Input-to-state stability for discrete-time time-varying systems with applications to robust stabilization of systems in power form

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Abstract

Input-to-state stability (ISS) of a parameterized family of discrete-time time-varying nonlinear systems is investigated. A converse Lyapunov theorem for such systems is developed. We consider parameterized families of discrete-time systems and concentrate on a semiglobal practical type of stability that naturally arises when an approximate discrete-time model is used to design a controller for a sampled-data system. An application of our main result to time-varying periodic systems is presented, and this is used to solve a robust stabilization problem, namely to design a control law for systems in power form yielding semiglobal practical ISS (SP-ISS).

Key words: Converse Lyapunov theorem; Discrete-time system; Input-to-state stability; Nonholonomic systems; Nonlinear systems; Power forms; Time-varying system.

1 Introduction

The prevalence of computer controlled systems and the fact that nonlinearities that arise naturally in most plants dynamics often can not be neglected in controller design, have driven people to study and investigate nonlinear sampled-data control systems. A framework for discrete-time control design via approximate models of the plant has been proposed in [19]. Within this framework, a parameterized family of discrete-time models of the plant is used to perform the controller design, aiming at stabilizing the original continuous-time plant. As indicated in [19], time-invariant models that are usually used in design are often inadequate in practice. There is a class of controllable nonlinear systems that may not be stabilizable using time-invariant control, but there exist time-varying controls to stabilize such systems [2, 26]. Since there are many systems in applications that belong to this class, the stabilization problem using time-varying control has become an important topic of study. In [22], a systematic design of time-varying controllers for a class of controllable systems without drift has been proposed. Stabilization using sinusoids for nonholonomic systems in power form was studied in [31]. A number of more recent works were based on these early results, e.g. [4] that studied exponential stabilization using Lyapunov approach, and [12] in which exponential stabilization for homogeneous systems was thoroughly investigated.

Among the results that are available in the literature, there are only few that consider input-to-state stabilization using time-varying control. Input-to-state stability (ISS) is a type of robust stability for nonlinear systems with inputs (see [25, 27]). Indeed, ISS is very important, especially when dealing with systems in the presence of disturbances. The first papers presenting Lyapunov characterization of ISS for time-varying nonlinear systems are [3, 8]. More recently, the authors of [18] have studied the problem using averaging technique. All the aforementioned works consider continuous-time systems. To the best of the authors knowledge, the only results on discrete-time systems are given in [6, 17], where asymptotic stability for discrete-time time-varying systems is studied. In [5], the same authors have used the results of [6] to prove a converse Lyapunov theorem for ISS for discrete-time time-invariant systems.

The importance of ISS and the scarcity of existing results considering this property in the context of discrete-time time-varying systems are the main motivations to study the Lyapunov characterization of ISS for discrete-time time-varying systems. We consider a general parameterized family of discrete-time time-varying nonlinear systems, which commonly arises in sampled-data control design as discussed in [19]. We are particularly interested in the ISS property in a semiglobal practical sense (SP-

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ISS). Our main result is a converse Lyapunov theorem that can be seen as a discrete-time counterpart of the result of [3], and at the same time as a generalization of the results of [5,6]. We also present an application of our main result to time-varying periodic systems.

Moreover, to illustrate the applicability of the main results, we address a robust stability design for a subclass of driftless control systems, which have a special structure called *power form*. We opted to focus on the sampled-data stabilization of this class of systems to achieve SP-ISS of the closed-loop system, for the following reasons. First, we can find a simple (strict) SP-ISS Lyapunov function for this class of systems, and second, because of the prevalence of using discrete-time controllers in real life applications. Systems in power form are commonly used to model the kinematic equations of *nonholonomic* systems such as mobile robots. Due to the functionalities of mobile robots, it is preferable to use a digital computer to steer and drive such systems, as the device requires much less space and it can handle other tasks simultaneously, such as data collection, while it is controlling the position of the robots. Since a mobile robot is a mechanical - therefore analog - plant, designing a digital controller for this system is a sampled-data control design problem. Considering the semiglobal practical property also makes sense, since we may assume that in practice the state space for the system is bounded.

The paper is organized as follows. In Section 2, we present preliminaries where notation and definitions are introduced. The main result is presented in Section 3. Section 4 is dedicated to an application of the main result to SP-ISS design problem for systems in power form. Design examples are presented in Section 5, and we conclude in Section 6.

2 Preliminaries

2.1 Definitions and notation

The set of real and natural numbers (including 0) are denoted respectively by $\mathbb{R}$ and $\mathbb{N}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and zero at zero. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Functions of class $\mathcal{K}_\infty$ are invertible. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Given two functions $\alpha(\cdot)$ and $\gamma(\cdot)$, we denote their composition and multiplication as $\alpha \circ \gamma(\cdot)$ and $\alpha(\cdot) \times \gamma(\cdot)$, respectively.

To begin with, we consider nonlinear time-varying systems described by

$$\dot{x} = f(t, x(t), d(t)) , \tag{1}$$

where $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ are the states and exogenous disturbances, respectively. Assume that the system (1) is between a sampler and zero order hold. The parameterized family of discrete-time model of (1) is written as

$$x(k+1) = F_T(k, x(k), d(k)) , \tag{2}$$

where the free parameter $T > 0$ is the sampling period. Assume that the function f is locally Lipschitz and $f(t, 0, 0) = 0$. Without loss of generality we may assume the same conditions for F_T. For any exogenous disturbance $d : \mathbb{N} \to \mathbb{R}^m$, we denote $\|d\|_\infty := \sup_{k\in\mathbb{N}} \|d(k)\|$. We use the notation $\mathcal{U}_{\bar{B}}$, for the set of disturbances d such that $\|d\|_\infty \leq 1$. We denote $x_\circ := x(k_\circ)$, $k_\circ \geq 0$, and Id the identity function, i.e. $\mathrm{Id}(s) = s$. For any functions or variables h we use the simplified notation $h(k, \cdot) := h(kT, \cdot)$.

We emphasize that for nonlinear systems the exact discrete-time model $F^e_T(k, x(k), d(k))$ is usually not known, since it requires solving a nonlinear initial value problem which is almost impossible in general (see [16] for more details). Although for some classes of nonlinear systems with special structure it is possible to compute explicitly the exact discrete-time model, the discretization usually destroys the special structure of the systems, which left the discrete-time model not useful for design purposes. Moreover, for systems with disturbances, the exact discrete-time model may not be computable anymore (see Section 5.2 for example). For these reasons, throughout the paper we assume that (2) is obtained by approximating the exact discrete-time model of (1). To guarantee that (2) is a good approximation of (1), we assume that F_T satisfies the following consistency property that is used to limit the mismatch.

Definition 2.1 (One-step consistency) *[16] The family of approximate discrete-time models F_T is said to be one-step consistent with the exact discrete-time models F^e_T if given any strictly positive real numbers Δ_x, Δ_d, there exist a function $\varrho \in \mathcal{K}_\infty$ and $T^* > 0$ such that*

$$|F^e_T - F_T| \leq T\varrho(T) , \tag{3}$$

for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x_\circ| \leq \Delta_x$ and $\|d\|_\infty \leq \Delta_d$. ■

One-step consistency is commonly used in numerical analysis literature (see for instance [7, 16, 20, 30]). Although F^e_T is not known, the consistency property is checkable [16]. Moreover, since we consider a semiglobal property, we assume that F^e_T and F_T are globally defined. We will use the following definitions and technicalities to construct and prove our main results. Note that the following definitions are modifications of those given in [5, 6].

