G$ (right).

We show our novel hardness result by reducing from the (DIRECTED) Disjoint Paths Problem, which is a graph problem defined as follows. Let us consider a directed graph $G=(V, E)$, and a set $P \subseteq V \times V$ of disjoint pairs of the form $\left(s_{i}, d_{i}\right)$ consisting of source $s_{i}$ and destination $d_{i}$, where $s_{i}, d_{i} \in V$ such that each vertex occurs at most once in $P$, i.e., $\left|\bigcup_{\left(s_{i}, d_{i}\right) \in P}\left\{s_{i}, d_{i}\right\}\right|=2 \cdot|P|$. Then, $(G, P)$ is an instance of the Disjoint Paths Problem, asking whether there exist $|P|$ many (vertex-disjoint) paths from $s_{i}$ to $d_{i}$ for $1 \leq i \leq|P|$. Concretely, each vertex of $G$ is allowed to appear in at most one of these paths. For the ease of presentation, we assume without loss of generality [76] that sources $s_{i}$ have no incoming edge $\left(x, s_{i}\right)$, and destinations $d_{i}$ have no outgoing edge $\left(d_{i}, x\right)$.

Example 8. Figure 4 (left) shows an instance $I=(G, P)$ of the Disjoint Paths Problem, where $P$ consists of pairs of the form $\left(s_{i}, d_{i}\right)$. The only solution to $I$ is both emphasized and colored in red. Figure 4 (right) depicts a $T D$ of $G$.

While under ETH, SAT cannot be solved in time $2^{o(k)} \cdot \operatorname{poly}(|\operatorname{at}(F)|)$, where $k$ is the treewidth of the primal graph of a given propositional formula $F$, the Disjoint Paths Problem is considered to be even harder. Concretely, the problem has been shown to be slightly superexponential as stated in the following proposition.

Proposition 1 ([76]). Under ETH, there is no algorithm solving the DISJoINT Paths Problem in time $2^{o(k \cdot \log (k))} \cdot \operatorname{poly}(|V|)$, where $(G, P)$ is any instance with $k=t w(G)$.

It turns out that the Disjoint Paths Problem is a suitable problem candidate for showing the hardness of ASP. In our reduction, we use the following notation of open pairs, which leads to the result below. Let $(G, P)$ be an instance of the Disjoint Paths Problem, $\mathcal{T}=(T, \chi)$ be a TD of $G$, and $t$ be a node of $T$. Then, a pair $(s, d) \in P$ is open in node $t$, if either $s \in \chi_{\leq t}$ ("open due to source $s$ ") or $d \in \chi \leq t$ ("open due to destination $d$ "), but not both.

Proposition 2 ( 87 ). An instance $(G, P)$ of the Disjoint Paths Problem does not have a solution if there is a $T D \mathcal{T}=(T, \chi)$ of $G$ and a bag $\chi(t)$ with more than $|\chi(t)|$ many pairs in $P$ that are open in a node $t$ of $T$.

Proof. The result, cf., [87, boils down to the fact that each bag $\chi(t)$, when removed from $G$, results in a disconnected graph consisting of two components. Between these components can be at most $|\chi(t)|$ different paths.


Figure 5: A pair-respecting TD (left), and a pair-connected TD $\mathcal{T}$ (right) of ( $G, P$ ) of Figure 4

Preparing pair-connected TDs. Before we present the actual reduction, we need to define a pair-respecting tree decomposition of an instance $(G, P)$ of the Disjoint Paths Problem. Intuitively, such a TD of $G$ additionally ensures that each pair in $P$ is encountered together in some TD bag.

Definition 2. $A T D \mathcal{T}=(T, \chi)$ of $G$ is a pair-respecting TD of $(G, P)$ if for every pair $p=(s, d)$ with $p \in P$, (1) whenever $p$ is open in a node $t$ due to $s$, or due to $d$, then $s \in \chi(t)$, or $d \in \chi(t)$, respectively. Further, (2) whenever $p$ is open in a node $t$, but not open in the parent $t^{\prime}$ of $t$ (" $p$ is closed in $t^{\prime}$ "), both $s, d \in \chi\left(t^{\prime}\right)$.

We observe that such a pair-respecting TD can be computed with only a linear increase in the (tree)width in the worst-case. Concretely, we can turn any $\mathrm{TD} \mathcal{T}=(T, \chi)$ of $G$ into a pair-respecting $\mathrm{TD} \mathcal{T}^{\prime}=\left(T, \chi^{\prime}\right)$ of $(G, P)$. Thereby, the tree $T$ is traversed for each $t$ of $T$ in post-order, and vertices of $P$ are added to $\chi(t)$ accordingly, resulting in $\chi^{\prime}(t)$, such that conditions (1) and (2) of pair-respecting TDs are met. Observe that this doubles the sizes of the bags in the worst-case, since by Proposition 2 there can be at most bag-size many open pairs.

Example 9. Figure 5 (left) shows a pair-respecting TD of $(G, P)$ of Figure 4. which can be obtained by transforming the TD of Figure 4 (right), followed by simplifications.

For a given sequence $\sigma$ of pairs of $P$ in the order of closure with respect to the post-order of $T$, we refer to $\sigma$ by the closure sequence of $\mathcal{T}$. We denote by $p \in_{i} \sigma$ that pair $p$ is the pair closed $i$-th in the order of $\sigma$. Intuitively, e.g., the first pair $p \epsilon_{1} \sigma$ indicates that pair $p \in P$ is the first to be closed when traversing $T$ in post-order.

Definition 3. $A$ pair-connected TD $\mathcal{T}=(T, \chi)$ of $(G, P)$ is a pair-respecting $T D$ of $(G, P)$, if, whenever a pair $p \in_{i} \sigma$ with $i>1$ is closed in a node $t$ of $T$, also for the pair $(s, d) \in_{i-1} \sigma$ closed directly before $p$ in $\sigma$, both $s, d \in \chi(t)$.

We can turn any pair-respecting, nice $\mathrm{TD} \mathcal{T}^{\prime}=\left(T, \chi^{\prime}\right)$ of width $k$ into a pairconnected TD $\mathcal{T}^{\prime \prime}=\left(T, \chi^{\prime \prime}\right)$ with constant increase in the width. Let therefore pair $p \in_{i} \sigma$ be closed ( $i>1$ ) in a node $t$, and pair $(s, d) \in_{i-1}$ be closed before $p$ in node $t^{\prime}$. Intuitively, we need to add $s, d$ to all bags $\chi^{\prime}\left(t^{\prime}\right), \ldots, \chi^{\prime}(t)$ of nodes encountered after node $t^{\prime}$ and before node $t$ of the post-order tree traversal, resulting in $\chi^{\prime \prime}$. However, the width of $\mathcal{T}^{\prime \prime}$ is at most $k+3 \cdot|\{s, d\}|=k+6$,
since in the tree traversal each node of $T$ is passed at most 3 times, namely when traversing down, when going from the left branch to the right branch, and then also when going upwards. Indeed, to ensure $\mathcal{T}^{\prime \prime}$ is a TD (connectedness condition), we add at most 6 additional atoms to every bag.

Example 10. Figure 5 (right) depicts a pair-connected TD of $(G, P)$ of Figure 4, obtainable by transforming the pair-respecting TD of Figure 5 (left), followed by simplifications.

## Reducing from Disjoint Paths to ASP

In this section, we show the main reduction $R$ of this work, assuming any instance $I=(G, P)$ of the Disjoint Paths Problem. Before we construct our program $\Pi$, we require a nice, pair-connected TD $\mathcal{T}=(T, \chi)$ of $G$, whose width is $k$ and a corresponding closure sequence $\sigma$. By Proposition 2, for each node $t$ of $\mathcal{T}$, there can be at most $k$ many open pairs of $P$, which we assume in the following. If this was not the case, we can immediately output, e.g., $\{a \leftarrow \neg a\}$.

Then, we use the following atoms in our reduction. Atoms $e_{u, v}$, or $n e_{u, v}$ indicate that edge $(u, v) \in E$ is used, or unused, respectively. Further, atom $r_{u}$ for any vertex $u \in V$ indicates that $u$ is reached via used edges. Finally, we also need atom $f_{t}^{u}$ for a node $t$ of $T$, and vertex $u \in \chi(t)$, to indicate that vertex $u$ is already finished in node $t$, i.e., $u$ has one used, outgoing edge. The presence of this atom $f_{t}^{u}$ in an answer set prohibits to take additional edges of $u$ in parent nodes of $t$, which is needed due to the need of disjoint paths of the Disjoint Paths Problem.

The instance $\Pi=R(I, \mathcal{T})$ constructed by reduction $R$ consists of three program parts, namely reachability $\Pi_{\mathcal{R}}$, linking $\Pi_{\mathcal{L}}$ of two pairs in $P$, as well as checking $\Pi_{\mathcal{C}}$ of disjointness of constructed paths. Consequently, $\Pi=\Pi_{\mathcal{R}} \cup$ $\Pi_{\mathcal{L}} \cup \Pi_{\mathcal{C}}$. All three programs $\Pi_{\mathcal{R}}, \Pi_{\mathcal{L}}$, and $\Pi_{\mathcal{C}}$ are guided along TD $\mathcal{T}$, which ensures that the width of $\Pi$ is only linearly increased. Note that this has to be carried out carefully, since, e.g., the number of atoms of the form $e_{u, v}$ using only vertices $u, v$ that appear in one bag, can be already quadratic in the bag size. The goal of this reduction, however, admits only a linear overhead in the bag size. Consequently, we are, e.g., not allowed to construct rules in $\Pi$ that require more than $\mathcal{O}(k)$ edges in one bag of a TD of $\mathcal{G}_{\Pi}$.

To this end, let the ready edges $E_{t}^{r e}$ in node $t$ be the set of edges $(u, v) \in E$ not present in $t$ anymore, i.e., $\{u, v\} \subseteq \chi\left(t^{\prime}\right) \backslash \chi(t)$ for any child node $t^{\prime} \in \operatorname{chldr}(t)$. Further, let $E_{n}^{\text {re }}$ for the root node $n=\operatorname{root}(T)$ additionally contain also all edges of $n$, i.e., $E \cap(\chi(n) \times \chi(n))$. Intuitively, ready edges for $t$ will be processed in node $t$. Note that each edge occurs in exactly one set of ready edges. Further, for nice TDs $\mathcal{T}$, we always have $\left|E_{t}^{\mathrm{re}}\right| \leq k$, i.e., ready edges are linear in $k$.

Example 11. Recall instance $I=(G, P)$ with $G=(V, E)$ of Figure 4, and pairconnected TD $\mathcal{T}=(T, \chi)$ of I of Figure 5 (right). Then, $E_{t_{1}}^{r e}=\emptyset, E_{t_{2}}^{r e}=\{(y, z),(z$, $\left.y),\left(z, d_{3}\right),\left(s_{3}, z\right)\right\}$, since $z \notin \chi\left(t_{2}\right)$, and $E_{t_{3}}^{r e}=E \backslash E_{t_{2}}^{r e}$ for root $t_{3}$ of $\mathcal{T}$.

Reachability $\Pi_{\mathcal{R}}$. Program $\Pi_{\mathcal{R}}$ is constructed as follows.

$$
\begin{array}{ll}
e_{u, v} \leftarrow r_{u}, \neg n e_{u, v} & \text { for each }(u, v) \in E_{t}^{\mathrm{re}} \\
n e_{u, v} \leftarrow \neg e_{u, v} & \text { for each }(u, v) \in E_{t}^{\mathrm{re}} \\
r_{v} \leftarrow e_{u, v} & \text { for each }(u, v) \in E_{t}^{\mathrm{re}} \tag{18}
\end{array}
$$

Rules 16 and 17 ensure that there is a partition of edges in used edges $e_{u, v}$ and unused edges $n e_{u, v}$. Additionally, Rules (16) make sure that only edges of adjacent, reachable vertices are used. Naturally, this requires that initially at least one vertex is reachable (constructed below). Rules (18) ensure reachability $r_{v}$ over used edges $e_{u, v}$ for a vertex $v$.

