Analysis of the Variability of Joint Input-Output Estimation Methods

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Abstract

It has been recently established that, when estimating parametric models on the basis of closed loop data, the frequency domain variability of direct and various indirect methods may significantly differ from one another. This paper continues this work by analysing the performance of certain common joint input-output estimation methods.

1 Introduction and Preliminary Results

This paper examines the frequency domain variability of estimates obtained from closed loop data. It is a continuation of work begun in [7] and contains results that could not be presented there due to space restrictions. Both papers are in recognition of the fact that, over the last decade, there has been significant interest in the development and analysis of estimation methods tailored to closed loop data settings, with [1, 3, 2, 10] representing some of the more recent contributions.

In particular, the recent work [2] has provided a survey of techniques. This highlights that they may be divided into classes of 'direct', 'indirect' and 'joint input-output' methods. Furthermore, it has been argued in [3] that, asymptotically in both data length and model order, all these methods provide (essentially) the same estimate variability.

The contribution of the companion work [7] has been to employ new results from [5] to quantify the variability of various direct and indirect estimation schemes in a manner that is exact for finite model order. Contrary to the asymptotic in model order conclusion made in [3], this has established important variability differences, and in a manner that exposes what features of the estimation problem contribute to estimation inaccuracies.

The purpose of this paper is to complete this study by considering the class of joint input-output methods. Necessarily, the division of work across this paper and [7] results in the need to cross-reference some results and expressions. We recommend that the papers be read together for maximum clarity. Nevertheless, in order to provide a self contained exposition, this paper reviews and very briefly represents certain key notations, assumptions and results from [7] that are essential to the work here.

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To begin this overview, it is assumed here that available data is collected under a closed loop scenario of

$$\begin{aligned} \mathcal{S}: \quad y_t &= G(q)u_t + e_t, &(1)\\ u_t &= K(q)\,[r_t - y_t] &(2)\end{aligned}$$

for some underlying true system $\mathcal{S}$, and where the data is used to find an estimate of the transfer function $G(q)$.

According to (1) and (2), the system $G(q)$ is under the influence of a linear time invariant controller $K(q)$ and an external set point signal $\{r_t\}$. Here $G(q) = B(q)/A(q)$ and $K(q) = kP(q)/L(q)$ are all rational in the backward shift operator q^{-1} and $\{e_t\}$ is a zero-mean white noise sequence that satisfies $\mathsf{E}\{e_t^2\} = \sigma^2, \mathsf{E}\{|e_t|^8\} < \infty$.

For the purposes of estimating $G(q)$, this paper employs a model $G(q,\theta)$ parametrized by a vector $\theta \in \mathbf{R}^n$, and then studies the noise induced frequency domain error $G(e^{j\omega}, \widehat{\theta}_N) - G(e^{j\omega})$, where $\widehat{\theta}_N$ is a prediction-error based estimate of the true parameters $\theta_\circ$ found on the basis of N samples of data.

To facilitate this study of variance error, the following assumptions were imposed in [7], and will also be taken to hold in this paper.

Standing Assumptions 1.1.

1. *The reference $\{r_t\}$ is quasi-stationary with power spectral density $\Phi_r(\omega) = \mu$ a constant. According to (2), this implies that the power spectral density $\Phi_u^r(\omega)$ of the component of $\{u_t\}$ that derives solely from $\{r_t\}$ is given as ($S(q)$ is defined below in (11))*

$$\Phi_u^r(\omega) = \mu|K(e^{j\omega})S(e^{j\omega})|^2. \tag{3}$$

2. *The controller $K(q)$ is of the form*

$$K(q) = k\,\frac{P(q)}{L(q)}, \qquad L(q) = \prod_{k=1}^{m_\ell}(1 - \ell_k q^{-1}), \qquad k \in \mathbf{R} \tag{4}$$

   and where $P(q)$ is formed as a subset of the open loop poles of $A(q)$ (a pole cancelling design). That is for some polynomial $\widetilde{A}(q)$ in q^{-1}

$$\widetilde{A}(q)P(q) = A(q), \quad A_c(q) = \widetilde{A}(q)L(q) + k\,B(q) \tag{5}$$

   with $A_c(q)$ being a polynomial whose zeros are the closed loop poles of (1),(2).

3. *The model structure is chosen such that the prediction error $\varepsilon_t(\theta)$ associated with a model parametrized by θ satisfies $\varepsilon_t(\theta_\circ) = e_t$ (ie. there is no undermodelling) and such that there are no pole-zero cancellations in any of the limiting transfer functions parametrized by $\theta_\circ$ (ie. the system state dimension is not overmodelled);*

4. *The underlying true system $G(q)$ is asymptotically stable;*

5. *Given a set $\{\xi_1, \cdots, \xi_m\}$ of points strictly within the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$, the function $\kappa(\omega) : [-\pi, \pi] \to \mathbf{R}$ is defined as*

$$\kappa(\omega) \triangleq \sum_{k=1}^{m} \frac{1 - |\xi_k|^2}{|e^{j\omega} - \xi_k|^2}. \tag{6}$$

$\square$

Under these assumptions, the previous work [7] has established that the variability of a direct estimate $G_{\rm di}(e^{j\omega},\widehat{\theta}_N)$, which simply uses input $\{u_t\}$ and output $\{y_t\}$ for estimation [7, 3] satisfies

$$\lim_{N\to\infty} N \,\mathrm{Var}\left\{G_{\rm di}(e^{j\omega},\widehat{\theta}_N)\right\} = \frac{\sigma^2}{\Phi_u(\omega)}\,\kappa_{\rm di}(\omega) \tag{7}$$

where $\kappa_{\rm di}(\omega)$ is given by (6) with the associated zeros $\{\xi_1,\cdots,\xi_{m_a+m_b}\}$ being defined as those of the polynomial $z^{m_a+m_b}A(z)A_c(z)$ (m_b and m_a are respectively the numerator and denominator orders of $G_{\rm di}(q,\theta)$).

Furthermore, when employing an indirect method in which the closed loop transfer function $T = GK/(1+GK)$ is first estimated, and then G is subsequently estimated as $G_{\rm id} = T/K(1-T)$, the previous work [7] has established that

$$\lim_{N\to\infty} N \,\mathrm{Var}\left\{G_{\rm id}(e^{j\omega},\widehat{\theta}_N)\right\} = \frac{\sigma^2}{\Phi_u^r(\omega)}\,\kappa_{\rm id}(\omega) \tag{8}$$

where $\kappa_{\rm id}(\omega)$ is given by (6) with the associated zeros $\{\xi_1,\cdots,\xi_{m_\alpha+m_\beta}\}$ being defined as those of the polynomial $z^{m_\alpha+m_\beta}A_c(z)\widetilde{A}(z)L(z)$ (m_β and m_α are respectively the numerator and denominator orders of the closed loop transfer function $T(q)$ defined below).

