

# On stabilizability and exact observability of stochastic systems with their applications <sup>1</sup>

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## Abstract

This paper discusses the stabilizability, weak stabilizability, exact observability and robust quadratic stabilizability of linear stochastic control systems. By means of the spectrum technique of the generalized Lyapunov operator, a necessary and sufficient condition is given for stabilizability and weak stabilizability of stochastic systems, respectively. Some new concepts called unremovable spectrums, strong solutions, and weakly feedback stabilizing solutions are introduced. An unremovable spectrum theorem is given, which generalizes the corresponding theorem of deterministic systems to stochastic systems. A stochastic Popov-Belevith-Hautus (PBH) criterion for exact observability is obtained. For applications, we give a comparison theorem for generalized algebraic Riccati equations (GAREs), and two results on Lyapunov-type equations are obtained, which improve the previous works. Finally, we also discuss robust quadratic stabilization of uncertain stochastic systems, and a necessary and sufficient condition is given for quadratic stabilization via a linear matrix inequality (LMI).

Keywords: Stabilizability; exact observability; quadratic stabilizability; strong solution; spectrum

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# 1 Introduction

Stabilizability and observability is essential and important concepts in modern control theory, especially, in system analysis and synthesis. Stochastic stabilizability (in mean square sense) plays a crucial role and is an essential assumption in many problems, such as infinite horizon stochastic optimal control problem (J. L. Willems & J. C. Willems, 1976; Ichikawa, 1979; Ait Rami & Zhou, 2000), robust and stochastic  $H^\infty$  problems (Ugrinovskii, 1998; Hinrichsen & Pritchard, 1998), filtering problems (Bensoussan, 1992, and the reference therein), control and stabilization problems for jump system (Ghaoui & Ait Rami, 1998, Gao et al., 2001 and the references therein). In this paper, we mainly study the stabilizability and exact observability of the following linear stochastic controlled system

$$\begin{cases} dx(t) = (Ax(t) + Bu(t)) dt + (Cx + Du) dw(t), x(0) = x_0 \in \mathcal{R}^n, \\ y = Qx, \end{cases} \quad (1)$$

where  $A, B, C, D, Q \in \mathcal{R}^{n \times n} \times \mathcal{R}^{n \times m} \times \mathcal{R}^{n \times n} \times \mathcal{R}^{n \times m} \times \mathcal{R}^{l \times n}$  are real constant matrices;  $w(\cdot)$  is a standard Wiener process with  $w(0) = 0$  defined on the filtered probability space  $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$  with  $\mathcal{F}_t = \sigma\{w(s), 0 \leq s \leq t\}$ . Without loss of generality, we assume  $w(\cdot)$  is one-dimensional for simplicity.

On the stabilizability of system (1) (For short, we also call  $(A, B; C, D)$  stabilizable in the context), some results were obtained, for example, Zhang (2000) proved that  $(A, B; C, D)$  is stabilizable if and only if (iff) the following GARE

$$PA + A'P + C'PC + I - (PB + C'PD)(I + D'PD)^{-1}(B'P + D'PC) = 0 \quad (2)$$

has a positive definite solution  $P > 0$ . By means of linear matrix inequalities (LMIs) and Lyapunov-type inequalities, Ait Rami and Zhou (2000) gave some other necessary and sufficient conditions. In addition, J. L. Willems and J. C. Willems (1976) also presented many criteria to test the stabilizability for some special stochastic systems with arbitrary noise intensities.

Exact observability was first introduced by Zhang (1998) by means of finite-dimensional invariance subspace, and then by Liu (1999) from the physical viewpoint, i.e., the zero output must correspond to zero state, which is the generalized version of complete observability of deterministic linear system theory, and has many applications in studying GAREs, and stochastic stability; see Zhang (1998) and Liu (1999).

In this present paper, we try to give a spectral or eigenvalue descriptions for the stabilizability and exact observability of stochastic system (1). In order to illustrate our goal, let us recall some

results with deterministic controlled system

$$\begin{cases} \dot{x} = Ax + Bu, & x(t_0) = x_0, \\ y(t) = Qx(t). \end{cases} \quad (3)$$

It is well known that for complete controllability, complete observability, stabilizability and detectability of system (3), we have the following criteria:

1)  $(A, B)$  is completely controllable iff there does not exist a nonzero vector  $\xi$  and  $\lambda \in \mathcal{C}$  satisfying

$$\xi' A = \lambda \xi', \quad \xi' B = 0. \quad (4)$$

2)  $(A, B)$  is stabilizable iff there does not exist a nonzero vector  $\xi$ ,  $Re(\lambda) \geq 0$  satisfying (4).

3)  $(Q, A)$  is completely observable iff there does not exist a nonzero vector  $\xi$  and  $\lambda \in \mathcal{C}$  satisfying

$$A\xi = \lambda\xi, \quad Q\xi = 0. \quad (5)$$

4)  $(Q, A)$  is detectable iff there does not exist a nonzero vector  $\xi$ ,  $Re(\lambda) \geq 0$  satisfying (5).

The above criteria for complete controllability (stabilizability, complete observability, detectability, resp.) are the so-called Popov-Belevith-Hautus (PBH) Criteria, and the  $\lambda$  satisfying (4) can be called an unremovable spectrum of system (3); see Proposition 1 of this paper. So PBH criterion tells us that system (3) is stabilizable iff its all unremovable spectra belong to left hand side complex plane. PBH criterion is important in the pole assignment of linear systems.

In this paper, we try to develop an analogous theory in stochastic systems. By defining the closed-loop operator  $\mathcal{L}_K$ , the necessary and sufficient conditions for stabilizability and weak stabilizability of system (1) are presented via the spectrum of operator  $\mathcal{L}_K$ . The proposed spectrum-based technique makes it possible to generalize some concepts of deterministic systems to stochastic case. For instance, we can define weak stabilizability of stochastic systems, strong solution of GARE, etc.. We obtain a stochastic unremovable spectrum theorem, however, PBH criterion is only a necessary but not a sufficient condition for the stochastic stabilizability. It is very interesting that PBH criterion still holds for exact observability. Moreover, we find that even if system (1) is exactly-terminal controllable (Peng 1994), the spectrum of system (1) cannot be assigned arbitrarily. All these reveal the essential differences between the deterministic and stochastic systems.

As our theoretical applications, some results on GAREs and Lyapunov-type equations are improved. This paper also discusses the quadratic stabilization of stochastic uncertain systems, for which, a necessary and sufficient condition is given.

This paper is organized as follows: In section 2, we discuss the relation between stabilizability and spectrum. Some definitions are introduced, then necessary and sufficient theorems are obtained for the stabilizability and unremovable spectrum. The stochastic PBH Criteria are also presented for some special systems. In section 3, a stochastic PBH criterion is obtained for exact observability, by duality, we also define stochastic detectability, various implication relations with complete observability, exact observability, detectability and stochastic detectability have been clarified. In section 4, we deal with the weak stabilizability of stochastic systems, and a necessary and sufficient theorem is given in form of spectrum, and some sufficient conditions are given for the weak stabilizability in LMIs and Lyapunov-type inequalities. In section 5, some theoretical applications to GAREs and Lyapunov-type equations are developed. Section 6 gives a necessary and sufficient condition for quadratic stabilization of stochastic uncertain systems. Section 7 concludes this paper with some comments.

## 2 Stabilizability and spectrum of stochastic systems

For convenience, we adopt the following notations:

$\mathcal{S}_n$ : the set of all  $n \times n$  symmetric matrices, its entries may be complex numbers;

$A'$  (  $\text{Ker}(A)$ ): the transpose ( kernel space) of the matrix  $A$ ;

$A \geq 0$  ( $A > 0$ ):  $A$  is a positive semidefinite (positive definite) symmetric matrix;

$I$ : identity matrix;

$\sigma(L)$ : spectral set of the operator or matrix  $L$ ;

$\mathcal{C}^-$  ( $\mathcal{C}^{-,0}$ ): the open left (closed left ) hand side complex plane.

In this section, we mainly study the stabilizability and spectral properties of stochastic system (1). Firstly, we give the following definitions:

**Definition 1.** Stochastic system (1) is called stabilizable (in the mean square sense), if there exists a feedback control  $u(t) = Kx(t)$ , such that for any  $x_0 \in \mathcal{R}^n$ , the closed-loop system

$$dx(t) = (A + BK)x(t) dt + (C + DK)x(t) dw(t), x(0) = x_0 \quad (6)$$

is asymptotically mean square stable, i.e.,  $\lim_{t \rightarrow \infty} E[x(t)x'(t)] = 0$ , where  $K \in \mathcal{R}^{m \times n}$  is a constant matrix.

Below, we define the spectrum of the closed-loop system (6).

**Definition 2.** For any given feedback gain matrix  $K$ , let  $\mathcal{L}_K$  be a linear operator from  $\mathcal{S}_n$

to  $\mathcal{S}_n$  defined as follows:

$$\mathcal{L}_K : X \in \mathcal{S}_n \mapsto (A + BK)X + X(A + BK)' + (C + DK)X(C + DK)'$$

The spectrum of  $\mathcal{L}_K$  is the set defined by  $\sigma(\mathcal{L}_K) = \{\lambda \in \mathcal{C} : \mathcal{L}_K(X) = \lambda X, X \in \mathcal{S}_n, X \neq 0\}$ , which is also called the spectrum of system (6).

Now, we give the first theorem for stabilizability of system (1).

**Theorem 1.** System (1) is stabilizable iff there exists a  $K \in \mathcal{R}^{m \times n}$ , such that the spectrum of (6) belongs to  $\mathcal{C}^-$ .