Linking of pairs $\Pi_{\mathcal{L}}$. Program $\Pi_{\mathcal{L}}$ is constructed as follows.

$$
\begin{array}{ll}
\leftarrow \neg r_{d} & \text { for each }(s, d) \in P \\
r_{s_{1}} \leftarrow & \text { for }\left(s_{1}, d\right) \in_{1} \sigma \\
r_{s_{i}} \leftarrow r_{s_{i-1}}, r_{d_{i-1}} &  \tag{21}\\
\text { for each }\left(s_{i}, d\right) \in_{i} \sigma,\left(s, d_{i-1}\right) \in_{i-1} \sigma
\end{array}
$$

Rules (19) make sure that, ultimately, destination vertices of all pairs are reached. As an initial, reachable vertex, Rule 20 sets the source vertex $s$ reachable, whose pair is closed first. Then, the linking of pairs is carried out along the TD in the order of closure, as given by $\sigma$. Thereby, Rules 21) conceptually construct auxiliary links (similar to edges) between different pairs, in the order of $\sigma$, which is guided along the TD to ensure only a linear increase in treewidth of $\mathcal{G}_{\Pi}$ of the resulting program $\Pi$. Interestingly, these additional dependencies, since guided along the TD, do not increase the treewidth by much as we will see in the next subsection.

Then, it is crucial that we prevent a source vertex $s_{i}$ of a pair $\left(s_{i}, d_{i}\right) \in_{i} \sigma$ from reaching a destination vertex $d_{j}$ of a pair $\left(s_{j}, d_{j}\right) \in_{j} \sigma$ preceding $\left(s_{i}, d_{i}\right)$ in $\sigma$, i.e., $j<i$. To this end, we need to construct parts of cycles that prevent this. Concretely, if some source $s_{i}$ reaches $d_{j}$, i.e., $d_{j}$ is reachable via $s_{i}$, the goal is to have a cyclic reachability from $d_{j}$ to $s_{i}$, with no provability for corresponding reachability atoms of the cycle. Actually, Rules 21) also have the purpose of aiding in the construction of these potential positive cycles. Thereby we achieve that if $d_{j}$ is reachable, this cannot be due to $s_{i}$, since reachability of $d_{j}, s_{j+1}, \ldots, s_{i}$ (therefore $s_{i}$ itself) is required for reachability of $s_{i}$. Consequently, assuming that there is no further rule proving any of these reachability atoms, which we will ensure in the construction of program $\Pi_{\mathcal{C}}$ below, we end up with cyclic reachability if $s_{i}$ is reached by $d_{j}$, such that none of the atoms of the cycle are proven. Figure 6 shows the positive dependency graph $D_{R_{\mathcal{L}}}$ of Rules (21), where pairs $\left(s_{i}, d_{i}\right) \in_{i} \sigma$, as discussed in the following example.

Example 12. Consider the dependency graph $D_{R_{\mathcal{L}}}$ of Rules (21), as depicted in Figure 6. Observe that whenever $s_{i}$ reaches some $d_{j}$ with $j<i$, this causes a cycle $C=r_{s_{i}}, \ldots, r_{d_{j}}, r_{s_{j+1}}, \ldots, r_{s_{i-1}}, r_{s_{i}}$ over reachability atoms in $D_{R_{\mathcal{L}}}$ (cyclic dependency).

If each vertex $u$ of $G$ can have at most one outgoing edge, i.e., only one atom $e_{u, v}$ in an answer set of $\Pi=R(I, \mathcal{T})$, no atom of $C$ can be proven (no


Figure 6: Positive dependency graph $D_{R_{\mathcal{L}}}$ (indicated by solid red edges) of Rules 21) constructed for any closure sequence $\sigma$ such that $\left(s_{i}, d_{i}\right) \in_{i} \sigma$. If a source $s_{i}$ reaches a destination $d_{j}$ of a preceding pair, i.e., $j<i$, (depicted via the dashed red edge), this results in a cycle (consisting of all bold-faced edges) such that none of the atoms of the cycle can be proven.
further rule allows provability). Note that $C$ could also be constructed by causing in the positive dependency graph $\mathcal{O}\left(|P|^{2}\right)$ many edges from $r_{d_{j}}$ to $r_{s_{i}}$ for $j<i$. This could be achieved, e.g., by constructing large rules, where reachability $r_{d_{j}}$ of every preceding destination vertex is required in the positive body in order to reach a certain source vertex $s_{i}$, i.e., in order to obtain reachability $r_{s_{i}}$. However, this would cause an increase of structural dependency, and in fact, the treewidth increase would be beyond linear.

Checking of disjointness $\Pi_{\mathcal{C}}$. Finally, we create rules in $\Pi$ that enforce at most one outgoing, used edge per vertex. This is required to ensure that we do not use a vertex twice, as required by the Disjoint Paths Problem. We do this by guiding the information, whether the corresponding outgoing edge was used, via atoms $f_{t}^{u}$ along the TD to ensure that the treewidth is not increased significantly. Having at most one outgoing, used edge per vertex of $G$ further ensures that when a source of a pair $p$ reaches a destination of a pair preceding $p$ in $\sigma$, then no atom of the resulting cycle as constructed in $\Pi_{\mathcal{L}}$ will be provable. Consequently, in the end every source of $p$ has to reach the destination of $p$ by the pigeon hole principle. Program $\Pi_{\mathcal{C}}$ is constructed for every node $t$ with $t^{\prime}, t^{\prime \prime} \in \operatorname{chldr}(t)$, if $t$ has child nodes, as follows.

$$
\begin{array}{ll}
f_{t}^{u} \leftarrow e_{u, v} & \text { for each }(u, v) \in E_{t}^{\mathrm{re}}, u \in \chi(t) \\
f_{t}^{u} \leftarrow f_{t^{\prime}}^{u} & \text { for each } u \in \chi(t) \cap \chi\left(t^{\prime}\right) \\
\leftarrow f_{t^{\prime}}^{u}, f_{t^{\prime \prime}}^{u} & \text { for each } u \in \chi\left(t^{\prime}\right) \cap \chi\left(t^{\prime \prime}\right), t^{\prime} \neq t^{\prime \prime} \\
\leftarrow f_{t^{\prime}}^{u}, e_{u, v} & \text { for each }(u, v) \in E_{t}^{\mathrm{re}}, u \in \chi\left(t^{\prime}\right) \\
\leftarrow e_{u, v}, e_{u, w} & \text { for each }(u, v),(u, w) \in E_{t}^{\mathrm{re}}, v \neq w
\end{array}
$$

Rules $(22)$ ensure that the finished flag $f_{t}^{u}$ is set for used edges $e_{u, v}$. Then, this information of $f_{t^{\prime}}^{u}$ is guided along the TD from child node $t^{\prime}$ to parent node $t$ by Rules (23). If for a vertex $u \in V$ we have $f_{t^{\prime}}^{u}$ and $f_{t^{\prime \prime}}^{u}$ for two different child nodes $t^{\prime}, t^{\prime \prime} \in \operatorname{chldr}(t)$, this indicates that two different edges were encountered both below $t^{\prime}$ and below $t^{\prime \prime}$. Consequently, this situation is avoided by Rules (24). Rules (25) make sure to disallow additional edges for vertex $u$ in a TD node $t$, if the flag $f_{t^{\prime}}^{u}$ of child node $t^{\prime}$ is set. Finally, Rules 26) prohibit two different edges for the same vertex $u$ within a TD node.

Example 13. Recall instance $I=(G, P)$ with $G=(V, E)$ of Figure 4, pairconnected $T D \mathcal{T}=(T, \chi)$ of I of Figure 5 (right), and $E_{t_{2}}^{r e}=\left\{(y, z),(z, y),\left(z, d_{3}\right)\right.$, $\left.\left(s_{3}, z\right)\right\}$. We briefly present the construction of $\Pi_{\mathcal{C}}$ for node $t_{2}$.
Rules $\Pi_{\mathcal{L}}$
(22) $f_{t_{2}}^{y} \leftarrow e_{y, z} ; f_{t_{2}}^{s_{3}} \leftarrow e_{s_{3}, z}$
(23) $f_{t_{2}}^{s_{3}} \leftarrow f_{t_{1}}^{s_{3}} ; f_{t_{2}}^{d_{3}} \leftarrow f_{t_{1}}^{d_{3}} ; f_{t_{2}}^{y} \leftarrow f_{t_{1}}^{y}$
(25) $\leftarrow f_{t_{1}}^{y}, e_{y, z} ; \leftarrow f_{t_{1}}^{z}, e_{z, y} ; \leftarrow f_{t_{1}}^{z}, e_{z, d_{3}} ; \leftarrow f_{t_{1}}^{s_{3}}, e_{s_{3}, z}$
(26) $\leftarrow e_{z, y}, e_{z, d_{3}}$

Correctness and Runtime Analysis. First, we show that the reduction is correct, followed by a result stating that the treewidth of the reduction is at most linearly worsened, which is crucial for the runtime lower bound to hold. Then, we present the runtime and the (combined) main result of this work.

Theorem 4 (Correctness). Reduction $R$ as proposed in this section is correct. Let us consider an instance $I=(G, P)$ of the Disjoint Paths Problem, and a pair-connected $T D \mathcal{T}=(T, \chi)$ of $G$. Then, $I$ has a solution if and only if the program $R(I, \mathcal{T})$ admits an answer set.

Proof. The proof is given in D.
Lemma 1 (Treewidth-awareness). Let $I=(G, P)$ be any instance of the Disjoint Paths Problem, and $\mathcal{T}$ be a nice, pair-connected TD of $I$ of width $k$. Then, the treewidth of $\mathcal{G}_{\Pi}$, where $\Pi=R(I, \mathcal{T})$ is obtained by $R$, is at most $\mathcal{O}(k)$.

Proof. Assume any pair-connected, nice TD $\mathcal{T}=(T, \chi)$ of $I=(G, P)$. Since $\mathcal{T}$ is nice, a node in $T$ has at most $\ell=2$ many child nodes. From $\mathcal{T}$ we construct a $\mathrm{TD} \mathcal{T}^{\prime}=\left(T, \chi^{\prime}\right)$ of $\mathcal{G}_{\Pi}$. Thereby we set for every node $t$ of $T, \chi^{\prime}(t):=$ $\left\{r_{u}, f_{t}^{u} \mid u \in \chi(t)\right\} \cup\left\{e_{u, v}, n e_{u, v}, r_{u}, r_{v}, f_{t^{\prime}}^{u} \mid(u, v) \in E_{t}^{\mathrm{re}}, t^{\prime} \in \operatorname{chldr}(t), u \in\right.$ $\left.\chi\left(t^{\prime}\right)\right\} \cup\left\{f_{t^{\prime}}^{u}, f_{t}^{u} \mid t^{\prime} \in \operatorname{chldr}(t), u \in \chi(t) \cap \chi\left(t^{\prime}\right)\right\}$. Observe that $\mathcal{T}^{\prime}$ is a valid TD of $\mathcal{G}_{\Pi}$. Further, by construction we have $\left|\chi^{\prime}(t)\right| \leq 2 \cdot|\chi(t)|+(4+\ell) \cdot k+(\ell+1) \cdot|\chi(t)|$, since $\left|E_{t}^{\mathrm{re}}\right| \leq k$. The claim sustains for nice TDs $(\ell=2)$.

Corollary 4 (Runtime). Reduction $R$ as proposed in this section runs for a given instance $I=(G, P)$ of the Disjoint Paths Problem with $G=(V, E)$, and a pair-connected, nice TD $\mathcal{T}$ of $I$ of width $k$ and $h$ many nodes, in time $\mathcal{O}(k \cdot h)$.

Next, we are in the position of showing the main result, namely the normal ASP lower bound.

