# Sparsifying Parity-Check Matrices ${ }^{\text {th }}$ 

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#### Abstract

Parity check matrices (PCMs) are used to define linear error correcting codes and ensure reliable information transmission over noisy channels. The set of codewords of such a code is the null space of this binary matrix. We consider the problem of minimizing the number of one-entries in parity-check matrices. In the maximumlikelihood (ML) decoding method, the number of ones in PCMs is directly related to


[^0]the time required to decode messages. We propose a simple matrix row manipulation heuristic which alters the PCM, but not the code itself. We apply simulated annealing and greedy local searches to obtain PCMs with a small number of one entries quickly, i.e. in a couple of minutes or hours when using mainstream hardware. The resulting matrices provide faster ML decoding procedures, especially for large codes.
Keywords: Parity-check matrix, Sparsifying matrices, Minimum decoders, Greedy search, Simulated annealing, Integer programming

## 1. Introduction

In today's world, a fast and reliable wireless internet connection is essential. During data transmission, the data may become perturbed due to weather conditions, obstacles, or other data traffic. To achieve robustness, usually all data is encoded before the transmission and decoded after reception. While encoding is an easy task once a suitable code is chosen, decoding may be extremely costly in terms of time and it may produce errors. These two problems can for example occur when streaming of a video: The video may need much time to load and buffer or the video may be of bad quality.

The optimal way to decode a message is the so-called Maximum-Likelihood (ML) decoding (see Helmling et al. (2012) for an introduction). It is well-known that ML decoding is NP-hard (Berlekamp et al. (1978)). In this paper, we aim to reduce the ML decoding time by altering the representation of the given code. In detail, we alter the code's underlying PCM such that the number of ones is reduced without changing the code itself. This can be done by adding one or more rows to another row which does not change the kernel of the matrix and thus does not change the code. Gensheimer et al. (2018) show that ML decoding works faster on sparse matrices. The authors also show that the number of one-entries of a PCM can be minimized by
solving an integer program for every row of the matrix. Although this optimization has to be done only once for each code, the computation time increases rapidly with increasing matrix size. Since each integer problem considered by Gensheimer et al. (2018) consists of $\mathcal{O}(n)$ variables and constraints, where $n$ is the number of columns, the authors did not compute an optimal solution for large matrices. Instead, approximations are given. In this paper, we focus on obtaining fast algorithms that approximate an optimal matrix by applying simulated annealing.

The remainder of this article is organized as follows. Section 2 presents basic concepts, definitions, and notation. Section 3 is devoted to matrix transitions and the problem is formulated in the context of simulated annealing meta-heuristics with the aim of modifying the current PCM quickly. Section 4 contains computational results which show that our approach efficiently yields very good approximations of the optimal sparse PCM in reasonable time frames, from a few minutes to a couple of hours, with commodity hardware (in some cases even greedy local searches obtain good results). Moreover, we also provide practical guidelines on how to select temperatures and cooling schedules for annealing. Section 5 presents other approaches used to deal with the same problem, namely those based in binary linear programming and solved by powerful linear programming solvers. Finally, Section 6, presents the main conclusions and provides future research ideas.

## 2. Coding Theory Basics

This section introduces basic notations, a fundamental result, and an illustrative example.

Definition 1 (Binary Linear Code). A binary linear code $C \subseteq\{0,1\}^{n}$ of size $n$ is a linear subspace of $\{0,1\}^{n}$. The elements of $C$ are called codewords.

In particular, $0^{n} \in C$ and, for all $y, y^{\prime} \in C$, it holds that $y+y^{\prime} \in C$ where the addition is performed modulo 2 .

Definition 2 (Parity-Check Matrix). A binary linear code $C$ can be represented by a parity-check matrix $(P C M) H \in\{0,1\}^{(n-k) \times n}$ where $C=\left\{y \in\{0,1\}^{n}: H y \equiv 0 \bmod 2\right\}$. If the rows of $H$ are linearly independent, $k$ is the dimension of $C$.

Thus, a binary linear code can be seen as the kernel of a binary matrix. With a parity-check matrix $H$ given, it can be easily decided if a given word $y \in\{0,1\}^{n}$ belongs to the code $C$.

Theorem 1 (see (Gensheimer et al., 2018)). Let $H \in\{0,1\}^{(n-k) \times n}$ be a paritycheck matrix of a code $C$ of dimension $k$. Then $H^{\prime} \in\{0,1\}^{(n-k) \times n}$ is a parity-check matrix of $C$ if and only if all rows of $H$ are elements of the span of the rows of $H^{\prime}$.

In particular, the code does not change if one or more rows are added to another row.

The following matrix is a PCM $H$ of a BCH cod $\AA^{1}$,

$$
H=\left(\begin{array}{lllllllllllllll}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0  \tag{1}\\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right) .
$$

[^1]The codewords are the vectors $y$ for which $H \cdot y \equiv \overrightarrow{0} \bmod 2$, where $\overrightarrow{0}$ is a vector of zeros. Therefore, the word 000000000000000 is a codeword, likewise 000000100010111 is also a codeword, since $H \cdot y=(00000000)^{T}$, when $y$ is the vector representation of this word. Note that for latter, $H \cdot y$ corresponds to summing (modulo 2) the columns highlighted in gray. This matrix contains 6 ones in the first row and 4 ones in each of the following 7 rows, yielding a total of 34 ones.

From basic linear algebra, it follows that a PCM can be modified in such a way that the underlying code is not changed. More precisely, given any invertible matrix $S$, the matrix $H^{\prime}=S \cdot H$ yields an alternative PCM for the same code. For example, the first row of $H$ can be replaced by adding the second row to it. This yields $H^{\prime}$ where the first row is $(110100010000000)$ while the remaining rows are identical to those in $H$. This first row contains 4 ones and, therefore, $H^{\prime}$ contains 32 ones in total.