Finally, an elementary principle, which is important for this paper, is that the relationship between the signals in (1),(2) may also be expressed as

$$\begin{aligned} y_t &= T(q)r_t + S(q)e_t && (9)\\ u_t &= S(q)K(q)r_t - S(q)K(q)e_t && (10)\end{aligned}$$

where $S(q)$ and $T(q)$ are the sensitivity and complementary sensitivity functions given (respectively) as

$$S(q) = \frac{1}{1+G(q)K(q)}, \qquad T(q) = \frac{G(q)K(q)}{1+G(q)K(q)}. \tag{11}$$

2 Main Results

As already mentioned, this paper completes a study begun in [7], by considering so-called'joint input-output' system identification methods. These are techniques in which two model structures parametrized separately by vectors θ and β are used as follows (here, and in what follows, notation such as $SK(q,\beta)$ will denote an appropriate order parametrization of compound transfer functions, whose elements have been defined previously):

$$\begin{aligned} y_t &= T(q,\theta)r_t + S(q,\theta)e_t && (12)\\ u_t &= SK(q,\beta)r_t - SK(q,\beta)e_t. && (13)\end{aligned}$$

These structures imply the following one-step ahead prediction errors

$$\begin{aligned} \varepsilon_t^y(\theta) &= S^{-1}(q,\theta)\left[y_t - T(q,\theta)r_t\right]\\ \varepsilon_t^u(\beta) &= SK^{-1}(q,\beta)\left[u_t - SK(q,\beta)r_t\right]\end{aligned}$$

which are used to find estimates $\widehat{\theta}_N$, $\widehat{\beta}_N$ according to

$$[\widehat{\theta}_N, \widehat{\beta}_N] = \arg\min_{\theta,\beta} V_N(\theta,\beta), \qquad V_N(\theta,\beta) \triangleq \frac{1}{2N}\sum_{t=1}^{N}[\varepsilon_t^y(\theta)]^2 + [\varepsilon_t^u(\beta)]^2. \tag{14}$$

Since $\{r_t\}$ is not correlated with $\{e_t\}$, then this implies the equivalent of two open loop estimation problems, with all the attendant advantages of this scenario [2, 7]. The estimate of the open loop dynamics is then found via this joint input-output method as

$$G_{\rm jio}(q,\widehat{\theta}_N,\widehat{\beta}_N) = \frac{T(q,\widehat{\theta}_N)}{SK(q,\widehat{\beta}_N)}. \tag{15}$$

Note that an advantage of this method in comparison to indirect methods [2, 7] is that knowledge of the controller $K(q)$ is not required. Furthermore, there exists a range of possible joint input-output methods that depend on the model structures used in (12), (13). In this paper, two are considered, the 'basic' joint input-output method and the 'co-prime factor' method.

2.1 Basic Joint Input-Output Identification

The so-called 'basic' joint input-output approach involves using a model structure for $T(q,\theta)$ given by

$$T(q,\theta) = \frac{\beta_0 + \beta_1 q^{-1} + \cdots + \beta_{m_\beta} q^{-m_\beta}}{1 + \alpha_1 q^{-1} + \cdots + \alpha_{m_\alpha} q^{-m_\alpha}} \tag{16}$$

and for $SK(q,\beta)$ given by

$$SK(q,\beta) = \frac{\rho_0 + \rho_1 q^{-1} + \cdots + \rho_{m_\rho} q^{-m_\rho}}{1 + \delta_1 q^{-1} + \cdots + \delta_{m_\delta} q^{-m_\delta}}.$$

The model structures for the noise models $S(q,\theta)$ and $SK(q,\beta)$ are assumed to be independently parametrized from the dynamics models $T(q,\theta)$, $SK(q,\beta)$, and such that they can completely describe the true noise models $S(q)$, $S(q)K(q)$ without any pole-zero cancellations.

The variance properties of the resulting basic joint input-output estimate $G_{\rm jio}(q,\widehat{\theta}_N,\widehat{\beta}_N)$ derived from (15), and under the same conditions considered in the previous sections, are now established via the following theorem.

Theorem 2.1. *Suppose that the Standing Assumptions 1.1 are satisfied. Suppose further that the model orders chosen for $T(q,\theta)$ satisfy the conditions of [7, Theorem 3.1] so that $A_c(z)\widetilde{A}(z)L(z)$ is a polynomial in z^{-1} of order less than $m_\alpha + m_\beta$. Finally, suppose that the model orders chosen for $SK(q,\beta)$ are such that $A_c(z)A(z)$ is a polynomial in z^{-1} of order less than $m_\delta + m_\rho$. Then using the joint input-output identification method, it holds that*

$$\begin{aligned}\lim_{N\to\infty} N\,\mathit{Var}\left\{G_{\rm jio}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\right\} \;&=\; |S(e^{j\omega})|^2 \lim_{N\to\infty} N\,\mathit{Var}\left\{G_{\rm id}(e^{j\omega},\widehat{\theta}_N)\right\} + \\ &\left(\frac{\mu}{\mu+\sigma^2}\right)|T(e^{j\omega})|^2 \lim_{N\to\infty} N\,\mathit{Var}\left\{G_{\rm di}(e^{j\omega},\widehat{\theta}_N)\right\} + \\ &\frac{2\sigma^2}{\Phi_u^r(\omega)}\mathit{Re}\left\{T(e^{j\omega})\overline{S(e^{j\omega})}\Delta(\omega)\right\}\end{aligned} \tag{17}$$

where $\Delta(\omega)$ is specified in the following equation (19) and

$$\lim_{N\to\infty} N\, \mathit{Var}\left\{G_{\rm id}(e^{j\omega},\widehat{\theta}_N)\right\}, \qquad \lim_{N\to\infty} N\, \mathit{Var}\left\{G_{\rm di}(e^{j\omega},\widehat{\theta}_N)\right\}$$

are given respectively by (8) with m_α, m_β the same as here, and (7) with $m_a = m_\delta$, $m_b = m_\rho$.

Proof. See Appendix A.1. □

This result suggests the following approximate quantification which is 'exact' for finite model order, but is of increasing accuracy with increasing data length N

$$\begin{aligned}\mathrm{Cov}\left\{G_{\rm jio}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\right\} \;\approx\;& |S(e^{j\omega})|^2\mathrm{Var}\left\{G_{\rm id}(e^{j\omega},\widehat{\theta}_N)\right\} + \left(\frac{\mu+\sigma^2}{\mu}\right)|T(e^{j\omega})|^2\left\{G_{\rm di}(e^{j\omega},\widehat{\theta}_N)\right\}\\ &+\frac{2}{N}\frac{\sigma^2}{\Phi_u^r(\omega)}\mathrm{Re}\left\{T(e^{j\omega})\overline{S(e^{j\omega})}\Delta(\omega)\right\}.\end{aligned} \tag{18}$$

Some comments about this result are clearly in order.

1. Firstly, the quantification (18) clearly shows that, in comparison with (7), (8) the variance $\mathrm{Var}\{G_{\rm jio}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\}$ for basic joint input-output methods is *not*, in general, equal to that associated with alternative direct and indirect estimation techniques.