Theorem 5 (Lower bound). Consider an arbitrary normal or HCF program $\Pi$, where $k$ is the treewidth of the primal graph of $\Pi$. Then, unless ETH fails, the consistency problem for $\Pi$ cannot be solved in time $2^{o(k \cdot \log (k))} \cdot \operatorname{poly}(|\operatorname{at}(\Pi)|)$.

Proof. Let $(G, P)$ be an instance of the Disjoint Paths Problem. First, we construct [16] a nice TD $\mathcal{T}$ of $G=(V, E)$ of treewidth $k$ in time $c^{k} \cdot|V|$ for some constant $c$ such that the width of $\mathcal{T}$ is at most $5 k+4$. Then, we turn the result into a pair-connected $\operatorname{TD} \mathcal{T}^{\prime}=\left(T^{\prime}, \chi^{\prime}\right)$, thereby having width at most $k^{\prime}=$
$2 \cdot(5 k+4)+6$. Then, we construct program $\Pi=R\left(I, \mathcal{T}^{\prime}\right)$. By Lemma 1 , the treewidth of $\mathcal{G}_{\Pi}$ is in $\mathcal{O}\left(k^{\prime}\right)$, which is in $\mathcal{O}(k)$. Assume towards a contradiction that consistency of $\Pi$ can be decided in time $2^{o(k \cdot \log (k))} \cdot \operatorname{poly}(|\operatorname{at}(\Pi)|)$. By correctness of $R$ (Theorem 4), this solves $(G, P)$, contradicting Proposition 1 .

Our reduction works by construction for any pair-connected TD. Consequently, this immediately yields a lower bound for the larger parameter pathwidth, which is similar to treewidth, as defined in the preliminaries.

Corollary 5 (Pathwidth lower bound). Consider any normal or HCF program $\Pi$, where $k$ is the pathwidth of the primal graph of $\Pi$. Then, unless ETH fails, the consistency problem for $\Pi$ cannot be solved in time $2^{o(k \cdot \log (k))}$. poly $(|a t(\Pi)|)$.

From Theorem 5, we infer that a general reduction from normal or HCF programs to SAT formulas cannot (uner ETH) avoid the treewidth (pathwidth) overhead, which renders our reduction from the previous section ETH-tight.

Corollary 6 (ETH-tightness of the Reduction to SAT). Under ETH, the increase of treewidth of the reduction using Formulas (1)-(6) cannot be significantly improved.

Proof. Assume towards a contradiction that one can reduce from an arbitrary normal ASP program $\Pi$, where $k$ is the treewidth of $\mathcal{G}_{\Pi}$ to a SAT formula, whose treewidth is in $o(k \cdot \log (k))$. Then, this contradicts Theorem 5, as we can use an algorithm [85, 49 for SAT being single exponential in the treewidth, thereby deciding consistency of $\Pi$ in time $2^{o(k \cdot \log (k))} \cdot \operatorname{poly}(\mid$ at $(\Pi) \mid)$.

Further consequences of our construction are discussed in E.

## 5 An Empirical Study of Treewidth-Aware Reductions

Despite the lower bound of the previous section, we show that the reductions of this work could still have practical impact. Recall the formalization of our treewidth-aware reduction of Section 3.1, which is guided along a tree decomposition. We implemented this translation in Python, resulting in the prototypical tool asp2sat $4^{4}$, in order to compare and study the effect on the (tree)width in practice.

So the overall goal of this section is to provide a neutral comparison, without explicitly focusing on beating existing ASP solvers. The reason for this goal lies in two reasons. First of all, there is a recent observation that modern (SAT) solvers have been highly optimized for current hardware for many years,

[^3]cf., 47, which also takes time for treewidth-based solvers. Further, finding ways to efficiently utilize treewidth seems to be still an ongoing process. While over the last couple of years, efficient solvers emerged, a clear set of techniques has not yet been fully settled. This can be witnessed by a couple of solvers that adhere to parameterized complexity and in particular to treewidth. Some techniques for solvers on Boolean formulas at least seem to be particular wellsuited for counting problems, e.g., based on dynamic programming 49, 48, hybrid solving [62, 10], which also includes the winner [69, 70, 42] of two tracks of the most recent model counting competition [43], as well as compact knowledge compilation 32, 33. However, treewidth also allows to improve solving hard decision problems with the help of knowledge compilation 25 .

Despite this still incomplete picture on potential applications for treewidth, we present the usefulness and potential impact of our reduction in the form of a neutral comparison based on treewidth upper bounds. Note that computing treewidth itself is NP-complete in general [5], so in this section we only focus on approximating treewidth upper bounds. So, in order to show that the reduction of Section 3.1 indeed bounds the treewidth, we compare treewidth upper bounds on our reduction with treewidth upper bounds on an established tool of the literature 65]. These upper bounds are obtained with an efficient decomposer, called $h t d$, which aims at obtaining decent tree decompositions fast [2]. However, especially if the instances are of high (tree)widths, the obtained upper bounds via the heuristics are oftentimes imprecise, cf. [28, 51]. Note that compared to treewidth, htd only provides upper bounds, where sometimes causing additional edges in the graph representation might strangely yield smaller widths. Nevertheless, for initial estimations and for studying trends, such decomposers still proved valueable in many treewidth-based solvers.

Our implementation asp2sat works as follows:

1. First, we parse the input program $\Pi$, using the Python API of clingo 55.
2. Then, we compute an initial treewidth upper bound for the primal graph of $\Pi$ using htd [2], which heuristically provides us a tree decomposition.
3. Next, we perform our TD-guided reduction of Section 3.1 along the lines of the TD of the previous step.
4. Finally, we use again htd in order to obtain a treewidth upper bound on the result.

Compared Translations In our experiments, we mainly compare treewidth upper bounds of the following translations.

- asp2sat: we use version 1.2 of htd as well as version 4.5 of clingo in order to translate from ASP to SAT.
- lp2sat: instances are translated [17] to CNFs by lp2normal 2.18 in combination with lp2atomic 1.17 and lp2sat 1.24. This allows us to compare the effect of global level mappings with the local level mappings of asp2sat.

Benchmark Scenarios In order to evaluate asp2sat, we considered the following scenarios.

S1 Acyclic Instances: Comparing treewidth overhead on instances, whose positive dependency graph is acyclic. For asp2sat this still leaves the guidance of proofs along a tree decomposition.

S2 Cyclic Instances: Evaluating the treewidth overhead on instances with cycles, which means for asp2sat that local level mappings are used.

S2b Cyclic Instances: Studying the treewidth overhead on cyclic instances as in S2, but here we use global level mappings for asp2sat and still guide proofs along decompositions.

For Scenarios S1 and S2, asp2sat uses local level mappings as defined in Section 3.1, whereas for S2b the proofs are still guided along decompositions, which is then, however, mixed with global level mappings similar as in lp2sat.

Benchmark Instances For Scenario S1, we used instances from recent applications that where heavily trained in competitions. In fact there are still active competitions that utilize answer set programming, which is in contrast to the ASP competition (ASPCOMP), whose last edition dates back to 2017. One such active competitition is the international competition on computational models of argumentation [77], which has a long tradition of applying ASP. So, for Scenario S1 our instances stem from the abstract argumentation competition [77] from 2019, since at the time of writing the instances for 2021 were not available. Then, on top we used ASP encodings for three canonical problems in that area [77: Computing admissible, complete, and stable extensions, whose ASP encodings were taken from the ASPARTIX suite [34]. The obtained programs of those instances with the resulting encodings are acyclic.

For Scenario S2 (S2b) we use real-world graphs, more specifically, public transport networks of several transport agencies over the world 40 as instances. They also have been used in the so-called PACE challenge competitions 2016 and 2017 [28]. In total these instances amount to 561 graph networks and 2553 subgraphs with a focus on different transportation modes. On top of these networks, we encoded reachability, where for each instances we assume the station with the smallest and largest index to be the start and end stations, respectively.

Note that all our instances including the corresponding ASP encoding and raw data of our benchmark runs are open source and are publicly available on github at github.com/hmarkus/asp2sat_translator/tree/benchmarks/results.

Benchmark Hardware All our solvers ran on a cluster consisting of 12 nodes. Each node of the cluster is equipped with two Intel Xeon E5-2650 CPUs, where each of these 12 physical cores runs at 2.2 GHz clock speed and has access to 256 GB shared RAM. Results are gathered on Ubuntu 16.04.1 LTS powered on kernel 4.4.0-139 with hyperthreading disabled using version 3.7.6 of Python3.


Figure 7: Scatter plot of asp2sat and lp2sat for Scenario S1, which compares the obtained treewidth upper bounds after translation. Observe that dots above the diagonal indicate instances, where asp2sat yields better treewidth upper bounds.

Since we consider these translation to be used in a preprocessing step, we allow for each instance and translation up to 200 seconds (timeout) and 8 GB of main memory. We compare the quality of the obtained treewidth upper bounds in the range from 0 to 80 , which is backed up by the maximal width that is reasonable for being utilized by state-of-the-art solvers, e.g., [8, 25, 49, 32, 33, 62 .

Benchmark Results First, we discuss Scenario S1, for which we present in Figure 7 a scatter plot of the treewidth upper bounds of asp2sat in comparison with lp2sat. It is easy to see that, despite the need for level mappings in S1, the usage of asp2sat yields better upper bounds in almost all the cases. Of course there are some outliers in Figure 7 below the diagonal, which could, however, also be due to the heuristical decomposer. Detailed statistical measures are given in Table 1, which confirms our observations that asp2sat slightly decreases also treewidth upper bounds on acyclic instances like the ones used in Scenario S1. Indeed, the overall increase compared to the initial treewidth upper bounds is not large for S1.

For Scenario S2, we first observe in Figure 8 that asp2sat yields significantly smaller treewidth upper bounds compared to lpsat in the range from 0 to 35 . Then, up to values of 50 , asp2sat still delivers good performance. Finally for larger bounds of asp2sat, the heuristics of htd probably suffer from larger primal graphs due to auxiliary variables, revealing that the treewidth approximation does not scale. Table 1 shows that compared to lp2sat, our translation asp2sat significantly decreases the mean and median among treewidth upper bounds for

| Scenario | $\sum$ | Translation | Treewidth upper bounds |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | min | max | mean | median | stddev |
| S1 | 478 | - | 2 | 82 | 19.7 | 14 | 18.5 |
|  |  | asp2sat | 3 | 79 | 24.5 | 17 | 19.9 |
|  |  | lp2sat | 4 | 80 | 25.5 | 18 | 19.9 |
| S2 | 861 | - | 1 | 8 | 3.7 | 4 | 1.5 |
|  |  | asp2sat | 3 | 80 | 25.6 | 19 | 19.2 |
|  |  | lp2sat | 3 | 79 | 32.4 | 29 | 17.6 |
| S2b | 1053 | - | 1 | 35 | 5.0 | 4 | 3.9 |
|  |  | asp2sat | 3 | 80 | 32.0 | 29 | 17.0 |
|  |  | lp2sat | 3 | 80 | 39.1 | 37 | 21.7 |

Table 1: Detailed results over Scenarios S1, S2, and S2b, where we depict statistical data (min, max, mean, median, stddev) among the number of instances ( $\sum$ ) solved by both lp2sat and asp2sat such that the treewith upper bound is below 80 . Note that translation "-" refers to initial treewidth upper bounds, i.e., bounds obtained by using no translation. Overall, asp2sat yields significantly lower bounds than lp2sat.

| Translation |  | Measures |  |  |  |
| :--- | ---: | ---: | ---: | ---: | ---: |
|  | Number of Clauses | Number of Atoms | Treewidth | Program Size |  |
|  | $m$ | $n$ | $k$ | $s \in \mathcal{O}(n m)$ | Runtime |
| - | $\mathcal{O}(k \log (k) \cdot(n+m))$ | $\mathcal{O}(k \log (k))$ | $\mathcal{O}\left(k^{2} \log (k) \cdot(n+m)\right)$ | $\mathcal{O}\left(k^{2} \log (k) \cdot(n+m)\right)$ |  |
| asp2sat | $\mathcal{O}\left(k^{2} \log (k) \cdot(n+m)\right)$ |  |  |  |  |
| lp2sat 65] | $\mathcal{O}(s \log (n))$ | $\mathcal{O}(n \log (n))$ | $\mathcal{O}(k \log (n))$ | $\mathcal{O}(s \log (n))$ | $\mathcal{O}(s \log (n))$ |

Table 2: Comparison of worst-case measures in terms of number of clauses (atoms), treewidth, program size, and runtime for both asp2sat and lp2sat. Note that translation "-" refers to initial values of the given input program. Overall, there are cases, where asp2sat yields smaller values than lp2sat, which however is not true in general.
a majority of the benchmark instances.
To mitigate this disadvantage of estimating treewidth in case of many auxiliary variables, we slightly adapted asp2sat such that it uses global level mappings as in lp2sat. Comparing the scatter plots of Figure 9 with Figure 8 reveals that for treewidth bounds up to about 30 local level mappings pay off. Consequently, in these cases the heuristics of htd still efficiently utilize local level mappings. However, these heuristics are still rather limited: Beyond treewidth upper bounds of 30 , Figure 9 reveals that global level mappings work better with state-of-the-art decomposers. This still leaves room for improvement for future decomposition heuristics that improve on large instances. In any case, the treewidth upper bound improvement of asp2sat compared to lp2sat is still significat in Scenario S2b, as indicated by statistical measures of Table 1 .