Given some code, our goal is to compute a PCM $H^{\prime}$ with the minimal number of one entries. In the example above, $H^{\prime}$ is sparser than $H$. In fact, for this particular code, it can be proven that the minimal number of ones of any PCM equals 32 and, therefore, $H^{\prime}$ is one of the sparsest PCMs.

## 3. Constructing Alternative Parity-Check Matrices

There may be several natural approaches to the problem of finding an invertible matrix $S$ such that $H^{\prime}=S \cdot H$ is sparse ${ }^{2}$.

[^2]The approach proposed in this article consists of selecting an origin row $i$ and a destination row $j$ and adding row $i$ to row $j$. The matrix $S$ representing this kind of elementary row operation is the unit matrix with an additional 1 in the entry $(j, i)$. This process is repeated several times, until the resulting matrix is sufficiently close to the optimal value.

### 3.1. Row Selection

A successful realization of this strategy entails several open issues, namely how to efficiently select the rows $i$ and $j$.

In particular, adding one row to another may sometimes increase the number of ones in $H^{\prime}$, although this process seems to be contradicting the objective. However, it is sometimes necessary to escape from local minima in the search space. We show that the simulated annealing meta-heuristic provides a good policy to guide the search process for a sparse parity-check matrix.

The main steps of the simulated annealing algorithm take into account the following aspects:

1. In the context of simulated annealing, we use $E\left(H^{\prime}\right)$ to represent the number of ones in $H^{\prime}$, i.e., the energy of the current state.
2. We associate with each transition (i.e. adding one row to another) a probability $e^{-d / T}$, where $d$ is the variation of $E\left(H^{\prime}\right)$ and $T$ is the current temperature of the process.
3. If the transition maintains the value $E\left(H^{\prime}\right)$ constant then $d=0$ and this probability is 1 , in which case the transition is accepted. Likewise, if $d<0$, then the probability formula yields a value greater than 1 and the transition is also accepted.
4. If $d>0$, then the transition represents an uphill movement and is not always accepted. If $T$ approximates 0 , then the formula approximates 0 and the movement is rejected. Hence, $T$ close to 0 yields a greedy local search.
5. If $d>0$ and if $T$ is an adequate value, then the formula yields a value between 0 and 1 . In this case, we choose a random number uniformly from $[0,1]$. If this number is smaller than the value $e^{-d / T}$ the transition is accepted, otherwise the transition is rejected.

An efficient implementation requires some tweaking, particularly, it is better to use an efficient strategy to select rows $i$ and $j$. There are several possible approaches:

1. Choosing $i$ and $j$ uniformly at random, but distinct;
2. Analyzing all pairs $i$ and $j$ and selecting the best existing transition;
3. Using dirty flags, in each row, to speed-up the search of good existing transitions.

The following paragraphs provide more details for the previously mentioned approaches.

### 3.1.1. Random Choice

This is the simplest approach and consists of choosing $i$ and $j$ uniformly at random, but distinct. We use this approach if there are no candidates for transitions which reduce $E\left(H^{\prime}\right)$.

### 3.1.2. Selecting Best Transitions

An alternative to the previous approach consists of analyzing all pairs $i$ and $j$ and selecting the pair leading to the best existing transition. This requires $O\left(n m^{2}\right)$
time, where $n$ is the number of columns in $H^{\prime}$ and $m$ is the number of rows. This effort has significant impact in the performance of the resulting algorithm, the time bound is excessive for the resulting gain. Therefore our algorithm never uses this selection procedure. Instead, we propose an heuristic to obtain similar results in at most $O(n m)$ time per analysis. Moreover, our approach only spends this time if there is a chance of reducing the number of ones in $H^{\prime}$. Most of the time, it certifies that no such move exists in $O(1)$ time.

### 3.1.3. Assigning Flags to Rows

Our strategy is to assign a dirty flag to each row. If this flag is set, then the row is considered dirty, otherwise it is considered clean. If $i$ and $j$ are clean rows, then neither adding row $i$ to row $j$ nor adding row $j$ to row $i$ produces a row with fewer ones. The smaller the number of ones in a matrix, the more rows become clean. Recall, for example, the matrix $H^{\prime}$ obtained from $H$ in Equation (1). We can safely consider all rows of $H^{\prime}$ as clean, because adding any two rows results in a row which contains more ones. When a given row $i$ is flagged as dirty, we can test this status by adding $i$ to every other row in $H^{\prime}$. Note that this is only a testing procedure and, thus, we do not alter $H^{\prime}$. If there exists a row $j$ such that adding $i$ to $j$ reduces the number of ones, then the pair of rows $i$ and $j$ is sent to the simulated annealing decision process, which, as discussed above, accepts this transition. The dirty flag of $i$ is kept by the simulating annealing procedure and, moreover, row $j$ is also flagged as dirty. Note that only the selection procedure is able to assign clean flags. A dirty flag is assigned by the simulated annealing process, to row $j$, whenever row $i$ is added to row $j$, no matter what the original flags of $i$ and $j$ are. Also in this case the flag of $i$ is kept.

Row $i$ can only be flagged as clean if there is no row $j$ such that adding $i$ to $j$
produces a row with less ones. In this case this selection procedure flags $i$ as clean. This process requires at most $O(n m)$ time, as mentioned before.