**Proof.** By definition, we only need to prove there exists a  $K \in \mathcal{R}^{m \times n}$ , such that (6) is asymptotically mean square stable. Let  $X(t) = E[x(t)x'(t)]$ , where  $x(t)$  is the trajectory of (6), then by Ito's formula,

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + X(A + BK)' + (C + DK)X(C + DK)' = \mathcal{L}_K(X(t)), \\ X(0) = X_0 = x_0x_0'. \end{cases} \quad (7)$$

Since  $X(t) \in \mathcal{S}_n$ , (7) is a symmetric matrix-valued equation including  $n(n + 1)/2$  different variables. If we write  $X = (Ex_ix_j)_{n \times n} = (X_{ij})_{n \times n}$ , and define a map  $\tilde{\mathcal{L}}$  from  $\mathcal{S}_n$  to  $\mathcal{R}^{\frac{n(n+1)}{2}}$  as follows:

$$\tilde{X} = \tilde{\mathcal{L}}(X) = (X_{11}, X_{12}, \dots, X_{1n}, X_{22}, X_{23}, \dots, X_{2n}, \dots, X_{n-1, n-1}, X_{n-1, n}, X_{nn})',$$

then there exists a unique matrix<sup>2</sup>  $L(K) \in \mathcal{R}^{\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}}$ , such that equation (7) is equivalent to

$$\dot{\tilde{X}} = L(K)\tilde{X}, \quad \tilde{X}(0) = \tilde{X}_0. \quad (8)$$

Obviously,

$$\lim_{t \rightarrow \infty} Ex(t)x'(t) = 0 \iff \lim_{t \rightarrow \infty} \tilde{X}(t) = 0 \iff \sigma(L(K)) \subset \mathcal{C}^-. \quad (9)$$

By Definition 2, it is not difficult to prove  $\sigma(L(K)) = \sigma(\mathcal{L}_K)$ . Hence, from the above, the proof of Theorem 1 is completed.

Below, we say  $L(K)$  is a matrix induced by  $\mathcal{L}_K$ .

**Remark 1.** It is easily seen that the following operator

$$\mathcal{L}_K^* : X \in \mathcal{S}_n \mapsto X(A + BK) + (A + BK)'X + (C + DK)'X(C + DK)$$

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<sup>2</sup>In order to give the exact expression of  $L(0)$ , we developed a new technique called  $\mathcal{H}$ -representation technique in 2012; see W. Zhang and B. S. Chen,  $\mathcal{H}$ -representation and applications to generalized Lyapunov equations and linear stochastic systems, IEEE Trans. Automatic Control, 57(12): pp. 3009-3022, 2012.

is the adjoint operator of  $\mathcal{L}_K$  with the inner product  $\langle X, Y \rangle = \text{trace}(X^*Y)$  for any  $X, Y \in \mathcal{S}_n$ . So system (1) is stabilizable iff there exists an  $K \in \mathcal{R}^{m \times n}$ , such that  $\sigma(\mathcal{L}_K^*) \subset \mathcal{C}^-$ . As we limit the coefficient matrices to be real, so  $\sigma(\mathcal{L}_K^*) = \sigma(\mathcal{L}_K)$ . If we denote the induced matrix of the operator  $\mathcal{L}_K^*$  by  $L^*(K)$ , then from the proof of Theorem 1, we have

$$\sigma(\mathcal{L}_K^*) = \sigma(\mathcal{L}_K) = \sigma(L(K)) = \sigma(L^*(K)),$$

where any one of them can characterize the stabilizability of system (1).

To illustrate the meaning of the above notations, we give an example as follows:

**Example 1:** In system (6), take  $K = I$ ,

$$A = \begin{bmatrix} -3 & 1/2 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easily tested that (7) becomes

$$\begin{bmatrix} \dot{X}_{11} & \dot{X}_{12} \\ \dot{X}_{12} & \dot{X}_{22} \end{bmatrix} = \begin{bmatrix} X_{12} & -X_{11} + \frac{1}{2}X_{22} \\ -X_{11} + \frac{1}{2}X_{22} & -2X_{12} + X_{22} \end{bmatrix},$$

which is equivalent to

$$\dot{\tilde{X}} = \begin{bmatrix} \dot{X}_{11} \\ \dot{X}_{12} \\ \dot{X}_{22} \end{bmatrix} = L(K)\tilde{X} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \frac{1}{2} \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12} \\ X_{22} \end{bmatrix}.$$

So

$$L(K) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \frac{1}{2} \\ 0 & -2 & 1 \end{bmatrix}.$$

Since  $\mathcal{L}_{K=I}(X) = \lambda X$ ,  $X \in \mathcal{S}^n$ , is equivalent to

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \frac{1}{2} \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12} \\ X_{22} \end{bmatrix} = \lambda \begin{bmatrix} X_{11} \\ X_{12} \\ X_{22} \end{bmatrix},$$

$\sigma(\mathcal{L}_{K=I}) = \sigma(L(K=I))$ . From  $\mathcal{L}_{K=I}^*(X) = \lambda X$ , we have

$$\begin{bmatrix} 0 & -2 & 0 \\ \frac{1}{2} & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12} \\ X_{22} \end{bmatrix} = \lambda \begin{bmatrix} X_{11} \\ X_{12} \\ X_{22} \end{bmatrix}.$$

Via a simple computation, we have  $\sigma(\mathcal{K}_{K=I}^*) = \sigma(L^*(K = I)) = \sigma(\mathcal{L}_{K=I}) = \sigma(L(K = I)) = \{\lambda_1, \lambda_2, \lambda_3\}$ , where

$$L^*(K = I) = \begin{bmatrix} 0 & -2 & 0 \\ \frac{1}{2} & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix},$$

and  $\lambda_i$  are the roots of the following characteristic polynomial of  $L^*(K = I)$  or  $L(K = I)$ , which is  $f(\lambda) = \lambda^3 - \lambda^2 + 2\lambda - 1$ .

**Definition 3.** We say that  $\lambda$  is an unremovable spectrum of system (1) with state feedback  $u = Kx$ , if there exists  $0 \neq X \in \mathcal{S}_n$ , such that for any  $K \in \mathcal{R}^{m \times n}$ ,

$$X(A + BK) + (A + BK)'X + (C + DK)'X(C + DK) = \lambda X \quad (10)$$

holds.

Below, we give a theorem with respect to the unremovable spectrum.

**Theorem 2**<sup>3</sup>.  $\lambda$  is an unremovable spectrum of system (1) iff there exists  $0 \neq X \in \mathcal{S}_n$ , such that the following three equalities

$$XA + A'X + C'XC = \lambda X, \quad XB + C'XD = 0, \quad D'XD = 0 \quad (11)$$

hold.

**Proof.** Note that (10) can be written as

$$XA + A'X + C'XC + (XB + C'XD)K + K'(XB + C'XD)' + K'D'XDK = \lambda X, \quad (12)$$

so if (11) holds, then (10) automatically holds, and the sufficiency is proved.

To prove the necessity of Theorem 2, we first take  $K = 0$  in (10), then

$$XA + A'X + C'XC = \lambda X$$

holds. Again, from (12), it follows that

$$(XB + C'XD)K + K'(XB + C'XD)' + K'D'XDK = 0. \quad (13)$$

Let  $XB + C'XD = F, D'XD = G$ , then (13) becomes

$$FK + K'F' = -K'GK. \quad (14)$$

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<sup>3</sup>This theorem was first pointed out as a conjecture in “W. Zhang, Study on algebraic Riccati equation arising from infinite horizon stochastic LQ optimal control, Ph.D Thesis, Hangzhou: Zhejiang University, 1998.”

Since the left hand side of (14) is linear with  $K$ , we must have  $G = 0$ . In fact, due to the linearity of  $K'GK$ ,

$$(K + K)'G(K + K) = 4K'GK = K'GK + K'GK = 2K'GK.$$

So  $K'GK = 0$  because of the arbitrariness of  $K$ , which is necessary that  $G = 0$ , i.e.,  $D'XD = 0$ . To prove  $F = 0$  or  $XB + C'XD = 0$ , we note that (14) becomes  $K'F' = -FK$ . Denote  $F = (f_{ij})_{n \times m}$ , and take

$$K = K_{ij} = (k_{ls})_{m \times n} = \begin{cases} 1, & \text{if } l = i, s = j, \\ 0, & \text{otherwise.} \end{cases}$$

From  $K'F' = -FK$ , one knows  $f_{1i} = f_{2i} = \cdots = f_{ni} = 0$ . Set  $i = 1, 2, \cdots, n$ , then  $f_{ij} = 0$  for  $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$ , that is  $F = 0$ . The proof of Theorem 2 is completed.

**Remark 2.** Commonly, we call  $\lambda$  an unremovable spectrum of (3) with state feedback  $u = Kx$ , if for any feedback gain  $K$ ,  $\lambda \in \sigma(A + BK)$ . We should point out that, when  $C = 0, D = 0$ , Definition 3 coincides with the conventional one of system (3) illustrated by the following Proposition, which may be new even in linear system theory.

**Proposition 1.** For  $C = 0, D = 0$ , the following conditions are equivalent:

- 1) There exists  $0 \neq X \in \mathcal{S}_n$  satisfying (11).
- 2) There exists a nonzero complex vector  $\xi$  satisfying (4).
- 3) For any  $K \in \mathcal{R}^{m \times n}$ ,  $\lambda \in \sigma(A + BK)$ .

**Proof.** 1) $\Leftrightarrow$  2). If 1) holds, then there exists  $0 \neq X \in \mathcal{S}_n$  satisfying

$$XA + A'X = \lambda X, \quad XB = 0,$$

which is equivalent to

$$XA + A'X = \lambda X, \quad XBB' = 0. \quad (15)$$

(15) can be written as

$$(I \otimes A' + A' \otimes I)\vec{X} = \lambda\vec{X}, \quad (I \otimes BB')\vec{X} = 0, \quad (16)$$

where  $\vec{X}$  denotes the vector formed by stacking the rows of  $X$  into one long vector, i.e.,

$$\vec{X} = [X_{11}, X_{12}, \cdots, X_{1n}, X_{21}, X_{22}, \cdots, X_{2n}, \cdots, X_{n1}, X_{n2}, \cdots, X_{nn}]',$$

$F \otimes G$  denotes the Kronecker product of two matrices  $F$  and  $G$ . Applying the above two facts, one can immediately show 1) $\Rightarrow$  2).