To provide some basic decision guideline, whether for some specific setting asp2sat or lp2sat might be more appropriate, we briefly summarize the hard facts of both approaches in Table 2. Note that there are also further flavors and extensions of both approaches, cf. , e.g., [53, 18, 39].


Figure 8: Scatter plot of asp2sat and lp2sat for Scenario S2, comparing treewidth upper bounds, similar to Figure 7 Observe that asp2sat performs best up to about 50.

## 6 Discussion, Conclusion, and Future Work

Understanding the hardness of ASP and the underlying reasons has attracted the attention of the KR community for a long time. This paper discusses this question from a different angle, which hopefully will provide new insights into the hardness of ASP and foster follow-up work. The results in this paper indicate that, at least from a structural point of view, deciding the consistency of ASP is already harder than SAT, since ASP programs might compactly represent structural dependencies within the formalism. More concretely, compiling the hidden structural dependencies of a program to a SAT formula, measured in terms of the well-studied parameter treewidth, causes a blow-up of the treewidth of the resulting formula. In the light of a known result [6] on the correspondence of treewidth and the resolution width applied in SAT solving, this reveals that ASP consistency might be indeed harder than solving SAT. We further presented a reduction from ASP to SAT that is aware of the treewidth in the sense that the reduction causes not more than this inevitable blow-up of the treewidth in the worst-case. Finally, we present an empirical evaluation of treewidth upper bounds obtained via standard heuristics for treewidth, where we compare our reduction against existing translations. In both cyclic as well as acyclic scenarios, our reduction and adaptions thereof seems promising and one might consider treewidth-aware reductions as a preprocessing tool in a portfolio setting. Interestingly, the reduction developed in this work already lead to follow-up studies [39] and treewidth-aware cycle breaking [36].

Our paper raises several questions for future work. While in this work, we


Figure 9: Scatter plot of asp2sat and lp2sat for Scenario S2b, which is similar to S2, but without local level mappings. Observe that compared to Figure 8 , asp2sat improves the result in almost all the cases.
consider treewidth-aware reductions for head-cycle-free programs, the construction can be also extended to cover the SModels intermediate format [37, which can be obtained via typical grounders that are capable of the ASP-Core-2 format [24] including, e.g., aggregates and choice rules. Intuitively, the treatment of choice rules is rather similar to existing approaches for treewidth 44 and the SModels format does not allow aggregates (so it needs to be removed during grounding). Still, we think it is an interesting question for future work, how more advanced formats like aspif [56] that also considers hybrid solving can be treated for treewidth. However, for weight rules, treewidth alone is intractable, i.e., it is not expected that such a reduction exists [83]. Further, for disjunctive rules one might expect a double exponential runtime, cf. [64, 41]. In terms of lower bounds, our results naturally carry over for more expressive fragments of logic programs. It might be still interesting to generalize our results to extended formalisms like epistemic logic programs, for which only certain lower bound results are known [46, 61].

Currently, we are working on different treewidth-aware reductions to SAT and further variants thereof, and how these variants perform in different settings (consistency vs. counting). Moreover, we are curious about treewidth-aware reductions to SAT, which preserve answer sets bijectively or are modular 65]. We hope this work might reopen the quest to study the correspondence of treewidth and ASP solving similarly to [6] for SAT. Also investigating further structural parameters "between" treewidth and directed variants of treewidth could lead to new insights, since for ASP, directed measures [15] often do not yield ef-
ficient algorithms. Given the fine-grained expressiveness results for different (sub-)classes of normal programs and the resulting expressive power hierarchy [65], we are curious to see also studies in this direction and to which extent results might differ, when further restricting to treewidth-aware reductions. Of particular interest might be the question of whether one can devise a different hardness proof for normal ASP and treewidth (cf., Section 4), such that only unary rules are used.

## Acknowledgements

We would like to thank the reviewers of the conference paper [60] as well as all involved reviewers of this work for their detailed, constructive and therefore extremely valuable feedback. Special appreciation goes to Andreas Pfandler for early discussions as well as to Jorge Fandinno for spotting a technical issue in an earlier version of the conference paper [60]. Finally, we acknowledge Rafael Kiesel for his contribution in an early version of asp2sat. This work has been supported by the Austrian Science Fund (FWF), Grants P32830 and Y698, as well as the Vienna Science and Technology Fund, Grant WWTF ICT19-065.