Let us now just highlight an important nuance. It is not sufficient to add $i$ to $j$ and verify if this reduces the number of ones compared to $j$. We must also verify if it reduces the number of ones compared to $i$. If both decrease the number of ones, we choose the transition which results in the sparser matrix. If the number of ones never decreases this selection procedure flags row $i$ as clean. Moreover, the rows $j$ to test are not considered in order, i.e., from 1 to $m$. This avoids a bias towards the first rows. Instead this procedure generates a random uniform permutation of the numbers 1 to $m$.

### 3.1.4. Additional Considerations

In the initialization, all rows are flagged as dirty since none of them was verified. As the algorithm evolves, the number of dirty rows decreases quickly to 0 and, in fact, most of the time the number of dirty rows is 0 . In this case, no analysis is performed and we select $i$ and $j$ uniformly at random, as explained before. This means that most of the time, we avoid the $O(n m)$ time operation and, instead, spend $O(1)$ time only, albeit we also do not obtain decreasing transitions. Still, this process is valuable since when the annealing algorithm escapes a local minima it quickly moves downhill to another minima. In some cases, we do pay the $O(n m)$ time cost but row $i$ gets flagged as clean. In these cases, we also do not find a row $j$ which decreases the number of ones and again choose $i$ and $j$ uniformly at random.

### 3.2. Choosing the Temperature

Another important issue in simulated annealing is the selection of the value of the temperature degrees $T$. This value is not constant during the execution of the
algorithm, but it is kept for around 100 iterations. After these iterations, the temperature is updated using a geometric rule, i.e., we change $T$ to $\alpha T$ with $\alpha<1$. Calculating the value of $\alpha$ is straightforward once we decide the initial temperature $T_{0}$, the final temperature $F$, as well as the number of steps $s$ desired to transform $T_{0}$ into $F$. Hence, the only issue consists in determining the numerical values of $T_{0}$ and $F$. This is challenging without further insight into the chosen parameters. We provide an intuitive approach to this choice, which is actually very robust for different problems.

## 4. Algorithmic Aspects

In this section, we review the simulated annealing algorithm and present its application to the problem of determining the sparsest parity-check matrix. We also discuss strategies for selecting the temperature and cooling rate. We finish this section by showing several experimental results of our approach and surveying the state of the art.

### 4.1. Simulated annealing for determining the sparsest PCM

The simulated annealing algorithm is stated in Algorithm 1. The current state of the algorithm is represented by the matrix $H^{\prime}$ which starts off equal to $H$. The initial temperature is set to $T_{0}$ and the final temperature is set to $F$. When the temperature reaches $F$, the while-loop is executed one last time. Each time this loop is executed, the for-loop is also executed. The value of Iter is fixed to 100 in all our tests. We kept it at a low value within the recommended range. Recall that, for all these 100 values of $k$, the algorithm iterates at a constant temperature. The temperature is decreased in line 12. To control how long the algorithm runs, we
choose a proper value for $\alpha$. When we want that the external while-loop is executed $s+1$ (assuming $\left.T_{0}<F\right)$ times, we set $\alpha=\sqrt[s]{F / T_{0}}$.
input : $\left\langle H, T_{0}, F, s\right\rangle\{$ Matrix $H$ is in alist format. $\}$
output: $\left\langle H^{\prime}\right\rangle$
$H^{\prime} \leftarrow H ;$
$T \leftarrow T_{0} ;$
3 $\alpha \leftarrow \sqrt[s]{F / T_{0}}$;
4 while $T \leqslant F$ do
5 for $k=1$ to Iter do
$(i, j, d) \leftarrow$ Analyze $\left(H^{\prime}\right) ;$
if Random $(0,1) \leqslant e^{-d / T}$ then
$H^{\prime}[j] \leftarrow H^{\prime}[i]+H^{\prime}[j] ;$
end
$k \leftarrow k+1 ;$
end
$T \leftarrow \alpha T ;$
end
return $H^{\prime}$;
Algorithm 1: Pseudo code for simulated annealing algorithm.

The algorithm comprises two functions.

1. Analyze ( $H^{\prime}$ ) follows the procedure described in Section 3. If all the rows are marked as clean, then row indexes $i$ and $j$ are chosen uniformly at random. We denote the respective rows by $H^{\prime}[i]$ and $H^{\prime}[j]$. If there is at least one row flagged dirty, then $i$ is chosen uniformly among the dirty rows and $j$ scans
through all the other rows to find one which decreases the number of ones. If such a row is found, then $j$ becomes the index of that row. In fact, $i$ and $j$ might need to be swapped to maximize the reduction in the number of ones. If no such row is found, then the flag of $i$ is set to clean and new $i$ and $j$ values are chosen uniformly at random and passed as the output of Analyze. The value $d$ in line 6 represents the variation in the number of ones the current transformation will imply. If $d$ is negative, the number of ones decreases; if $d$ is positive, it increases. The + symbol in line 8 represents the row addition in $\mathbb{Z}_{2}$.
2. The function Random $(0,1)$ returns a random number in $[0,1]$, chosen uniformly at random. To obtain this value and test the condition in line 7, we rewrite the condition. The goal is to avoid the loss of precision that results from division and exponentiation, and to guarantee sound random numbers. Instead, we test the following condition

$$
d \leqslant T(30 \ln 2-\ln R)
$$

In this condition, $R$ is an integer chosen uniformly at random from 1 to $2^{30}$, using the arc4random_uniform function of the BSD stdlib.

