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Brief paper

Analysis, design, and performance limitations of H_2 optimal filtering in the presence of an additional input with known frequency ☆

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Abstract

In this paper, the inputs are considered to be of two types. The first type of input, as in standard H_2 optimal filtering, is a zero mean wide sense stationary white noise, while the second type is a linear combination of sinusoidal signals each of which has an unknown amplitude and phase but known frequency. The generalized H_2 optimal filtering problem seeks to find a *linear stable* filter that estimates a desired output such that the H_2 norm of the transfer matrix from the white noise input to the estimation error is minimized subject to the constraint that the mean of the error converges to zero for all initial conditions of the given system and filter and for all possible external sinusoidal signals. The analysis, design, and performance limitations of generalized H_2 optimal filters are presented here.

Keywords: White noise input; Sinusoidal input; Unbiased filtering; H_2 optimal filtering; Performance limitations

1. Introduction

A crucial component of the celebrated Kalman filtering problem (see, for instance, Sorenson, 1985) or otherwise known as the H_2 optimal filtering problem is that it assumes that the noise (external input) is zero mean. For the case, when the noise has a non-zero constant (DC) mean, as discussed in Blight (1989) and as discussed in the body of this paper, a modification to the standard H_2 optimal filter is necessary. In this paper, our model for the external inputs consists of two different types. One type is a white noise while the other type is a linear combination of sinusoidal signals each of which has an unknown amplitude and phase but known frequency. The latter can of course be used to represent the unknown mean of the noise but this setting is clearly much more general than that. Under such inputs, we seek here a '*linear stable unbiased* (*in a generalized sense*)' filter that renders the steady state performance measure (namely, the RMS norm of the estimation error signal) as small as *possible*. We call such filtering problems generalized H_2 optimal filtering problems. After formulating such generalized H_2 optimal filtering problems, we show that these problems can be reduced to (standard) H_2 optimal filtering problems for an expanded system constructed from the data of the given system. We will then study the cost incurred by the additional requirement of rejecting a sinusoidal signal of known frequency but unknown amplitude and phase. We will show that the infimum of the RMS norm is not affected by the additional requirement. In general, the solvability conditions of the generalized H_2 optimal filtering problem might be stronger than those of the standard H_2 optimal filtering problem but for a large class of problems they are identical.

As well known, the RMS norm of a signal is a steady state performance measure. Such a performance measure is blind to the transient aspect of estimation error signal. As such, whenever the requirement of generalized unbiasedness (see the text of the paper for details) in the presence of sinusoidal signals of known frequency but unknown amplitude and phase is met,

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we note that, in the absence of the white noise, the estimation error is an energy signal. This lets us define the energy of the error signal as the transient performance measure. In this paper, we will compute both the steady state and the transient performance measures, and show that the non-minimum-phase dynamics of the given system plays a significant role in dictating both these measures. In fact, we will uncover a peculiar property: the minimal steady state performance measure (namely the minimal RMS norm of the error signal) *reduces* when the non-minimum-phase zeros are moved closer to the imaginary axis; however in contrast the newly defined transient performance measure increases, and actually *could be unbounded* when the non-minimum-phase zeros are moved closer to the modes of the second type of input which, by our assumption, are on the boundary of the stability domain, namely the imaginary axis.

In what follows, the entire complex plane and the open left-half complex plane are, respectively, denoted by $\mathbb{C}$ and $\mathbb{C}^-$.

2. Preliminaries

Let us consider the plant or system model

$$\Sigma : \dot{x} = Ax + Bu, \quad y = Cx + Du, \quad z = Ex + Fu. \tag{1}$$

Here, $u \in \mathbb{R}^m$ is the input, $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the measured output, and $z \in \mathbb{R}^q$ is the desired output signal to be estimated. We decompose the input u into two parts, $u' = (u_1' \ u_2')'$, where $u_1 \in \mathbb{R}^{m_1}$ and $u_2 \in \mathbb{R}^{m_2}$. As mentioned in the Introduction, the first type of input, denoted by u_1, is assumed to be a zero mean wide sense stationary white noise of unit intensity. On the other hand, the second type of input, denoted by u_2, is assumed to be a linear combination of sinusoidal signals each of which has an unknown amplitude and phase but known frequency. Clearly, such a signal u_2 can be modeled as the output of a known linear autonomous system with unknown initial conditions. Such a system is called an exogenous system or for short exosystem. Thus, consider