## References

[1] D. Abels, J. Jordi, M. Ostrowski, T. Schaub, A. Toletti, and P. Wanko. Train scheduling with hybrid ASP. In $L P N M R$, volume 11481 of $L N C S$, pages 3-17. Springer, 2019.
[2] M. Abseher, N. Musliu, and S. Woltran. htd - A free, open-source framework for (customized) tree decompositions and beyond. In CPAIOR'17, volume 10335 of $L N C S$, pages 376-386. Springer, 2017.
[3] M. Alviano, G. Amendola, C. Dodaro, N. Leone, M. Maratea, and F. Ricca. Evaluation of disjunctive programs in WASP. In LPNMR'19, volume 11481 of $L N C S$, pages 241-255. Springer, 2019.
[4] M. Alviano and C. Dodaro. Completion of disjunctive logic programs. In IJCAI'16, pages 886-892. IJCAI/AAAI Press, 2016.
[5] S. Arnborg, D. G. Corneil, and A. Proskurowski. Complexity of finding embeddings in a k-tree. SIAM J. Alg. Disc. Meth., 8:277-284, 1987.
[6] A. Atserias, J. K. Fichte, and M. Thurley. Clause-learning algorithms with many restarts and bounded-width resolution. J. Artif. Intell. Res., 40:353373, 2011.
[7] M. Balduccini, M. Gelfond, and M. Nogueira. Answer set based design of knowledge systems. Ann. Math. Artif. Intell., 47(1-2):183-219, 2006.
[8] M. Bannach and S. Berndt. Practical access to dynamic programming on tree decompositions. Algorithms, 12(8):172, 2019.
[9] R. Ben-Eliyahu and R. Dechter. Propositional semantics for disjunctive logic programs. Ann. Math. Artif. Intell., 12(1):53-87, 1994.
[10] V. Besin, M. Hecher, and S. Woltran. Utilizing Treewidth for Quantitative Reasoning on Epistemic Logic Programs. Theory Pract. Log. Program., 21(5):575-592, 2021. Winner of the Best Paper Award as well as the Best Student Paper Award at ICLP'21.
[11] M. Bichler, M. Morak, and S. Woltran. selp: A single-shot epistemic logic program solver. Theory Pract. Log. Program., 20(4):435-455, 2020.
[12] N. Bidoít and C. Froidevaux. Negation by default and unstratifiable logic programs. Theo. Comput. Science, 78(1):85-112, 1991.
[13] A. Biere, M. Heule, H. van Maaren, and T. Walsh, editors. Handbook of Satisfiability, volume 185 of Frontiers in Artif. Intellig. and Applications. IOS Press, Feb. 2009.
[14] B. Bliem, M. Morak, M. Moldovan, and S. Woltran. The impact of treewidth on grounding and solving of answer set programs. J. Artif. Intell. Res., 67:35-80, 2020.
[15] B. Bliem, S. Ordyniak, and S. Woltran. Clique-width and directed width measures for answer-set programming. In ECAI'16, volume 285 of FAIA, pages 1105-1113. IOS Press, 2016.
[16] H. L. Bodlaender, P. G. Drange, M. S. Dregi, F. V. Fomin, D. Lokshtanov, and M. Pilipczuk. A c ${ }^{\mathrm{k}} \mathrm{n}$ 5-Approximation Algorithm for Treewidth. SIAM J. Comput., 45(2):317-378, 2016.
[17] J. Bomanson. lp2normal - A normalization tool for extended logic programs. In LPNMR'17, volume 10377 of $L N C S$, pages 222-228. Springer, 2017.
[18] J. Bomanson, M. Gebser, T. Janhunen, B. Kaufmann, and T. Schaub. Answer set programming modulo acyclicity. Fundam. Inform., 147(1):6391, 2016.
[19] J. Bomanson and T. Janhunen. Normalizing cardinality rules using merging and sorting constructions. In $L P N M R$ '13, volume 8148 of $L N C S$, pages 187-199. Springer, 2013.
[20] G. Brewka, T. Eiter, and M. Truszczyński. Answer set programming at a glance. Comm. of the ACM, 54(12):92-103, 2011.
[21] P. Cabalar, J. Fandinno, J. Garea, J. Romero, and T. Schaub. eclingo : A solver for epistemic logic programs. Theory Pract. Log. Program., 20(6):834-847, 2020.
[22] P. Cabalar, J. Fandinno, T. Schaub, and P. Wanko. A Uniform Treatment of Aggregates and Constraints in Hybrid ASP. In KR'20, pages 193-202, 92020.
[23] P. Cabalar, J. Fandinno, T. Schaub, and P. Wanko. An ASP semantics for constraints involving conditional aggregates. In ECAI'20, volume 325 of Frontiers in Artif. Intellig. and Applications, pages 664-671. IOS Press, 2020.
[24] F. Calimeri, W. Faber, M. Gebser, G. Ianni, R. Kaminski, T. Krennwallner, N. Leone, M. Maratea, F. Ricca, and T. Schaub. Asp-core-2 input language format. Theory Pract. Log. Program., 20(2):294-309, 2020.
[25] G. Charwat and S. Woltran. Expansion-based QBF solving on tree decompositions. Fundam. Inform., 167(1-2):59-92, 2019.
[26] K. L. Clark. Negation as failure. In Logic and Data Bases, Advances in Data Base Theory, pages 293-322. Plemum Press, 1977.
[27] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, M. P. Dániel Marx, M. Pilipczuk, and S. Saurabh. Parameterized Algorithms. Springer, 2015.
[28] H. Dell, C. Komusiewicz, N. Talmon, and M. Weller. The pace 2017 parameterized algorithms and computational experiments challenge: The second iteration. In IPEC'17, LIPIcs, pages 30:1-30:13. Dagstuhl Publishing, 2017.
[29] R. Diestel. Graph Theory, 4th Edition, volume 173 of Graduate Texts in Mathematics. Springer, 2012.
[30] J. Dix, G. Gottlob, and V. W. Marek. Reducing disjunctive to nondisjunctive semantics by shift-operations. Fundam. Inform., 28(1-2):87100, 1996.
[31] R. G. Downey and M. R. Fellows. Fundamentals of Parameterized Complexity. Texts in Computer Science. Springer, 2013.
[32] J. M. Dudek, V. Phan, and M. Y. Vardi. ADDMC: weighted model counting with algebraic decision diagrams. In $A A A I^{\prime} 20$, pages 1468-1476. AAAI Press, 2020.
[33] J. M. Dudek, V. H. N. Phan, and M. Y. Vardi. DPMC: weighted model counting by dynamic programming on project-join trees. In CP'20, volume 12333 of $L N C S$, pages 211-230. Springer, 2020.
[34] W. Dvorák, S. A. Gaggl, A. Rapberger, J. P. Wallner, and S. Woltran. The ASPARTIX system suite. In H. Prakken, S. Bistarelli, F. Santini, and C. Taticchi, editors, Computational Models of Argument (COMMA), volume 326 of Frontiers in Artificial Intelligence and Applications, pages 461-462. IOS Press, 2020.
[35] T. Eiter and G. Gottlob. On the computational cost of disjunctive logic programming: Propositional case. Ann. Math. Artif. Intell., 15(3-4):289323, 1995.
[36] T. Eiter, M. Hecher, and R. Kiesel. Treewidth-aware cycle breaking for algebraic answer set counting. In $K R^{\prime} 21$, pages 269-279, 2021.
[37] I. Elkabani, E. Pontelli, and T. C. Son. Smodels ${ }^{\text {a }}$ - A system for computing answer sets of logic programs with aggregates. In LPNMR'05, volume 3662 of $L N C S$, pages 427-431. Springer, 2005.
[38] F. Fages. Consistency of clark's completion and existence of stable models. Methods Log. Comput. Sci., 1(1):51-60, 1994.
[39] J. Fandinno and M. Hecher. Treewidth-aware complexity in ASP: not all positive cycles are equally hard. In $A A A I^{\prime} 21$, pages $6312-6320$. AAAI Press, 2021.
[40] J. K. Fichte. daajoe/gtfs2graphs - a GTFS transit feed to graph format converter. https://github.com/daajoe/gtfs2graphs at commit 219944893f874b365de1ed87fc265fd5d19d5972, 2016.
[41] J. K. Fichte and M. Hecher. Treewidth and counting projected answer sets. In LPNMR'19, volume 11481 of $L N C S$, pages 105-119. Springer, 2019.
[42] J. K. Fichte and M. Hecher. Model Counting Competition 2021 Awards, 2021. Available at: https://mccompetition.org/assets/files/2021/MC2021_ awards.pdf.
[43] J. K. Fichte, M. Hecher, and F. Hamiti. The Model Counting Competition 2020. ACM J. Exp. Algorithmics, 26, 2021.
[44] J. K. Fichte, M. Hecher, M. Morak, and S. Woltran. Answer set solving with bounded treewidth revisited. In LPNMR'17, volume 10377 of $L N C S$, pages 132-145. Springer, July 2017.
[45] J. K. Fichte, M. Hecher, M. Morak, and S. Woltran. Exploiting treewidth for projected model counting and its limits. In SAT'18, volume 10929 of $L N C S$, pages 165-184. Springer, July 2018.
[46] J. K. Fichte, M. Hecher, and A. Pfandler. Lower Bounds for QBFs of Bounded Treewidth. In LICSㅇ20, pages 410-424. Assoc. Comput. Mach., 2020.
[47] J. K. Fichte, M. Hecher, and S. Szeider. A Time Leap Challenge for SATSolving. In CP'20, volume 12333 of $L N C S$, pages 267-285. Springer, 2020.
[48] J. K. Fichte, M. Hecher, P. Thier, and S. Woltran. Exploiting Database Management Systems and Treewidth for Counting. Theory Pract. Log. Program., pages 1—30, 2021.
[49] J. K. Fichte, M. Hecher, and M. Zisser. An Improved GPU-Based SAT Model Counter. In CP'19, volume 11802 of $L N C S$, pages 491-509. Springer, 2019.
[50] J. K. Fichte, M. Kronegger, and S. Woltran. A multiparametric view on answer set programming. Ann. Math. Artif. Intell., 86(1-3):121-147, 2019.
[51] J. K. Fichte, N. Lodha, and S. Szeider. Sat-based local improvement for finding tree decompositions of small width. In SAT, volume 10491 of $L N C S$, pages 401-411. Springer, 2017.
[52] J. Flum and M. Grohe. Parameterized Complexity Theory, volume XIV of Theo. Comput. Science. Springer, 2006.
[53] M. Gebser, T. Janhunen, and J. Rintanen. Answer set programming as SAT modulo acyclicity. In ECAI'14, volume 263 of Frontiers in Artif. Intellig. and Applications, pages 351-356. IOS Press, 2014.
[54] M. Gebser, R. Kaminski, B. Kaufmann, and T. Schaub. Answer Set Solving in Practice. Morgan \& Claypool, 2012.
[55] M. Gebser, R. Kaminski, B. Kaufmann, and T. Schaub. Clingo = ASP + control: Preliminary report. CoRR, abs/1405.3694, 2014.
[56] M. Gebser, R. Kaminski, B. Kaufmann, and T. Schaub. Multi-shot ASP solving with clingo. Theory Pract. Log. Program., 19(1):27-82, 2019.
[57] M. Gebser, B. Kaufmann, and T. Schaub. Solution enumeration for projected boolean search problems. In CPAIOR'09, volume 5547 of $L N C S$, pages 71-86. Springer, 2009.
[58] M. Gelfond and V. Lifschitz. Classical negation in logic programs and disjunctive databases. New Generation Comput., 9(3/4):365-386, 1991.
[59] C. Guziolowski, S. Videla, F. Eduati, S. Thiele, T. Cokelaer, A. Siegel, and J. Saez-Rodriguez. Exhaustively characterizing feasible logic models of a signaling network using answer set programming. Bioinformatics, 29(18):2320-2326, 2013. Erratum see Bioinformatics 30, 13, 1942.
[60] M. Hecher. Treewidth-aware Reductions of Normal ASP to SAT - Is Normal ASP Harder than SAT after All? In KR'20, pages 485-495, 9 2020. Winner of the Best Student Paper Award at KR'20.
[61] M. Hecher, M. Morak, and S. Woltran. Structural decompositions of epistemic logic programs. In AAAI'20, pages 2830-2837. AAAI Press, 2020.
[62] M. Hecher, P. Thier, and S. Woltran. Taming High Treewidth with Abstraction, Nested Dynamic Programming, and Database Technology. In SAT'20, volume 12178 of LNCS, pages 343-360. Springer, 2020.
[63] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? J. of Computer and System Sciences, 63(4):512530, 2001.
[64] M. Jakl, R. Pichler, and S. Woltran. Answer-set programming with bounded treewidth. In IJCAI'09, volume 2, pages 816-822, 2009.
[65] T. Janhunen. Some (in)translatability results for normal logic programs and propositional theories. J. of Applied Non-Classical Logics, 16(1-2):3586, 2006.
[66] T. Janhunen and I. Niemelä. The answer set programming paradigm. AI Magazine, 37(3):13-24, 2016.
[67] H. Kleine Büning and T. Lettman. Propositional logic: deduction and algorithms. Cambridge University Press, 1999.
[68] T. Kloks. Treewidth. Computations and Approximations, volume 842 of LNCS. Springer, 1994.
[69] T. Korhonen. Github repository of sharpsat-td, 2021. Available at: https: //github.com/Laakeri/sharpsat-td.
[70] T. Korhonen and M. Järvisalo. Integrating Tree Decompositions into Decision Heuristics of Propositional Model Counters. In CP'21, volume 210 of LIPIcs, pages 8:1-8:11. Dagstuhl Publishing.
[71] M. Lackner and A. Pfandler. Fixed-parameter algorithms for finding minimal models. In KR'12. AAAI Press, 2012.
[72] M. Lampis and V. Mitsou. Treewidth with a quantifier alternation revisited. In IPEC'17, volume 89, pages 26:1-26:12. Dagstuhl Publishing, 2017.
[73] V. Lifschitz and A. A. Razborov. Why are there so many loop formulas? ACM Trans. Comput. Log., 7(2):261-268, 2006.
[74] F. Lin and J. Zhao. On tight logic programs and yet another translation from normal logic programs to propositional logic. In IJCAI'03, pages 853-858. Morgan Kaufmann, August 2003.
[75] F. Lin and Y. Zhao. ASSAT: computing answer sets of a logic program by SAT solvers. Artif. Intell., 157(1-2):115-137, 2004.
[76] D. Lokshtanov, D. Marx, and S. Saurabh. Slightly superexponential parameterized problems. In SODA'11, pages 760-776. SIAM, 2011.
[77] J.-G. Mailly, E. Lonca, J.-M. Lagniez, and J. Rossit. International Competition on Computational Models of Argumentation 2021, 2021. Website: http://argumentationcompetition.org/2021/rules.html.
[78] W. Marek and M. Truszczyński. Autoepistemic logic. J. of the ACM, 38(3):588-619, 1991.
[79] D. Marx and V. Mitsou. Double-Exponential and Triple-Exponential Bounds for Choosability Problems Parameterized by Treewidth. In ICALP'16, volume 55 of LIPIcs, pages 28:1-28:15. Dagstuhl Publishing, 2016.
[80] R. Niedermeier. Invitation to Fixed-Parameter Algorithms, volume 31 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, 2006.
[81] I. Niemelä, P. Simons, and T. Soininen. Stable model semantics of weight constraint rules. In $L P N M R$ '99, volume 1730 of $L N C S$, pages $317-331$. Springer, Dec. 1999.
[82] M. Nogueira, M. Balduccini, M. Gelfond, R. Watson, and M. Barry. An AProlog decision support system for the Space Shuttle. In $P A D L^{\prime} 01$, volume 1990 of LNCS, pages 169-183. Springer, 2001.
[83] R. Pichler, S. Rümmele, S. Szeider, and S. Woltran. Tractable answer-set programming with weight constraints: bounded treewidth is not enough. Theory Pract. Log. Program., 14(2), 2014.
[84] N. Robertson and P. D. Seymour. Graph minors II: Algorithmic aspects of tree-width. J. Algorithms, 7:309-322, 1986.
[85] M. Samer and S. Szeider. Algorithms for propositional model counting. $J$. Discrete Algorithms, 8(1):50-64, 2010.
[86] T. Schaub and S. Woltran. Special issue on answer set programming. KI, $32(2-3): 101-103,2018$.
[87] P. Scheffler. A Practical Linear Time Algorithm for Disjoint Paths in Graphs with Bounded Tree-width. Technical Report Report-396-1994, TU Berlin, 1994. Available at: http://www.redaktion.tu-berlin.de/ fileadmin/i26/download/AG_DiskAlg/FG_KombOptGraphAlg/preprints/ 1994/Report-396-1994.ps.gz.
[88] M. Truszczynski. Trichotomy and dichotomy results on the complexity of reasoning with disjunctive logic programs. Theory Pract. Log. Program., 11(6):881-904, 2011.

## A Correctness of the Reduction of Section 3.1

Theorem 2 (Correctness). The reduction from an HCF program $\Pi$ and a $T D \mathcal{T}=(\widetilde{T}, \chi)$ of $\mathcal{G}_{\Pi}$ to SAT formula $F$ consisting of Formulas (1)- is correct. Precisely, for each answer set of $\Pi$ there is a model of $F$ and vice versa.

Proof. " $\Rightarrow$ ": Assume an answer set $M$ of $\Pi$. Then, there is an ordering $\varphi$ over $\operatorname{at}(\Pi)$, where every atom of $M$ is proven. Next, we construct a model $I$ of $F$ as follows. For each $x \in \operatorname{at}(\Pi)$, we let (c1) $x \in I$ if $x \in M$. For each node $t$ of $T$, and $x \in \chi(t)$ : (c2) For every $l \in \llbracket x \rrbracket_{t, i}$ with $i=\hat{\varphi}_{t}(x)$, we set $l \in I$ if $l$ is a variable. (c3) If there is a rule $r \in \Pi_{t}$ proving $x$, we let both $p_{<t}^{x}, p_{t}^{x} \in I$. Finally, (c4) we set $p_{<t}^{x} \in I$, if $p_{<t^{\prime}}^{x} \in I$ for $t^{\prime} \in \operatorname{chldr}(t)$.