It remains to discuss the selection of the temperature parameters $T_{0}$ and $F$. Like in several parametric methods, determining these values is largely an experimental procedure, which depends heavily on the application at hand. This is obviously true for our application. However, we wish to convey some insight into the choice of these parameters. Let us recall the acceptance condition in line 7 of Algorithm 1. The following equation captures this condition, using $p$ to denote the random number,

$$
\begin{equation*}
p \leqslant e^{-d / T} \tag{2}
\end{equation*}
$$

We may consider the extreme cases, when the inequality becomes an equality, and we rewrite the condition to obtain $T$. The resulting equation is

$$
\begin{equation*}
T=-d / \ln p \tag{3}
\end{equation*}
$$

This means that a temperature can be defined by specifying $d$ and $p$. We prefer to specify these parameters as they lead to a more intuitive notion of temperature. For example, for the matrix $H$ in Equation (1) in Section 1, we can decide to accept an increase of two one-entries, i.e., $d=2$, in $4 \%$ of the transitions considered, this means $p=0.04$. Hence, we obtain a numerical value for $T$ of approximately 0.62133 . Note that a temperature applies to all the tests of line 7, so we might inquire what is the probability that this temperature accepts a value of $d=4$. Using standard calculus, it turns out that the resulting probability is $0.04^{2}=0.0016$. In general, if we define the values $d_{1}$ and $p_{1}$, then the probability $p_{2}$ for a delta $d_{2}$ is given as $p_{2}=p_{1}^{d_{2} / d_{1}}$. Hence, for $d_{2}=1$, we obtain a probability of $20 \%$ with $p_{2}=\sqrt{0.04}=0.2$. The following plot illustrates the resulting curve.


Hence, it is possible to specify the same temperature by inputting any one of the three points indicated above. In general, defining a temperature in this way
is more intuitive than the single numerical value. In our prototype, temperatures are specified by providing parameters $p$ and $f$. The value $d$ is then obtained as $d=f \times N$, where $N$ is the number of columns of the corresponding matrix. We keep the value $f$ constant at 0.01 , meaning that uphill movements that increase the number of ones by $d$ are only accepted $1 \%$ of the time. Hence, the input parameters are fairly intuitive. Naturally the parameters are chosen by trial and error, still this approach gives some reasonable initial values.

Another important aspect of the simulated annealing algorithm is the cooling schedule of temperature degrees, which is affected by the final temperature $F$. Again, we specify $F$ by choosing $d$ and $p$. In this case, it is sensible to maintain one of the parameters constant. For example, we could choose $d=1$ and $p=0.04$, meaning that from $T_{0}$ to $F$ we maintain the $4 \%$ uphill probability, but reduce the value of $d$ from 2 to 1 .

An example of how this approach can simplify the temperature definition is shown in Table 1, where the selection of hot and cold values for $f$ and $p$ is fairly straightforward, but the resulting temperatures $T_{0}$ and $F$ are fairly peculiar. Note that, without the insight we have just described, the initial order of magnitude for $T_{0}$ and amplitude $F-T_{0}$ are mysteries that need to be solved by trial and error. Let us now proceed to the experimental evaluation.

### 4.2. Experimental results and discussion

In this section, we describe the experimental setup used to test our algorithm. We used several PCMs in alist format 3 . The matrices are obtained from a channel code databasc ${ }^{4}$. Our prototype is available at "https://github.com/LuisRusso-INESC-

[^3]
## ID/SPCM.

### 4.2.1. The Design of the Experiments

We selected reasonable initial temperature $T_{0}$ and final temperature $F$. We also executed our algorithm with extremely low temperature settings. At these temperatures, the simulated annealing heuristic reduces to a greedy local search procedure, which always reduces the number of ones and never accepts any transitions which increase this value.

The result of these executions is shown in plots of time versus the number of ones in the underlying matrix. Moreover, because the algorithm is probabilistic, the results vary. Therefore, we present the results of several repetitions of the algorithm. The local search algorithm is repeated 32 times and the simulated annealing algorithm is repeated 128 times. The number of times that the while-loop in Algorithm 1 is executed, can be very big, i.e., $s$ can be as high as several millions. Therefore, we sample data points from some of those executions.

To execute these tests, we used a dedicated server and executed the tests in parallel, one per core, but without using hyper-threading. Therefore, we executed 32 parallel tests at a time. Note that this kind of hardware is necessary only because we want to study the performance of the proposed algorithm. For the goal of obtaining a sparse PCM of a given code, it is possible to use commodity hardware.

The server contained an $\operatorname{Intel}(\mathrm{R})$ Xeon(R) CPU E7 4830 running at 2.13 GHz , with 32 physical cores, the architecture is 64 -bit. The server contains 4 sockets each containing 8 cores. The system reported 4255.86 BogoMIPS, and has the following cache sizes L1d 32KB, L1i 32KN, L2 256KB, L3 24576KB. This means that the matrices considered fitted in cache. The overall system memory is 251 GB and it has 7 GB of swap, but as we just pointed out, this memory was not crucial to the
algorithm. In fact, some of the tests were initially performed on an Eee PC with an Intel Atom CPU N270 running at 1.60 GHz and with 1.96 GB of main memory.

### 4.2.2. Results

In Figure 1, we show how the number of ones can be reduced by our algorithm. The green bars occupy $100 \%$ of the initial number of ones. The blue bars indicate the percentage of the number of ones obtained by the greedy algorithm compared to the initial number of ones. This value is the one obtained by the best run of the greedy search. The red bars indicate the percentage of ones obtained by the simulated annealing algorithm compared to the initial number of ones.