Definition 2.2 (Semiglobal practical ISS) *The family of systems (2) is semiglobally practically input-to-state stable (SP-ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that for any strictly positive real numbers $\Delta_x, \Delta_d, \delta$ there exists $T^* > 0$ such that the solutions of the system satisfy*

$$|x(k, k_\circ, x_\circ, d)| \leq \beta(|x_\circ|, (k - k_\circ)) + \gamma(\|d\|_\infty) + \delta, \tag{4}$$

for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x_\circ| \leq \Delta_x$, and $\|d\|_\infty \leq \Delta_d$. Moreover, if there is no disturbance ($d = 0$), the system is semiglobally practically asymptotically stable (SP-AS). ■

Definition 2.3 (SP-ISS Lyapunov function) *A family of functions $V_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a family of SP-ISS Lyapunov functions for the family of systems (2) if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, a positive definite function α and a function $\chi \in \mathcal{K}$, and for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_1, \nu_2, \delta$ there exists strictly positive numbers T^* and L, such that*

$$\underline{\alpha}(|x|) \leq V_T(k, x) \leq \overline{\alpha}(|x|) \ , \tag{5}$$

$$|x| \geq \chi(|d|) + \nu_1 \ \Rightarrow$$

$$V_T(k+1, F_T) - V_T(k, x) \leq -T\alpha(|x|) \ , \tag{6}$$

$$V_T(k+1, F_T) \leq V_T(k, x) + \nu_2 \ , \tag{7}$$

for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x| \leq \Delta_x$, and $\|d\|_\infty \leq \Delta_d$, and

$$|V_T(k, x_1) - V_T(k, x_2)| \leq L\,|x_1 - x_2| \ , \tag{8}$$

for all $k \geq k_\circ$, $T \in (0, T^*)$ and $x_i \in [\delta, \Delta_x]$, $i = \{1, 2\}$. Moreover, if $d = 0$, the family of functions V_T is called a family of SP-AS Lyapunov functions and V_T is called a smooth Lyapunov function if it is smooth in $x \in \mathbb{R}^n$. ■

Consider nonlinear time-varying systems with control input:

$$\dot{x} = f(t, x(t), u(t), d(t)) \ , \tag{9}$$

where $u \in \mathbb{R}^l$ is a feedback control $u(t) := u(x(t))$. The parameterized family of discrete-time model of (9) is

$$x(k+1) = F_T(k, x(k), u(k), d(k)) \ . \tag{10}$$

We use the following assumption for F_T in (10).

Assumption 2.1 *There exists $T^* > 0$ sufficiently small, such that for all $T \in (0, T^*)$, and all $k \geq k_\circ$, F_T is continuous and*
$\lim_{T\to 0} F_T(k+1, x(k), u(k), d(k)) \to x(k)$. ■

Note that the continuity assumption of F_T does not necessarily require continuity of the control signal $u(k)$. If we use the approximate model (10) to design a discrete-time controller, we can obtain a discrete-time controller $u(k) := u_T(x(k))$ that is parameterized by T.

Definition 2.4 *Let $\hat{T} > 0$ be given and for each $T \in (0, \hat{T})$ let the functions $V_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $u_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ be defined. The pair (u_T, V_T) is a semiglobally practically input-to-state stabilizing (SP-ISS) pair for the system (10) if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, a positive definite function α and a function $\chi \in \mathcal{K}$ such that for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_1, \nu_2$ there exist strictly positive real numbers M and T^*, with $T^* \leq \hat{T}$, such that (5), (6), (7), (8) and*

$$|u_T(k, x)| \leq M \ , \tag{11}$$

hold, for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x| \leq \Delta_x$, and $\|d\|_\infty \leq \Delta_d$. Moreover, if $d = 0$, the pair (V_T, u_T) is called a SP-AS pair. ■

Remark 2.1 While due to continuity of solutions condition (7) is not needed in the continuous-time context, we require this condition to guarantee boundedness of trajectories (see [19] for more details). ■

Definition 2.5 (Δ-UBIBS) *The family of systems (2) is Δ- uniformly bounded input bounded state (Δ-UBIBS) if there exist functions $\sigma_1, \sigma_2 \in \mathcal{K}$, and for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_b, \delta_b$ there exists $T^* > 0$ such that*

$$\sup_{k \geq k_\circ} |x(k, k_\circ, x_\circ, d)| \leq \max\{\sigma_1(|x_\circ|) + \nu_b, \sigma_2(\|d\|_\infty)\} + \delta_b \ , \tag{12}$$

for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x_\circ| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$. By causality property, (12) is equivalent to $\sigma_1(s) \geq s$ and

$$|x(k, k_\circ, x_\circ, d)| \leq \max_{k_\circ \leq j \leq k-1} \{\sigma_1(|x_\circ|) + \nu_b, \sigma_2(|d(j)|)\} + \delta_b \ . \tag{13}$$

■

Remark 2.2 Instead of (12), we could write

$$\sup_{k \geq k_\circ} |x(k, k_\circ, x_\circ, d)| \leq \max\{\sigma_1(|x_\circ|), \sigma_2(\|d\|_\infty)\} + \delta,$$

where $\delta := \nu_b + \delta_b$ (similarly for (13)). However, we have chosen to use (12) (respectively (13)) for convenience in proving our main result. ■

Definition 2.6 ($\mathcal{K}$-asymptotic gain) *The family of systems (2) has a $\mathcal{K}$-asymptotic gain if there exists a function $\gamma_a \in \mathcal{K}$ and for any strictly positive real numbers Δ_x, Δ_d, π there exists $T^* > 0$, such that*

$$\overline{\lim_{k\to\infty}} |x(k, k_\circ, x_\circ, d)| \leq \gamma_a(\overline{\lim_{k\to\infty}} |d(k)|) + \pi,$$

for all $T \in (0, T^*)$, $|x_\circ| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$. ■

Definition 2.7 (SPRS) *The system (2) is semiglobally practically robustly stable (SPRS), if there exists a function $\rho \in \mathcal{K}_\infty$ and for any strictly positive real numbers $\Delta_x, \Delta_d, \delta$ there exists $T^* > 0$, such that for all $k \geq k_\circ$, $T \in (0, T^*)$, $x \in \mathbb{R}^n$ with $|x_\circ| \leq \Delta_x$, and $d \in \mathcal{U}_{\bar{B}}$ such that $\|d\rho(|x|)\|_\infty \leq \Delta_d$, the system $x(k+1) = F_T(k, x, d\rho(|x|)) =: G_T(k, x, d)$ is SP-AS.* ■

Remark 2.3 *[6]* For any $\mathcal{KL}$ function β, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that $\beta(s, r) \leq \rho_1(\rho_2(s)e^{-r})$, for all $s \geq 0$ and all $r \geq 0$. ■

2.2 Nonholonomic systems in power form

Consider systems in power form, modelled as:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1 u_2 \\
\dot{x}_4 &= \frac{1}{2} x_1^2 u_2 \\
&\vdots \\
\dot{x}_n &= \frac{1}{(n-2)!} x_1^{n-2} u_2 \ .
\end{aligned} \tag{14}$$

The model (14) can be obtained from a diffeomorphic transformation [23] of systems in chained form, which are usually used to model the dynamics of car-like mobile robots with $(n-3)$ trailers. The transformation from a kinematic model of mobile robots to a chained form is presented in [28]. The system (14) can be written in a compact form as

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2 \ , \tag{15}$$

with the vector fields

$$f_1 = \frac{\partial}{\partial x_1}; \quad f_2 = \sum_{j=2}^{n} \frac{x_1^{j-2}}{(j-2)!} \frac{\partial}{\partial x_j} \ .$$

In the presence of disturbances we have

$$\dot{x} = \sum_{i=1}^{l=2} f_i(x)u_i + \sum_{j=1}^{m} e_j(x)d_j \ . \tag{16}$$

The nominal systems (14) belongs to the class of nonlinear systems whose exact discrete-time model can be explicitly computed. However, as indicated in Section 2, the exact discrete-time model of this class is not in power form and is also not affine in u. Moreover, when we consider systems with disturbances, the availability of the exact model is no more guaranteed. In this paper we use Euler approximation in order to preserve the power form structure of the system and to deal with disturbances. The Euler model of (16) is written as

$$x(k+1) = x(k) + T\left[\sum_{i=1}^{l=2} f_i(x)u_i + \sum_{j=1}^{m} e_j(x)d_j\right] \ . \tag{17}$$

3 ISS Lyapunov converse theorem for time-varying systems

In this section we state and prove our main result. The main result (Theorem 3.1) is a converse Lyapunov theorem of ISS for parameterized discrete-time time-varying nonlinear systems. We provide necessary and sufficient conditions for which a parameterized family of discrete-time time-varying nonlinear systems is input to state stable in a semiglobal practical sense. This result is a discrete-time counterpart of [3], and it generalizes the main result of [5, 6]. The technique used in proving our results is similar to the technique that has been used in [6]. However, there are more technicalities needed to deal with the semiglobal practical property we consider. We are now ready to state our main result.