It remains to show that $I$ is indeed a model of $F$. By (c1), Formulas (1) are satisfied by $I$. Further, by (c2) of $I$, the order of $\varphi$ is preserved among $\chi(t)$ for each node $t$ of $T$, therefore Formulas (2) are satisfied by $I$. Further, by definition of TDs, for each rule $r \in \Pi$ there is a node $t$ with $r \in \Pi_{t}$. Consequently, $M$ is proven with ordering $\varphi$, for each $x \in M$ there is a node $t$ and a rule $r \in \Pi_{t}$ proving $x$. Then, Formulas (6) are satisfied by $I$ due to (c3), and Formulas (5) are satisfied by $I$ due to (c4). Finally, by connectedness of TDs, also Formulas (3) and (4) are satisfied.
" $\Leftarrow$ ": Assume any model $I$ of $F$. Then, we construct an answer set $M$ of $\Pi$ as follows. We set $a \in M$ if $a \in I$ for any $a \in \operatorname{at}(\Pi)$. We define for each node $t$ a $t$-local ordering $\varphi_{t}$, where we set $\varphi_{t}(x)$ to $j$ for each $x \in \chi(t)$ such that $j$ is the decimal number of the binary number for $x$ in $t$ given by $I$. Concretely, $\varphi_{t}(x):=j$, where $j$ is such $I \models \llbracket x \rrbracket_{t, j}$. Then, we define an ordering $\varphi$ iteratively as follows. We set $\varphi(a):=0$ for each $a \in \operatorname{at}(\Pi)$, where there is no node $t$ of $T$ with $\varphi_{t}(b)<\varphi_{t}(a)$. Then, we set $\varphi(a):=1$ for each $a \in \operatorname{at}(\Pi)$, where there is no node $t$ of $T$ with $\varphi_{t}(b)<\varphi_{t}(a)$ for some $b \in \chi(t)$ not already assigned in the previous iteration, and so on. In turn, we construct $\varphi$ iteratively by assigning increasing values to $\varphi$. Observe that $\varphi$ is well-defined, i.e., each atom $a \in \operatorname{at}(\Pi)$ gets a unique value since it cannot be the case for two nodes $t, t^{\prime}$ and atoms $x, x^{\prime} \in \chi(t) \cap \chi\left(t^{\prime}\right)$ that $\varphi_{t}(x)<\varphi_{t}\left(x^{\prime}\right)$, but $\varphi_{t^{\prime}}(x) \geq \varphi_{t^{\prime}}\left(x^{\prime}\right)$. Indeed, this is prohibited by Formulas (2) and connectedness of $\mathcal{T}$ ensuring that $\mathcal{T}$ restricted to $x$ is still connected.

It remains to show that $\varphi$ is an ordering for $\Pi$ proving $M$. Assume towards a contradiction that there is an atom $a \in M$ that is not proven. Observe that either $a$ is in the bag $\chi(n)$ of the root node $n$ of $T$, or it is forgotten below $n$. In both cases we require a node $t$ such that $p_{<t}^{x} \notin I$ by Formulas (4) and (3), respectively. Consequently, by connectedness of $\mathcal{T}$ and Formulas (5) there is a node $t^{\prime}$, where $p_{t^{\prime}}^{x} \in I$. But then, since Formulas (6) are satisfied by $I$, there is a rule $r \in \Pi_{t^{\prime}}$ proving $a$ with $\varphi_{t^{\prime}}$. Therefore, since by construction of $\varphi$, there cannot be a node $t$ of $T$ with $x, x^{\prime} \in \chi(t), \varphi_{t}(x)<\varphi_{t}\left(x^{\prime}\right)$, but $\varphi(x) \geq \varphi\left(x^{\prime}\right), r$ is proving $a$ with $\varphi$.


Figure 10: Graph $\mathcal{G}_{\Pi^{\prime}}$ (left) and a tree decomposition $\mathcal{T}^{\prime}$ of $\mathcal{G}_{\Pi^{\prime}}$ (right), where program $\Pi^{\prime}$ is given in Example 14 .

## B Strengthening the Reduction of Section 3.1

Next, we strengthen the reduction of Section 3.1 to remove some duplicate $\mathcal{T}$ local orderings for a particular answer set of $\Pi$.

In Formulas (27), we ensure that if a variable $x \in \operatorname{at}(\Pi)$ is set to false, then its ordering position is zero. Formulas $(28)$ make sure that if the position of $x$ is set to $i \geq 1$ in node $t$, there has to be a bag atom $y$ having position $i-1$. Intuitively, if this is not the case we could shift the position of $x$ from $i$ to $i-1$. Finally, Formulas (29) ensure that whenever in a node $t$ there is a rule $r \in \Pi_{t}$ with $x \in H_{r}$ and $x$ has position $i \geq 1$, either there is at least one atom $y \in B_{r}^{+}$ having position $i-1$, or $r$ is not proving $x$.

$$
\begin{align*}
& \neg x \longrightarrow \bigwedge_{1 \leq j \leq\lceil\log (|\chi(t)|)\rceil} \neg b_{x_{t}}^{j} \quad \text { for each } x \in \chi(t)  \tag{27}\\
& \llbracket x \rrbracket_{t, i} \longrightarrow \bigvee \llbracket y \rrbracket_{t, i-1} \quad \text { for each } x \in \chi(t), 1 \leq i<|\chi(t)|  \tag{28}\\
& y \in \chi(t) \backslash\{x\} \\
& \bigwedge_{H_{r}, 1 \leq i<|\chi(t)|}\left(\llbracket x \rrbracket_{t, i} \longrightarrow \bigvee_{b \in B_{r}^{+}} \neg b \vee\left(b \not_{t} x\right) \vee\right. \\
& \left.\bigvee_{\cup\left(H_{r} \backslash\{x\}\right)} a \vee \bigvee_{y \in B_{r}^{+}} \llbracket y \rrbracket_{t, i-1}\right) \quad \text { for each } x \in \chi(t) \tag{29}
\end{align*}
$$

In general, we do not expect to get rid of all redundant $\mathcal{T}$-local orderings for an answer set, though. The reason for this expectation lies in the fact that the different (chains of) rules required for setting the position for an atom $a$ that is part of cycles of $D_{\Pi}$ might be spread across the whole tree decomposition. Therefore, these local orderings might not provide the same information that we get from global orderings [65], where we have absolute values. Instead, these local orderings are insufficient to conclude absolute positions without further information. This is clarified in the following example.

Example 14. Consider the program $\Pi^{\prime}:=\{b \vee n b \leftarrow ; c \leftarrow b ; a \leftarrow c ; d \vee n d \leftarrow$ $; a \leftarrow d\}$. Observe that program $\Pi^{\prime}$ has four answer sets $\{n b, n d\},\{a, d, n b\}$, $\{a, b, c, n d\}$, as well as $\{a, b, c, d\}$. Assume the $T D \mathcal{T}^{\prime}=\left(T^{\prime}, \chi^{\prime}\right)$ of Figure 10 . whose width is 1 and equals the treewidth of $\mathcal{G}_{\Pi}$. This $T D \mathcal{T}^{\prime}$ is such that $\Pi_{t_{3}}=$
$\{a \leftarrow c\}$ and $\Pi_{t_{5}}=\{a \leftarrow d\}$. The particular issue is that node $t_{3}$ only considers atoms $a, c$ and node $t_{5}$ only considers atoms $a, d$. Now, assume answer set $M=\{a, b, c, d\}$. Then, given only $t_{3}$-local orderings and $t_{5}$-local orderings, we cannot conclude a unique, canonical global ordering for $\{a, c, d\} \subseteq M$. In particular, one could prove $a$ with either $a \leftarrow c$ or with $a \leftarrow d$ (or both). From $a$ global perspective the latter rule would be preferred to prove a, since it allows an ordering with a smaller position for $a$. This is witnessed by the corresponding ordering $\varphi:=\{b \mapsto 0, c \mapsto 1, d \mapsto 0, a \mapsto 1\}$ for $M$. If instead we use the rule $a \leftarrow c$ for proving $a$, this would require ordering $\varphi^{\prime}:=\{b \mapsto 0, c \mapsto 1, a \mapsto$ $2, d \mapsto 0\}$, i.e., $\varphi$ is preferred since $\varphi(a)<\varphi^{\prime}(a)$. However, this information is "lost" due to the usage of local orderings, which makes it hard to define canonical orderings. Therefore our constructed SAT formula yields two satisfying assignments for $M$ in this case, corresponding to proving a either with $a \leftarrow d$ or $a \leftarrow c$. In general, a TD similar to $\mathcal{T}^{\prime}$ cannot be avoided. In particular, one can construct programs, where similar situations have to occur in every TD of smallest width.

One can even devise further cases, where without absolute orders it is hard to verify whether it is indeed required that an atom precedes an other atom. This is still the case, if for each answer set $M$ of $\Pi$, and every $a \in M$, there can be only one rule $r \in \Pi$ suitable for proving $a$. From now on, we refer to such HCF programs $\Pi$ by uniquely provable. Note that even for uniquely provable programs, there might be several cycles in its positive dependency graph. In fact, the program that will be used for the hardness result of normal ASP and treewidth in Section 4 is uniquely provable. However, even for uniquely provable programs and any TD of $\mathcal{G}_{\Pi}$, there is in general no bijective correspondence between answer sets of $\Pi$ and models of Formulas (1)-(29). Consequently, one could compare different, absolute positions of orderings, cf., 65], instead of the ordering positions relative to one TD node as presented here, which requires to store for each atom in the worst-case numbers up to $\mid$ at $(\Pi) \mid-1$. Obviously, this number is then not bounded by the treewidth, and one cannot encode it without increasing the treewidth in general. Observe that even if one uses orderings on a component-by-component basis, similar to related work [65], this issue still persists in general since the whole program could be one large component.

## C Correctness of the Reduction of Section 3.2

In order to discuss correctness, we rely on the following lemma, which establishes that the reduction actually ensures and preserves the transitive closure over $\prec$ for each node of any tree decomposition.

Lemma 2 (Transitive Closure). Let $\Pi$ be an HCF program, $\mathcal{T}=(T, \chi)$ be a $T D$ of $\mathcal{G}_{\Pi}$, and let $t$ be any node of $T$. Further, let $F_{\leq t}$ consist of all Formulas (9) constructed for node $t$ and every node below $t$ in $T$, and let $\chi_{\leq t}$ be the union of bag $\chi(t)$ and all bag contents for every node below $t$ in $T$. Then,
for any model $M$ of $F_{\leq t}$ the following invariant holds: All transitive consequences of $\prec$ in $F_{\leq t}$ are preserved for $t$, i.e., we have $M \supseteq\{(x \prec y) \mid\{x, y\} \subseteq$ $\chi(t)$, there is a path $x, p_{1}, \ldots, p_{o}, y$ from $x$ to $y$ in $\mathcal{G}_{\Pi}$ with $\left\{p_{1}, \ldots, p_{o}\right\} \subseteq \chi_{\leq t}$, $\left.\left\{\left(x \prec p_{1}\right),\left(p_{1} \prec p_{2}\right), \ldots,\left(p_{o} \prec y\right)\right\} \subseteq M\right\}$.

Proof. The proof proceeds by induction. Thereby we assume the invariant holds for every child node of $t$ and show that then it also is ensured for $t$. For simplicity, we only show the ideas for nice TDs, as for non-nice TDs the following cases just overlap. Recall that a nice TD of $\mathcal{G}_{\Pi}$ of width $k=t w\left(\mathcal{G}_{\Pi}\right)$, having only $h=\mathcal{O}(|\operatorname{at}(\Pi)|)$ many nodes [68] [Lem. 13.1.2] always exists. We distinguish the following cases.

Case (type $(t)=l e a f)$ : The invariant vacuously holds since $\chi(t)=\emptyset$.
Case $(\operatorname{type}(t)=$ forget or type $(t)=$ join $)$ : In these cases, we do not encounter any new auxiliary variable, nor do we add a new formula (that has not been constructed for a node below $t$ ) of the form (9). Therefore, since the invariant holds for the child node of $t$, it is also valid for $t$.