Fact 1 (Ortega, 1987). If  $\xi_1, \xi_2, \dots, \xi_p$  are linearly independent eigenvectors of  $A'$ , then the Kronecker product

$$\xi_i \otimes \xi_j, \quad i, j = 1, 2, \dots, p$$

are linearly independent eigenvectors of  $(I \otimes A' + A' \otimes I)$ .

Fact 2 (Ortega, 1987). If  $e_1, e_2, \dots, e_n$  are the coordinate vectors of  $\mathcal{R}^n$ ,  $\zeta_1, \zeta_2, \dots, \zeta_q$  are linearly independent eigenvectors of  $BB'$  corresponding to zero eigenvalue, then the Kronecker product of  $e_i$  and  $\zeta_j$

$$e_i \otimes \zeta_j, \quad i = 1, 2, \dots, n, j = 1, \dots, q$$

are linearly independent eigenvectors of  $I \otimes BB'$  corresponding to zero eigenvalue.

2) $\Rightarrow$  1) is simple, we only need to take  $X = \xi\xi'$ . So 1) $\Leftrightarrow$  2).

2) $\Leftrightarrow$  3). From 2), it is easy to show for any  $K$ ,  $\lambda \in \sigma(A' + K'B') = \sigma(A + BK)$ , so 2) $\Rightarrow$  3).

3) $\Rightarrow$  2). If  $Re(\lambda) \geq 0$ , and  $\lambda \in \sigma(A + BK)$  with any  $K$ , but 2) does not hold, then by PBH criterion,  $(A, B)$  is stabilizable. So there exists  $K_1$  such that  $\sigma(A + BK_1) \subset \mathcal{C}^-$ , accordingly, it is impossible that  $\lambda \in \sigma(A + BK)$  with  $Re(\lambda) \geq 0$ . If  $Re(\lambda) < 0$ , then  $\mu = -\lambda \in \sigma(-A - BK)$  with  $Re(\mu) > 0$ . By the same discussion, we can still show that 3) implies 2). Therefore, 2) $\Leftrightarrow$  3). Proposition 1 is proved.

By Theorem 2 and Proposition 1, deterministic PBH criterion can be stated in another form as follows:

**Corollary 1.** When  $C = 0, D = 0$  in (1),  $(A, B)$  is stabilizable iff all the unremovable spectra of system (3) belong to  $\mathcal{C}^-$ , that is, there does not exist a nonzero  $X \in \mathcal{S}_n$ ,  $Re(\lambda) \geq 0$ , satisfying

$$XA + A'X = \lambda X, \quad XB = 0. \quad (17)$$

From Corollary 1, it is natural to conjecture that system (1) is stabilizable iff its all unremovable spectra in  $\mathcal{C}^-$ , or there does not exist  $0 \neq X \in \mathcal{S}^n$ ,  $Re(\lambda) \geq 0$  satisfying (11). Unfortunately, the following example shows that this conjecture is not true, so PBH criterion cannot be generalized to the stabilizability of stochastic systems, which reveals the essential difference between deterministic and stochastic systems with control entering into diffusion term.

**Example 2.** Consider a one-dimensional case of system (1). Take  $D \neq 0$ , then there does not exist  $X \neq 0$ ,  $Re(\lambda) \geq 0$ , satisfying (11). But from Theorem 1, system (1) is stabilizable iff  $B^2 + 2BCD - 2AD^2 > 0$ . Obviously,  $B^2 + 2BCD - 2AD^2 > 0$  is not equivalent to  $D \neq 0$ .

But can we be sure that stochastic PBH criterion holds with  $D = 0$ ? The answer is still no; see the following example:

**Example 3.** In system (1), take  $D = 0$ ,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

One can test that for any  $X \in \mathcal{S}_2$ , the following equation

$$XA + A'X + C'XC = \lambda X, \quad XB = 0, \quad Re(\lambda) \geq 0$$

does not have nonzero solution  $X$ , but  $(A, B; C, 0)$  is not stabilizable. Because by Zhang (2000),  $(A, B; C, 0)$  is stabilizable iff GARE

$$PA + A'P + C'PC - PBR^{-1}B'P + Q = 0 \quad (18)$$

has a positive definite solution  $P$ , but for our given data, the solutions of GARE (18) is

$$P_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

When does the PBH criterion hold for stochastic stabilizability? For the special case, we obtain the following theorem.

**Theorem 3 (Stochastic PBH criterion).** For  $D = 0$ , if there exists a matrix  $C_1$ , such that for any  $X \in \mathcal{S}_2$ ,  $C'XC = XC_1 + C_1'X$ , then system (1) is stabilizable iff its all unremovable spectra belong to  $\mathcal{C}^-$ , i.e., there does not exist nonzero  $X \in \mathcal{S}^n$ , such that

$$XA + A'X + C'XC = \lambda X, \quad XB = 0, \quad Re(\lambda) \geq 0. \quad (19)$$

**Proof.** By Theorem 1 and Remark 1, system (1) is stabilizable iff there exists an  $K \in \mathcal{R}^{m \times n}$ ,  $\sigma(\mathcal{L}_K^*) \subset \mathcal{C}^-$ . Note that for  $X \in \mathcal{S}_n$ ,

$$\begin{aligned} \mathcal{L}_K^*(X) &= X(A + BK) + (A + BK)'X + C'XC \\ &= X(A + C_1 + BK) + (A + BK + C_1X)'X. \end{aligned}$$

We know that system (1) is stabilizable iff the following deterministic linear system

$$\dot{z} = (A + C_1)z + Bu$$

is stabilizable, which, from Corollary 1, is equivalent to that there does not exist nonzero  $X$  satisfying

$$X(A + C_1) + (A + C_1)'X = XA + A'X + C'XC = \lambda X, \quad XB = 0.$$

Theorem 3 is proved.

Another problem is on the spectral placement. It is well known that in deterministic systems, a necessary and sufficient condition for complete controllability of  $(A, B)$  is that the spectrum of (3) can be arbitrarily assigned ( De Carlo 1989). But for stochastic systems, it is hard to relate controllability with spectrum in many existing definitions of stochastic controllability; see Peng (1994), Bashirov and Kerimov (1997), Mahmudov (2001). To illustrate the complexity, consider system (1) with ungiven initial state, on which the exactly-terminal controllability was introduced by Peng (1994) as follows:

**Definition 4.** System (1) is called exactly terminal-controllable, if for any  $\xi \in L^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_T)$ , there exists at least one admissible control  $u(t) \in L^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$ , and initial state  $x_0 \in \mathcal{R}^n$ , such that the corresponding trajectory satisfies  $x(T) = \xi$ , where  $L^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_T)$  denotes all  $\mathcal{F}_t$ -adapted, measurable and square integrable processes.

The following example shows that even if system (1) is exactly-terminal controllable, its spectrum cannot be assigned arbitrarily.

**Example 4.** We still consider one-dimensional case. Assume (1) is exactly-terminal controllable, then  $D \neq 0$  (Peng, 1994). By a simple computation, for any  $K$ ,

$$\sigma(\mathcal{L}_K^*) = \lambda = D^2K^2 + 2(CD + B)K + 2A + C^2 \geq 2A - \frac{(CD + B)^2}{D^2}.$$

So the spectrum cannot take arbitrary value in  $\mathcal{R}$ .

We note that (7) can also be written as

$$\dot{\vec{X}} = [(A + BK) \otimes I + I \otimes (A + BK) + (C + DK) \otimes (C + DK)]\vec{X} := L^0(K)\vec{X}. \quad (20)$$

Since  $\lim_{t \rightarrow \infty} E[x(t)x'(t)] = 0 \Leftrightarrow \lim_{t \rightarrow \infty} \vec{X} = 0$ ,  $(A, B; C, D)$  is stabilizable iff there exists a  $K$ , such that  $\sigma(L^0(K)) \subset \mathcal{C}^-$ . Kleinman (1969) further asserted that  $L^0(K)$  will have several repeated eigenvalues, this assertion is not true; see Zhang (1998). While each of  $L^0(K)$  and  $L(K)$  describes the stabilizability of system (1), it is natural to ask how relation with the spectrum between  $L^0(K)$  and  $L(K)$ ? In general, we have the following result.

**Proposition 2.**

- 1) If  $\lambda \in \sigma(L(K))$ , then  $\lambda \in \sigma(L^0(K))$ ;

2) If  $\lambda \in \sigma(L^0(K))$ , but  $\lambda$  does not belong to  $\sigma(L(K))$ , then there exists  $X \in \mathcal{C}^{n \times n}$ , such that  $L^0(K)\vec{X} = \lambda\vec{X}$ ,  $X = -X'$ .

**Proof.** If we Define linear operator  $\mathcal{L}_K^0$  as follows:

$$\mathcal{L}_K^0 : X \in \mathcal{C}^{n \times n} \mapsto (A + BK)X + X(A + BK)' + (C + DK)X(C + DK)'$$

then it is easily seen  $\sigma(L^0(K)) = \sigma(\mathcal{L}_K^0)$ . If  $\lambda \in \sigma(L(K))$ , then there exists  $X \in \mathcal{S}_n$ , such that  $\mathcal{L}_K(X) = \lambda X$ , therefore,  $\mathcal{L}_K^0(X) = \lambda X$ , i.e,  $\lambda \in \sigma(L^0(K))$ .