Theorem 3.1 *The parameterized family of discrete-time time-varying systems (2) is SP-ISS if and only if it admits a (smooth) SP-ISS Lyapunov function V_T.* ■

Before we proceed with proving Theorem 3.1, we first state some lemmas that are instrumental in constructing the proof of the theorem. The proof of Lemmas 3.2 and 3.3 are given in [6].

Lemma 3.1 *If the family of systems (2) is SP-ISS, then it is Δ-UBIBS and it admits a $\mathcal{K}$-asymptotic gain. Moreover, the system is SPRS, and hence SP-AS.* ■

Lemma 3.2 *[6, Lemma 2.7] If there exists a continuous SP-ISS Lyapunov function V_T with respect to a compact set $\mathbb{X}$, then there exists also a smooth one W_T with respect to the same set. Moreover, if V_T is periodic with period $\lambda > 0$, then W_T can also be chosen to be periodic with period λ.* ■

Lemma 3.3 *[6, Lemma 2.8] Assume that system (2) admits a SP-ISS Lyapunov function V_T. Then there exists a smooth function $\tau \in \mathcal{K}_\infty$ such that $W_T = \tau \circ V_T$ is also a SP-ISS Lyapunov function for (2), and (6) holds for some $\alpha \in \mathcal{K}_\infty$.* ■

Proof of Lemma 3.1:
SP-ISS $\Rightarrow$ Δ-UBIBS + $\mathcal{K}$-asymptotic gain: Suppose that the system (2) is SP-ISS. Let $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ be as in Definition 2.2. By the properties of $\mathcal{KL}$ functions, if we fix the second argument, then β is a $\mathcal{K}$ function in its first argument. Hence, the Δ-UBIBS property is directly implied. Also, by definition, the function γ is the $\mathcal{K}$-asymptotic gain of the system (2).

Δ-UBIBS + $\mathcal{K}$-asymptotic gain $\Rightarrow$ SPRS $\Rightarrow$ SP-AS: Suppose that the system (2) is Δ-UBIBS and it admits a $\mathcal{K}$-asymptotic gain. Let $\sigma_1, \sigma_2 \in \mathcal{K}$ be as in Definition 2.5. Given any strictly positive numbers $\Delta_x, \Delta_d, \nu_b, \delta_b$, there exists $T^* > 0$, such that (12) holds for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x| \leq \Delta_x$ and $\|d\|_\infty \leq \Delta_d$. Without loss of generality, we assume that the $\mathcal{K}$-asymptotic gain γ is such that

$$\gamma = \sigma_2 \ , \tag{18}$$

and the offset $\pi = \delta_b$. Let the positive numbers ν_c and ν_d be such that

$$\nu_c \geq \nu_b + \delta_b \ , \tag{19}$$

$$\nu_d \leq \min_{s \in [0,1)} [\sigma_2(\rho(|x_\rho|)) - \sigma_2(s\rho(|x_\rho|))] \tag{20}$$

$$\nu_c - \nu_d < \nu_b \ . \tag{21}$$

We have from Definition 2.5 that $\sigma_1 \geq \mathrm{Id}$ for all $s \geq 0$. Pick any function $\rho \in \mathcal{K}_\infty$ such that

$$\gamma \circ \rho(s) \leq s/2, \quad \forall s \geq 0. \tag{22}$$

We will show that, with this choice of ρ, the system

$$x(k+1) = F_T(k, x, d\rho(|x|)) \tag{23}$$

is SP-AS. Pick any initial condition such that $|x_\circ| \leq \Delta_x$. Let $x_\rho(k)$ denote the corresponding trajectory of system (23). We **claim** the following:

$$\sigma_2 \circ \rho(|x_\rho(k, k_\circ)|) \leq \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c \ , \quad \forall k \geq 0. \tag{24}$$

Proof of claim: The claim is trivially true for $x_\circ = 0$. Assume now we have nonzero initial states, $x_\circ \neq 0$. It is then obvious that the claim is true for $k = 0$, since

$$\sigma_2(\rho(|x_\rho(0)|)) = \gamma(\rho(|x_\rho(0)|)) \le \frac{1}{2}|x_\circ| \le \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c.$$

The last part to prove is for $k > 0$. Let

$$k_1 = \min\left\{k \in \mathbb{N} \mid \sigma_2 \circ \rho(|x_\rho(k, k_\circ)|) \ge \frac{\sigma_1(|x_\circ|)}{2} + \nu_c\right\},$$

which means that $k_1 > 0$. Suppose that the claim is false and hence $k_1 < \infty$. Then for $0 \le k \le k_1 - 1$, (24) holds. From (20) and (21), we have that

$$\begin{aligned}\sigma_2(|d(k)\rho(|x_\rho(k, k_\circ)|)|) &\le \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c - \nu_d \\ &\le \frac{1}{2}\sigma_1(|x_\circ|) + \nu_b \ ,\end{aligned} \tag{25}$$

for $0 \le k \le k_1 - 1$. Then it follows from the Δ-UBIBS property of the system, and particularly from (12), that

$$\begin{aligned}|x_\rho(k_1)| \le \max_{0\le j\le k_1-1} &\{\sigma_1(|x_\circ|) + \nu_b, \\ &\sigma_2(|d(j)\rho(|x_\rho(j)|)|)\} + \delta_b \\ &\le \sigma_1(|x_\circ|) + \nu_b + \delta_b \le \sigma_1(|x_\circ|) + \nu_c \ ,\end{aligned} \tag{26}$$

which, by (18) and (22), implies that

$$\begin{aligned}\sigma_2(\rho(|x_\rho(k_1)|) &\le \frac{1}{2}|x_\rho(k_1)| \le \frac{1}{2}\sigma_1(|x_\circ|) + \frac{\nu_c}{2} \\ &< \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c \ ,\end{aligned} \tag{27}$$

which contradict the definition of k_1. Hence, the claim is true.

An immediate consequence of the claim is that (26) holds for all $k \in \mathbb{N}$ and that $\overline{\lim}_{k\to\infty}|x_\rho(k)|$ is finite. By (18), $\sigma_2 = \gamma$ is the $\mathcal{K}$-asymptotic gain. From Definition 2.6, we have

$$\begin{aligned}\overline{\lim_{k\to\infty}}|x_\rho(k)| &\le \overline{\lim_{k\to\infty}}\,\gamma(|d(k)\rho(|x_\rho(k)|)|) + \pi \\ &\le \overline{\lim_{k\to\infty}}|x_\rho(k)|/2 + \nu_c + \pi \ ,\end{aligned} \tag{28}$$

which shows that $\overline{\lim}_{k\to\infty}|x_\rho(k)| \le 2(\nu_c + \pi)$, which is bounded for each trajectory, for all $k \ge k_\circ$. This shows that (23) is SPRS and hence SP-AS. Therefore, this completes the proof of Lemma 3.1. ■

Proof of Theorem 3.1 In proving this theorem, we follow the technique that has been used to prove the converse Lyapunov theorem in [9], combined with Theorem 1 of [5].

Proof of sufficiency: From the statement of the theorem, suppose that for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_1, \nu_2$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$, $|x| \le \Delta_x$, $\|d\|_\infty \le \Delta_d$, a smooth radially unbounded continuous function $V_T(k, x)$ is a SP-ISS Lyapunov function for the family of systems (2). Let the functions $\underline{\alpha}, \overline{\alpha}, \alpha$ and χ be as in Definition 2.3, and let $\delta > 0$ be such that

$$\max_{s\in(0,\Delta_d)}\{\underline{\alpha}^{-1}(\overline{\alpha}(\chi(s)+\nu_1)) - \underline{\alpha}^{-1}(\overline{\alpha}(\chi(s)))\} \le \delta \ . \tag{29}$$

We consider two cases:

Case 1: $|x| \ge \chi(|d|) + \nu_1$**.**
Using (5) and (6), it is obvious that we can write

$$\begin{aligned}&V_T(k, x) \ge \tilde{\chi}(|d|) + \tilde{\nu}_1 \ \Rightarrow \\ &\quad V_T(k+1, F_T) - V_T(k, x) \le -T\tilde{\alpha}(V_T(k, x)) \ ,\end{aligned} \tag{30}$$

by choosing $\tilde{\chi} = \underline{\alpha} \circ \chi$ and $\tilde{\alpha} = \alpha \circ \overline{\alpha}^{-1}$. By Lemma 3.3, since V_T is a smooth Lyapunov function, we can have $\alpha \in \mathcal{K}_\infty$. Using (7), and applying the comparison principle [19, Proposition 1], there exists $\beta_{\tilde{\alpha}} \in \mathcal{KL}$, such that

$$\begin{aligned}&V_T(k, x) \ge \tilde{\chi}(|d|) + \tilde{\nu}_1 \ \Rightarrow \\ &\qquad V_T(k, x) \le \beta_{\tilde{\alpha}}(V_T(k_\circ, x_\circ), k) + \nu_2 \ .\end{aligned} \tag{31}$$