   In fact, consideration of only the first two terms in (18) illustrates that under the assumptions of Theorem 2.1, $\mathrm{Var}\{G_{\rm co}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\}$ is a pseudo-convex combination of $\mathrm{Var}\{G_{\rm id}(e^{j\omega},\widehat{\theta}_N)\}$ and $\mathrm{Var}\{G_{\rm di}(e^{j\omega},\widehat{\theta}_N)\}$, where the epithet 'pseudo' is used since $S+T=1$ and hence if $\mu\gg\sigma^2$ then

$$|S(e^{j\omega})|^2 + \left(\frac{\mu+\sigma^2}{\mu}\right)|T(e^{j\omega})|^2 \approx 1.$$

   Therefore (again ignoring the last $\Delta(\omega)$ term) the expression (18) shows that while at some frequencies where $|S|\approx 1$ or $|T|\approx 1$ the variance of joint input-output methods estimates might be approximately the same as either a direct method or basic indirect method estimate, it is certainly not in general equal to either of them.

2. Turning now to the last component of (18), the term $\Delta(\omega)$ needs to be defined. In fact it is given as

$$\Delta(\omega) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_1(\lambda,\omega)\overline{\Delta_2(\lambda,\omega)}\,\mathrm{d}\lambda, \tag{19}$$

   where

$$\Delta_1(\lambda,\omega) = \sum_{k=1}^{m_\alpha+m_\beta}\mathcal{B}_k(e^{j\lambda})\overline{\mathcal{B}_k(e^{j\omega})}, \qquad \Delta_2(\lambda,\omega) = \sum_{k=1}^{m_\delta+m_\rho}\mathcal{F}_k(e^{j\lambda})\overline{\mathcal{F}_k(e^{j\omega})}, \tag{20}$$

   and with $\{\zeta_k\}$ being the zeros of $z^{m_\alpha+m_\beta}A_c(z)\widetilde{A}(z)L(z)$ and $\{\tau_k\}$ being the zeros of $z^{m_\delta+m_\rho}A_c(z)A(z)$

$$\mathcal{B}_k(z) \triangleq \frac{z\sqrt{1-|\zeta_k|^2}}{z-\zeta_k}\,\phi_{k-1}(z), \quad \phi_k(z)\triangleq\prod_{\ell=1}^{k}\frac{1-\overline{\zeta_\ell}z}{z-\zeta_\ell}, \quad \phi_0(z)\triangleq 1$$

   while

$$\mathcal{F}_k(z) \triangleq \frac{z\sqrt{1-|\tau_k|^2}}{z-\tau_k}\,\psi_{k-1}(z), \quad \psi_k(z)\triangleq\prod_{\ell=1}^{k}\frac{1-\overline{\tau_\ell}z}{z-\tau_\ell}, \quad \psi_0(z)\triangleq 1.$$

Although this definition of $\Delta(\omega)$ appears quite complicated, it has a simple geometric interpretation, that is obtained by considering the function spaces

$$V_\zeta \triangleq \text{Span}\left\{\frac{1}{\prod_{k=1}^{m_\alpha+m_\beta}(1-\zeta_k z^{-1})}, \cdots, \frac{z^{-(m_\alpha+m_\beta))}}{\prod_{k=1}^{m_\alpha+m_\beta}(1-\zeta_k z^{-1})}\right\} \tag{21}$$

$$V_\tau \triangleq \text{Span}\left\{\frac{1}{\prod_{k=1}^{m_\delta+m_\rho}(1-\tau_k z^{-1})}, \cdots, \frac{z^{-(m_\delta+m_\rho))}}{\prod_{k=1}^{m_\delta+m_\rho}(1-\tau_k z^{-1})}\right\}. \tag{22}$$

With this in mind, define $P_{V_\zeta} : H_2 \to V_\zeta$ as the orthogonal projection of an arbitrary function $f \in H_2$ onto V_ζ:

$$P_{V_\zeta} f \mapsto \widehat{f} \in V_\zeta \text{ such that } \left\langle \widehat{f} - f, g \right\rangle = 0 \ \forall g \in V_\zeta.$$

Similarly $P_{V_\tau} : H_2 \to V_\tau$ is the orthogonal projection onto V_τ. Then projections can be given an explicit formulation via $\Delta_1(\lambda,\omega)$ and $\Delta_2(\lambda,\omega)$ as [6]

$$[P_{V_\zeta} f](\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\overline{\Delta_1(\lambda,\omega)}\,\mathrm{d}\lambda, \qquad [P_{V_\tau} f](\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\overline{\Delta_2(\lambda,\omega)}\,\mathrm{d}\lambda.$$

Therefore, according to (19), $\Delta(\omega)$ is the best (in the sense of minimal H_2 norm of error) approximation of $\Delta_1(\lambda,\omega)$ (which, as a function of λ is in V_ζ for all ω) in terms of an element $f \in V_\tau$.

As such, an explicit expression for $\Delta(\omega)$ arises by noting that $\{\mathcal{F}_k\}$ is an orthonormal basis for V_τ so that

$$\Delta(\omega) = \sum_{n=1}^{m_\delta+m_\rho}\sum_{k=1}^{m_\alpha+m_\beta}\left\langle \mathcal{B}_k(e^{j\lambda}), \mathcal{F}_n(e^{j\lambda})\right\rangle \overline{\mathcal{B}_k(e^{j\omega})}\mathcal{F}_n(e^{j\omega}).$$

This representation highlights that $\Delta(\omega)$ is a rational function that contains poles equal to the open and closed loop poles, and at the complex conjugate of the open loop poles and the controller poles. Unfortunately since, in general, the inner product term has quite a complicated (but computable) closed form expression, then a simple formulation of $\Delta(\omega)$ does not seem possible. However, since $\Delta(\omega)$ is the sum of terms that are of indefinite sign, while the first two terms in (17) involve sums of $|\mathcal{B}_k(e^{j\omega})|^2, |\mathcal{F}_k(e^{j\omega})|^2$ which are all positive, then these first two terms could be expected to dominate the quantification. It is the experience of the authors that this is generally true.

3. Consider the special case of proportional control in which $K(q) = k \in \mathbf{R}$ so that $\widetilde{A}(z) = A(z)$ and hence, provided that $\mu \gg \sigma^2$ so $\Phi_u^r(\omega) \approx \Phi_u(\omega)$ then

$$\text{Var}\left\{G_{\text{id}}(e^{j\omega}, \widehat{\theta}_N)\right\} = \text{Var}\left\{G_{\text{di}}(e^{j\omega}, \widehat{\theta}_N)\right\} \approx \frac{\sigma^2}{N\Phi_u(\omega)}\,\kappa_{\text{di}}(\omega). \tag{23}$$

In this same situation, the spaces V_ζ and V_τ in (21), (22) become the same, so that $\Delta_2(\lambda,\omega) = \Delta_1(\lambda,\omega)$ and hence using the reproducing kernel property of $\Delta_1(\lambda,\omega)$

$$\Delta(\omega) = \Delta_1(\omega,\omega) = \left[(m_\beta - m_\alpha) + \sum_{k=1}^{m_\alpha}\frac{1-|\eta_k|^2}{|e^{j\omega}-\eta_k|^2} + \sum_{k=1}^{m_a}\frac{1-|\xi_k|^2}{|e^{j\omega}-\xi_k|^2}\right] = \kappa_{\text{di}}(\omega). \tag{24}$$

Consequently, substituting (23) and (24) into (18) indicates that in this special case of proportional control

$$\begin{aligned}\text{Var}\left\{G_{\text{jio}}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\right\} &= \text{Var}\left\{G_{\text{di}}(e^{j\omega},\widehat{\theta}_N)\right\}\left[|S|^2+|T|^2+T\overline{S}\right] \\ &= \text{Var}\left\{G_{\text{di}}(e^{j\omega},\widehat{\theta}_N)\right\} = \text{Var}\left\{G_{\text{id}}(e^{j\omega},\widehat{\theta}_N)\right\}.\end{aligned}$$

While this last comment establishes that there may be situations, such as that of simple proportional control, in which direct and indirect methods deliver estimates of the same accuracy, the quantification (17) establishes that this is by no means the general situation.