Figures 2, 3, 4, and 5 show the results from the time experiments. The $x$ axis indicates time, as the algorithm proceeds. The $y$ axis has a double scale, on the right we indicate the actual number of ones, on the left, we indicate the ratio between the current solution and the overall minimum attained value. Note that this minimum might be larger than the number of ones in the code's sparsest parity matrix. The remaining experimental results are shown in Appendix A. The blue points are sampled from the greedy algorithm and the red points are sampled from the simulated annealing algorithm. If you have a black and white version of this paper, then the greedy points are still easy to identify, because they seem to form constant lines, whereas the simulated annealing points are usually decreasing.

In Table 1, we show the parameters that we used for the simulated annealing algorithm. The number $N$ indicates the number of columns of the corresponding matrix. The value of $d$ is obtained as $f \cdot N$, where $f$ is also given in the table. Likewise, the value $p$ is also stated in the Table 1. We use Equation (3) to determine both, the initial $T_{0}$ and the final temperature $F$. These values are also shown in the table. The number of iterations $S$ is given in the last column of the table. As


Figure 1: Bar chart illustrating the relative number of ones obtained with the greedy and simulated annealing algorithms.
discussed in Section 4.1, we keep the value $f$ constant at 0.01 , meaning that uphill movements that increase the number of ones by $d$ are only accepted $1 \%$ of the time.

### 4.2.3. Comments and Discussion

The results for the LTE Turbo code with $n=396$ and $k=128$ show the best illustration for the methods we present. As shown by the bars in Figure 1, for this particular code, the number of ones in greedy and simulated annealing algorithms is significantly smaller than the original number of ones. Moreover, there is even

Table 1: Algorithm parametrization.

|  | Start |  |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Code | $N$ | $f$ | $p$ | $T_{0}$ | $f$ | $p$ | $F$ | $S$ |
| LTE-TC-N396-K128 | 396 | 0.05 | 0.01 | 4.30 | 0.01 | 0.01 | 0.86 | $5.12 \mathrm{E}+6$ |
| BCH-127-92-5-strip | 127 | 0.05 | 0.01 | 2.77 | 0.01 | 0.01 | 0.28 | $5.12 \mathrm{E}+6$ |
| BCH-255-207-6-strip | 255 | 0.05 | 0.01 | 1.38 | 0.03 | 0.01 | 1.66 | $5.12 \mathrm{E}+8$ |
| BCH-7200-7032-12-strip | 7200 | 0.004 | 0.01 | 6.25 | 0.003 | 0.01 | 4.69 | $1.28 \mathrm{E}+6$ |

a significant difference between these two algorithms. The points from the greedy algorithm form straight lines, showing clearly that this algorithm quickly gets captured in local minima. The number of ones initially decreases quickly for both, the greedy and simulated annealing algorithm. It is omitted from the plots to keep the scale range smaller and make it easier to compare the performance of the greedy and simulated annealing algorithms. However, for the LTE Turbo code, it is significant as shown by the bars in Figure 1.

For the remaining codes, the difference between the greedy and the simulated annealing is smaller, mainly because the initial number of ones seems to be closer to the optimal sparsest PCM. Note that the ratios in the scale of Figures 3, 4, and 5 are very close to 1 and the blue and red bars in Figure 1 are close to the green bar. The time plots still show the simulated annealing algorithm improving over time, as it decreases to reach a factor close to 1 , at which point the searches seem to stabilize, and possibly the optimal value was obtained. For the code BCH-255-207-6, the procedure takes much longer, indicating that for this code better results could be achieved by the simulated annealing algorithm, given more time. However, since the running time exceeded 50 hours, the test was stopped. Note that to produce the information in the plot, we required 5 times this period.


Figure 2: Time ( $x$ axis) versus number of ones (right $y$ axis) or factor to overall minimum number of ones (left $y$ axis) for LTE_TC_N396_K128.

It is also interesting to note that the difficulty of determining the sparsest paritycheck matrix is intrinsic to the code in consideration and cannot be directly inferred from the number of columns in the code matrix. The code BCH-7200-7032-12 has 7200 columns but seems to stabilize much quicker, in a matter of minutes, whereas the code BCH-255-207-6 has only 255 columns but is much more challenging.

In Appendix A, we show further experimental results. We show several BCH and LTE codes. Our algorithm seems to consistently and significantly improve the LTE codes. This is a good result although the improvement ratio seems to degrade as the size of the LTE codes increases. We believe that this is a parameter problem, as all these tests are conducted with the same $p$ and $f$ parameters. It seems that this generic scheduling is too cold for the smaller LTE codes, as the cloud of red dots does not seem to contract to a point and, instead, remains wide. When $N$ is between


Figure 3: Time ( $x$ axis) versus number of ones (right $y$ axis) or factor to overall minimum number of ones (left $y$ axis) for BCH_127_92_5_strip.

276 and 612 , the cooling seems to be adequate, although for the larger codes it may benefit from more iterations. For the larger codes, the concavity of the cloud seems to change, thus indicating that, for these codes, the overall scheduling is too hot. We plan to experiment with cooler schedules.

As a final consideration, we discuss the practical consequences of reducing the number of ones in PCMs. As mentioned in Section 1, our main motivation for sparsifying a PCM is to decrease the time necessary for ML decoding. A systematic study of ML decoding is beyond the scope of this paper, and moreover Gensheimer et al. (2018) have already established that ML decoding is indeed faster for sparser matrices. Instead we focus on code checking, instead of decoding. By checking we mean that the procedure can only determine whether a codeword $y$ belongs to a certain code $C$. However when $y$ does not belong to $C$, the checking procedure can


Figure 4: Time ( $x$ axis) versus number of ones (right $y$ axis) or factor to overall minimum number of ones (left $y$ axis) for BCH_127_92_6_strip.
not determine the most likely $y^{\prime}$ that belongs to $C$ and that got distorted into $y$.
Restricting our analysis to a checking procedure, instead of a decoder, is motivated by three reasons. First checking procedures are much simpler than decoders. Second the number of ones of the PCM has a significant correlation to the performance of the simple checking procedure we present. Third, and most importantly, checking is enough for most of the words. Note that all received words must be checked, and checking should be enough for most of them, as most of them should be codewords. Otherwise there will be a significant portion of received words that will be distorted beyond recovery. This means that checking amounts to the majority of the time that is necessary to process the received words. Therefore a reduction in the checking time is guaranteed to translate into a significant reduction of the overall time, even before the improvement in the decoding procedure is accounted for.