As to prove 2), we note that if  $\lambda \in \sigma(L^0(K))$ , then there exists  $X \in \mathcal{C}^{n \times n}$  satisfying  $\mathcal{L}_K^0(X) = \lambda X$ . By symmetry, there also has  $\mathcal{L}_K^0(X') = \lambda X'$ . If  $X \neq -X'$ , then from

$$\mathcal{L}_K^0(X + X') = \mathcal{L}_K^0(X) + \mathcal{L}_K^0(X') = \lambda X + \lambda X' = \lambda(X + X'),$$

$(X + X') \neq 0 \in \mathcal{S}_n$  is an eigenvector with respect to the eigenvalue  $\lambda$ . Therefore,  $\lambda \in \sigma(L(K))$ , which results in a contradiction, and, accordingly,  $X = -X'$ .

Theorem 1 or Proposition 1 has theoretical value in studying the spectral allocation and the solutions of GARE. Obviously, from the practical point of view, both of them are not convenient for testing the stabilizability of  $(A, B; C, D)$ . Ait Rami and Zhou (2000) presented an efficient method expressed by an LMI.

### 3 The spectral characterization for stochastic observability and detectability

In this section, we apply the spectral technique to study the observability and detectability of system (1). We first give a spectral criterion for exact observability of stochastic systems, and then by duality, we define stochastic detectability. All these concepts play critical roles in many fields, and some applicable examples can be found in section 5 of this paper.

**Definition 5:** Consider the following stochastic system with measurement equation:

$$\begin{cases} dx(t) = (Ax(t) + Bu(t)) dt + (Cx + Du) dw(t), x(0) = x_0 \in \mathcal{R}^n, \\ y(t) = Qx(t). \end{cases} \quad (21)$$

We call  $x_0 \in \mathcal{R}^n$  an unobservable state, if let  $u(t) \equiv 0$ , then, for any  $T > 0$ , the output response with respect to  $x_0$ , is always equal to zero, i.e.,

$$y(t) \equiv 0, \quad a.s., \quad t \in [0, T].$$

**Definition 6:** System (21) is called exactly observable, if there is no unobservable initial state (except zero initial state).

**Remark 3.** Definition 6 can also be equivalently expressed as follows: We call system (21) exactly observable, if for arbitrary  $x_0 \in \mathcal{R}^n$ ,  $x_0 \neq 0$ , there exists a  $t > 0$  such that  $y(t) = Qx(t) \neq 0$ , where  $x(t)$  is the solution to stochastic differential equation (SDE)

$$dx = Ax dt + Cx dw, x(0) = x_0 \in \mathcal{R}^n. \quad (22)$$

For simplicity, when system (21) is exactly observable, we also call  $[A, C|Q]$  exactly observable.

**Theorem 4.**  $[A, C|Q]$  is exactly observable iff there does not exist nonzero  $X \in \mathcal{S}_n$ , such that

$$XA' + AX + CXC' = \lambda X, \quad QX = 0, \quad \lambda \in \mathcal{C}. \quad (23)$$

**Proof.** Let  $X(t) = E[x(t)x'(t)], Y(t) = E[y(t)y'(t)]$ , where  $x(t)$  is the solution of (22). As in the previous discussion,  $X(t)$  satisfies

$$\dot{X}(t) = AX + XA' + CXC' := \mathcal{L}_{A,C}(X(t)), \quad X(0) = x_0x_0'. \quad (24)$$

By Remark 3, we know that  $[A, C|Q]$  is exactly observable iff for arbitrary  $X_0 = x_0x_0' \neq 0$ , there exists a  $t > 0$  such that

$$Y(t) = Ey(t)y'(t) = QX(t)Q' \neq 0. \quad (25)$$

From the proof of Theorem 3, (24) is equivalent to

$$\dot{\tilde{X}} = L(A, C)\tilde{X}, \quad (26)$$

where  $L(A, C)$  is the induced matrix by operator  $\mathcal{L}_{A,C}$ . Secondly, due to  $X(t) \geq 0$  for all  $t \geq 0$ , (25) is equivalent to

$$Y_1 = QX(t) \neq 0, \quad (27)$$

which is equivalent to

$$\vec{Y}_1 = \vec{L}(Q)\tilde{X} \neq 0. \quad (28)$$

$\vec{L}(Q)$  is one  $n^2 \times \frac{n(n+1)}{2}$ -order matrix, which is uniquely determined by  $Q$ . So (21) is exactly observable iff the deterministic system

$$\begin{cases} \dot{\tilde{X}} = L(A, C)\tilde{X}, \\ \vec{Y}_1 = \vec{L}(Q)\tilde{X} \end{cases} \quad (29)$$

is completely observable. By PBH criterion for observability, (29) is completely observable iff there does not exist  $0 \neq \xi \in \mathcal{C}^{\frac{n(n+1)}{2}}$ , such that

$$L(A, C)\xi = \lambda\xi, \vec{L}(Q)\xi = 0, \lambda \in \mathcal{C}.$$

By our definition with  $L(A, C), \vec{L}(Q)$ , which is equivalent to that there does not exist  $0 \neq X \in \mathcal{S}_n$  satisfying (23), the proof of Theorem 4 is completed.

We give the following example to illustrate the notion of  $\vec{L}$ .

**Example 5.** In (27), taking

$$Y_1 = \begin{bmatrix} y_{11}^1 & y_{12}^1 \\ y_{21}^1 & y_{22}^1 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}.$$

From  $Y_1 = QX$ , we have

$$\begin{cases} y_{11}^1 = 3x_{11} + 2x_{12}, \\ y_{12}^1 = 3x_{12} + 2x_{22}, \\ y_{21}^1 = 5x_{11} + 7x_{12}, \\ y_{22}^1 = 5x_{12} + 7x_{22}. \end{cases} \quad (30)$$

(30) can be written in the matrix form as

$$\vec{Y}_1 = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 5 & 7 & 0 \\ 0 & 5 & 7 \end{bmatrix} \tilde{X} := \vec{L}(Q)\tilde{X}.$$

So

$$\vec{L}(Q) = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 5 & 7 & 0 \\ 0 & 5 & 7 \end{bmatrix}.$$

**Remark 4.** Liu (1999) proved a dual principle, which asserted that  $[A, C|Q]$  is exactly observable iff the following system is exactly controllable.

$$-dx = (A'x + C'z + Q'u) dt - zdw.$$

**Corollary 2.** If there does not exists nonzero  $X \in \mathcal{S}_n$  satisfying

$$XA' + AX = \lambda X, \quad QX = 0, \quad \lambda \in \mathcal{C}, \quad (31)$$

then for any real matrix  $C$  of suitable dimension,

$$XA' + AX + CXC' = \lambda X, \quad QX = 0, \quad \lambda \in \mathcal{C} \quad (32)$$

does not have nonzero solution  $X \in \mathcal{S}_n$ .

**Proof.** By the same discussion as in Theorem 4, we can prove that  $(Q, A)$  is completely observable iff there does not exist nonzero  $X \in \mathcal{S}_n$  satisfying (31). Secondly, from Liu (1999), we know that  $[A, C|Q]$  is exactly observable iff  $\text{Rank}(P_0) = n$ , where

$$P_0 = [Q', A'Q', C'Q', A'C'Q', C'A'Q', (A')^2Q', (C')^2Q', \dots]'$$

Obviously,  $\text{Rank}(Q', A'Q', (A')^2Q', \dots, (A')^{n-1}Q')' \leq \text{Rank}(P_0)$ , so if  $(Q, A)$  is completely observable, then so does  $[A, C|Q]$ . Applying Theorem 4, Corollary 2 is immediately derived.

The following proposition may be useful in some cases, it is a generalized version of Lemma 4.1 of Wonham (1968).

**Proposition 3.** Let

$$Q'Q + D'D = F'F \quad (33)$$

and  $G_1, G_2$  are any real matrices of suitable dimension. If  $[A, C|Q]$  is exactly observable, then so does  $[A + G_1D, C + G_2D|F]$ .

**Proof.** If  $[A + G_1D, C + G_2D|F]$  is not exactly observable, then there exists  $0 \neq X \in \mathcal{S}_n$ , such that

$$X(A + G_1D)' + (A + G_1D)X + (C + G_2D)X(C + G_2D)' = \lambda X, \quad FX = 0, \quad \lambda \in \mathcal{C}.$$

From  $FX = 0$  and (33), we have  $QX = 0, DX = 0$ , together with the above equations, one has

$$XA' + AX + CXC' = \lambda X, \quad QX = 0, \quad \lambda \in \mathcal{C}.$$

By Theorem 4,  $[A, C|Q]$  is not exactly observable, which contradicts the given conditions.

Based on stabilizability, we can define stochastic detectability via duality.

**Definition 7.** We say that  $[A, C|Q]$  is stochastic detectable, if  $(A', Q'; C', 0)$  is stabilizable.

From Theorem 3, we have

**Corollary 3.** If  $[A, C|Q]$  is stochastic detectable, then there does not exist nonzero  $X \in \mathcal{S}_n$ , such that

$$XA' + AX + CXC' = \lambda X, \quad QX = 0, \quad \text{Re}(\lambda) \geq 0. \quad (34)$$

If there exists  $C_1$  such that for any  $X \in \mathcal{S}_n$ ,  $CXC' = XC'_1 + C_1X$ , then (34) is also a necessary condition for stochastic detectability.

There does not have any implication between exact observability and stochastic detectability; see the following examples:

**Example 6.** Take

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

One can easily test that (23) does not have nonzero solution  $X \in \mathcal{S}_2$ , so  $[A, C|Q]$  is exactly observable. However,  $[A, C|Q]$  is not stochastic detectable, because  $(A', Q', C', 0)$  is not stabilizable as shown in Example 3.

**Example 7.** Take  $Q = 0$ ,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

then it is easily tested that  $[A, C|Q]$  is stochastic detectable, but it is not exactly observable.