2.2 Coprime Factor Identification

A generalisation of the basic joint input-output approach is the so-called 'co-prime factor method' [9, 8, 10], which starts from the relationships (12), (13) and introduces the filtered signal

$$x_t = X(q)\,r_t, \qquad X(q) = \frac{X_N(q)}{X_D(q)} \tag{25}$$

where $X_N(q)$, $X_D(q)$ are polynomials in q^{-1} and which leads to

$$\begin{aligned} y_t &= N(q,\theta)x_t + S(q,\theta)e_t \\ u_t &= M(q,\beta)x_t - SK(q,\beta)e_t \end{aligned} \tag{26}$$

where

$$N(q,\theta) = T(q,\theta)X^{-1}(q), \qquad M(q,\beta) = SK(q,\beta)X^{-1}(q).$$

This is nothing more than a re-parametrization of (12), (13) and suggests the use of the prediction error residuals

$$\varepsilon_t^y(\theta) = S(q,\theta)[y_t - N(q,\theta)x_t], \qquad \varepsilon_t^u(\beta) = SK{-}1(q,\beta)[u_t - M(q,\beta)x_t], \tag{27}$$

as a method of obtaining estimates $\widehat{\theta}_N$, $\widehat{\beta}_N$ according to (14) which then delivers an estimate $G_{\text{co}}(q,\widehat{\theta}_N,\widehat{\beta}_N)$ of the input-output dynamics according to

$$G_{\text{co}}(q,\widehat{\theta}_N,\widehat{\beta}_N) = \frac{N(q,\widehat{\theta}_N)}{M(q,\widehat{\beta}_N)}. \tag{28}$$

The appended subscript 'co' denotes that estimates obtained in this way have often been termed 'co-prime factor method' estimates [8, 2, 10, 3] on account of the fact that it is possible to choose $X(q)$ such that $N(q,\widehat{\theta}_N)$, $M(q,\widehat{\beta})$ are a normalised co-prime pair, in which case $G_{\text{co}}(q,\widehat{\theta}_N,\widehat{\beta}_N)$ is of minimal order.

In order to study the accuracy of $G_{\text{co}}(q,\widehat{\theta}_N,\widehat{\beta}_N)$, it is necessary to also consider a 'generalised' direct estimate $G_{\text{di}x}(q,\widehat{\theta}_N)$ which also depends on the use of a pre-filter $X(q)$ of the rational form specified in (25). This estimate has originally been detailed in [7] as being given by

$$G_{\text{di}x}(q,\widehat{\theta}_N) = G_x(q,\widehat{\theta}_N)X(q) \tag{29}$$

where $G_x(q,\widehat{\theta}_N)$ is estimated on the basis of the residual

$$\varepsilon_t(\theta) = H^{-1}(q,\theta)[y_t - G_x(q,\theta)x_t], \qquad x_t = X(q)u_t.$$

In what follows shortly, this paper will employ the result established in [7]

$$\lim_{N\to\infty} N\,\mathrm{Var}\{G_{\mathrm{di}x}(e^{j\omega},\widehat{\theta}_N)\} = \frac{\sigma^2}{\Phi_u(\omega)}\,\kappa_{\mathrm{di}x}(\omega) \tag{30}$$

where $\kappa_{\mathrm{di}x}(\omega)$ is given by (6) with the $\{\xi_k\}$ in that expression being the zeros of

$$z^{m_a+m_b}A(z)A_c(z)X_N(z)X_D(z). \tag{31}$$

Similarly, consider a generalisation of the preceding 'basic' indirect identification method (detailed earlier in this paper, and also in [7]) in which the estimate $N(q,\widehat{\theta}_N)$ is obtained via the model structure (26),(27) with

$$N(q,\theta) = \frac{\beta_0+\beta_1 q^{-1}+\cdots+\beta_{m_\beta}q^{-m_\beta}}{1+\alpha_1 q^{-1}+\cdots+\alpha_{m_\alpha}q^{-m_\alpha}}$$

which is then used to form a 'generalised' indirect estimate $G_{\mathrm{id}x}(q,\widehat{\theta}_N)$ of $G(q)$ according to

$$G_{\mathrm{id}x}(q,\widehat{\theta}_N) = \frac{X(q)N(q,\widehat{\theta}_N)}{K(q)[1-X(q)N(q,\widehat{\theta}_N)]}. \tag{32}$$

Therefore, since

$$\mathrm{d}G_{\mathrm{id}x} = \frac{X}{K^2S}\,\mathrm{d}N$$

it follows using the arguments in the proof of the following Theorem 2.2 that

$$\lim_{N\to\infty} N\,\mathrm{Var}\{G_{\mathrm{id}x}(e^{j\omega},\widehat{\theta}_N)\} = \frac{\sigma^2}{\Phi_u^r(\omega)}\,\kappa_{\mathrm{id}x}(\omega) \tag{33}$$

where $\kappa_{\mathrm{id}x}(\omega)$ is defined according to (6) with the $\{\xi_k\}$ in that expression being the zeros of

$$z^{m_\alpha+m_\beta}A_c(z)\widetilde{A}(z)L(z)X_N(z)X_D(z).$$

The purpose of considering these hypothetical estimates $\mathrm{Var}\{G_{\mathrm{di}x}(q,\widehat{\theta}_N)\}$ and $\mathrm{Var}\{G_{\mathrm{id}x}(q,\widehat{\theta}_N)\}$ together with their frequency domain variances given by (29), (32) is that they arise naturally in the quantification of the variance of system estimates formed via the co-prime factor method.

This is established in the following theorem, for which it is also necessary to specify that the model structure employed for estimation of $M(q)$ is of the form

$$M(q,\beta) = \frac{\rho_0+\rho_1 q^{-1}+\cdots+\rho_{m_\rho}q^{-m_\rho}}{1+\delta_1 q^{-1}+\cdots+\delta_{m_\delta}q^{-m_\delta}}.$$

With this in mind, the accuracy of the co-prime factor method of indirect identification may be quantified via the following result.