Case $(\operatorname{type}(t)=i n t)$ : Let $\{a\}=\chi(t) \backslash \chi\left(t^{\prime}\right)$ with $\operatorname{chldr}(t)=\left\{t^{\prime}\right\}$. Assume towards a contradiction that a transitive consequence of $F_{\leq t}$ is missing, i.e., we have a path $x, p_{1}, \ldots, p_{o}, y$ from $x$ to $y$ in $\mathcal{G}_{\Pi}$ with $\left\{p_{1}, \ldots, p_{o}\right\} \subseteq \chi_{\leq t}$, but $(x \prec y) \notin M$. Then, we have either $x=a$ or $y=a$ since otherwise the consequence would have been already missing for $t^{\prime}$. We continue with the case $x=a$ since the other case works analogously. Consequently, $p_{1} \in \chi(t)$, since otherwise $\mathcal{T}$ would not be a TD of $\mathcal{G}_{\Pi}$ due to the fact that the edge $\left\{x, p_{1}\right\}$ would not occur in any TD node of $\mathcal{T}$. Then, by the induction hypothesis, we have $\left(p_{1} \prec y\right) \in M$. As a result, we obtain $(x \prec y) \in M$ since $x, p_{1}, y \in \chi(t)$, which contradicts our assumption.

Corollary 7 (Acyclicity). Let $\Pi$ be an $H C F$ program, $\mathcal{T}=(T, \chi)$ be a $T D$ of $\mathcal{G}_{\Pi}$, and let $t$ be any node of $T$. Then, the formula $F_{\leq t}^{\prime}$ consisting of all Formulas (9) and (10) for $t$ and every node below $t$, ensures that there cannot exist a model $M$ of $F_{\leq t}^{\prime}$, where the transitive consequences of $\prec$ in $F_{\leq t}^{\prime}$ as in Lemma 2 form a cycle.

Proof. The proof is a direct consequence of Lemma 2 and the observation that if there is a path $x, p_{1}, \ldots, p_{o}, x$ from $x$ to $x$ in $\mathcal{G}_{\Pi}$ with $\left(x \prec p_{1}\right),\left(p_{1} \prec\right.$ $\left.p_{2}\right), \ldots,\left(p_{o} \prec x\right) \in M$, one can reverse any $\left(y \prec y^{\prime}\right) \in M$. More concretely, for any $\left(y \prec y^{\prime}\right) \in M$ with $y, y^{\prime} \in \chi(t)$, by Formulas (9), one can derive $\left(y^{\prime} \prec y\right) \in M$, which is prohibited by Formulas 10 . Consequently, cyclic, transitive consequences over variables in $\chi_{\leq t}$ cannot occur in any model of $F_{\leq t}^{\prime}$. The claim follows since $\chi_{\leq n}=\operatorname{at}(\Pi)$ for root $n=\operatorname{root}(T)$.

Theorem 6 (Correctness and Bijectivity). The reduction from a uniquely provable program $\Pi$ and a TD $\mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$ to SAT formula $F^{\prime}$ consisting of Formulas (1), (3)-(5), and (7)-(15) is correct. Concretely, for each answer set of $\Pi$ there is exactly one model of $F^{\prime}$ and vice versa.

Proof. " $\Rightarrow$ ": Assume an answer set $M$ of $\Pi$. Then, there is a minima ${ }^{5}$ ordering $\varphi$ over at $(\Pi)$, where every atom of $M$ is proven. Next, we construct a model $I$ of $F^{\prime}$ as follows. For each $x \in \operatorname{at}(\Pi)$, we let (c1) $x \in I$ if $x \in M$. For each node $t$ of $T$, and $x, y \in \chi(t):(c 2)$ We set $(x \prec y) \in I$ if and only if $\varphi(x)<\varphi(y)$. (c3) If there is a rule $r \in \Pi_{t}$ proving $x$ using $\varphi$, we let $p_{t, r}^{x}, p_{<t}^{x}, p_{t}^{x} \in I$ as well as $p_{t}^{y \prec x}, p_{<t}^{y \prec x} \in I$ for $y \in B_{r}^{+}$. Further (c4) if there is $z \in \chi(t)$ with $|\{x, y, z\}|=3$ and $(x \prec y),(y \prec z) \in I$, we set $p_{t}^{x \prec z}, p_{\ll}^{x \prec z} \in I$. Finally, (c5) for $t^{\prime} \in \operatorname{chldr}(t)$, we set $p_{<t}^{x} \in I$ if $p_{<t^{\prime}}^{x} \in I$, as well as $p_{<t}^{y \prec x} \in I$ if $p_{<t^{\prime}}^{y \prec x} \in I$.

Observe that $I$ is indeed a model of $F^{\prime}$. By (c1), Formulas (1) are satisfied by $I$. Further, by (c2) of $I$, the order of $\varphi$ is preserved among $\chi(t)$ for each node $t$ of $T$, therefore Formulas (9) and 10 ) are satisfied by $I$. Then, due to (c3), Formulas $(7)$ and Formulas (8) are satisfied and so are Formulas (11). Further, due to both (c3) and (c4), Formulas (12) are satisfied. Further, by definition of TDs, for each rule $r \in \Pi$ there is a node $t$ with $r \in \Pi_{t}$. Consequently, $M$ is proven with ordering $\varphi$, for each $x \in M$ there is a node $t$ and a rule $r \in \Pi_{t}$ proving $x$. Then, Formulas (5) and satisfied by $I$ due to (c5). Finally, by connectedness of TDs, also Formulas (3) and (4) as well as (13) and (14) are satisfied.

It remains to show that there cannot be a model $I^{\prime} \neq I$ of $F^{\prime}$ with $I^{\prime} \cap a t(\Pi)=$ $M$. Assume towards a contradiction that such a model $I^{\prime}$ of $F^{\prime}$ indeed exists. Then, since $\Pi$ is uniquely provable, for each atom $x \in$ at $(\Pi)$, there is only one rule $r$ suitable for proving $x$. Observe that therefore $I^{\prime}$ agrees with $I$ on variables of the form $p_{t, r}^{x}$ by Formulas (7). Consequently, $I^{\prime}$ agrees with $I$ on provability variables of the form $p_{t}^{x}, p_{<t}^{x}$ by Formulas (11) and (3)-(5). Analogously $I^{\prime}$ also sets provability variables of the form $p_{t}^{x \prec y}, p_{<t}^{x \prec y}$ as $I$ by Formulas 12 and 13 (15). Then, since relation $\prec$ for $I^{\prime}$ is transitively closed by Lemma 2 and due to acyclicity of $\prec$ by Corollary 7 , we have that $I^{\prime}$ precisely gives rise to ordering $\varphi$, which contradicts the assumption.
" $\Leftarrow$ ": Assume any model $I$ of $F^{\prime}$. Then, we construct an answer set $M$ of $\Pi$ as follows. We set $a \in M$ if $a \in I$ for any $a \in \operatorname{at}(\Pi)$. Then, we define an ordering $\varphi$ iteratively as follows. We set $\varphi(a):=0$ for each $a \in$ at( $\Pi)$, where there is no atom $b \in \operatorname{at}(\Pi)$ with $(b \prec a) \in I$. Then, we set $\varphi(a):=1$ for each $a \in \operatorname{at}(\Pi)$, where there is no atom $b \in \operatorname{at}(\Pi)$ with $(b \prec a) \in I$ such that $b$ has not been already assigned in the previous notation. In turn, we construct $\varphi$ iteratively by assigning increasing values to $\varphi$. Observe that $\varphi$ is well-defined, i.e., each atom $a \in \operatorname{at}(\Pi)$ gets a unique value, since $\prec$ is transitively closed by Lemma 2 and by Corollary 7 there cannot be a cycle over relation $\prec$ for $I$.

Obviously, $M$ is a model of $\Pi$ by Formulas (1). It remains to show that $\varphi$ is an ordering for $\Pi$ proving $M$. Assume towards a contradiction that there is an atom $a \in M$ that is not proven. Observe that either $a$ is in the bag $\chi(n)$ of the root node $n$ of $T$, or it is forgotten below $n$. In both cases we require a node $t$ such that $p_{<t}^{x} \notin I$ by Formulas (4) and (3), respectively. Consequently, by connectedness of $\mathcal{T}$ and Formulas (5) there is a node $t^{\prime}$, where $p_{t^{\prime}}^{x} \in I$. But

[^4]then, since Formulas (12) are satisfied by $I$, there is a rule $r \in \Pi_{t^{\prime}}$ proving $a$ with $\varphi$.

Note that the bijectivity result above can be generalized. Let therefore $\Pi$ be an HCF program and $M$ be an answer set of $\Pi$. Then, we define the dependency graph $D_{\Pi}^{M}$ for $M$ by $D_{\Pi}^{M}:=(M, E)$ where we have an edge $(x, y) \in E$ if and only if there is a rule $r \in \Pi$ suitable for proving $y$ with $x \in B_{r}^{+}$. Observe that Theorem 6 also holds for HCF programs $\Pi$, as long as for each answer set $M$ of $\Pi$ the graph $D_{\Pi}^{M}$ is acyclic. We call such programs $\Pi$ deterministically provable and it is easy to see that every uniquely provable program is also deterministically provable. While for such deterministically provable programs there might be several rules suitable for proving an atom $a$ of such an answer set $M$, a model for $M$ of the SAT formula constructed above is still unique as it greedily aims to prove $a$ with every applicable rule, cf., Formulas (8).

Corollary 8. The reduction from a deterministically provable program $\Pi$ and a $T D \mathcal{T}=(T, \chi)$ of $\mathcal{G}_{\Pi}$ to SAT formula $F^{\prime}$ consisting of Formulas (1), (3)-(5), and (7)-15) is correct. Concretely, for each answer set of $\Pi$ there is exactly one model of $F^{\prime}$ and vice versa.

Proof. " $\Rightarrow$ ": Assume an answer set $M$ of $\Pi$. Then, there is a greedy-minimal ${ }^{6}$ ordering $\varphi$ over at $(\Pi)$, where every atom of $M$ is proven. Next, we construct a model $I$ of $F^{\prime}$ as in Theorem 6. Observe that $I$ is a model of $F^{\prime}$ by the same arguments as in Theorem 6 .

It remains to show that there cannot be a model $I^{\prime} \neq I$ of $F^{\prime}$ with $I^{\prime} \cap a t(\Pi)=$ $M$. Assume towards a contradiction that such a model $I^{\prime}$ of $F^{\prime}$ exists. Then, since $\Pi$ is deterministically provable, i.e., there is no cycle in $D_{\Pi}^{M}$ for each atom $x \in M$, every rule suitable for proving $x$ proves $x$, cf., Formulas (8). Observe that therefore $I^{\prime}$ agrees with $I$ on variables of the form $p_{t, r}^{x}$ by Formulas $(7)$. Consequently, $I^{\prime}$ agrees with $I$ on provability variables of the form $p_{t}^{x}, p_{<t}^{x}$ by Formulas (11) and (3)-(5). Analogously $I^{\prime}$ also sets provability variables of the form $p_{t}^{x \prec y}, p_{<t}^{x} \prec$ as $I$ by Formulas 12 and 13 . Then, since relation $\prec$ for $I^{\prime}$ is transitively closed by Lemma 2 and due to acyclicity of $\prec$ by Corollary 7, we have that $I^{\prime}$ precisely gives rise to ordering $\varphi$, which contradicts the assumption.
" $\Leftarrow$ ": The other direction holds by the same argument as in Theorem6.
Example 15. Recall program $\Pi$ from Example 1 and answer sets $M_{1}=\{b, c, d\}$, $M_{2}=\{b, e\}, M_{3}=\{a, c, d\}$ as well as $M_{4}=\{a, d, e\}$ of $\Pi$. Observe that the graph $D_{\Pi}^{M}$ is acyclic for any answer set $M \in\left\{M_{1}, M_{3}, M_{4}\right\}$, whereas $D_{\Pi}^{M_{2}}$ contains a cycle. Consequently, the formula $F^{\prime}$ constructed by the reduction above is guaranteed to have exactly one model for each of the answer sets $M_{1}, M_{3}, M_{4}$, which is a priori not guaranteed for $M_{2}$. However, in this case it is easy to

[^5]see that also for $M_{2}$ formula $F^{\prime}$ has only one corresponding model since, although $D_{\Pi}^{M_{2}}$ is not acyclic, there is no rule suitable for proving e without having b first.