$$\Sigma_a : \dot{x}_a = Sx_a, \quad u_2 = C_a x_a, \tag{2}$$

where $x_a \in \mathbb{R}^{n_a}$ for some n_a. An important special case where this type of problem arises is the case of having a system driven by a wide sense stationary white noise input with unknown intensity (variance) and mean. It is easily verified that the H_2 optimal filters are independent of the intensity level of the noise. After all if we change B into BV with V invertible then the class of H_2 optimal filters remains the same even though V does effect the RMS gain. However, having a non-zero mean of the external input requires a modification of our filter as we will see later on.

Let us next partition the matrices, B, D, and F in conformity with the partitioning of u,

$$B = (B_1 \ B_2), \quad D = (D_1 \ D_2), \quad F = (F_1 \ F_2). \tag{3}$$

The system Σ then has the structure

$$\Sigma : \begin{cases} \dot{x} = Ax + B_1u_1 + B_2u_2, \\ y = Cx + D_1u_1 + D_2u_2, \\ z = Ex + F_1u_1 + F_2u_2. \end{cases} \tag{4}$$

Our interest lies in estimating the desired output signal z using only the output y but not the input u. As usual, let $\hat{z}$ be the estimate of z as given by a filter, and let e_z be the estimation error, $e_z = z - \hat{z}$.

It is natural to use the following assumption.

Assumption 1. The matrix pair (C, A) is detectable.

We consider a general proper filter of the form

$$\Sigma_{\mathrm{f}} : \dot{\xi} = L\xi + My, \quad \hat{z} = N\xi + Py. \tag{5}$$

Whenever $P = 0$, the above filter is said to be a strictly proper filter. We require that the filter (5) be internally stable.

3. Problem statement

For the case when $u_2 = 0$, we get the standard H_2 optimal filtering problem in which a linear stable unbiased filter is sought that minimizes the RMS norm of the error signal e_z. In this section, we formulate a generalized H_2 optimal filtering problem. We first have the following definition.

Definition 2. Consider the given system Σ along with the exosystem Σ_a. We say a linear stable strictly proper (or proper) filter (5) *is generalized unbiased* if, in the absence of the input u_1, the estimation error e_z decays asymptotically to zero for all possible initial conditions of the given system (4) and the filter (5), and for all possible input signals u_2.

The above definition, whenever $u_2 = 0$, reduces to the familiar notion of unbiasedness of filters. The *generalized optimal filtering problem under white noise input* can be defined now as the problem of finding, whenever it exists, a linear stable strictly proper (or proper) filter which is generalized unbiased while the RMS norm of the error signal $\|e_z\|_{\mathrm{rms}}$ is as small as possible. Also, the infimum of the RMS norm of the error signal e_z over the set of all linear stable strictly proper (or proper) unbiased filters can be called the *generalized optimal filtering performance measure under white noise input* via linear stable strictly proper (or proper) filters, and can be denoted by $\gamma^*_{g,\mathrm{sp}}$ (or $\gamma^*_{g,\mathrm{p}}$). We note that the generalized optimal filtering problem under white noise input can be given a deterministic interpretation since the RMS norm of the error signal, e_z, is equal to the H_2 norm of the transfer matrix from the input u_1 to the error e_z. That is, we can interpret the generalized optimal filtering problem under white noise input as the generalized H_2 optimal filtering problem, and similarly $\gamma^*_{g,\mathrm{sp}}$ (or $\gamma^*_{g,\mathrm{p}}$) as the generalized H_2 optimal filtering performance measure via linear stable strictly proper (or proper) filters.

Whenever the input u_2 is set to zero, the generalized H_2 optimal filtering problem for the given system Σ reduces to the celebrated H_2 optimal filtering problem (Kalman filtering problem) for a system Σ_0 given by

$$\Sigma_0 : \dot{x} = Ax + B_1u_1, \ y = Cx + D_1u_1, \ z = Ex + F_1u_1. \tag{6}$$

Also, in this case, we denote the infimum of the RMS norm of the error signal over all the linear unbiased stable filters for the system Σ_0 by $\gamma^*_{\rm sp}$ or $\gamma^*_{\rm p}$ depending on whether we use strictly proper or proper filters.