Theorem 2.2. *Suppose that the Standing Assumption 1.1 are satisfied. Suppose further that the model orders chosen for $N(q,\theta)$ are such that $X_N(z)X_D(z)A_c(z)\widetilde{A}(z)L(z)$ is a polynomial in z^{-1} of order less than $m_\alpha+m_\beta$. Finally, suppose that the model orders chosen for $M(q,\beta)$ are such that*

$X_N(q)X_D(z)A_c(z)A(z)$ is a polynomial in z^{-1} of order less than $m_\delta + m_\rho$. Then using the co-prime factor identification method (25)-(28)

$$\begin{aligned}\lim_{N\to\infty} N\, \mathit{Var}\left\{G_{\rm co}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\right\} &= |S(e^{j\omega})|^2 \lim_{N\to\infty} N\, \mathit{Var}\left\{G_{{\rm id}x}(e^{j\omega},\widehat{\theta}_N)\right\} + \\ &\left(\frac{\mu}{\mu+\sigma^2}\right)|T(e^{j\omega})|^2 \lim_{N\to\infty} N\, \mathit{Var}\left\{G_{{\rm di}x}(e^{j\omega},\widehat{\theta}_N)\right\} + \\ &\frac{2\sigma^2}{\Phi_u^r(\omega)} \mathit{Re}\left\{T(e^{j\omega})\overline{S(e^{j\omega})}\Delta(\omega)\right\}\end{aligned}$$

where the asymptotic in N values of Var$\{G_{{\rm di}x}(e^{j\omega},\widehat{\theta}_N)$ and Var$\{G_{{\rm id}x}(e^{j\omega},\widehat{\theta}_N)$ are given by (30) and (33) respectively, and $\Delta(\omega)$ is as defined via (19)-(22) save that the spaces V_τ and V_ζ involved in that definition are both augmented to also include the poles and zeros of $X(z)$.

Proof. See Appendix A.2. □

Again, via the associated approximation suggested by this result of

$$\begin{aligned}\mathrm{Var}\left\{G_{\rm co}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\right\} &\approx \frac{|S(e^{j\omega})|^2}{N}\mathrm{Var}\left\{G_{{\rm id}x}(e^{j\omega},\widehat{\theta}_N)\right\} + \\ &\left(\frac{\mu}{\mu+\sigma^2}\right)\frac{|T(e^{j\omega})|^2}{N}\mathrm{Var}\left\{G_{{\rm di}x}(e^{j\omega},\widehat{\theta}_N)\right\} + \\ &\frac{2\sigma^2}{N\Phi_u^r(\omega)}\mathrm{Re}\left\{T(e^{j\omega})\overline{S(e^{j\omega})}\Delta(\omega)\right\}\end{aligned} \quad (34)$$

this Theorem highlights that there can be appreciable differences in the accuracy of estimates obtained in closed loop, depending on the estimation strategy employed.

In particular, note that all the terms on the right hand side of the quantification (34) depend on both the poles and zeros of the chosen pre-filter $X(q)$, which (of course) was not a factor in direct and indirect estimation methods addressed via (7), (8) and studied in [7].

Again, there is a special case worth mentioning of strictly proportional control, in which case (via an identical argument as used in the previous section), the quantification (34) reduces to

$$\mathrm{Var}\left\{G_{\rm co}(e^{j\omega},\widehat{\theta}_N,\widehat{\beta}_N)\right\} = \mathrm{Var}\left\{G_{{\rm di}x}(e^{j\omega},\widehat{\theta}_N)\right\} = \mathrm{Var}\left\{G_{{\rm id}x}(e^{j\omega},\widehat{\theta}_N)\right\}.$$

However, again because of their dependence on the poles and zeros of $X(q)$, these latter two quantities are not equal to the variances for direct and basic indirect methods.

3 Simulation Examples

In order to emphasise possible ramifications of the new variance quantifications developed in this paper, this penultimate section provides an illustrative simulation example. For this purpose the following system (introduced in[7]) is considered

$$y_t = G(q)u_t + e_t, \qquad G(q) = \frac{1.6177q^2 - 0.74q}{q^2 - 1.8q + 0.81} \quad (35)$$

where, according to (2), it is under the influence of the pole cancelling controller

$$K(q) = \frac{q^2 - 1.8q + 0.81}{q^3 - 1.9801q^2 + 0.99q}. \quad (36)$$

This choice of $K(q)$ implies $\widetilde{A}(z) = 1$ according to (5) and delivers

$$T(q) = \frac{1.618q - 0.74}{q^2 - 0.3624q + 0.25}, \qquad S(q) = \frac{q^2 - 1.9801q + 0.99}{q^2 - 0.3624q + 0.25}$$

which implies closed loop poles at $\eta_1, \eta_2 = 0.5e^{\pm j1.2}$.

Therefore, according to (7), (8), (18) and the discussion following Theorem 2.1, the estimation error involved with the joint input-output method (15) should be different from that involved with direct or basic indirect methods as studied in [7].

Indeed, as shown in Figure 1, simulation confirms this. In that figure the solid line is the 'true' variability as computed via Monte–Carlo simulation over 1000 estimate realisations, each derived from $N = 10000$ data points with measurement noise variance $\sigma^2 = 10$ and input $\Phi_r(\omega) = 1$. This is seen to be in very close agreement with the smooth dashed line which is the approximate quantification (18) derived in this paper.

Also shown in Figure 1 as a dash-dot and a (peaked) dashed line are, respectively, the quantifications (7), (8) pertaining to direct and basic indirect methods, whose accuracy was established in Figure 2 of [7].

In relation to this, note the qualitative and quantitative difference between these direct, and basic indirect variances when compared to the basic joint input-output method variability. This illustrates a main theme of this paper and [7]. The choice of closed loop identification method can have a significant impact on estimation accuracy in the frequency domain.

Finally, in order to illustrate the results obtained here for the co-prime factor method, consider the use of the pre-filter

$$X(q) = \frac{1}{q^2 - 1.8237q + 0.9801}$$

which is of bandpass type with centre frequency at 0.4 rad/s. In this case, the true frequency response variability Var$\{G_{\rm co}(e^{j\omega}, \widehat{\theta}_N, \widehat{\beta}_N)\}$, again as estimated via Monte–Carlo simulation over 1000 data realisations, is shown as the solid line in Figure 2. Almost exactly matching this is the dashed line which is the new quantification (34). Note the presence of the peak at 0.4 rad/s introduced, as predicted, by the associated poles of $X(q)$. Note also, the significant difference (an order of magnitude at some frequencies) between the accuracy of the co-prime factor approach and that of the direct, and basic indirect approaches also shown as the indicated dashed and dash-dot lines in Figure 2.

4 Conclusion

By the examination of joint input-output estimation techniques, this paper has completed a study begun in [7] of the frequency domain accuracy of various approaches to estimtion on the basis of closed loop data. The key feature discriminating this joint work from previous contributions has been to derive variance quantifications that are exact for finite model order. This has necessitated the imposition of certain assumptions that preclude the most general closed loop scenario, but with the dividends of exposing new principles affecting estimation accuracy, and in fact of establishing by theoretical argument that differencies in frequency domain accuracy may actually exist.