Therefore, for all $k \ge k_\circ$, we can write

$$V_T(k, x(k + k_\circ, k_\circ, x_\circ, d)) \le \beta_{\tilde{\alpha}}(V_T(k_\circ, x_\circ), k) + \nu_2 \ .$$

Further, using (5) we obtain

$$\begin{aligned}|x(k{+}k_\circ, k_\circ, x_\circ, d)| &\le \underline{\alpha}^{-1}\left(\beta_{\tilde{\alpha}}(V_T(k_\circ, x_\circ), k) + \nu_2\right) \\ &\le \underline{\alpha}^{-1}\left(\beta_{\tilde{\alpha}}(\overline{\alpha}(|x_\circ|), k) + \nu_2\right) \\ &\le \underline{\alpha}^{-1} \circ \beta_{\tilde{\alpha}}(\overline{\alpha}(|x_\circ|), k) + \delta_2 =: \beta(|x_\circ|, k) + \delta_2.\end{aligned}$$

Hence, $|x(k, k_\circ, x_\circ, d)| \le \beta(|x_\circ|, k - k_\circ)$.

Case 2: $|x| < \chi(|d|) + \nu_1$**.**
From (5), we have that

$$\underline{\alpha}(|x|) \le V_T(k, x) \le \overline{\alpha}(|x|) \le \overline{\alpha}(\chi(|d|) + \nu_1) \ , \tag{32}$$

which implies that

$$\begin{aligned}|x(k, k_\circ, x_\circ, d)| &\le \underline{\alpha}^{-1}(\overline{\alpha}(\chi(|d|) + \nu_1)) \\ &\le \gamma(|d|) + \delta_1 \le \gamma(\|d\|_\infty) + \delta_1 \ ,\end{aligned} \tag{33}$$

where $\gamma := \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi$.
From Case 1 and Case 2, and defining $\delta := \max\{\delta_1, \delta_2\}$, we conclude that for any $|x| \le \Delta_x$ and $\|d\|_\infty \le \Delta_d$ the following holds:

$$|x(k, k_\circ, x_\circ, d)| \le \beta(|x_\circ|, k - k_\circ) + \gamma(\|d\|_\infty) + \delta \ , \tag{34}$$

and this completes the proof of the sufficiency.

Proof of necessity: Suppose that the system (2) is SP-ISS. Let arbitrary strictly positive numbers $\Delta_x, \Delta_d, \delta$ be given. We have shown in Lemma 3.1 that SP-ISS implies SPRS with input $d\rho(|x|)$, where $d \in \mathcal{U}_{\bar{B}}$ and $\rho \in \mathcal{K}_\infty$. This further implies that the system is SP-AS. Let the numbers $\Delta_x, \Delta_d, \delta$ *generate* $T_1^* > 0$.[3] From the SP-AS property, we have that for all $|x| \le \Delta_x$, $d \in \mathcal{U}_{\bar{B}}$, $k \ge k_\circ$ and $T \in (0, T_1^*)$

$$|x(k + k_\circ, k_\circ, x_\circ, d\rho(|x_\circ|))| \le \beta(|x_\circ|, k) + \delta \ , \tag{35}$$

holds. By Remark 4.1, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that

[3] Given the numbers $\Delta_x, \Delta_d, \delta$, we can compute a sampling period $T_1^* > 0$.

$$|x(k+k_\circ,k_\circ,x_\circ,d\rho(|x_\circ|))| \leq \rho_1(\rho_2(|x_\circ|)e^{-k})+\delta\,. \quad (36)$$

Define $\omega := \rho_1^{-1}$, and let $\delta_\rho > 0$ be such that

$$\max_{s\in[0,\Delta_x]} \Big(\omega(\rho_1(\rho_2(s)e^{-k})+\delta) - \rho_2(s)e^{-k}\Big) \leq \delta_\rho\,. \quad (37)$$

From (36) and (37) we obtain

$$\omega(|x(k+k_\circ,k_\circ,x_\circ,d\rho(|x_\circ|))|) \leq \rho_2(|x_\circ|)e^{-k}+\delta_\rho\,.$$

Since ω and ρ_2 are $\mathcal{K}_\infty$ functions, we can always find $\tilde\rho_2 \in \mathcal{K}_\infty$ such that

$$\begin{aligned}\omega(|x(k+k_\circ,&k_\circ,x_\circ,d\rho(|x_\circ|))|)\\ &\leq \tilde\rho_2(|x_\circ|)e^{-k} \leq \rho_2(|x_\circ|)e^{-k}+\delta_\rho\,. \end{aligned}\quad (38)$$

Let

$$\begin{aligned}V_{0T}(k_\circ,&x_\circ,d\rho(|x_\circ|))\\ &= \sum_{k=0}^{\infty}\omega(|x(k+k_\circ,k_\circ,x_\circ,d\rho(|x_\circ|))|)\,. \end{aligned}\quad (39)$$

It follows from (38) that

$$\begin{aligned}\omega(|x_\circ|) &\leq V_{0T}(k_\circ,x_\circ,d\rho(|x_\circ|))\\ &\leq \sum_{k=0}^{\infty}\tilde\rho_2(|x_\circ|)e^{-k} \leq \frac{e}{e-1}\tilde\rho_2(|x_\circ|)\,. \end{aligned}\quad (40)$$

This shows that the series in (39) is convergent, uniformly in $x_\circ$ with $|x_\circ| \leq \Delta_x$ and in $d \in \mathcal{U}_{\bar B}$. Since for each k and $k_\circ \in \mathbb{N}$, the function ω is uniformly continuous on $d \in \mathcal{U}_{\bar B}$, then so is V_{0T}. Define V_T by

$$V_T(k_\circ,x_\circ) = \sup_{d\in\mathcal{U}_{\bar B}} V_{0T}(k_\circ,x_\circ,d\rho(|x_\circ|))\,. \quad (41)$$

It then follows immediately from (40) that

$$\omega(|x_\circ|) \leq V_T(k_\circ,x_\circ) \leq \frac{e}{e-1}\tilde\rho_2(|x_\circ|)\,. \quad (42)$$

Hence, selecting $\underline{\alpha}(s) := \omega(s)$ and $\overline{\alpha}(s) := \frac{e}{e-1}\tilde\rho_2(s)$ proves that (5) holds.

Next, we show that V_T admits the desired decay estimate (6). Pick any $k_\circ$, $x_\circ$ such that $|x_\circ| \leq \Delta_x$, and any $\mu \in \mathcal{U}_{\bar B}$. Consider the exact solution $x_f := F_T^e(k_\circ,x_\circ,\mu\rho(|x_\circ|))$ and the approximate solution $x_F := F_T(k_\circ,x_\circ,\mu\rho(|x_\circ|))$. Since F_T is one-step consistent with F_T^e, we have that

$$|x_f - x_F| \leq T\varrho(T)\,, \quad \varrho \in \mathcal{K}_\infty\,. \quad (43)$$

Let $T^* \leq \min\{1, T_1^*, T_2^*\}$ be sufficiently small such that, by continuity of V_T and the one-step consistency property of F_T, we may assume the existence of $\tilde\varrho \in \mathcal{K}_\infty$ such that

$$|V_T(k_\circ+1,x_F) - V_T(k_\circ+1,x_f)| \leq T\tilde\varrho(T) \quad (44)$$

holds for all $T \in (0,T^*)$. Let $\nu > 0$ be such that

$$\tilde\varrho(T^*) \leq \nu. \quad (45)$$

By uniqueness of exact solutions, we see that for any $d \in \mathcal{U}_{\bar B}$ such that $d(k_\circ) = \mu$, it holds that

$$\begin{aligned}x(k+&k_\circ+1,k_\circ+1,x_f,d\rho(|x_f|))\\ &= x(k+k_\circ+1,k_\circ,x_\circ,d\rho(|x_\circ|)), \quad \forall k \geq 0. \end{aligned}\quad (46)$$