**Proposition 4.** If  $[A, C|Q]$  is stochastic detectable, then  $(Q, A)$  is detectable.

**Proof.** By Definition 7, if  $[A, C|Q]$  is exactly detectable, then there exists a constant matrix  $H'$ , such that

$$dx = (A' + Q'H')x dt + C'x dw \quad (35)$$

is asymptotically mean square stable, which implies (Has'minskii, 1980) that

$$dx = (A' + Q'H')x dt \quad (36)$$

is asymptotically stable, so  $(Q, A)$  is detectable.

## 4 On weak stabilizability of stochastic systems

In this section, we study the weak stabilizability of system (1), which describes a class of weak stability, and has close relationship with strong solutions of GAREs.

**Definition 8.** We say system (1) is weakly stabilizable, if there exists a matrix  $K \in \mathcal{R}^{m \times n}$ , via the state feedback  $u(t) = Kx(t)$ , the closed-loop system (6) is two-stable (Has'minskii, 1980), i.e., for each  $\varepsilon > 0$ , there exist an  $\delta > 0$ , such that

$$E|x(t, x_0)|^2 < \varepsilon$$

whenever  $t \geq 0$  and  $|x_0| < \delta$ .

Our result is given as follows:

**Proposition 5.** System (1) is weakly stabilizable iff one of the following conditions holds:

- 1) There exists  $K \in \mathcal{R}^{m \times n}$ , such that  $\sigma(L(K)) = \{\lambda_i, i = 1, 2, \dots, n(n+1)/2\} \subset \mathcal{C}^{-,0}$ , and whenever  $Re\lambda_i = 0$ , all associated Jordan blocks of  $\lambda_i = 0$  are  $1 \times 1$ ;
- 2) There exists  $P > 0$ , such that

$$PL(K) + L'(K)P \leq 0. \quad (37)$$

**Proof.** By Definition 8, system (1) is weakly stabilizable iff there exists  $K$  with that system (6) is weakly stable. As done in Theorem 1, noting that for each  $\varepsilon > 0$ , there exists a  $\delta > 0$ , such that

$$E|x(t, x_0)|^2 < \varepsilon$$

whenever  $t \geq 0$  and  $|x_0| < \delta$ , which is equivalent to that for each  $\varepsilon > 0$ , there exists a  $\delta > 0$ , such that

$$\|\tilde{X}(t, x_0)\| < \varepsilon$$

whenever  $t \geq 0$  and  $\|\tilde{X}_0\| < \delta$ . From (8) and Theorem 5.2.3 of Ortega (1987), the latter is equivalent to that  $L(K)$  is weakly negative stable, which completes the proof of 1).

2) is a simple corollary of Theorem 5.4.3 of Ortega (1987).

The following theorem is a sufficient condition for weak stabilizability expressed by LMIs and Lyapunov-type inequalities, which will be used later. Analogous results for stabilizability can be found in Ait Rami and Zhou (2000).

**Theorem 5.** System (1) is weakly stabilizable if one of the following conditions holds.

- 1) There exist  $K \in \mathcal{R}^{m \times n}$  and  $P > 0$ , such that

$$P(A + BK) + (A + BK)'P + (C + DK)'P(C + DK) \leq 0. \quad (38)$$

- 2) There exist  $K \in \mathcal{R}^{m \times n}$  and  $P > 0$ , such that

$$(A + BK)P + P(A + BK)' + (C + DK)P(C + DK)' \leq 0. \quad (39)$$

- 3) There exist matrices  $P$  and  $Y$ , such that the following LMI holds.

$$\begin{bmatrix} AP + PA' + BY + Y'B' & CP + DY \\ PC' + Y'D' & -P \end{bmatrix} \leq 0. \quad (40)$$

**Proof.** If 1) holds, then by Dynkin's formula (Oksendal,1998), we have

$$\begin{aligned}
& E x'(t) P x(t) = x'_0 P x_0 + E \int_0^t \mathcal{A}(x'(s) P x(s)) ds = x'_0 P x_0 \\
& + E \int_0^t \left( \langle (A + BK)x, \frac{\partial(x' P X)}{\partial x} \rangle + \frac{1}{2} \langle (C + DK)x, \frac{\partial^2(x' P X)}{\partial x^2} (C + DK)x \rangle \right) ds \\
& = x'_0 P x_0 + E \int_0^t x' (P(A + BK) + (A + BK)'P + (C + DK)'P(C + DK)) x ds \\
& \leq x'_0 P x_0,
\end{aligned} \tag{41}$$

where  $\mathcal{A}$  is the infinitesimal generator of  $x(t)$ , i.e., the trajectory of stochastic system (6). From (41), for each  $\varepsilon > 0$ ,

$$\begin{aligned}
E|x(t)|^2 & \leq \max(1, \lambda_{max}(P)) E|x(t)|^2 \\
& \leq \max(1, \lambda_{max}(P)) \frac{\lambda_{max}(P)}{\lambda_{min}(P)} |x_0|^2 \\
& = \max\left(\frac{\lambda_{max}(P)}{\lambda_{min}(P)}, \frac{\lambda_{max}^2(P)}{\lambda_{min}(P)}\right) |x_0|^2 := C_0 |x_0|^2 < \varepsilon
\end{aligned} \tag{42}$$

whenever  $|x_0| < \delta := \frac{\varepsilon^{1/2}}{C_0^{1/2}}$ . So system (1) is weakly stabilizable.

If 2) holds, by the same discussion as in 1), we can prove that the dual system of (6)

$$dx(t) = (A + BK)'x(t) dt + (C + DK)'x(t) dw(t), x(0) = x_0 \tag{43}$$

is weakly stable. From Theorem 1 and Remark 1, this is equivalent to the weak stabilizability of (6).

As to that 3) implies (1) stabilizable, this is due to the equivalence of 2) and 3). Set  $Y = KP, P > 0$ , then (39) can be written as

$$AP + BY + PA' + Y'B' + (CP + DY)P^{-1}(CP + DY)' \leq 0. \tag{44}$$

Applying Schur's lemma, (44) is equivalent to (40), which ends our proof.

**Remark 5.** From Proposition 5, we have every reason to conjecture that Theorem 5 should be not only a sufficient, but also a necessary condition for weak stabilizability, but how do we prove its true?

**Remark 6.** Taking  $P = I$  in (41), apparently,  $(A, B; C, D)$  being weak stabilizable implies that  $(A, B)$  is weakly stabilizable.

The following Proposition will be used in Section 5.

**Proposition 6.** For any real matrix  $K \in \mathcal{R}^{m \times n}$ , if  $\sigma(\mathcal{L}_K^*) \subset \mathcal{C}^{-,0}$ , then  $\sigma(A + BK) \subset \mathcal{C}^{-,0}$ .

**Proof.** Since  $\sigma(\mathcal{L}_K^*) \subset \mathcal{C}^{-,0}$ , so for any  $\varepsilon > 0$ ,  $\sigma(\mathcal{L}_K^{*,\varepsilon}) \subset \mathcal{C}^-$ , where  $\mathcal{L}_K^{*,\varepsilon}(\cdot)$  is defined by

$$\mathcal{L}_K^{*,\varepsilon} : X \in \mathcal{S}^n \longmapsto X(A - \varepsilon I + BK) + (A - \varepsilon I + BK)'X + (C + DK)X(C + DK)'.$$

By Theorem 1 and Remark 1, the stochastic system

$$dx(t) = (A - \varepsilon I + BK)x dt + (C + DK)x dw(t), x(0) = x_0 \in \mathcal{R}^n$$

is stable in mean square sense, which implies (Has'minskii, 1980)  $\sigma(A - \varepsilon I + BK) \subset \mathcal{C}^-$ . Let  $\varepsilon \rightarrow 0$ , by the continuity of spectrum (Sontag,1990), we have  $\sigma(A + BK) \subset \mathcal{C}^{-,0}$ .

We should point out that the inverse of Proposition 6 does not hold even if  $\sigma(A + BK) \subset \mathcal{C}^-$ ; see the following example.

**Example 8.** Take  $A = -I, C = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, K = 0, B$  and  $D$  are arbitrary, then  $\sigma(A + BK) = \sigma(A) = \{-1, -1\}$ , but one can compute  $\sigma(\mathcal{L}_{K=0}^*) = \{7, 10, 14\}$ .

## 5 Some applications

### 5.1 Applications to GAREs

In this section we apply spectrum technique to study GARE

$$\begin{cases} PA + A'P + C'PC + Q - (PB + C'PD)(R + D'PD)^{-1}(B'P + D'PC) = 0, \\ R + D'PD > 0. \end{cases} \quad (45)$$

This equation has many applications in infinite horizon linear quadratic optimal control, stochastic stability, filtering, etc., see Bensoussan (1982,1992), Liu (1999), Gao and Ahmed (1987), Wonham (1968), Zhang (1998), and the references therein. Equation (45) is a generalized version of deterministic algebraic Riccati equation (DARE)

$$PA + A'P + Q - PBR^{-1}B'P = 0. \quad (46)$$

It is well known that if  $P \in \mathcal{S}^n$  is a solution of (46), and  $\sigma(A - BR^{-1}B'P) \subset \mathcal{C}^-$ , then  $P$  is called a feedback stabilizing solution; If  $\sigma(A - BR^{-1}B'P) \subset \mathcal{C}^{-,0}$ , then  $P$  is called a strong solution (Park and Kailath 1997). In other words, the classification of solutions of algebraic Riccati equations is according to the stability of the closed-loop system (taking  $u = -R^{-1}B'Px$ )

$$\dot{x} = (A - BR^{-1}B'P)x.$$

But how do we make a classification for the solutions of GARE (45)? Especially, how do we define its strong solution? We note that some authors ( De Souza and Fragoso, 1990) took  $P$  as a strong solution of GARE (45), if the spectral set of the following deterministic system

$$\dot{x} = (A - B(R + D'PD)^{-1}(B'P + D'PC))x \quad (47)$$

satisfying  $\sigma(A - B(R + D'PD)^{-1}(B'P + D'PC)) \subset \mathcal{C}^{-,0}$ . However, this definition about strong solution of GARE (45) is unreasonable, since from  $\sigma(A - B(R + D'PD)^{-1}(B'P + D'PC)) \subset \mathcal{C}^{-,0}$ , we do not know anything about the stability of the closed-loop system (set the feedback gain  $K = -(R + D'PD)^{-1}(B'P + D'PC)$ )

$$dx = (A + BK)x dt + (C + DK)x dw. \quad (48)$$

Now we give a more reasonable definition based on spectrum as follows:

**Definition 9.** A solution  $P \in \mathcal{S}_n$  of GARE (45) is called a feedback stabilizing solution, if  $\sigma(\mathcal{L}_K^*) \subset \mathcal{C}^-$ ;  $P$  is called a strong solution, if  $\sigma(\mathcal{L}_K^*) \subset \mathcal{C}^{-,0}$ , where

$$K = -(R + D'PD)^{-1}(B'P + D'PC).$$

The following theorem is a comparison theorem for GARE (45), which improves the corresponding results of De Souza and Fragoso (1990), Ait Rami and Zhou (2000). Firstly, we give a lemma to be used later.