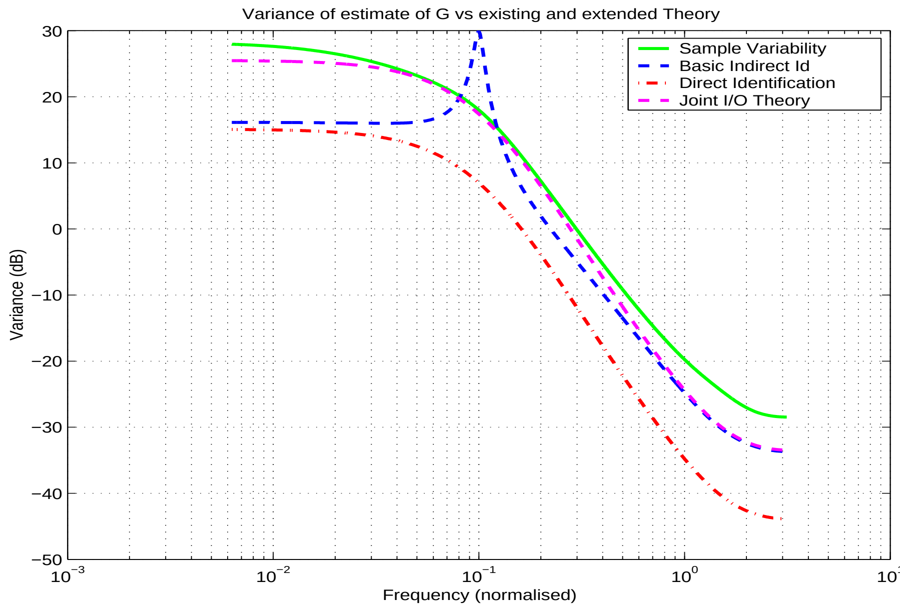

Figure 1: *Variability of joint input-output method compared with direct and indirect approaches under the same experimental conditions.*

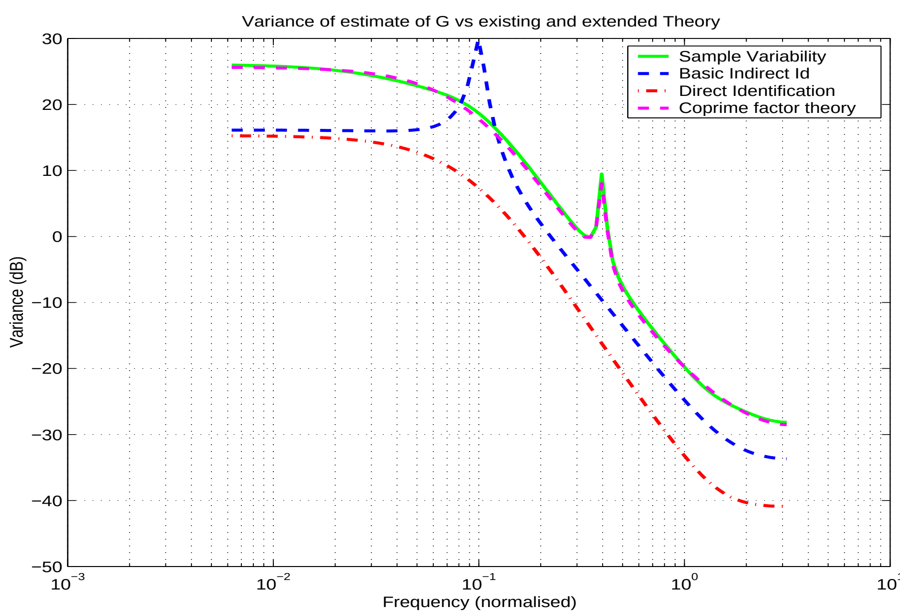

Figure 2: *Variability of co-prime factor method estimate compared with direct and indirect approaches under the same experimental conditions.*

A Proof of Theorems

A.1 Proof of Theorem 2.1

Proof. Since $G = T/SK$ then

$$\mathrm{d}G = \frac{1}{SK}\mathrm{d}T - \frac{T}{(SK)^2}\mathrm{d}SK = \frac{1}{SK}\left[\mathrm{d}T - G\,\mathrm{d}SK\right]$$

and therefore

$$\begin{aligned}\text{Var}\{G(e^{j\omega},\widehat{\theta}_N)\} &= \frac{1}{|SK|^2}\Big[\text{Var}\{T(e^{j\omega},\widehat{\theta}_N)\} + |G|^2\text{Var}\{SK(e^{j\omega},\widehat{\beta}_N)\} \\ &\quad -2\text{Re}\left\{G(e^{j\omega})\,\text{Cov}\left\{SK(e^{j\omega},\widehat{\beta}_N)T(e^{j\omega},\widehat{\theta}_N)\right\}\right\}\Big].\end{aligned} \tag{A.1}$$

However, as already established in equation (A.1) of [7]

$$\lim_{N\to\infty} N\,\text{Var}\left\{T(e^{j\omega},\widehat{\theta}_N)\right\} = \sigma^2\frac{|S(e^{j\omega},\theta_\circ)|^2}{\mu}\,\kappa_{\text{id}}(\omega).$$

where $\kappa_{\text{id}}(\omega)$ is defined by (6) with the $\{\xi_k\}$ being the zeros of $z^{m_\alpha+m_\beta}A_c(z)\widetilde{A}(z)L(z)$. Turning now to the quantification of Var$\{T(e^{j\omega},\widehat{\beta}_N)\}$, note that the assumption of $H(q)=1$ implies that the noise model in (13) is SK and hence the corresponding $A_\dagger(z)$ in equation (22) of [7] is

$$A_\dagger(z) = A_c^2(z)\frac{A(z)}{A_c(z)} = A_c(z)A(z)$$

which, under the assumption of the theorem is a polynomial of order less than $m_\delta + m_\rho$. Therefore, Theorem 3.1 of [7] can be applied to quantify the variance of the estimate $SK(q,\widehat{\beta}_N)$ as

$$\lim_{N\to\infty} N\,\text{Var}\left\{SK(e^{j\omega},\widehat{\beta}_N)\right\} = \sigma^2\frac{|SK(e^{j\omega},\beta_\circ)|^2}{\mu}\,\kappa_{\text{di}}(\omega)$$

where the zeros defining $\kappa_{\text{di}}(\omega)$ are those of $z^{m_\rho+m_\delta}A_c(z)A(z)$, and hence are the same as those used to in the case of direct identification studied earlier. Finally, via Lemma B.1

$$\lim_{N\to\infty} N\,\text{Cov}\left\{SK(e^{j\omega},\widehat{\beta}_N)T(e^{j\omega},\widehat{\theta}_N)\right\} = -\frac{\sigma^2}{\mu}K(e^{j\omega})|S(e^{j\omega})|^2\Delta(\omega).$$

Substituting these variance expressions into (A.1) then implies that

$$\begin{aligned}\lim_{N\to\infty} N\,\text{Var}\{G(e^{j\omega},\widehat{\theta}_N)\} &= \frac{\sigma^2}{\mu|SK|^2}\left[|S|^2\kappa_{\text{id}}(\omega) + |T|^2\kappa_{\text{di}}(\omega) + 2|S|^2\text{Re}\left\{GK\Delta\right\}\right] \\ &= |S|^2\,\frac{\sigma^2}{\Phi_u^r(\omega)}\kappa_{\text{id}}(\omega) + |T|^2\left(\frac{\mu+\sigma^2}{\mu}\right)\frac{\sigma^2}{\Phi_u(\omega)}\kappa_{\text{di}}(\omega) + \\ &\quad \frac{2\sigma^2}{\Phi_u^r(\omega)}\text{Re}\left\{T\overline{S}\Delta(\omega)\right\}.\end{aligned}$$