Hence, using (44), (45) and $T^* \leq 1$, we have

$$\begin{aligned}&V_T(k_\circ+1,x_F) \leq V_T(k_\circ+1,x_f) + T\tilde\varrho(T)\\ &\leq \sup_{d\in\mathcal{U}_{\bar B}} \sum_{k=0}^{\infty}\omega(|x(k+k_\circ+1,k_\circ+1,x_f,d\rho(|x_f|))|)+T\nu\\ &\leq \sup_{d\in\mathcal{U}_{\bar B}} \sum_{k=0}^{\infty}\omega(|x(k+k_\circ+1,k_\circ,x_\circ,d\rho(|x_\circ|))|)+T\nu\\ &\leq \sup_{d\in\mathcal{U}_{\bar B}} \sum_{k=1}^{\infty}\omega(|x(k+k_\circ,k_\circ,x_\circ,d\rho(|x_\circ|))|)+T\nu\\ &\leq \sup_{d\in\mathcal{U}_{\bar B}} \sum_{k=0}^{\infty}\omega(|x(k+k_\circ,k_\circ,x_\circ,d\rho(|x_\circ|))|)\\ &\quad -\omega(|x(k_\circ,k_\circ,x_\circ,d\rho(|x_\circ|))|)+T\nu\\ &\leq V_T(k_\circ,x_\circ) - T\omega(|x_\circ|) + T\nu\,,\end{aligned}$$

which implies that

$$\begin{aligned}V_T(k_\circ+1,F_T(k_\circ,x_\circ,\mu\rho(|x_\circ|)) - V_T(k_\circ,x_\circ)&\\ \leq -T\omega(|x_\circ|)+T\nu\,,&\end{aligned}\quad (47)$$

for all $|x| \leq \Delta_x$ and $d \in \mathcal{U}_{\bar B}$, which is equivalent to

$$\begin{aligned}|u| \leq \rho(|x|) \;\;\Rightarrow\;\; &V_T(k_\circ+1,F_T(k_\circ,x_\circ,u))\\ &-V_T(k_\circ,x_\circ) \leq -T\omega(|x_\circ|)+T\nu\,,\end{aligned}\quad (48)$$

and it is obviously also equivalent to

$$\begin{aligned}|x| \geq \chi(|u|)+\nu_1 \;\;\Rightarrow\;\; &V_T(k_\circ+1,F_T(k_\circ,x_\circ,u))\\ &-V_T(k_\circ,x_\circ) \leq -T\alpha(|x_\circ|)\,,\end{aligned}\quad (49)$$

where $\chi := \rho^{-1}$ and $\alpha := \frac{3}{4}\omega$ and $\nu_1 \leq \omega^{-1}(4\nu)$. Therefore, (6) is satisfied.

For (8) to hold, the Lyapunov function needs to be uniformly continuous in the state space for all small sampling periods. To prove the continuity of V_T, we use the following lemma, which is a modification of [6, Lemma 4.4] that applies to parameterized discrete-time systems.

Lemma 3.4 *There exists $T_2^* > 0$ such that for all $T \in (0,T_2^*)$, the function V_T is continuous in $\mathbb{R}^n$, for each $k \in \mathbb{N}$.* ■

Proof of Lemma 3.4: Let $k_0 \in \mathbb{N}$ be given. Uniform continuity of $V_{0T}(k_\circ,x_\circ)$ and hence of $V_T(k_\circ,x_\circ)$, independently of T, follows directly from (40). Following the proof of [6, Lemma 4.4], we can show that $V_T(k_\circ,\cdot)$ is continuous on $\mathbb{R}^n$ independently of T. Moreover, by

Assumption 2.1, there exists $T_2^* > 0$ such that for all $T \in (0, T_2^*)$ there exists $\delta > 0$ such that for any $k \geq k_\circ$

$$|x(k+1) - x(k)| < \delta, \text{ and } \lim_{T\to 0} |x(k+1) - x(k)| \to 0 \ .$$

Taking $x(k) = x_\circ$, then $x(k+1) = F_T(k_\circ, x_\circ, d)$. By uniform continuity of $V_T(k_\circ, x_\circ)$, and since the composition of continuous functions is also continuous, for all $T \in (0, T_2^*)$, V_T is uniformly continuous on $\mathbb{R}^n$, for each $k \in \mathbb{N}$. ■

Note however that the continuous Lyapunov function obtained up to this step is not necessarily smooth. Using Lemma 3.2, we can show the existence of a smooth Lyapunov function W_T as a continuous Lyapunov function V_T exists, and Lemma 3.3 generates the smooth Lyapunov function, by assuming that $\alpha \in \mathcal{K}_\infty$.
The last thing to show is that (7) holds. We have assumed that F_T is globally defined for small T, so that F_T is finite for all $k \geq k_\circ$, $|x_\circ| \leq \Delta_x$ and $\|d\|_\infty \leq \Delta_d$. This guarantees the existence of $c > 0$ such that

$$|F_T - x_\circ| \leq c \ , \quad \forall k \geq k_\circ \ . \tag{50}$$

Moreover, by Lemma 3.2 we may assume that V_T is smooth. Then, using (50) and the smoothness of V_T, and letting L be the Lipschitz constant of V_T, we obtain that

$$\begin{aligned} V_T(k_\circ+1, F_T(k_\circ, x_\circ, u)) - V_T(k_\circ, x_\circ) \\ \leq L\,|F_T - x_\circ| \leq Lc := \nu_2 \ , \end{aligned} \tag{51}$$

Hence (7) holds, and this completes the proof of the necessity and of the theorem. ■

3.1 Application to periodic systems

In this section, we apply Theorem 3.1 to periodic discrete-time time-varying systems. Systems belonging to this class are very important in various engineering applications, particularly in tracking control problems (see for instance [12, 22, 29, 31]).

The system (2) is a periodic system if F_T is periodic in time with period $\lambda > 0$, i.e.

$$F_T(kT + m\lambda, x, d) = F_T(kT, x, d) \ , \quad m \in \mathbb{N} \ . \tag{52}$$

We can state the following result.

Corollary 3.1 *The parameterized family of discrete-time time-varying periodic system (2) with period $\lambda > 0$ is SP-ISS if and only if it admits a (smooth) SP-ISS periodic Lyapunov function with the same period λ.* ■

4 Semiglobal practical input-to-state stabilization for systems in power form

In this section we apply the results from Section 3 to the SP-ISS control design problem for systems in *power form*. Recall the system (16) with arbitrary $l \in \mathbb{N} - \{0\}$.

$$\dot{x} = \sum_{i=1}^{l} f_i(x)u_i + \sum_{j=1}^{m} e_j(x)d_j \ . \tag{53}$$

Robust stabilization using continuous feedback for systems (53) has been a difficult problem to solve. Let alone that the nominal system does not satisfy Brockett's necessary condition for smooth stabilization using pure state feedback [1], which makes it necessary to use either control that depends on time (time-varying control) or discontinuous control. The result of [10] states that there does not exist a continuous homogeneous controller that robustly stabilizes the system (53) against modeling uncertainties. Many researchers have been trying to solve this problem using discontinuous feedback (see [11, 14, 24]). Various results have also been obtained for asymptotic stabilization of the systems. Except a few works in multirate control such as [13, 15, 32], almost all available results concentrate on continuous-time design (see for example [12, 22, 31]). Moreover, the results that are based on Lyapunov approach mostly rely on periodicity and LaSalle Invariance Principle to complete the stability analysis, since finding a strict Lyapunov function for driftless systems is in general very difficult. Yet, due to the inapplicability of LaSalle Invariance Principle for systems with uncertainty, this approach cannot be used to solve a robust stabilization problem.

In this section we address a robust stabilization problem for systems in power form (16), which is a particular case of (53) with $l = 2$. We provide a pair of SP-ISS Lyapunov function and control law for the system. The control law is similar to the one proposed in [23], and the Lyapunov function is a modification of the one proposed in [22]. Our result can be seen as a discrete-time counterpart and to some extent a generalization of [22, Theorem 2]. We first focus on the SP-AS problem for the nominal system ($d = 0$), and since we have a strict Lyapunov function, we can naturally obtain a solution to the SP-ISS problem for the system with disturbance (17).