**Lemma 1 (Ichikawa (1979); Prato and Zabczyk (1992)).** The following Ito-type differential equation

$$dx = Fx + Gx dw, \quad x(0) = x_0 \quad (49)$$

is asymptotically mean square stable, iff for any  $Q > 0$ , the Lyapunov-type equation

$$PF + F'P + G'PG = -Q$$

has a positive solution  $P > 0$ .

When (49) is asymptotically mean square stable, we also call  $(F, G)$  stable for short.

**Theorem 6.** Suppose  $(A, B; C, D)$  is stabilizable with the weighting real matrices  $(Q, R) \in \mathcal{S}_n \times \mathcal{S}_m$ . Let  $\hat{P}$  be any real symmetric solution of the GARE

$$\begin{cases} PA + A'P + C'PC + \hat{Q} - (PB + C'PD)(\hat{R} + D'PD)^{-1}(B'P + D'PC) = 0, \\ \hat{R} + D'PD > 0. \end{cases} \quad (50)$$

If  $R \geq \hat{R}, Q \geq \hat{Q}$ , then GARE (45) has a maximal solution  $\bar{P}, \bar{P} \geq \hat{P}$ ; Moreover,  $\bar{P}$  is a strong solution.

**Proof.** Define an operator  $\mathcal{R}$  as follows:

$$\begin{aligned} & \mathcal{R}(P, M, N, \varepsilon) \\ &= PA + A'P + C'PC + M + \varepsilon I - (PB + C'PD)(N + D'PD)^{-1}(B'P + D'PC). \end{aligned}$$

From the given conditions, we know that  $\mathcal{R}(\hat{P}, \hat{Q}, \hat{R}, 0) = 0, \mathcal{R}(\hat{P}, Q, R, \varepsilon) > 0, R + D'\hat{P}D > 0$ , then by Theorem 10 and Theorem 12 of Ait Rami and Zhou (2000), there exist maximal solutions  $\hat{P}_{max}, \hat{P}_{max}^0, \hat{P}_{max}^\varepsilon$ , respectively, to GARE (50), (45) and GARE

$$\begin{cases} PA + A'P + C'PC + Q + \varepsilon I - (PB + C'PD)(R + D'PD)^{-1}(B'P + D'PC) = 0, \\ R + D'PD > 0. \end{cases} \quad (51)$$

Moreover, under the constraint of (1), we have

$$V^*(\hat{R}, \hat{Q}) = \inf_{u \in \mathcal{U}_{ad}^\infty} \left\{ E \int_0^\infty (x' \hat{Q} x + u' \hat{R} u) dt, \lim_{t \rightarrow \infty} Ex(t)x'(t) = 0 \right\} = x_0' \hat{P}_{max} x_0, \quad (52)$$

$$V^*(R, Q) = \inf_{u \in \mathcal{U}_{ad}^\infty} \left\{ E \int_0^\infty (x' Q x + u' R u) dt, \lim_{t \rightarrow \infty} Ex(t)x'(t) = 0 \right\} = x_0' \hat{P}_{max}^0 x_0, \quad (53)$$

$$V^*(R, Q + \varepsilon I) = \inf_{u \in \mathcal{U}_{ad}^\infty} \left\{ E \int_0^\infty (x' (Q + \varepsilon I) x + u' R u) dt, \lim_{t \rightarrow \infty} Ex(t)x'(t) = 0 \right\} = x_0' \hat{P}_{max}^\varepsilon x_0, \quad (54)$$

where  $\mathcal{U}_{ad}^\infty$  denotes all  $\mathcal{F}_t$ -adapted, measurable processes  $u(\cdot) : [0, \infty) \times \Omega \mapsto \mathcal{R}^m$ , satisfying

$$E \int_0^\infty |u(t)| dt < \infty.$$

From (52), (53), (54), one has

$$\hat{P}_{max}^\varepsilon \geq \bar{P} := \hat{P}_{max}^0 \geq \hat{P}_{max} \geq \hat{P}. \quad (55)$$

The first part of Theorem 6 is proved.

In what follows, we prove that  $\bar{P}$  is a strong solution. From (54),  $\hat{P}_{max}^\varepsilon$  is monotonic and bounded from below with respect to  $\varepsilon$ , it is easily derived

$$\lim_{\varepsilon \rightarrow 0} \hat{P}_{max}^\varepsilon = \bar{P}. \quad (56)$$

Now we prove that  $\hat{P}_{max}^\varepsilon$  is a feedback stabilizing solution of GARE (51). Denote  $A_\varepsilon = A + BK_\varepsilon, C_\varepsilon = C + DK_\varepsilon, \hat{A} = A + B\hat{K}, \hat{C} = C + D\hat{K}$ , where

$$K_\varepsilon = -(R + D'\hat{P}_{max}^\varepsilon D)^{-1}(B'\hat{P}_{max}^\varepsilon + D'\hat{P}_{max}^\varepsilon C),$$

$$\hat{K} = (R + D' \hat{P} D)^{-1} (B' \hat{P} + D' \hat{P} C).$$

Note that (50) and (51) can be respectively written as

$$\begin{cases} \hat{P}_{max}^\varepsilon A_\varepsilon + A'_\varepsilon \hat{P}_{max}^\varepsilon + C'_\varepsilon \hat{P}_{max}^\varepsilon C_\varepsilon + K'_\varepsilon R K_\varepsilon + Q + \varepsilon I = 0, \\ R + D' \hat{P}_{max}^\varepsilon D > 0, \end{cases} \quad (57)$$

and

$$\begin{cases} \hat{P} \hat{A} + \hat{A}' \hat{P} + \hat{C}' \hat{P} \hat{C} + \hat{K}' \hat{R} \hat{K} + \hat{Q} = 0, \\ R + D' \hat{P} D > 0. \end{cases} \quad (58)$$

Subtracting (58) from (62), by a series of computations, we have

$$\begin{aligned} (\hat{P}_{max}^\varepsilon - \hat{P}) A_\varepsilon &+ A'_\varepsilon (\hat{P}_{max}^\varepsilon - \hat{P}) + C'_\varepsilon (\hat{P}_{max}^\varepsilon - \hat{P}) C_\varepsilon + \varepsilon I \\ &+ (\hat{P} B + C' \hat{P} D) K_\varepsilon + (\hat{P} B + C' \hat{P} D) (\hat{R} + D' \hat{P} D)^{-1} (B' \hat{P} + D' \hat{P} C) \\ &+ K'_\varepsilon (B' \hat{P} + D' \hat{P} C) + K'_\varepsilon (R + D' \hat{P} D) K_\varepsilon = 0 \end{aligned} \quad (59)$$

From (59), especially noting the last term of the right hand side of the above equation with the given condition  $R \geq \hat{R}$ , we derive

$$\begin{aligned} (\hat{P}_{max}^\varepsilon - \hat{P}) A_\varepsilon &+ A'_\varepsilon (\hat{P}_{max}^\varepsilon - \hat{P}) + C'_\varepsilon (\hat{P}_{max}^\varepsilon - \hat{P}) C_\varepsilon \\ &\leq -\varepsilon I - (\hat{P} B + C' \hat{P} D) K_\varepsilon - (\hat{P} B + C' \hat{P} D) (\hat{R} + D' \hat{P} D)^{-1} (B' \hat{P} + D' \hat{P} C) \\ &- K'_\varepsilon (B' \hat{P} + D' \hat{P} C) - K'_\varepsilon (\hat{R} + D' \hat{P} D) K_\varepsilon \\ &= -\varepsilon I - [(\hat{P} B + C' \hat{P} D) + K'_\varepsilon (\hat{R} + D' \hat{P} D)] (\hat{R} + D' \hat{P} D)^{-1} \\ &\times [(\hat{P} B + C' \hat{P} D) + K'_\varepsilon (\hat{R} + D' \hat{P} D)]' < 0. \end{aligned} \quad (60)$$

We first assert that for any  $\varepsilon > 0$ ,  $\hat{P}_{max}^\varepsilon > \hat{P}$ . Otherwise, by (55),  $\hat{P}_{max}^\varepsilon - \hat{P} \geq 0$ ,  $Ker(\hat{P}_{max}^\varepsilon - \hat{P}) \neq \phi$ . Let  $0 \neq \xi \in Ker(\hat{P}_{max}^\varepsilon - \hat{P})$ , premultiplying by  $\xi'$  and postmultiplying by  $\xi$  in (60), then

$$0 \leq \xi' C'_\varepsilon (\hat{P}_{max}^\varepsilon - \hat{P}) C_\varepsilon \xi \leq -\varepsilon \xi' \xi < 0,$$

which is a contradiction, so  $\hat{P}_{max}^\varepsilon > \hat{P}$ . Together with (60),  $\hat{P}_{max}^\varepsilon - \hat{P}$  is a positive solution to the Lyapunov-type inequality

$$P A_\varepsilon + A'_\varepsilon P + C'_\varepsilon P C_\varepsilon < 0.$$

So by Lemma 1, we have  $(A_\varepsilon, C_\varepsilon)$  is stable, i.e.,  $\hat{P}_{max}^\varepsilon$  is a feedback stabilizing solution. So  $\sigma(L^*(K_\varepsilon)) = \sigma(\mathcal{L}_{K_\varepsilon}^*) \subset \mathcal{C}^-$ , while from (56), it follows that

$$\lim_{\varepsilon \rightarrow 0} K_\varepsilon = \bar{K} := -(R + D' \bar{P} D)^{-1} (B' \bar{P} + D' \bar{P} C).$$

According to the continuity of spectrum (Lemma A.4.1 of Sontag, 1990), we have  $\sigma(L^*(\bar{K})) = \sigma(\mathcal{L}_{\bar{K}}^*) \subset \mathcal{C}^{-,0}$ . The proof of Theorem 6 is completed.