4. Performance, existence conditions, and design

We need to investigate several issues pertaining to generalized H_2 optimal filtering, namely computing $\gamma^*_{g,\rm sp}$ or $\gamma^*_{g,\rm p}$, developing the existence and uniqueness conditions for the generalized H_2 optimal filters, and designing the generalized H_2 optimal filters. In this section, we relate these issues to those of standard H_2 optimal filtering, however, for an expanded system $\tilde{\Sigma}$ which is constructed by viewing together the given system Σ and the exosystem Σ_a as one system,

$$\tilde{\Sigma}: \begin{cases} \dot{\bar{x}} = A_e\bar{x} + B_e u_1, \\ y = C_e\bar{x} + D_1 u_1, \\ z = E_e\bar{x} + F_1 u_1, \end{cases} \tag{7}$$

$$A_e = \begin{pmatrix} A & B_2C_a \\ 0 & S \end{pmatrix}, \quad B_e = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$
$$C_e = (C \;\; D_2C_a), \quad E_e = (E \;\; F_2C_a). \tag{8}$$

We will impose the following assumption for $\tilde{\Sigma}$. This assumption implies that Assumption 1 for Σ is satisfied.

Assumption 3. The matrix pair (C_e, A_e) is detectable.

Theorem 4. *Consider the system Σ given in* (4) *and the exosystem Σ_a given in* (2). *Let Assumption* 3 *be satisfied for the expanded system $\tilde{\Sigma}$ given in* (7). *Then, $\gamma^*_{g,\rm sp}$* (*or $\gamma^*_{g,\rm p}$*) *equals the infimum of the RMS norm of the estimation error signal $\tilde{e}_z$ for the expanded system $\tilde{\Sigma}$ over all unbiased strictly proper* (*or proper*) *stable filters*.

Proof. The proof follows from the proof of Theorem 5. □

The theorem below provides a road-map to study existence, uniqueness, and design of the generalized H_2 optimal filters.

Theorem 5. *Consider the system Σ given in* (4) *and the exosystem Σ_a given in* (2). *Let Assumption* 3 *be satisfied for the expanded system $\tilde{\Sigma}$ of* (7). *Consider a filter $\Sigma_{\rm f}$ of the form* (5). *Then, the following two statements are equivalent*:

(i) *The filter $\Sigma_{\rm f}$ is a proper* (*or strictly proper*) *generalized H_2 optimal filter for Σ.*
(ii) *The filter $\Sigma_{\rm f}$ is a proper* (*or strictly proper*) *H_2 optimal filter for $\tilde{\Sigma}$.*

Proof. Assume a filter $\Sigma_{\rm f}$ of the form (5) is unbiased in the sense of Definition 2 for the system Σ along with the associated exosystem Σ_a, and yields a stable transfer matrix $G_{u_1 e_z}$ from u_1 to $e_z = z - \hat{z}$. Then it can be trivially verified that such a filter when applied to the expanded system is unbiased and results in the same stable transfer matrix $G_{u_1 e_z}$ from u_1 to $e_z = z - \hat{z}$. The converse of the above implication is also trivially satisfied. But then it is immediate that a filter is a generalized H_2 optimal filter for system (4) and the associated exosystem (2) if and only if it is an H_2 optimal filter for system (7). □

In view of Theorem 5, one can deal with various aspects of the generalized H_2 optimal filtering problem for a given system in terms of similar issues of a standard H_2 optimal filtering problem for an expanded system. In particular, we refer the interested reader to Saberi et al. (2000b) and Saberi et al. (1995) for the issues of design such as testing the solvability conditions, concerned architecture of filters, and design algorithms.