Figure 5: Time ( $x$ axis) versus number of ones (right $y$ axis) or factor to overall minimum number of ones (left $y$ axis) for BCH_127_92_12_strip.

Now recall that according to definition 2 checking whether a word $y$ is a code word is a matter computing a matrix multiplication in $\mathbb{Z}_{2}$, i.e., $H y \equiv 0 \bmod 2$. We will now explain how to efficiently perform this operation, in a way that depends on the number of ones in $H$, meaning that the performance of the algorithm benefits from the fact that $H$ is sparse. First note that we can use xor to compute addition in $\mathbb{Z}_{2}$. Second recall the formula for matrix multiplication, given in the following equation, where the elements of the resulting vector $v$ are indexed as $v_{i}$, with $i$ between 1 and $n-k$,

$$
\begin{equation*}
v_{i}=\sum_{j=1}^{n} H_{i, j} y_{j} . \tag{4}
\end{equation*}
$$

The elements where $H_{i, j}=0$ can simply be removed from the sum. Hence, in $\mathbb{Z}_{2}$,
this equation can be simplified to

$$
\begin{equation*}
v_{i}=\sum_{H_{i, j}=1} y_{j} . \tag{5}
\end{equation*}
$$

This equation is optimised by the fact that $H$ is fixed for several different words $y$, and depends on the number of ones in $H$. Let us illustrate this for $y=000000100010111$ and as defined in Equation (11). The computation in Equation 4 amounts to the following calculation, where the $H_{i, j}=1$ values are highlighted,

$$
\left(\begin{array}{l}
1 \times 0+0 \times 0+1 \times 0+1 \times 0+1 \times 0+0 \times 0+0 \times 1+1 \times 0+1 \times 0+0 \times 0+0 \times 1+0 \times 0+0 \times 1+0 \times 1+0 \times 1 \\
0 \times 0+1 \times 0+1 \times 0+0 \times 0+1 \times 0+0 \times 0+0 \times 1+0 \times 0+1 \times 0+0 \times 0+0 \times 1+0 \times 0+0 \times 1+0 \times 1+0 \times 1 \\
0 \times 0+0 \times 0+1 \times 0+1 \times 0+0 \times 0+1 \times 0+0 \times 1+0 \times 0+0 \times 0+1 \times 0+0 \times 1+0 \times 0+0 \times 1+0 \times 1+0 \times 1 \\
0 \times 0+0 \times 0+0 \times 0+1 \times 0+1 \times 0+0 \times 0+1 \times 1+0 \times 0+0 \times 0+0 \times 0+1 \times 1+0 \times 0+0 \times 1+0 \times 1+0 \times 1 \\
0 \times 0+0 \times 0+0 \times 0+0 \times 0+1 \times 0+1 \times 0+0 \times 1+1 \times 0+0 \times 0+0 \times 0+0 \times 1+1 \times 0+0 \times 1+0 \times 1+0 \times 1 \\
0 \times 0+0 \times 0+0 \times 0+0 \times 0+0 \times 0+1 \times 0+1 \times 1+0 \times 0+1 \times 0+0 \times 0+0 \times 1+0 \times 0+1 \times 1+0 \times 1+0 \times 1 \\
0 \times 0+0 \times 0+0 \times 0+0 \times 0+0 \times 0+0 \times 0+1 \times 1+1 \times 0+0 \times 0+1 \times 0+0 \times 1+0 \times 0+0 \times 1+1 \times 1+0 \times 1 \\
0 \times 0+0 \times 0+0 \times 0+0 \times 0+0 \times 0+0 \times 0+0 \times 1+1 \times 0+1 \times 0+0 \times 0+1 \times 1+0 \times 0+0 \times 1+0 \times 1+1 \times 1
\end{array}\right)
$$

On the other hand the computation of the same result according to Equation (5) is illustrated by the following calculation,

$$
\left(\begin{array}{c}
0+0+0+0+0+0 \\
0+0+0+0 \\
0+0+0+0 \\
0+0+1+1 \\
0+0+0+0 \\
0+1+0+1 \\
1+0+0+1 \\
0+0+1+1
\end{array}\right) .
$$

This latter calculation is much smaller than the previous one, thus making the checking procedure much faster. Hence we use Equation (5) to implement it. A hardware implementation of a checking procedure can definitely benefit from Equation (5), since every add operation is implemented with an xor gate and we only need to add the bits $y_{j}$ for which $H_{i, j}=1$. Reducing the number of xor gates reduces
both the cost and the time requirements of the resulting circuit. This application is clearly important and it will be the focus of further research. For now we will describe a computer architecture aware checker implementation.

Modern CPUs provide bitwise XOR operations, meaning that the operation is applied to all the bits in the computer word, i.e., to 64 bits at a time in contemporary CPUs. The bits in the computer word are processed essentially in parallel. We explore this parallelism by checking 64 independent received words at the same time. To clearly establish the relation between the number of ones in $H$ and the performance of a computation based on Equation (5), it would be enough to check only one word. However this would be a considerable waist of performance. Hence we chose to check 64 words at a time. This means that a batch of 64 codewords needs to be packed for checking, so that all the $y_{j}$ bits are stored in a single computer word.