4.1 Semiglobal practical asymptotic stabilization

We consider a stabilization problem in the absence of disturbances, and we state the following theorem.

Theorem 4.1 *Consider the Euler approximate model (17) with $d = 0$, namely*

$$x(k+1) = x(k) + T\sum_{i=1}^{2} f_i(x)u_i \ . \tag{54}$$

Suppose the functions $\rho : \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfy the following properties.
P1. The function W is continuously differentiable on $\mathbb{R}^{n-1}$ and of class C^2 on $\mathbb{R}^{n-1} - \{0\}$, and it is defined as

$$W(x) = \sum_{i=2}^{n} c_i\,|x_i|^{a_i} \ , \tag{55}$$

with $c_i > 0$, $a_i \in \{2, 3, \cdots\}$.

P2. The function ρ is of class C^1 on $(0, \infty)$, and it is defined as

$$\rho(s) = g_0\,|s|^b \ , \quad b > 0, \ g_0 > 0 \ . \tag{56}$$

Then there exists $T^* > 0$ such that for all $T \in (0, T^*)$, the controller $u_T := (u_{1T}, u_{2T})^T$, where

$$\begin{aligned} u_{1T} &= -g_1 x_1 - \rho(W)\Big(\cos((k+1)T) \\ &\quad - \frac{\epsilon}{2}\sin((k+1)T)\Big) + \frac{\epsilon}{2}\Delta_\rho \sin((k+1)T) \\ u_{2T} &= -g_2 \operatorname{sign}(L_{f_2}W)\,|L_{f_2}W|^a \Big(2\rho(W) \\ &\quad + 2(g_1x_1 + \rho(W)\cos((k+1)T))\cos((k+1)T) \\ &\quad - \epsilon g_1 x_1 \sin((k+1)T)\Big), \end{aligned} \tag{57}$$

with $\Delta_\rho = \rho(W(x(k+1))) - \rho(W(x(k)))$, $g_1 > 0$, $g_2 > 0$, a sufficiently small $\epsilon > 0$ and $a > 0$, is a SP-AS controller for the system (54) and the function

$$\begin{aligned} V_T(k,x) &= (g_1x_1 + \rho(W)\cos(kT))^2 + \rho(W)^2 \\ &\quad - \epsilon g_1 x_1 \rho(W)\sin(kT) \end{aligned} \tag{58}$$

is a SP-AS Lyapunov function for system (54), (57). ■

Proof of Theorem 4.1: Given the functions W and ρ satisfying P1 and P2 respectively. We prove that (u_T, V_T) is a SP-AS pair for the system (54) by showing the existence of the positive numbers (T^*, M) such that the inequalities (5)-(8) and (11) hold.

Fix strictly positive numbers Δ_x, ν_1 and ν_2. We consider arbitrary x with $|x| \le \Delta_x$. Let $T_1 > 0$ be such that for all $|x| \le \Delta_x$ and $T \in (0, T_1)$, we have $|x(k+1)| \le \Delta_x + 1$. Without loss of generality, assume that $T_1 < 1$. From P1 and P2 respectively, we see that the functions W and $\rho(W)$ are zero at zero, positive definite in $\mathbb{R}^{n-1}$ and radially unbounded. To show that the inequality (5) holds, we write the Lyapunov function (58) as

$$V_T(k,x) = \begin{bmatrix} x_1 & \rho(W) \end{bmatrix} P \begin{bmatrix} x_1 \\ \rho(W) \end{bmatrix} ,$$

with

$$P = \begin{bmatrix} g_1^2 & g_1(\cos(kT) - \epsilon\sin(kT)) \\ g_1(\cos(kT) - \epsilon\sin(kT)) & \cos^2(kT) + 1 \end{bmatrix}.$$

The determinant of the matrix P is

$$|P| = g_1^2\Big[1 - \Big(\epsilon^2\sin^2(kT) - 2\epsilon\cos(kT)\sin(kT)\Big)\Big] . \tag{59}$$

Let $\epsilon > 0$ be sufficiently small, such that

$$\epsilon^2\sin^2(kT) - 2\epsilon\cos(kT)\sin(kT) \le \tilde{\epsilon} < 1 . \tag{60}$$

Hence, the matrix P is positive definite, and this implies that $V_T(k,x)$ is positive definite and radially unbounded. Therefore, inequality (5) holds.

We now prove (6) by showing that with the controller (57), the Lyapunov difference is negative definite in a semiglobal practical sense. Using the Mean Value Theorem we obtain

$$\begin{aligned} \Delta_\rho &:= \rho(W(F_T)) - \rho(W(x(k))) \\ &\le \frac{d\rho(W)}{dW}\bigg|_{W=W^*} (W(F_T) - W(x(k))) \\ &\le b g_0 |W^*|^{b-1} \Delta_W, \end{aligned} \tag{61}$$

where $W^* = \theta_1 W(x(k+1)) + (1-\theta_1)W(x(k))$ for $\theta_1 \in [0,1]$, and

$$\begin{aligned} \Delta_W &:= W(F_T) - W(x(k)) \\ &\le \frac{dW}{dx}\bigg|_{x=x^\star} (F_T - x(k)) \le L_{f_2}W(x^\star) T u_{2T}, \end{aligned} \tag{62}$$

with $x^\star = \theta_2 x(k+1) + (1-\theta_2)x(k)$ for $\theta_2 \in [0,1]$. Let $T_2 > 0$ be sufficiently small, such that for $T \in (0, T_2)$ we can assume that $L_{f_2}W(x^\star) \approx L_{f_2}W(x)$. Moreover, we use the following approximation

$$\cos((k+1)T) - \cos(kT) \approx T\sin(kT) \approx O(T^2) , \tag{63}$$

$$\sin((k+1)T) - \sin(kT) \approx T\cos(kT) \approx O(T) . \tag{64}$$

The Lyapunov difference can then be written as

$$\begin{aligned} \Delta V_T &= V_T(k+1, x(k+1)) - V_T(k, x(k)) \\ &= \Big(g_1(x_1 + Tu_1) + (\rho(W) + \Delta_\rho)\cos((k+1)T)\Big)^2 \\ &\quad - (g_1x_1 + \rho(W)\cos(kT))^2 + 2\rho(W)\Delta_\rho \\ &\quad - \epsilon g_1(x_1 + Tu_1)\,(\rho(W) + \Delta_\rho)\sin((k+1)T) \\ &\quad + \epsilon g_1 x_1 \rho(W)\sin(kT) . \end{aligned}$$

We use (61), (62), (63), (64) and ϵ sufficiently small ($\epsilon = O(T)$) and substitute the controller (57) to obtain

$$\begin{aligned} \Delta V_T &\le O(T^2) - 2Tg_1\Big(g_1x_1 + \rho(W)\cos((k+1)T)\Big)^2 \\ &\quad - 2Tg_1(\frac{\epsilon}{2}\rho(W)\sin((k+1)T))^2 \\ &\quad - TA(\epsilon g_1 x_1 \sin((k+1)T))^2 \\ &\quad - 2Tg_1(\frac{\epsilon}{2}\Delta_\rho \sin((k+1)T))^2 - TA\Big(2\rho(W) \\ &\quad + 2(g_1x_1 + \rho(W)\cos((k+1)T))\cos((k+1)T)\Big)^2, \end{aligned}$$

where $A := g_2 b g_0 |W^*|^{b-1} |L_{f_2}W|^{a+1} \ge 0$. We now focus on the state x_1 in the first term, and the states $x_i,\ i = 2, 3, \cdots, n$ in the second term. The first term is negative definite for $x_1 \ne -\rho(W)\cos((k+1)T)/g_1$, but at these points, the third term is negative, and hence the sum of both terms is still negative. Moreover, the second term is negative definite for $(k+1)T \ne i\pi,\ i \in \mathbb{N}$. However, at these points the total quantity is still negative since $\cos((k+1)T)$ reaches its maximum and the non-trigonometric term is nonzero. Therefore, we can write