The above development is based on the assumption that the pair (C_e, A_e) is detectable. Then, to complete our study, we need to examine the implications when it is not so. It is natural indeed to assume that (C, A) is detectable. Moreover, if there are unstable dynamics which are not observable from y but which are observable from z, then clearly we will never be able to obtain an unbiased filter. Using the Hautus (1973) test for detectability, this can be formally expressed by the following necessary condition:

Assumption 6. For all $\lambda \in \mathbb{C}$ with Re $\lambda \geqslant 0$ we have

$$\operatorname{rank}\begin{pmatrix} \lambda I - A & -B_2C_a \\ 0 & \lambda I - S \\ C & D_2C_a \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \lambda I - A & -B_2C_a \\ 0 & \lambda I - S \\ C & D_2C_a \\ E & F_2C_a \end{pmatrix}.$$

If (C, A) is detectable and the above assumption is satisfied then we can use a reduction technique to get into a situation, where (C_e, A_e) is detectable. We first find V_1 and V_2 such that $\operatorname{im}(V_1' \; V_2')'$ represents the unstable, unobservable dynamics of the pair (C_e, A_e). Detectability of (C, A) implies that V_2 must be injective. Moreover, Assumption 6 implies that we must have $EV_1 + F_2C_aV_2 = 0$. Then there exists a suitable basis transformation for the exosystem such that

$$V_2 = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}, \quad C_a = (C_{a1} \;\; C_{a2}).$$

Consider the following exosystem:

$$\Sigma_{a1}: \dot{x}_{a1} = S_{11}x_{a1}, \quad u_2 = C_{a1}x_{a1}. \tag{9}$$

Then the above implies that system (4) with the original exosystem (2) and the same system (4) with the new exosystem (9) result in the same outputs y and z provided we modify the initial conditions of the system $x(0)$ and the exosystem $x_a(0)$ to the initial conditions

$$x(0) - V_1(0 \;\; I)x_a(0) \quad \text{and} \quad (I \;\; 0)x_a(0)$$

for the system and exosystem, respectively. From this it is clear that a filter design for the original system and exosystem can be reduced to a filter design for the same system but with a modified (reduced) exosystem. After this reduction we obtain a system and exosystem which when viewed together are detectable from y.

5. Dependence of performance and existence conditions on the input signal u_2

We ask ourselves here two fundamental questions:

(i) How does the performance of generalized H_2 optimal filtering for Σ differs from the performance of H_2 optimal filtering for Σ_0?
(ii) How do the solvability conditions of generalized H_2 optimal filtering problem for Σ differ from those of H_2 optimal filtering problem for Σ_0?

The following theorem answers the first question.

Theorem 7. *Consider the generalized H_2 optimal filtering problem of Section* 3 *for the system Σ of* (4) *along with the associated exosystem Σ_a of* (2) *whose performance measure is indicated by $\gamma^*_{g,\mathrm{sp}}$ or $\gamma^*_{g,\mathrm{p}}$ depending upon whether the class of strictly proper or proper filters are used. Also, consider the H_2 optimal filtering problem for the system Σ_0 of* (6) *whose performance measure is indicated by γ^*_{sp} or γ^*_{p} depending upon whether the class of strictly proper or proper filters are used. Then, under Assumptions* 1 *and* 6, *we have*

$$\gamma^*_{g,\mathrm{sp}} = \gamma^*_{\mathrm{sp}} \quad \textit{and} \quad \gamma^*_{g,\mathrm{p}} = \gamma^*_{\mathrm{p}}.$$

Proof. We can assume, without loss of generality, that Assumption 3 is satisfied. In view of Theorem 4, $\gamma^*_{g,\mathrm{sp}}$ (or $\gamma^*_{g,\mathrm{p}}$) is the infimum of the RMS norm of the estimation error $\tilde{e}_z$ over all the linear unbiased stable strictly proper (or proper) filters for the expanded system $\tilde{\Sigma}$ of (7). Also, γ^*_{sp} (or γ^*_{p}) is the infimum of the RMS norm of the error e_z over all the linear unbiased stable strictly proper (or proper) filters for the system Σ_0 of (6). To facilitate the comparison of $\gamma^*_{g,\mathrm{sp}}$ (or $\gamma^*_{g,\mathrm{p}}$) with γ^*_{sp} (or γ^*_{p}), consider the semi-stabilizing solution Q of the continuous-time linear matrix inequality (CLMI),

$$\begin{pmatrix} AQ + QA' + B_1B_1' & QC' + B_1D_1' \\ CQ + D_1B_1' & D_1D_1' \end{pmatrix} \geqslant 0, \tag{10}$$

and the semi-stabilizing solution Q_e of the CLMI,

$$\begin{pmatrix} A_eQ_e + Q_eA_e' + B_eB_e' & Q_eC_e' + B_eD_1' \\ C_eQ_e + D_1B_e' & D_1D_1' \end{pmatrix} \geqslant 0.$$