## D Correctness of the Reduction of Section 4

For proving correctness of Theorem 4, we rely on the following key lemma.
Lemma 3 ( $\leq 1$ Outgoing Edge). Let us consider any instance $I=(G, P)$ of the Disjoint Paths Problem, and any answer set $M$ of $R(I, \mathcal{T})$ using any pair-connected TD $\mathcal{T}$ of $(G, P)$. Then, there cannot be two edges of the form $e_{u, v}, e_{u, w} \in M$.

Proof. Assume towards a contradiction that there are three different vertices $u, v$, $w \in V$ with $e_{u, v}, e_{u, w} \in M$. Then, by Rules (26) there cannot be a node $t$ with $(u, v),(u, w) \in E_{t}^{\text {re }}$. However, by the definition of TDs, there are nodes $t^{\prime}, t^{\prime \prime}$ with $(u, v) \in E_{t^{\prime}}^{\mathrm{re}}$ and $(u, w) \in E_{t^{\prime \prime}}^{\mathrm{re}}$. By connectedness of TDs, $u$ appears in each bag of any node of the path $X$ between $t^{\prime}$ and $t^{\prime \prime}$. Then, either $t^{\prime}$ is an ancestor of $t^{\prime \prime}$ (or vice versa, symmetrical) or there is a common ancestor $t$. In the former case, $f_{t^{\prime \prime}}^{u}$ is justified by Rules 22) and so is $f_{\hat{t}}^{u}$ on each node $\hat{t}$ of $X$ by Rules 23) and therefore ultimately Rules 25 fail due to $f_{t^{\prime}}^{u}, e_{u, w} \in M$. In the latter case, $f_{t^{\prime}}^{u}, f_{t^{\prime}}^{u}$ is justified by Rules 22 and so is $f_{\hat{t}}^{u}$ on each node $\hat{t}$ of $X$ by Rules 23). Then, Rules (24) fail due to $f_{t^{\prime}}^{u}, f_{t^{\prime \prime}}^{u} \in M$.

Theorem 4 (Correctness). Reduction $R$ as proposed in this section is correct. Let us consider an instance $I=(G, P)$ of the Disjoint Paths Problem, and a pair-connected $T D \mathcal{T}=(T, \chi)$ of $G$. Then, $I$ has a solution if and only if the program $R(I, \mathcal{T})$ admits an answer set.

Proof. " $\Rightarrow$ ": Assume any positive instance $I$ of Disjoint Paths Problem. Then, there are disjoint paths $P_{1}, \ldots, P_{i}, \ldots P_{|P|}$ from $s_{1}$ to $d_{1}, \ldots, s_{i}$ to $d_{i}, \ldots$, $s_{|P|}$ to $d_{|P|}$ for each pair $\left(s_{i}, d_{i}\right) \in P$. Assuming further pair-connected TD $\mathcal{T}$ of $I$, we construct in the following an answer set $M$ of $\Pi=R(I, \mathcal{T})$. To this end, we collect reachable atoms $A:=\left\{u \mid u\right.$ appears in some $\left.P_{i}, 1 \leq i \leq|P|\right\}$ and used edges $U:=\left\{(u, v) \mid v\right.$ appears immediately after $u$ in some $P_{i}, 1 \leq$ $i \leq|P|\}$. Then, we construct answer set candidate $M:=\left\{r_{u} \mid u \in A\right\} \cup\left\{e_{u, v} \mid\right.$ $(u, v) \in U\} \cup\left\{n e_{u, v} \mid(u, v) \in E \backslash U\right\} \cup\left\{f_{t}^{u} \mid(u, v) \in U \cap E_{t}^{\mathrm{re}}\right\} \cup\left\{f_{t}^{u} \mid\right.$ $(u, v) \in U \cap E_{t^{\prime}}^{\mathrm{r}}, u \in \chi(t), t^{\prime}$ is a descendant of $t$ in $\left.T\right\}$. It remains to show that $M$ is an answer set of $\Pi$. Observe that $M$ satisfies all the rules of $\Pi_{\mathcal{R}}$. In particular, by construction, we have reachability $r_{v}$ for every vertex $v$ of every pair in $P$, and the partition in used edges $e_{u, v}$ and unused edges $n e_{u, v}$ is ensured. Further, $\Pi_{\mathcal{L}}$ is satisfied, as, again by construction, for each vertex $v$ of every pair in $P$, we have $r_{v} \in M$. Finally, $\Pi_{\mathcal{C}}$ is satisfied as by construction $f_{t}^{u} \in M$ iff $e_{u, v} \in M \cap E_{t}^{\mathrm{re}}$ or $e_{u, v} \in M \cap E_{t^{\prime}}^{\mathrm{re}}$ for any descendant node $t^{\prime}$ of $t$ with $u \in \chi(t)$. It is easy to see that $M$ is indeed a $\subseteq$-smallest model of the reduct $\Pi^{M}$, since, atoms for used and unused edges form a partition of $E$.
" $\Leftarrow$ ": Assume any answer set $M$ of $\Pi$. First, we observe that we can only build paths from sources towards destinations, as sources have only outgoing edges and destinations allow only incoming edges. Further, by construction, vertices can only have one used, outgoing edge, cf., Lemma 3. Consequently, if a vertex had more than one used, incoming edge, one cannot match at least one pair of $P$ (by combinatorial pigeon hole principle). Hence, in an answer set $M$ of $\Pi$, there is at most one incoming edge per vertex. By construction of $\Pi$, in order to reach each $d_{i}$ with $\left(s_{i}, d_{i}\right) \in_{i} \sigma, s_{i}$ cannot reach some $d_{j^{\prime}}$ with $j^{\prime}<i$. Towards a contradiction assume otherwise, i.e., $s_{i}$ reaches $d_{j^{\prime}}$. But then, by construction of the reduction, we also have a reachable path from $d_{j^{\prime}}$ to $s_{i}$, consisting of $d_{j^{\prime}}, d_{j^{\prime}+1}, \ldots, d_{i-1}, s_{i}$. Since every vertex has at most one incoming edge, $d_{j^{\prime}}$ cannot have any other justification for being reachable, nor does any source on this path. Hence, this forms a cycle such that no atom of the cycle is proven, which cannot be present in an answer set. Therefore, $s_{i}$ only reaches $d_{i}$, since otherwise there would be at least one vertex $s_{j}$ required to reach $s_{i^{\prime}}$ with $\left(s_{i^{\prime}}, d_{i^{\prime}}\right) \in_{i^{\prime}} \sigma, i^{\prime}<j$. Consequently, we construct a witnessing path $P_{i}$ for each pair $(s, d) \in_{i} \sigma$ as follows: $P_{i}:=s, p_{1}, \ldots, p_{m}, d$ where $\left\{e_{s, p_{1}}, e_{p_{1}, p_{2}}, \ldots, e_{p_{m-1}, p_{m}}, e_{p_{m}, d}\right\} \subseteq M$. Thus, $P_{i}$ starts with $s$, follows used edges in $M$ and reaches $d$.

## E Properties and Consequences of Section 4

The resulting program of the reduction consisting of Rules (16)-26 is not unary. However, only Rules (21) as well as $\sqrt{24}-\sqrt{26}$ are not unary. Still, Rules (25) and (26) can be turned unary by replacing the occurrence of $e_{u, v}$ in these two rules by $\neg n e_{u, v}$. Further, Rules 24 can be replaced by the following rules, which use an additional auxiliary atom "bad".

$$
\begin{array}{ll}
\operatorname{bad} \leftarrow f_{t}^{u}, f_{t^{\prime}}^{u} & \text { for each } u \in \chi\left(t^{\prime}\right) \cap \chi\left(t^{\prime \prime}\right), t^{\prime} \neq t^{\prime \prime} \\
\leftarrow \operatorname{bad} & \tag{31}
\end{array}
$$

On the other hand, for Rules (21) the resulting (positive) cycles of the dependency graph are required for the whole construction, cf., Figure 6. More precisely, it is indeed essential for the whole construction that reachability of a source $s_{i}$ requires both reachability of the preceding source $s_{i-1}$ and destination $d_{i-1}$. Otherwise we cannot prevent a source from reaching a preceding destination via cyclic reachability without provability and still linearly preserve the treewidth. Consequently, Rules (21) are not unary and we expect that this is crucial. Nevertheless, it was shown that non-unary programs are more expressive than unary programs 65]. Still, we are convinced that exploiting cyclic, unproven reachability such that the treewidth is not increased more than linearly, actually requires the usage of non-unary rules.

Example 16. Consider again Figure 6, depicting the positive dependency graph $D_{R_{\mathcal{L}}}$ of Rules (21), as well as Example 12. More concretely, consider the same situation of Example 12, where a source $s_{i}$ reaches some destination $d_{j}$
with $j<i$, which causes a cycle $C=r_{s_{i}}, \ldots, r_{d_{j}}, r_{s_{j+1}}, \ldots, r_{s_{i}}$ over reachability atoms. Then, it is crucial for the construction that Rules (21) are not unary. To be more concrete, for the instantiated rule $r$ with $r_{s_{j+1}} \in H_{r}$, we require that both $r_{s_{j}}, r_{d_{j}} \in B_{r}^{+}$. If instead of $r$ we constructed two rules $r_{s_{j+1}} \leftarrow r_{s_{j}}$ and $r_{s_{j+1}} \leftarrow r_{d_{j}}$, every atom of the cycle $C$ could be provable since $r_{s_{j+1}}$ can already be proven by the former rule. Further, also for the instantiated rule $r^{\prime}$ of Rules (21) with $r_{s_{o}} \in H_{r^{\prime}}$ for every $j+1<o \leq i$, we require that the body is not unary. If instead of such a rule $r^{\prime}$, we constructed two rules $r_{d_{o}} \leftarrow r_{s_{o-1}}$ and $r_{d_{o}} \leftarrow r_{d_{o-1}}$, every atom of the cycle $C$ could be provable since $r_{d_{o}}$ is already proven by the latter rule. Since, in particular the result should hold for any such cycle $C$, we rely on non-unary rules for our reduction to work.


[^0]:    ${ }^{1}$ Double exponentiality refers to runtimes of the form $2^{2 \mathcal{O}(k)} \cdot n$.

[^1]:    ${ }^{2}$ Let $G=(V, E)$ be a digraph and $W \subseteq V$. Then, a (directed) cycle in $G$ is a $W$-cycle if it contains all vertices from $W$.

[^2]:    ${ }^{3}$ Analogously, the primal graph $\mathcal{G}_{F}$ of a propositional Formula $F$ (in CNF) uses variables of $F$ as vertices and adjoins two vertices $a, b$ by an edge, if there is a clause in $F$ containing $a, b$.

[^3]:    ${ }^{4}$ Our translator asp2sat is open source and readily available at github.com/hmarkus/asp2sat_translator

[^4]:    ${ }^{5}$ An ordering $\varphi$ over at $(\Pi)$ is minimal (for $M$ ) if there is no atom $a \in$ at $(\Pi)$ such that when decreasing $\varphi(a)$, interpretation $M$ can still be proved with the resulting (modified) ordering.

[^5]:    ${ }^{6}$ A greedy-minimal ordering $\varphi$ over at $(\Pi)$ for $M$ is an ordering that serves in proving every atom $a \in M$ with every rule $r \in \Pi$ suitable for proving $a$, which is not the case anymore when decreasing $\varphi(b)$ for some $b \in \operatorname{at}(\Pi)$.