Our test consisted in generating random bit words and checking with several PCMs for the same code, in particular PCMs obtained with greedy search and with simulated annealing. The results are shown in the plots in Figures 6 and 7 .

In these plots the $x$ axis corresponds to the number of ones in the respective PCMs, and the $y$ axis corresponds to the average time to check a given word. Like before the results for the LTE Turbo code with $n=396$ and $k=128$ show the best illustration of the methods we present. In this example the PCMs with most ones are clearly slower than the PCMs with less ones. Moreover because the number of ones varies significantly, between 1500 and more than 4000 , this effect is even more notorious. This case clearly shows an important improvement that results from obtaining sparse PCMs of a given code.

On the other hand for the code BCH-127-92-5-strip this effect is not as notorious as the variance of the checking time is very big.


Figure 6: Number of ones ( $x$ axis) versus time ( $y$ axis).

### 4.2.4. Related Work

In this section, we discuss some related work to give perspective on our work. Simulated annealing is a probabilistic technique for approximating the global minimum of a function. The name derives from the metallurgy technique of heating and controlled cooling of metal. The method was initially used to approximate the global minimum of a function with several variables (Khachaturyan et al., 1979, 1981). It was then formulated in the context of optimization by Kirkpatrick et al. (1983). One of its initial-and most well-known-applications was to the travelling salesman problem (Černỳ, 1985). Since then, it has been applied to a wide range of applications, as surveyed by Koulamas et al. (1994).

Information theory lies at the core of modern computers and communication technologies (Shannon, 1948) and can be traced back to the 1940s. The two main applications of information theory are data compression and error correcting codes.


Figure 7: Number of ones ( $x$ axis) versus time ( $y$ axis).

The pioneering work on error correcting codes was made by Hamming (1950) with the introduction of the Hamming $(7,4)$ code. This code is also a linear code, as the codewords form a linear subspace. The codewords are 7 bits long. The distance among codewords is 3 meaning that is possible to correct errors in a single bit, or detect errors in at most 2 bits. For a nice introduction to the subject, we refer to the book of Hill (1986). Several linear codes followed this initial breakthrough Golay, 1949). The BCH codes that we tested extensively in this work were discovered independently by Hocquenghem (1959) and Bose and Ray-Chaudhuri (1960). A personal description of this time is presented by Reed $(2000)$, which naturally describes the history of Reed-Solomon codes (Reed and Solomon, 1960), also a class of linear codes. Turbo codes are a much more recent discovery and are the first practical codes to approximate the channel capacity (Berrou et al., 1993). They are currently in use in 3 G and 4G mobile communication standards and deep space communication, as
well as in other applications where it is necessary to achieve reliable information transmission over bandwidth or latency constrained channels.

The main advantage of linear codes is that, given their parity-check matrix, it is straightforward to check if a given word is a codeword or not. Cancellieri (2015) gives an extensive discussion on the relation between generator matrices and PCMs. Detecting errors is simple as it uses matrix multiplication only. Note that even matrix multiplication becomes faster for sparser matrices. Still, the main advantage of sparser matrices is in error correction, for which maximum likelihood approaches are used (Breitbach et al., 1998; Feldman, 2003; Helmling et al., 2014; Tanatmis et al., 2010; Vontobel and Koetter, 2007, Zhang and Siegel, 2012). Other applications of sparse PCMs (in a row sense) along with a theoretical analysis of lower bounds is presented by Naor and Verstraëte (2008).

Recently, Gensheimer et al. (2018) pointed out that minimizing the number of ones in the PCM reduces the run time of ML decoding. The authors presented a method of obtaining sparse PCMs based on integer programming, which can be solved with the Gurobi or CPLEX solvers. Here, we partially reproduce their result tables for a couple of BCH and LTE codes and show that the simulated annealing algorithm obtains comparable final results. Table 2 shows these results, where the time reflects the amount of time each algorithm was allowed to run. The best result was often found earlier. The IP refers to the algorithm by Gensheimer et al., the bound is a lower bound on the number of ones. We only computed the lower bound for the smaller BCH codes. As expected, the greedy algorithm performs very well, but does not always obtain the best value. In most cases, the number of ones matches the lower bound and are therefore known to be optimal. In the case of the code BCH-6336 , the lower bound is smaller than the results obtained by the simulated annealing and the IP algorithm, with both obtaining the same result of 384 . The results for

LTE codes were obtained using the same simulated annealing parameterization as for BCH codes. Even though the results obtained with simulated annealing (in much less time) are comparable to those obtained by IP, a more tailored parameterization would allow to attain even better results.

Table 2: Comparison of the number of ones obtained by lower bound, IP, the greedy algorithm and the simulated annealing.

|  | bound | IP |  | Greedy |  | Annealing |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Code | $\# 1 \mathrm{~s}$ | $\# 1 \mathrm{~s}$ | time (s) | \#1s | time (s) | $\# 1 \mathrm{~s}$ | time (s) |
| BCH-63-30 | 396 | 396 | 990 | 406 | 191 | 396 | 210 |
| BCH-63-36 | 378 | 384 | 810 | 402 | 216 | 384 | 212 |
| BCH-63-39 | 336 | 336 | 720 | 344 | 211 | 336 | 221 |
| BCH-63-45 | 288 | 288 | 540 | 288 | 301 | 288 | 287 |
| BCH-63-51 | 288 | 288 | 360 | 288 | 299 | 288 | 366 |
| BCH-63-57 | 192 | 192 | 180 | 192 | 539 | 192 | 630 |
| LTE-132-40 | - | 472 | 82800 | 743 | 5.76 | 562 | 7.71 |
| LTE-156-48 | - | 568 | 97200 | 940 | 5.83 | 662 | 10.8 |
| LTE-180-56 | - | 663 | 37200 | 1180 | 6.59 | 776 | 13.8 |
| LTE-204-64 | - | 760 | 42000 | 1401 | 6.34 | 865 | 18.9 |
| LTE-396-128 | - | 1594 | 16080 | 3543 | 10.4 | 2030 | 127 |
| LTE-780-256 | - | 3377 | 31440 | 10933 | 20.9 | 5564 | 808 |