Then it is easily verified that

$$Q_e = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, we recall from Saberi et al. (2000b) that $\gamma^*_{\mathrm{sp}} = (\mathrm{trace}(EQE'))^{1/2}$ and

$$\gamma^*_{\mathrm{p}} = (\mathrm{trace}((E - P^*C)Q(E - P^*C)'))^{1/2},$$

where P^* is any solution of the equation $F_1 - PD_1 = 0$ for P, and where Q is the unique semi-stabilizing solution of the CLMI (10). Then we have

$$\gamma^*_{\mathrm{sp}} = (\mathrm{trace}\, EQE')^{1/2} = (\mathrm{trace}\, E_eQ_eE_e')^{1/2} = \gamma^*_{g,\mathrm{sp}}.$$

On the other hand, for proper filters we need to find P^* such that $F_1 - P^*D_1 = 0$. But then again,

$$\begin{aligned}\gamma^*_{\mathrm{p}} &= [\mathrm{trace}(E - P^*C)Q(E - P^*C)']^{1/2} \\ &= [\mathrm{trace}(E_e - P^*C_e)Q_e(E_e - P^*C_e)']^{1/2} = \gamma^*_{g,\mathrm{p}}.\end{aligned}$$

This completes the proof. □

The following theorem answers the second question regarding the dependency of the solvability conditions of the generalized H_2 optimal filtering problem on the input signal u_2.

Theorem 8. *Consider the generalized H_2 optimal filtering problem of Section* 3 *for the system Σ of* (4) *along with the associated exosystem Σ_a of* (2). *Let Assumption* 6 *be satisfied. Also, consider the H_2 optimal filtering problem for the system Σ_0 of* (6), *and let Assumption* 1 *be satisfied. We have the following statements*:

(i) *For the case when $F_2 = 0$, the generalized H_2 optimal filtering problem is solvable via strictly proper filters if and only if the H_2 optimal filtering problem for Σ_0 is solvable via strictly proper filters.*
(ii) *For the case when $F_2 = 0$ and additionally $D_2 = 0$, the said generalized H_2 optimal filtering problem is solvable via proper filters if and only if the H_2 optimal filtering problem for Σ_0 is solvable via proper filters.*

Proof. Using the reduction technique presented earlier, we can assume without loss of generality that Assumption 3 is satisfied. By Theorem 5, we need to compare the conditions for the solvability of the H_2 optimal filtering problem for the system Σ_0 of (6) with those of $\tilde{\Sigma}$ of (7). Note that we already studied the relevant CLMIs necessary for such a comparison in the proof of Theorem 7. Let Q be the semi-stabilizing solution of the CLMI (10) and define B_Q and D_Q by

$$\begin{pmatrix} AQ + QA' + B_1B_1' & QC' + B_1D_1' \\ CQ + D_1B_1' & D_1D_1' \end{pmatrix} = \begin{pmatrix} B_Q \\ D_Q \end{pmatrix} (B_Q' \;\; D_Q').$$

We consider first strictly proper filters. Strictly proper H_2 optimal filters exist only if $F_1 = 0$. Then, in view of Saberi et al. (2000b), we need to relate the solvability of the exact input decoupling (EID) filtering problems of the following two systems:

$$\dot{\tilde{x}} = A\tilde{x} + B_Q\tilde{u}, \quad \tilde{y} = C\tilde{x} + D_Q\tilde{u}, \quad \tilde{z} = E\tilde{x} \tag{11}$$

and

$$\begin{cases} \dot{\bar{x}} = \begin{pmatrix} A & B_2C_a \\ 0 & S \end{pmatrix}\bar{x} + \begin{pmatrix} B_Q \\ 0 \end{pmatrix} u_1, \\ y = (C \;\; D_2C_a)\bar{x} + D_Qu_1, \\ z = (E \;\; 0)\bar{x}. \end{cases} \tag{12}$$