Noting that from Theorems 4.1 and 4.2 of [7]

$$\lim_{N\to\infty} N\,\text{Var}\left\{G_{\text{id}}(e^{j\omega},\widehat{\theta}_N)\right\} = \frac{\sigma^2}{\Phi_u^r(\omega)}\kappa_{\text{id}}(\omega), \qquad \lim_{N\to\infty} N\,\text{Var}\left\{G_{\text{di}}(e^{j\omega},\widehat{\theta}_N)\right\} = \frac{\sigma^2}{\Phi_u(\omega)}\kappa_{\text{di}}(\omega)$$

then completes the proof. □

A.2 Proof of Theorem 2.2

Proof. Following along the same lines as the proof of Theorem 2.1, since $G = NM^{-1}$ then

$$\mathrm{d}G = \frac{1}{M}\,\mathrm{d}N - \frac{N}{M^2}\,\mathrm{d}M = \frac{1}{M}\left[\mathrm{d}N - G\,\mathrm{d}M\right]$$

and therefore

$$\begin{aligned}\mathrm{Var}\{G(e^{j\omega},\widehat{\theta}_N)\} \quad &= \quad \frac{1}{|M|^2}\Big[\mathrm{Var}\{N(e^{j\omega},\widehat{\theta}_N)\} + |G|^2\mathrm{Var}\{M(e^{j\omega},\widehat{\beta}_N)\} \\ &\quad -2\mathrm{Re}\left\{G(e^{j\omega})\,\mathrm{Cov}\left\{M(e^{j\omega},\widehat{\beta}_N)N(e^{j\omega},\widehat{\theta}_N)\right\}\right]\Big\}. \end{aligned} \tag{A.2}$$

Considering first $\mathrm{Var}\{N(e^{j\omega},\widehat{\theta}_N)\}$, note that

$$N(q,\theta_\circ) = \frac{B(q)X_D(q)}{A_c(q)X_N(q)}, \qquad S(q,\theta_\circ) = \frac{\widetilde{A}(q)L(q)}{A_c(q)}$$

and therefore, for the purposes of employing Theorem 3.1 of [7], it is necessary to consider

$$A_\dagger(z) = A_c^2(z)X_N^2(z)\frac{\widetilde{A}(z)L(z)}{A_c(z)}\frac{X_D(z)}{X_N(z)} = A_c(z)\widetilde{A}(z)L(z)X_N(z)X_D(z).$$

Clearly, this is a polynomial, which under the assumptions of the theorem is of order less than $m_\alpha + m_\beta$. Hence, according to Theorem 3.1 of [7]

$$\lim_{N\to\infty} N\,\mathrm{Var}\left\{N(e^{j\omega},\widehat{\theta}_N)\right\} = \sigma^2\frac{|S(e^{j\omega},\theta_\circ)|^2}{\Phi_x(\omega)}\,\kappa_{\mathrm{id}x}(\omega). \tag{A.3}$$

where $\kappa_{\mathrm{id}x}(\omega)$ is defined by (6) with the $\{\xi_k\}$ being the zeros of $z^{m_\alpha+m_\beta}A_c(z)\widetilde{A}(z)L(z)X_N(z)X_D(z)$. Turning now to the quantification of $\mathrm{Var}\{M(e^{j\omega},\widehat{\beta}_N)\}$, note that

$$M(q,\theta_\circ) = \frac{A(q)X_D(q)}{A_c(q)X_N(q)}, \qquad SK(q,\theta_\circ) = \frac{A(q)}{A_c(q)}$$

and therefore, for the purposes of employing Theorem 3.1 of [7], it is necessary to consider

$$A_\dagger(z) = A_c^2(z)X_N^2(z)\frac{A(z)}{A_c(z)}\frac{X_D(z)}{X_N(z)} = A_c(z)A(z)X_N(z)X_D(z).$$

Again, this is a polynomial, which under the assumptions on of the theorem is of order less than $m_\rho + m_\delta$. Hence, according to Theorem 3.1 of [7]

$$\lim_{N\to\infty} N\,\mathrm{Var}\left\{M(e^{j\omega},\widehat{\theta}_N)\right\} = \sigma^2\frac{|SK(e^{j\omega},\theta_\circ)|^2}{\Phi_x(\omega)}\,\kappa_{\mathrm{di}x}(\omega). \tag{A.4}$$

where $\kappa_{\mathrm{di}x}(\omega)$ is defined according to (6) with the $\{\xi_k\}$ being the zeros of $z^{m_\rho+m_\delta}A_c(z)A(z)X_N(z)X_D(z)$. Compared to the proof of Theorem 2.1, the inclusion of the pre-filter $X(q)$ has, relative to the joint input-output approach, added the poles and zeros of $X(q)$ into the associated polynomials $A_\dagger(z)$ that quantify variance. With this in mind, it is straightforward to see that the proof Lemma B.1 can be adapted in the same fashion to conclude that

$$\lim_{N\to\infty} N\,\mathrm{Cov}\left\{M(e^{j\omega},\widehat{\beta}_N)N(e^{j\omega},\widehat{\theta}_N)\right\} = -\frac{\sigma^2}{\Phi_x(\omega)}K(e^{j\omega})|S(e^{j\omega})|^2\Delta(\omega) \tag{A.5}$$

where $\Delta(\omega)$ is again given by (19) but now with $\Delta_1(\lambda,\omega)$ and $\Delta_2(\lambda,\omega)$ being (respectively) the reproducing kernels for space spanned by the elements of $S^{-1}(z)X(z)\mathrm{d}N(z,\theta)/\mathrm{d}\theta$ and $SK^{-1}(z)X(z)\mathrm{d}M(z,\theta)/\mathrm{d}\theta$ and with respect to the measure $\mu\mathrm{d}\lambda$, and it is then straightforward to see that these are the same spaces V_ζ and V_τ considered in (21), (22) for the joint input-output method after they are augmented to include the poles and zeros of $X(z)$. Substituting (A.5), (A.4) and (A.3) into (A.2) then implies that

$$\begin{aligned}\lim_{N\to\infty} N\,\mathrm{Var}\{G(e^{j\omega},\widehat{\theta}_N)\} &= \frac{\sigma^2}{|M|^2\Phi_x(\omega)}\left[|S|^2\kappa_{\mathrm{id}x}(\omega)+|T|^2\kappa_{\mathrm{di}x}(\omega)+2|S|^2\mathrm{Re}\left\{GK\Delta\right\}\right]\\ &= |S|^2\frac{\sigma^2}{\Phi_u^r(\omega)}\kappa_{\mathrm{id}x}(\omega)+|T|^2\left(\frac{\mu+\sigma^2}{\mu}\right)\frac{\sigma^2}{\Phi_u(\omega)}\kappa_{\mathrm{di}x}(\omega)+\\ &\quad\ \frac{2\sigma^2}{\Phi_u^r(\omega)}\mathrm{Re}\left\{T\overline{S}\Delta(\omega)\right\}.\end{aligned}$$

Substitution of (30) and (32) into the above then completes the proof. □

B Technical Lemma

Lemma B.1. *Under the conditions of Theorem 2.1*

$$\lim_{N\to\infty} N\,Cov\left\{SK(e^{j\omega},\widehat{\beta}_N)T(e^{j\omega},\widehat{\theta}_N)\right\} = -\frac{\sigma^2}{\mu}K(e^{j\omega})|S(e^{j\omega})|^2\,\Delta(\omega),$$

where $\Delta(\omega)$ is defined in (19).