**Corollary 4.** If  $Q \geq 0, R > 0$ , system (1) is stabilizable, then GARE (45) has a maximal solution, which is also a strong solution.

**Proof.** Take  $\hat{Q} = 0, \hat{R} = R$  in (50), then (50) also has a solution  $\hat{P} = 0$ . Corollary 4 is immediately deduced from Theorem 6.

**Remark 7.** Under the same condition of Corollary 4, Ait Rami and Zhou (2000) proved that GARE (45) has a maximal solution. Here, we further assert that it is also a strong solution.

**Remark 8.** Taking  $D = 0$ , under the stabilizable condition of  $(A, B)$ , and

$$\inf_K \left\| \int_0^\infty \exp(t(A + BK)')(C'C) \exp(t(A + BK)) dt \right\| < 1, \quad (61)$$

De Souza and Fragoso (1990) proved an analogous comparison theorem to Theorem 5, which asserted that (45) has a solution  $P \geq \hat{P}$  with  $\sigma(A + BK) \subset \mathcal{C}^{-,0}$ , where  $K = -R^{-1}B'\bar{P}$ . Theorem 6 improves their result because our given condition is weaker (see Zhang (1998)), but the consequence is stronger than that of De Souza and Fragoso (1990) from Proposition 6.

**Corollary 5.** For  $Q \geq 0, R \geq 0$ , if GARE (45) has a solution  $P > 0$ , then  $P$  is a weakly feedback stabilizing solution.

**Proof.** Note that GARE (45) can be written as

$$\begin{cases} P(A + BK) + (A + BK)'P + (C + DK)'P(C + DK) = -Q - K'RK, \\ R + D'PD > 0, K = -(R + D'PD)^{-1}(B'P + D'PC). \end{cases} \quad (62)$$

By applying 1) of Theorem 5, this corollary is immediately derived.

When does GARE (45) have a feedback stabilizing solution? This problem was discussed in detail in Zhang (1998).

**Example 9.** Assume all the given data are scalar in (45) with  $a, b, c, d, q$  and  $r$  substituting for  $A, B, C, D, Q$  and  $R$ , respectively. Taking  $q = 0, r = 1, d \neq 0$ ,  $(a, b; c, d)$  is stabilizable, then GARE (45) has two solutions  $p_1 = 0, p_2 = \frac{2a+c^2}{b^2+2bcd-2ad^2}$ .  $p_2$  exists due to the following fact: Since  $(a, b; c, d)$  is stabilizable,  $d \neq 0$ , so  $b^2 + 2bcd - 2ad^2 > 0$  from Theorem 1. For  $p_1 = 0$ , the feedback gain  $K_1 = 0$ , and  $\mathcal{L}_{K_1}^* = 2a + c^2$ . By Remark 1,  $p_1 = 0$  is a feedback stabilizing solution, if

$$\begin{cases} b^2 + 2bcd - 2ad^2 > 0, \\ 2a + c^2 < 0. \end{cases}$$

By Proposition 5 or direct computation,  $p_1 = 0$  is a weakly feedback stabilizing solution, if

$$\begin{cases} b^2 + 2bcd - 2ad^2 > 0, \\ 2a + c^2 = 0, \end{cases}$$

and  $p_1 = 0$  is a asymptotically feedback stabilizing solution (i.e., the closed-loop system is asymptotically stable in probability; see Has'miskii,1980), if

$$\begin{cases} b^2 + 2bcd - 2ad^2 > 0, \\ 2a - c^2 < 0. \end{cases}$$

Analogously, for  $p_2$ , the corresponding feedback gain  $K_2 = -\frac{2a+c^2}{b^2+2bcd-2ad^2}$ .  $p_2$  is a feedback stabilizing solution, if

$$\begin{cases} b^2 + 2bcd - 2ad^2 > 0, \\ 2(a + bK_2) + (c + dK_2)^2 < 0, \end{cases}$$

and  $p_2$  is a feedback weakly stabilizing solution, if

$$\begin{cases} b^2 + 2bcd - 2ad^2 > 0, \\ 2(a + bK_2) + (c + dK_2)^2 = 0. \end{cases}$$

Finally,  $p_2$  is a feedback asymptotically stabilizing solution, if

$$\begin{cases} b^2 + 2bcd - 2ad^2 > 0, \\ 2(a + bK_2) - (c + dK_2)^2 < 0. \end{cases}$$

**Remark 9.** From the above example, we can see even if  $\mathcal{L}_{K_i}^* > 0$ , the closed-loop system can still achieve some stochastic stability, which reveals the complexity about the study of solutions of GARE (45).

## 5.2 Applications to the Lyapunov-type equations

In this section, we study Lyapunov-type equation

$$PA + A'P + \sum_{i=1}^n C_i'PC_i = -Q. \quad (63)$$

It is well known that a matrix  $A$  is Hurwitz or deterministic system

$$\dot{x} = Ax, x(0) = x_0$$

is asymptotically stable iff the following Lyapunov equation

$$PA + A'P = -Q, Q > 0$$

has a positive definite solution  $P > 0$ . More generally, if  $Q \geq 0$ ,  $(Q^{1/2}, A)$  is completely observable, then the above assertion still holds; see Jacobson, Martin, Pachter & Geveci, (1980). In the same way, (63) has a close relation with the stability of the following system

$$dx = Ax dt + \sum_{i=1}^n C_i' x dw_i, \quad x(0) = x_0 \in \mathcal{R}^n, \quad (64)$$

where  $C_i, i = 1, 2, \dots, n$ , are  $n \times n$  real matrices,  $w_i, i = 1, 2, \dots, n$ , are independent standard Wiener processes. In order to discuss (63), Definitions 6 and 7 should be modified as follows:

**Definition 10.** We call  $[A, C_1, C_2, \dots, C_n|Q]$  exactly observable, if for any nonzero  $x_0$ , there exists  $t > 0$  such that  $y(t) = Qx(t) \neq 0$ , where  $x(t)$  is the solution of (64).

**Definition 11.** We call  $[A, C_1, C_2, \dots, C_n|Q]$  stochastic detectable, if there exists a real matrix  $H$ , such that

$$dx = (A + HQ)x dt + \sum_{i=1}^n C_i' x dw_i$$

is asymptotically mean square stable.

When (64) is asymptotically mean square stable, we also call  $(A, C_1, C_2, \dots, C_n)$  stable. In the following context, we will generalize Lemma 1 to  $Q \geq 0$  by means of exact observability and stochastic detectability.

**Theorem 7.** Assume  $Q \geq 0$ , then we have

1) If  $[A, C_1, C_2, \dots, C_n|Q^{1/2}]$  is stochastic detectable, and Lyapunov-type equation (63) has a solution  $P \geq 0$ , then  $(A, C_1, C_2, \dots, C_n)$  is stable.

2) If  $[A, C_1, C_2, \dots, C_n|Q^{1/2}]$  is exactly observable, then  $(A, C_1, C_2, \dots, C_n)$  is stable iff Lyapunov-type equation (63) has a positive solution  $P > 0$ .

**The proof of 1).** Consider stochastic system (64), by Dynkin's formula, we have

$$\begin{aligned} 0 \leq Ex'(t)Px(t) &= x_0'Px_0 + E \int_0^t x'(s)(PA + A'P + \sum_{i=1}^n C_i'PC_i)x(s) ds \\ &= x_0'Px_0 - E \int_0^t x'(s)Qx(s) ds, \end{aligned} \quad (65)$$

so

$$E \int_0^\infty x'Qx ds < \infty.$$

In addition, since  $[A, C|Q^{1/2}]$  is stochastically detectable, there exists  $H \in \mathcal{R}^{n \times n}$  such that

$$dx = (A + HQ^{1/2})x(t)dt + Cx(t) dw \quad (66)$$

is asymptotically mean square stable, which is also mean square exponentially stable because of time-invariance (Has'minskii, 1980). Let  $x_H$  be fundamental matrix solution associated with (66), then there exist  $\alpha, \beta > 0$ , such that

$$E|x_H(t, s)|^2 \leq \beta \exp(-\alpha(t - s)), t \geq s.$$

The solution of (64) can be written as

$$x(t) = x_H(t, 0) - \int_0^t x_H(t, s) H Q^{1/2} x(s) ds. \quad (67)$$

By a simple discussion, (67) results in  $E \int_0^\infty |x(t)|^2 dt < \infty$ , which derives  $\lim_{t \rightarrow \infty} E[x(t)x'(t)] = 0$  from Has'minskii (1980), 1) is proved.