$$\Delta V_T \le -T\tilde{\alpha}(|x|) + O(T^2) , \tag{65}$$

with $\tilde{\alpha}$ positive definite. Define $\tilde{\nu}_1 := \kappa\tilde{\alpha}(\nu_1)$, $0 < \kappa < 1$, and let $T_3 > 0$ be such that for all $T \in (0, T_3)$, the term $O(T^2) < T\tilde{\nu}_1$. Defining $T^* := \min\{\tilde{T}, T_1, T_2, T_3\}$, then for all $|x| \le \Delta_x$, and all $T \in (0, T^*)$, we have that

$$\Delta V_T \le -T\tilde{\alpha}(|x|) + T\tilde{\nu}_1 , \tag{66}$$

and hence, (6) holds. Inequality (7) follows directly from (66). Uniform continuity of the Lyapunov function (58) in $\mathbb{R}^n$, independently of T is obvious from its construction. Therefore, it implies that the Lipschitz condition (8) holds. The last thing is to show that (11) holds. From (55), (56), (61), (62), and $|x(k+1)| \le \Delta_x + 1$, we obtain

$$\begin{aligned}
u_{1T} &= -g_1 x_1 - \rho(W)(\cos((k+1)T) \\
&\quad - \frac{\epsilon}{2}\sin((k+1)T)) + \frac{\epsilon}{2}\Delta_\rho \sin((k+1)T) \\
&\le g_1 \Delta_x + g_0(c^*(n-1)\Delta_x^{a^*})^b(1+\frac{\epsilon}{2}) \\
&\quad + \epsilon b g_0 (c^*(n-1)(\Delta_x+1)^{a^*})^b =: M_1 \ , \\
u_{2T} &= -g_2 \,\mathrm{sign}(L_{f_2}W)\,|L_{f_2}W|^a \Big(2\rho(W) \\
&\quad + 2(g_1x_1 + \rho(W)\cos((k+1)T))\cos((k+1)T) \\
&\quad - \epsilon g_1 x_1 \sin((k+1)T)\Big) \\
&\le g_2 a^* c^*(n-1)(\Delta_x)^{(a^*+n-3)} \\
&\quad \times \Big(4g_0(c^*(n-1)\Delta_x^{a^*})^b + (2+\epsilon)g_1\Delta_x\Big) =: M_2 \ ,
\end{aligned}$$

with $c^* := \max\{c_i\}$ and $a^* := \max\{a_i\}$. Let $M = M_1 + M_2$ then (11) holds, and this completes the proof. ■

Remark 4.1 Comparing the structure of the controller (57) with the homogeneous controller proposed in [23], we can see that the former is a perturbed form of the latter. ■

4.2 Semiglobal practical input-to-state stabilization

In the presence of modeling uncertainties it has been proven in [10] that smooth control is not robust in stabilizing affine systems, of which systems in power form are a special case. Although the robustness definition of [10] is not general, it shows that robust stability design for this class of system is nontrivial. In Theorem 4.1, we have obtained V_T, a strict SP-AS Lyapunov function for the system. It is known that negative definiteness of ΔV_T makes possible to extend the result directly to the stabilization in the presence of disturbances. The following is an extension of Theorem 4.1 to SP-ISS using smooth feedback.

Theorem 4.2 *Consider the Euler approximate model (17). Suppose that the functions ρ and W satisfy properties P1 and P2 respectively. Then there exists $T^* > 0$ such that for all $T \in (0, T^*)$, the controller (57) is a SP-ISS controller for the system (17) and the function (58) is a SP-ISS Lyapunov function for the system (17), (57).* ■

Sketch of the proof of Theorem 4.2: The proof follows very similar steps as the proof of Theorem 4.1, by taking into account the disturbance $d \in \mathbb{R}^l$, affecting the system (17). Given a positive number $\Delta_d > 0$ such that the disturbance d satisfies $|d| \le \Delta_d$. Note that, while in the SP-AS case it is sufficient to show that (6) holds with a positive definite $\tilde{\alpha}$, for SP-ISS $\tilde{\alpha}$ is required to be a $\mathcal{K}_\infty$-function. Therefore we need to modify the last step in the following way. Note that by using Young's inequality we can split all terms containing the states and the disturbance. Through suitable majorization and since $\sin((k+1)T))^2 \le 1$ we obtain

$$\begin{aligned}
\Delta V_T &\le -T\bar{A}(x_1^2 + \rho(W)^2\sin((k+1)T)^2) + T\bar{\chi}(|d|) + T\bar{\nu}_1 \\
&\le -T\bar{A}(x_1^2 + \rho(W)^2)\sin((k+1)T)^2 + T\bar{\chi}(|d|) + T\bar{\nu}_1 \ ,
\end{aligned}$$

with $\bar{A} > 0$ and $\bar{\chi} \in \mathcal{K}$. We add and subtract the term $T\mu\bar{A}(x_1^2 + \rho(W)^2)$ with $0 < \mu \lll T$, so that $\mu\bar{A}(x_1^2 + \rho(W)^2) \le 0.1\bar{\nu}_1$ for all $|x| \le \Delta_x$. Hence,

$$\begin{aligned}
\Delta V_T &\le -T\bar{A}(x_1^2 + \rho(W)^2)(\sin((k+1)T)^2 + \mu) \\
&\quad + T\bar{\chi}(|d|) + T(\bar{\nu}_1 + 0.1\bar{\nu}_1) \\
&\le -T\tilde{\alpha}(|x|) + T\bar{\chi}(|d|) + T\tilde{\nu}_1 \ ,
\end{aligned}$$

with $\tilde{\alpha} \in \mathcal{K}_\infty$ and $\tilde{\nu}_1 = 1.1\bar{\nu}_1$, that implies that (6) holds. The rest follows exactly the proof of Theorem 4.1. ■

In practice, when designing a discrete-time controller for a continuous-time plant, the final goal is to achieve stability for the sampled-data system. We are then interested in knowing what can be achieved for the sampled-data system, if the discrete-time model of the system is SP-ISS with the designed controllers. For time invariant systems, the relation between SP-ISS of a discrete-time model and the sampled-data system follows from the results of [16] and [21]. In the present work, although we are dealing with time-varying systems, due to uniform boundedness of all signals with respect to time, the relation between SP-ISS of the closed-loop discrete-time model and the sampled-data system follows closely and lead to the same conclusions as the results of [16] and [21]. The following result is stated without proof, and it can further be shown that our design satisfies this result and the pair (u_T, V_T) given by (57) and (58) is a SP-ISS pair for systems in power form (16).

Proposition 4.1 *If the following conditions hold: (i) the pair (u_T, V_T) is an SP-ISS pair for the approximate model (10); (ii) F_T is one step consistent with F_T^e ((43) holds); (iii) u_T is uniformly locally bounded; (iv) the solution of the sampled-data system (9) with u_T is bounded over T, then the pair (u_T, V_T) is an SP-ISS pair for the sampled-data system.* ■

5 Design examples

5.1 SP-AS design for a car-like mobile robot

Consider a simple kinematic model of a car-like mobile robot moving on a plane [31]:

$$\begin{aligned}
\dot{x} &= v\cos\theta; \quad \dot{\phi} = \omega; \\
\dot{y} &= v\sin\theta; \quad \dot{\theta} = \tfrac{1}{L}\tan(\phi)v \ ,
\end{aligned} \tag{67}$$

with v - the forward velocity, ω - the steering velocity, (x, y) - the Cartesian position of the center of mass of the robot, ϕ - the angle of the front wheels with respect to the car (the steering angle) and θ - the orientation of the car with respect to some reference frames. Using a coordinate transformation [31], we obtain the model of system (67) in power form with $n = 4$. It has been shown in [23] that the controller [4]

$$
\begin{aligned}
u_1 &= -3x_1 + 0.4\sqrt[6]{W(x)}\cos t \\
u_2 &= -0.03\kappa\,\mathrm{sign}(L_{f_2}W(x))\sqrt[5]{|L_{f_2}W(x)|}\ ,
\end{aligned} \tag{68}
$$

with $W(x) = 0.5x_2^6 + 10^4|x_3|^3 + 1.5\times 10^6 x_4^2$ and $L_{f_2}W(x) = 3x_2^5 + 3\times 10^4\,\mathrm{sign}(x_3)x_3^2 x_1 + 1.5\times 10^6 x_4 x_1^2$, asymptotically stabilizes the mobile robot. Applying Theorem 4.1, we construct the controller

$$
\begin{aligned}
u_{1T} &= -3x_1 + \rho(W)(\cos((k+1)T) - \frac{\epsilon}{2}\sin((k+1)T)) \\
u_{2T} &= u_2\Big(2\rho(W) - 3\epsilon x_1 \sin((k+1)T) \\
&\quad + 2(3x_1 + \rho(W)\cos((k+1)T))\cos((k+1)T)\Big),
\end{aligned} \tag{69}
$$

with $\rho(W) = 0.4\sqrt[6]{W(x)}$ and u_2 given by (68) with $\kappa = 1$, which is a SP-AS controller for the Euler model of the system. As stated in Remark 4.1, the controller (69) is a perturbed version of (68).