Proof. Via a standard first order Taylor expansion argument [4]

$$\widehat{\theta}_N-\theta_\circ \approx R^{-1}(\theta_\circ)\frac{\mathrm{d}}{\mathrm{d}\theta}V_N(\theta_\circ,\beta_\circ)$$

where the accuracy in the above approximation increases with increasing N and

$$R(\theta)\triangleq \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathbf{E}\left\{\psi_t(\theta)\psi_t^T(\theta)\right\},\qquad \frac{\mathrm{d}}{\mathrm{d}\theta}V_N(\theta_\circ,\beta_\circ)=-\frac{1}{N}\sum_{t=1}^{N}\psi_t(\theta_\circ)e_t$$

with

$$\psi_t(\theta_\circ)\triangleq S^{-1}(q,\theta_\circ)\left[\frac{\mathrm{d}T(q,\theta_\circ)}{\mathrm{d}\theta},\frac{\mathrm{d}S(q,\theta_\circ)}{\mathrm{d}\theta}\right]\left[\begin{array}{c} r_t\\ e_t\end{array}\right].$$

Similarly

$$\widehat{\beta}_N-\beta_\circ \approx M^{-1}(\beta_\circ)\frac{\mathrm{d}}{\mathrm{d}\beta}V_N(\theta_\circ,\beta_\circ)$$

where

$$M(\beta)\triangleq \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathbf{E}\left\{\phi_t(\beta)\phi_t^T(\beta)\right\},\qquad \frac{\mathrm{d}}{\mathrm{d}\beta}V_N(\beta_\circ,\beta_\circ)=-\frac{1}{N}\sum_{t=1}^{N}\phi_t(\beta_\circ)e_t$$

with

$$\phi_t(\beta_\circ)\triangleq -SK^{-1}(q,\beta_\circ)\left[\frac{\mathrm{d}SK(q,\beta_\circ)}{\mathrm{d}\beta},-\frac{\mathrm{d}SK(q,\beta_\circ)}{\mathrm{d}\beta}\right]\left[\begin{array}{c} r_t\\ e_t\end{array}\right].$$

Therefore,

$$N\,\mathrm{Cov}\{\widehat{\theta}_N\widehat{\beta}_N^T\} \approx -\sigma^2 R^{-1}(\theta_\circ)\,\frac{1}{N}\sum_{t=1}^{N}\mathbf{E}\left\{\psi_t(\theta_\circ)\phi_t^T(\beta_\circ)\right\}\,M^{-1}(\beta_\circ) \tag{B.1}$$

with increasing accuracy in the approximation as $N\to\infty$. Now, defining

$$\Pi(q,\theta)\triangleq[T(q,\theta),S(q,\theta)]\,,\qquad \Gamma(q,\beta)\triangleq[SK(q,\beta),SK(q,\beta)]$$

$$Z(q,\theta_\circ)\triangleq\left[\frac{\mathrm{d}T(q,\theta_\circ)}{\mathrm{d}\theta},\frac{\mathrm{d}S(q,\theta_\circ)}{\mathrm{d}\theta}\right],\qquad S(q,\beta_\circ)\triangleq\left[\frac{\mathrm{d}SK(q,\beta_\circ)}{\mathrm{d}\beta},-\frac{\mathrm{d}SK(q,\beta_\circ)}{\mathrm{d}\beta}\right]$$

then again using a first order Taylor expansion

$$\Pi(e^{j\omega},\widehat{\theta}_N)-\Pi(e^{j\omega},\theta_\circ)\approx Z^T(e^{j\omega},\theta_\circ)[\widehat{\theta}_N-\theta_\circ],\quad \Gamma(e^{j\omega},\widehat{\beta}_N)-\Gamma(e^{j\omega},\beta_\circ)\approx S^T(e^{j\omega},\theta_\circ)[\widehat{\beta}_N-\beta_\circ].$$

Therefore, combining with (B.1) implies that

$$\lim_{N\to\infty}N\frac{1}{K(e^{j\omega})|S(e^{j\omega})|^2}\,\mathrm{Cov}\left\{\Gamma(e^{j\omega},\widehat{\theta}_N)\Pi(e^{j\omega},\widehat{\theta}_N)\right\}=$$

$$-\left[\frac{Z(e^{j\omega},\theta_\circ)}{S(e^{j\omega})}\right]^\star T_n^{-1}\left(\frac{Z\phi_\zeta Z^\star}{|S|^2}\right)T_n\left(\frac{Z\phi_\zeta S^\star}{|S|^2K^\star}\right)T_n^{-1}\left(\frac{S\phi_\zeta S^\star}{|SK|^2}\right)\frac{S(e^{j\omega},\beta_\circ)}{S(e^{j\omega})K(e^{j\omega})}=$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(\lambda,\omega)\gamma(\omega,\lambda)\mu\,\mathrm{d}\lambda$$

where

$$\varphi(\lambda,\omega)=\left[\frac{Z(e^{j\omega},\theta_\circ)}{S(e^{j\omega})}\right]^\star T_n^{-1}\left(\frac{Z\phi_\zeta Z^\star}{|S|^2}\right)\frac{Z(e^{j\lambda},\theta_\circ)}{S(e^{j\lambda})}$$

$$\gamma(\omega,\lambda)=\left[\frac{S(e^{j\lambda},\beta_\circ)}{S(e^{j\lambda})K(e^{j\lambda})}\right]^\star T_n^{-1}\left(\frac{S\phi_\zeta S^\star}{|SK|^2}\right)\frac{S(e^{j\omega},\beta_\circ)}{S(e^{j\omega})K(e^{j\omega})}$$

Now, according to (16) of [7], then by construction $\varphi(\lambda,\omega)$ and $\gamma(\lambda,\omega)$ are diagonal 2×2 matrix-valued functions. Denote the top left elements of them as $\Delta_1(\lambda,\omega)$ and $\Delta_2(\lambda,\omega)$. Then, since $\Phi_{re}(\omega)=0$

$$\lim_{N\to\infty}N\frac{1}{K(e^{j\omega})|S(e^{j\omega})|^2}\,\mathrm{Cov}\left\{SK(e^{j\omega},\widehat{\beta}_N)T(e^{j\omega},\widehat{\theta}_N)\right\}=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_1(\lambda,\omega)\Delta_2(\omega,\lambda)\mu\,\mathrm{d}\lambda$$

However, using the ideas developed in [6], $\Delta_1(\lambda,\omega)$ and $\Delta_2(\lambda,\omega)$ are both reproducing kernels for the space spanned by the elements of (respectively) of $S^{-1}(z)\mathrm{d}T(z,\theta)/\mathrm{d}\theta$ and $SK^{-1}(z)\mathrm{d}SK(z,\theta)/\mathrm{d}\theta$ and with respect to the measure $\mu\mathrm{d}\lambda$. Furthermore, these spaces are equal to those of V_ζ and V_τ defined in (21), (22) and hence again via the results in [6], the kernels can then be expressed via the quantities in (20) after division by μ. □

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