In view of Saberi et al. (2000b), it is easily checked that the EID filtering problem for system (11) is solvable if and only if the EID filtering problem for system (12) is solvable after we have established that

$$\mathscr{S}^-(A, B_Q, C, D_Q) = (I \;\; 0)\mathscr{S}^-\left(\begin{pmatrix} A & B_2C_a \\ 0 & S \end{pmatrix}, \begin{pmatrix} B_Q \\ 0 \end{pmatrix}, (C \;\; D_2C_a), D_Q\right).$$

The above can easily be verified. This proves the result (i). Result (ii) pertaining to proper filters follows similarly. In the above expression, $\mathscr{S}^-(A, B, C, D)$ denotes the strongly controllable subspace of a system characterized by the quadruple (A, B, C, D) (see, for instance, Trentelman et al., 2001). □

The above theorem begs the question whether u_2 affects the existence of a generalized H_2 optimal filter or not if the matrix F_2 is not zero. The following example answers this question.

Example 9. Consider a system

$$\Sigma : \dot{x} = u_1 + u_2, \quad y = x + u_1 + u_2, \quad z = x + f_2 u_2, \tag{13}$$

where f_2 is some constant. Also, let the exosystem be

$$\Sigma_a : \dot{x}_a = 0 \quad \text{and} \quad u_2 = x_a. \tag{14}$$

In view of Theorem 5, one can verify easily that the generalized H_2 optimal filtering problem for the above given systems Σ and Σ_a is not solvable when f_2 is non-zero. However, for the above system Σ in the absence of the input signal u_2, that is for the system Σ_0,

$$\Sigma_0 : \dot{x} = u_1, \quad y = x + u_1, \quad z = x, \tag{15}$$

one can easily verify that the EID filtering problem is solvable and hence, in particular, the H_2 optimal filtering problem is solvable. This demonstrates that, in general, the solvability of the generalized H_2 optimal filtering problem does depend on the input signal u_2 or equivalently on the exosystem Σ_a.

Remark 10. Example 9 demonstrates that Theorem 8 does not hold if we drop the condition of F_2 being zero. As a matter of fact, for all systems with (A, B, C_1, D_1) right-invertible, we can prove that the generalized H_2 optimal filtering problem is solvable only if $F_2C_a = 0$.

6. Transient performance measure

In the previous sections, we defined and discussed a method of computing the steady state generalized H_2 optimal filtering performance measure, namely $\gamma^*_{g,\mathrm{sp}}$ or $\gamma^*_{g,\mathrm{p}}$. In this section, we define and then compute the transient performance measure. As discussed in the Introduction, the transient performance measure is the energy of the error signal in the absence of the white noise input u_1. To define it clearly, consider the system Σ given in (4), the filter Σ_f given in (5), and the exosystem Σ_a given in (2). Using the matrix triple (A_e, C_e, E_e) as in (8), we can combine the given system Σ and the exosystem Σ_a together and form the expanded system $\tilde{\Sigma}$ as in (7) except that we set $u_1 = 0$, i.e., we have

$$\dot{\bar{x}} = A_e\bar{x}, \quad y = C_e\bar{x}, \quad z = E_e\bar{x}. \tag{16}$$

Whenever the generalized unbiased requirement is satisfied by the filter Σ_f, the error e_z is an energy signal, and thus we can define the transient performance measure J^g as follows:

$$J^g(\bar{x}_0, \xi_0, \Sigma_\mathrm{f}) = \int_0^\infty e_z(t)' e_z(t)\,\mathrm{d}t.$$

In the above equations, $\bar{x} = (x' \; x_a')'$, $\bar{x}_0 = \bar{x}(0)$, $\xi_0 = \xi(0)$. The initial condition of the filter can be chosen as zero. Let e_i $(i = 1, \ldots, n + n_a)$ be an orthonormal basis of $\mathbb{R}^{n+n_a}$. We note that the initial condition $\bar{x}_0$ of the given system is usually unknown. This suggests that one can generate an average transient performance measure as

$$\tilde{J}^g(\Sigma_\mathrm{f}) = \sum_{i=1}^{n+n_a} J^g(e_i, 0, \Sigma_\mathrm{f}). \tag{17}$$

Note that it can be shown that this criterion does not depend on the specific orthonormal basis used in its definition. In what follows, we will denote the infimum of $\tilde{J}^g(\Sigma_\mathrm{f})$ over all linear stable generalized unbiased strictly proper or proper filters by $\tilde{J}^{*g}_{\mathrm{sp}}$ or by $\tilde{J}^{*g}_{\mathrm{p}}$, respectively.