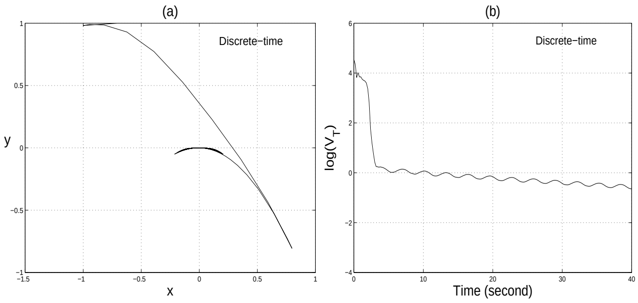

Figure 1. Response of the car model controlled using our proposed controller (69).

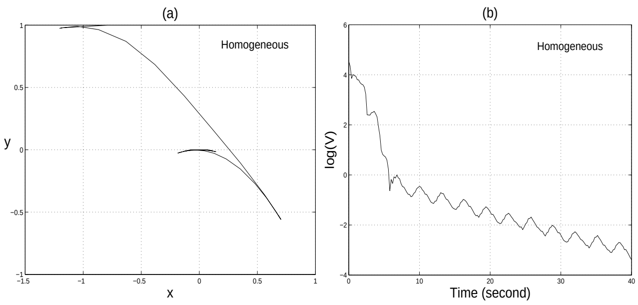

Figure 2. Response of the car model controlled using the sampled homogeneous controller (68).

Figure 1 shows the simulation results when the controller (69) is applied to control the plant in power form, and Figure 2 shows the response when the sample and hold version of controller (68), with $k = 25/6$, is applied. In the simulations we have used $x_\circ = (0,0,0,1)'$, $T = 0.2$ and $\epsilon = 0.35$. We display the (x,y) position of the car and the $\log(V_T)$ respectively. The (x,y) position of the car is given by the equations $x = x_1$ and $y = x_4 - x_1x_3 + \frac{1}{2}x_1^2x_2$. Moreover, for comparison to the graph of $\log(V_T)$ of Figure 1(b), we have plotted $\log(V)$, where $V = V_T$ with $\epsilon = 0$, in Figure 2(b). It is shown that the proposed perturbed controller (69) performs very similarly to the homogeneous controller (68) in the absence of disturbances. Note however that the controller (69) is in fact also a SP-ISS stabilizing controller for the same system with disturbance.

5.2 SP-ISS design for a unicycle mobile robot

Consider the model of a unicycle mobile robot moving on a plane, with two independent motorized wheels [22]:

$$
\dot{x} = v\cos\theta + d\sin\theta;\ \dot{y} = v\sin\theta - d\cos\theta;\ \dot{\theta} = \omega, \tag{70}
$$

with v - the forward velocity, ω - the steering velocity, (x,y) - the Cartesian position of the center of mass of the robot, θ - the heading angle from the horizontal axis, and d - a disturbance (exogenous force) perpendicular to the forward direction. Using the coordinate transformation $x_1 = x;\ x_2 = \tan\theta;\ x_3 = -y + x\tan\theta$, we obtain the model of system (70) in power form with disturbance:

$$
\begin{aligned}
\dot{x}_1 &= u_1 + \frac{d}{\sqrt{1+x_2^2}} \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1u_2 + d\sqrt{1+x_2^2}\ ,
\end{aligned} \tag{71}
$$

where $u_1 := v\cos\theta$, and $u_2 := \omega\sec^2\theta$. Note that for system (71), the exact discrete-time model is not computable explicitly. Choosing

$$
W = 0.5x_2^4 + 10^4x_3^2 \text{ and } \rho(W) = 0.1\sqrt[4]{|W|}\ ,
$$

it can be shown that the controller

$$
\begin{aligned}
u_{1T} &= -2x_1 + \rho(W)(\cos((k+1)T) - \frac{\epsilon}{2}\sin((k+1)T)) \\
u_{2T} &= -0.05\,\mathrm{sign}(L_{f_2}W)\,|L_{f_2}W|^{\alpha}\Big(2\rho(W) \\
&\quad + 2(2x_1 + \rho(W)\cos((k+1)T))\cos((k+1)T) \\
&\quad - 2\epsilon x_1\sin((k+1)T)\Big)\ ,
\end{aligned} \tag{72}
$$

and the Lyapunov function

$$
\begin{aligned}
V_T &= (2x_1 + \rho(W)\cos((k+1)T)^2 + \rho(W)^2 \\
&\quad - 2\epsilon x_1\rho(W)\sin((k+1)T)
\end{aligned} \tag{73}
$$

is a SP-ISS pair for the closed-loop system which consist of the Euler model of (71) with the controller (72).

Figure 3 shows the simulation results for the system controlled using our proposed robust controller (72), in comparison with the homogeneous controller [23], in the presence of a constant disturbance $d = 0.2$. We display

---

[4] For consistency with Theorem 4.1, we have replaced the term $\sin t$ with $\cos t$ in u_1 of [23].

the (x, y) position [5], which is given by the equations $x = x_1$ and $y = x_1 x_2 - x_3$. In the simulation, we use the initial condition $x_0 = (0,\ 0,\ -1)^T$, $T = 0.5$ and $\epsilon = \frac{1}{3}$..

The simulation shows that, for the chosen parameters, the position of the vehicle is closer to the origin when applying the proposed controller (72). This indicates that compared to the homogeneous controller, our proposed controller performs somewhat better in the presence of a disturbance. This behaviour is consistent for other simulation settings with a careful choice of parameters of the controller.

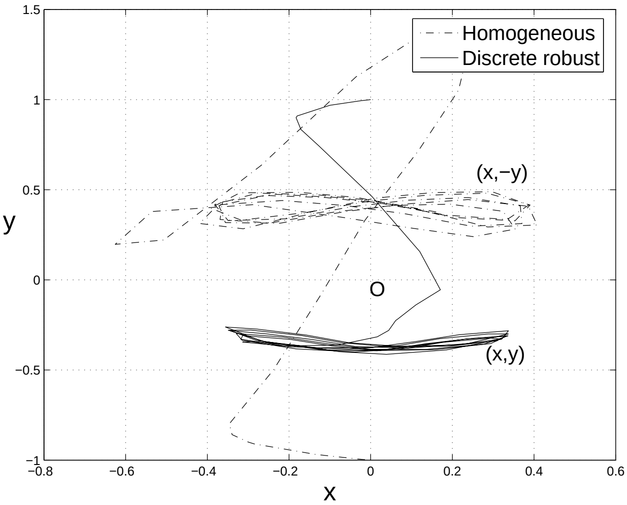

Figure 3. Position (x, y) with robust controller (72) vs position $(x, -y)$ with homogeneous controller (upside down), for $d = 0.2$.

6 Summary

We have presented a converse Lyapunov theorem for SP-ISS for parameterized discrete-time time-varying systems. We have also presented an application of our result to discrete-time time-varying periodic systems and have applied this result to solve a discrete-time robust stabilization problem for nonholonomic systems in power form. We have proposed a construction of a discrete-time SP-ISS control law and a strict Lyapunov function. Design examples show that robust stabilization using continuous control is possible, and this gives an alternative to emulation design for sampled-data stabilization for systems in power form. Moreover, since the exact discrete-time model of the nominal systems in power form can be explicitly computed, designing a discrete-time asymptotically (and robustly) stabilizing controller for the systems based on the exact discrete-time model and comparing the performance of the controller with the result of this paper would be an interesting topic for further research. Finally, the applicability of our method to more general classes of nonholonomic systems would also be an interesting direction to investigate.

[5] Note that in Figure 3, in order to give a clear comparison, we plot the (x, y) position of the mobile robot controlled by the robust controller, and plot the $(x, -y)$ position of the mobile robot when applying the homogeneous controller.

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