A recent survey on the applications of sparse binary matrices was given by Martinovic et al. (2005). The authors start by discussing how to represent sparse matrices to reduce space requirements and then focus on binary matrices and their applications in clustering, web graph computations, web link analysis and binary factor analysis Keprt (2014).

As a final note we point out the significant impact of the work on graph sparsification (Keprt, 2014, Benczúr and Karger, 1996; Spielman and Teng, 2013a, b; Koutis et al. (2014) which reduces a graph to a smaller graph that contains many of the same properties, but that requires much less space. This transformation is lossy in the sense that it may be impossible to recover the original graph from the sparse graph. This line of research is a good example of the advantages that can be obtained with sparser binary matrices, in this case graph adjacency matrices.

## 5. Conclusion and Future work

In this paper, we considered the problem of finding the sparsest parity-check matrix (PCM) for a given linear error correcting code. We proposed an algorithm which modifies this matrix by adding one row to another. We flagged rows as clean or dirty as a way to speed-up the choice of which rows to alter. For codes that contained excessive density, this process turns out to be effective. However, if this process is applied in a greedy fashion, i.e., never considering movements which make the underlying matrix denser, the resulting PCM may still be significantly far from the global optimum. Therefore, we studied a simulated annealing approach, which yields very good results.

In general, our experimental results indicate that the simulated annealing algorithm is very likely to achieve a global minimum, given a reasonable amount of time, with reasonable hardware requirements. Most of the codes tested seemed to stabilize within a couple of hours with one notable exception being the $\mathrm{BCH}-255-207-6$ code.

We also proposed a simple way to choose the temperature parameters for the simulated annealing algorithm, by specifying with what probability $p$ are we willing to accept an uphill transition that impacts our goal by a value of $d$. We also explained how this probability $p$ changes for different values of $d$.

We are currently working on checking the performance of our algorithm in other error correcting codes and on fine tuning the corresponding ideal temperatures and cooling schedules. The results of applying simulated annealing to error decoding were very positive. Moreover, Turbo codes are a particularly relevant class of codes, see Section 4.2.4 and Appendix A. In the future, we plan to investigate more applications of this technique. In particular, we want to apply this technique to the decoding process, thus presenting a possible alternative to ML decoding.

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## Appendix A. Experimental Results

This appendix provides more experimental results for our algorithm.
This first plot is similar to the plot in Figure 1, but for our full test set. The bars represent percentage of the number of ones in the resulting PCMs. The green bars are always $100 \%$. The blue bars correspond to the results obtained by the greedy approach and the red bars the results of the Simulated Annealing algorithm. The results are sorted in decreasing values of the greedy algorithm.

## Number of ones



Figure A.8: Bar chart illustrating the relative number of ones obtained with the greedy and simulated annealing algorithms.







BCH_255_231_3_strip


BCH_3240_3072_12_strip
$\left\{\begin{array}{l}251500 \\ 251000 \\ 250500 \\ 250000 \\ 249500\end{array}\right.$
00.511 .522 .533 .544 .55
Time (minutes)

BCH_6480_6312_12_strip


BCH_255_163_12_strip

| 4 |  |
| :---: | :---: |
| 1.035 | - |
| 1.03 | - |
| 1.025 | 7 ${ }^{2}$ |
| 1.02 |  |
| 1.015 |  |
| 1.01 | \%rewtertway |
| 1.005 | =-20.4a |
| 1.005 | - |
|  |  |
|  | 00.511 .522 .533 .544 .55 |
|  | Time (minutes) |

BCH_1023_993_3_strip


BCH_5400_5232_12_strip


BCH_16383_16215_12_strip





LTE_TC_N564_K184


LTE_TC_N612_K200


LTE_TC_N660_K216


LTE_TC_N540_K176


LTE_TC_N588_K192


LTE_TC_N636_K208


LTE_TC_N684_K224





LTE_TC_N996_K328


LTE_TC_N1044_K344


LTE_TC_N924_K304


LTE_TC_N972_K320


LTE_TC_N1020_K336


LTE_TC_N1068_K352






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[^1]:    ${ }^{1}$ In particular $n=15$ is the codeword size and $k=7$ is the dimension.

[^2]:    ${ }^{2}$ Note that, for any word $y$, we want to have $H y \equiv \overrightarrow{0}$ iff $H^{\prime} y \equiv \overrightarrow{0}$. The forward implication is straight forward, if $H y \equiv \overrightarrow{0}$ then $H^{\prime} y \equiv S \cdot H y \equiv S \overrightarrow{0}=\overrightarrow{0}$. The implication in the reverse direction can be proven by assuming that $H^{\prime} y \equiv \overrightarrow{0}$, and because $S$ is invertible, we can apply $S^{-1}$ to both sides of the equation and obtain that $H y \equiv \overrightarrow{0}$.

[^3]:    3 "http://www.inference.org.uk/mackay/codes/alist.html
    4 "https://www.uni-kl.de/channel-codes/channel-codes-database/