The proof of 2): If  $(A, C_1, C_2, \dots, C_n)$  is stable, then (63) has a solution  $P \geq 0$ ; see El Ghaoui and Ait Rami (1996). Now we show  $P > 0$ , otherwise, there exists  $x_0 \neq 0$ , such that  $Px_0 = 0$ . From (65), for any  $T > 0$ , we have

$$0 \leq E \int_0^T x'(s) Q x(s) ds = -E[x'(T) P x(T)] \leq 0,$$

which follows  $y(t) = Qx(t) \equiv 0, \forall t \in [0, T]$ , but this is impossible because of exact observability, so  $P > 0$ .

If (63) has a positive solution  $P > 0$ , from (65), we know that  $V(x(t)) := E[x'(t) P x(t)]$  is monotonically decreasing and bounded from below with respect to  $t$ , so  $\lim_{t \rightarrow \infty} V(x(t))$  exists. If we let  $t_n = nT$ , then

$$V(x(t_{n+1})) \leq V(x(t)) \leq V(x(t_n)), \quad t_n \leq t \leq t_{n+1}. \quad (68)$$

Again, by (65),

$$V(x(t_{n+1})) - V(x(t_n)) = -E \int_{t_n}^{t_{n+1}} x'(t) Q x(t) dt = -E[x'(t_n) H x(t_n)], \quad (69)$$

where  $H$  is some positive matrix (Zhang (1998)). Taking limit in the above, we have

$$\lim_{n \rightarrow \infty} V(x(t_n)) = \lim_{n \rightarrow \infty} E|x(t_n)|^2 = 0.$$

By (68),  $\lim_{n \rightarrow \infty} E[x(t)x'(t)] = \lim_{n \rightarrow \infty} V(x(t)) = 0$ , so  $(A, C_1, C_2, \dots, C_n)$  is stable.

## 6 Robust stabilization of stochastic systems

In this section, we study the robust quadratic stabilization of the following system

$$dx = ((A + \Delta A)x + Bu)dt + (Cx + Du)dw, \quad x(0) = x_0, \quad (70)$$

where  $\Delta A$  is an uncertain real matrix satisfying the “matching condition”

$$\Delta A = EFG, F \in \mathcal{F} = \{F : F'F \leq I, F \in \mathcal{R}^{k \times j}\}.$$

The analogous problems for deterministic systems were discussed by Khargonekar, Petersen and Zhou (1990), Petersen (1988), et al. Such problems have significant sense, since even for stochastic models, the coefficient matrices is not necessary to be obtained exactly. We first give the following definition, which is a generalized version of Definition 2.2 (Khargonekar, Petersen and Zhou (1990)).

**Definition 12.** System (70) (with  $u = 0$ ) is said to be quadratically stable, if there exists a positive definite matrix  $P > 0$  and a constant  $\alpha > 0$ , such that the differential generator of Lyapunov function  $V(x) = x'Px$  satisfies

$$\mathcal{L}[V(x)] = x'(PA + A'P + P\Delta A + (\Delta A)'P + C'PC)x \leq -\alpha\|x\|^2$$

for all  $x \in \mathcal{R}^n$ . (70) is called quadratically stabilizable, if there exists a state feedback  $u(t) = Kx(t)$ , such that the closed-loop system is quadratically stable.

The following theorem is a necessary and sufficient condition for quadratic stabilizability, and its proof needs two well known results as follows:

**Lemma 2 (Xie, 1996).** For any real matrices  $Y, H$  and  $E$  of suitable dimensions,  $Y \in \mathcal{S}_n$ , then for all  $F$  with  $F'F \leq I$ , we have

$$Y + HFE + E'F'H' < 0$$

iff for some  $\varepsilon > 0$ ,

$$Y + \varepsilon HH' + \varepsilon^{-1}E'E < 0.$$

**Lemma 3 (Schur's lemma).** For real matrices  $N, M = M', R = R' > 0$ , the following two conditions are equivalent:

- 1)  $M - NR^{-1}N' > 0$ ,
- 2)  $\begin{bmatrix} M & N \\ N' & R \end{bmatrix} > 0$ .

**Theorem 8.** System (70) is quadratically stabilizable iff there exist real matrices  $Y$  and  $X > 0$ , such that

$$\begin{bmatrix} AX + XA' + BY + Y'B' + CXC' + CY'D' + DYC' & DY & XG' \\ & Y'D' & -X & 0 \\ & GX & 0 & -I \end{bmatrix} < 0, \quad (71)$$

especially,  $u(t) = Kx(t) = YX^{-1}x(t)$  is a quadratically stabilizing control law.

**Proof.** By Definition 8, (70) is quadratically stabilizable iff there exist matrices  $K$  and  $P > 0$ , such that

$$P(A + \Delta A + BK) + (A + \Delta A + BK)'P + (C + DK)'P(C + DK) < 0. \quad (72)$$

From Remark 1,  $(A + \Delta A + BK, C + DK)$  is stable  $\Leftrightarrow ((A + \Delta A + BK)', (C + DK)')$  is stable, so by Lemma 1, (72) is equivalent to

$$P(A + \Delta A + BK)' + (A + \Delta A + BK)P + (C + DK)P(C + DK)' < 0.$$

Setting  $Y_1 = KP$ , by Lemma 2, the above is equivalent to

$$\begin{bmatrix} P(A + \Delta A)' + (A + \Delta A)P + BY_1 + Y_1'B' + CPC' + CY_1'D' + DY_1C' & DY_1 \\ Y_1'D' & -P \end{bmatrix} < 0. \quad (73)$$

Take

$$Z = \begin{bmatrix} PA' + AP + BY_1 + Y_1'B' + CPC' + CY_1'D' + DY_1C' & DY_1 \\ Y_1'D' & -P \end{bmatrix} < 0,$$

then (73) can be written as

$$\begin{aligned} Z + \begin{bmatrix} \Delta AP + P(\Delta A)' & 0 \\ 0 & 0 \end{bmatrix} &= Z + \begin{bmatrix} EFGP + PG'F'E' & 0 \\ 0 & 0 \end{bmatrix} \\ &= Z + \begin{bmatrix} E \\ 0 \end{bmatrix} F \begin{bmatrix} GP & 0 \end{bmatrix} + \begin{bmatrix} PG' \\ 0 \end{bmatrix} F' \begin{bmatrix} E' & 0 \end{bmatrix} < 0. \end{aligned} \quad (74)$$

By Lemma 2, (74) is equivalent to that for some  $\varepsilon > 0$ ,

$$\begin{aligned} Z + \varepsilon \begin{bmatrix} E \\ 0 \end{bmatrix} \begin{bmatrix} E' & 0 \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} PG' \\ 0 \end{bmatrix} \begin{bmatrix} GP & 0 \end{bmatrix} \\ &= \begin{bmatrix} PA' + AP + BY_1 + Y_1'B' + CPC' + CY_1'D' + DY_1C' & & DY_1 \\ +\varepsilon EE' + \varepsilon^{-1} PG'GP & & \\ Y_1'D' & & -P \end{bmatrix} < 0. \end{aligned}$$

The above divided by  $\varepsilon$  with  $X = \varepsilon^{-1}P, Y = \varepsilon^{-1}Y_1$ , results in

$$\begin{bmatrix} XA' + AX + BY + Y'B' + CXC' + CY'D' + DYC' & & DY \\ +EE' + XG'GX & & \\ Y'D' & & -X \end{bmatrix} < 0.$$

Again, by Lemma 3, the above is equivalent to

$$\begin{bmatrix} AX + XA' + BY + Y'B' + CXC' + CY'D' + DYC' & DY & XG' \\ & Y'D' & -X & 0 \\ & GX & 0 & -I \end{bmatrix} < 0.$$

From our proof,  $u(t) = Kx(t) = YX^{-1}x(t)$  is a quadratically stabilizing control law, the proof of Theorem 8 is completed.

Since (71) is an LMI, by some existing tools (Boyd, et al, 1994), one can easily test whether it is empty or not, so Theorem 8 has practical value.

**Remark 10.** By the same discussion as in Theorem 8, it is not difficult to deal with the quadratic stabilizability of

$$dx = ((A + \Delta A)x + (B + \Delta B)u)dt + ((C + \Delta C)x + Du)dw, \quad x(0) = x_0$$

where uncertain matrices  $\Delta A, \Delta B$  and  $\Delta C$  satisfy

$$[\Delta A, \Delta B, \Delta C] = EF[G_1, G_2, G_3].$$

Moreover, an analogous theorem to Theorem 8 can be obtained.

## 7 Conclusion

This paper studies the stabilizability (in mean square sense), weak stabilizability, exact observability of stochastic linear controlled systems by the aid of spectrum technique.

Firstly, some new concepts are introduced such as unremovable spectrums, strong solutions of GAREs, etc.. Based on the spectrum technique, we have obtained necessary and sufficient conditions for both stabilizability and weak stabilizability of stochastic systems, and have found some new phenomena different from deterministic systems. An important problem is that under what conditions with  $A, B, C$  and  $D$ , there always exists a feedback gain matrix  $K$ , such that for any given  $\lambda_1, \lambda_2, \dots, \lambda_{n(n+1)/2} \in \mathcal{C}$ ,  $\sigma(\mathcal{L}_K^*) = \{\lambda_1, \lambda_2, \dots, \lambda_{n(n+1)/2}\}$ ?

Secondly, we should point out that Theorem 7 will have important applications in the studies of GAREs, filtering, stochastic stability. further discussion will be included in our forthcoming paper.

Finally, there are many topics in robust quadratic stabilization of stochastic systems needed to be studied, including time-varying (the simplest case is  $F = F(t)$ ) and nonlinear cases. All the above problems are not only interesting but also important.

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