It is straightforward to show that $\tilde{J}^g(\Sigma_\mathrm{f})$ is related to the H_2 performance measure when using the same filter for an appropriately defined auxiliary system,

$$\Sigma_{au} : \dot{x}_{au} = A_e x_{au} + Iv, \; y_{au} = C_e x_{au}, \; z_{au} = E_e x_{au}, \tag{18}$$

where v is an unknown white noise input. Let the H_2 optimal filtering performance of the system Σ_{au} over all linear stable unbiased strictly proper or proper filters, respectively, be denoted by $\gamma^*_{\mathrm{sp}}(\Sigma_{au})$ or $\gamma^*_{\mathrm{p}}(\Sigma_{au})$. Then, the following result whose proof can be written easily relates $\tilde{J}^{*g}_{\mathrm{sp}}$ and $\tilde{J}^{*g}_{\mathrm{p}}$, respectively to $\gamma^*_{\mathrm{sp}}(\Sigma_{au})$ and $\gamma^*_{\mathrm{p}}(\Sigma_{au})$.

Lemma 11. *Consider the generalized H_2 optimal filtering problem of Section* 3 *for the system Σ of* (4) *along with the associated exosystem Σ_a of* (2). *Let Assumptions* 1 *and* 3 *be satisfied. Let Σ_{au} be given by* (18). *Then, we have*

$$\tilde{J}^{*g}_{\mathrm{sp}} = [\gamma^*_{\mathrm{sp}}(\Sigma_{au})]^2 \quad \text{and} \quad \tilde{J}^{*g}_{\mathrm{p}} = [\gamma^*_{\mathrm{p}}(\Sigma_{au})]^2.$$

Remark 12. We note from Saberi et al. (2000a) that only the unstable zero dynamics and the non-left invertible dynamics of the subsystem characterized by $(A_e, I, C_e, 0)$ contribute to the value of $\gamma^*_{\mathrm{sp}}(\Sigma_{au})$. However, it is easy to see that the said subsystem does not have any zero dynamics, and moreover it is left invertible only if rank $C_e = (n + n_a)$. Since rank $C_e \neq (n + n_a)$, the said subsystem is always non-left invertible. This implies that $\gamma^*_{\mathrm{sp}}(\Sigma_{au})$ is always non-zero. In other words, the average transient performance measure $\tilde{J}^{*g}_{\mathrm{sp}}$ is always non-zero. By a similar reasoning, it follows that $\tilde{J}^{*g}_{\mathrm{p}}$ is non-zero as well.

7. Limitations of generalized H_2 optimal filtering performance and transient performance due to the locations of the invariant zeros

In this section, we study the limitations on $\gamma^*_{g,\mathrm{sp}}$, $\gamma^*_{g,\mathrm{p}}$, $\tilde{J}^{*g}_{\mathrm{sp}}$, and $\tilde{J}^{*g}_{\mathrm{p}}$ due to the structural properties of the given system. Let us first focus on $\gamma^*_{g,\mathrm{sp}}$ or $\gamma^*_{g,\mathrm{p}}$. In this regard, we already know that $\gamma^*_{g,\mathrm{sp}}=\gamma^*_{\mathrm{sp}}$ and $\gamma^*_{g,\mathrm{p}}=\gamma^*_{\mathrm{p}}$. Hence, known results from, for instance, Saberi et al. (1995) tell us that when the invariant zeros of the given system Σ move closer to the imaginary axis, the achievable performance measure $\gamma^*_{g,\mathrm{sp}}$ or $\gamma^*_{g,\mathrm{p}}$ improves. If the system Σ is left-invertible then such a performance measure will even converge to zero when all the invariant zeros move towards the imaginary axis.

We consider next $\tilde{J}^{*g}_{\mathrm{sp}}$ and $\tilde{J}^{*g}_{\mathrm{p}}$. In this regard, a relevant question is under what circumstances $\tilde{J}^{*g}_{\mathrm{sp}}$ and $\tilde{J}^{*g}_{\mathrm{p}}$ are unbounded. Apparently, under such circumstances estimation is impossible. We focus here on developing a relationship between $\tilde{J}^{*g}_{\mathrm{sp}}$ or $\tilde{J}^{*g}_{\mathrm{p}}$ and the locations/direction of the invariant zeros of the subsystem characterized by $(A_e, I, C_e, 0)$. It turns out that the non-minimum phase dynamics and the exosystem dynamics play significant roles in dictating the behavior of $\tilde{J}^{*g}_{\mathrm{sp}}$ or $\tilde{J}^{*g}_{\mathrm{p}}$. Basically, we find out that $\tilde{J}^{*g}_{\mathrm{sp}}$ or $\tilde{J}^{*g}_{\mathrm{p}}$ is inversely related to the distance between the invariant zeros and the modes of the exosystem, and indeed it could go to infinity when the minimal distance of poles of the exosystem and the invariant zeros of the system goes to zero. There are two possible exceptions to this behavior. Firstly, when the effect of the invariant zeros of the system are asymptotically invisible from the output z (i.e., the non-minimum phase dynamics is asymptotically unobservable from the desired output to be estimated). Secondly, if u_2 is a vector then the input direction of an invariant zero and the direction of a pole of the exosystem need to be mis-aligned (to be made precise soon) in order to have the cost bounded when the pole and the invariant zero get close to each other. Due to lack of space, we illustrate our findings by considering two special but important cases and an example.

We proceed now to illustrate the above discussed results. Let λ be any unstable invariant zero of (A, B_2, C, D_2). Hence, there exists vectors $\tilde{p}$ and $\tilde{q}$ such that

$$\operatorname{rank}\begin{pmatrix}\lambda I - A & -B_2\\ C & D_2\end{pmatrix}\begin{pmatrix}\tilde{p}\\ \tilde{q}\end{pmatrix}=0.$$

Detectability of (C, A) guarantees that $\tilde{q}\neq 0$. As seen from (2) the exosystem is characterized by the matrices S and C_a. If for an eigenvalue μ of S, we can choose an eigenvector s with $Ss=\mu s$ such that $C_a s=\tilde{q}$, then we call the pole of the exosystem μ and the invariant zero λ of the system *aligned*, otherwise they are mis-aligned. Note that by scaling $\tilde{p}$ and $\tilde{q}$ we can guarantee, without loss of generality, that $\|s\|=1$. We will show that if an invariant zero of the system moves towards an aligned pole of the exosystem then the average transient performance measure will go to infinity. For two special cases,

Case 1: $S=0$ and $C_a=I$ (input u_2 is a vector DC signal),

Case 2: $m_2=1$ (input u_2 is a scalar signal),

all the poles of the exosystem are aligned to all the invariant zeros of the system characterized by (A, B_2, C, D_2). But in general this might clearly be not the case. Note that, in the above two cases, it can be shown that Assumption 3 implies that (A, B_2, C, D_2) is left-invertible.

As we discussed earlier, the minimal achievable H_2 norm of Σ_{au} is indeed the minimal average transient performance measure. To simplify our study of the H_2 norm of Σ_{au}, we restrict v by setting $v=(0\ \tilde{q}')'\omega$, and we add an additional measurement $y_1=(\tilde{m}\ 0)x_{aux}+\tilde{n}\omega$, where $\tilde{m}$ is such that $\tilde{m}\tilde{p}=1$ and $\tilde{n}=(\mu-\lambda)^{-1}$. Obviously, both these actions reduce the achievable H_2 norm and hence we are investigating a lower bound for the achievable H_2 norm of Σ_{au}. The above restrictions imply that we will study the design of an observer for the system

$$\bar{\Sigma}_{au}:\begin{cases}\dot{x}_{au}=\begin{pmatrix}A & BC_a\\ 0 & S\end{pmatrix}x_{au}+\begin{pmatrix}0\\ \tilde{q}\end{pmatrix}\omega,\\ \bar{y}_{au}=\begin{pmatrix}\tilde{m} & 0\\ C & D_2C_a\end{pmatrix}x_{au}+\begin{pmatrix}\tilde{n}\\ 0\end{pmatrix}\omega,\\ z_{au}=(E\ \ F_2C_a)x_{au}.\end{cases}$$

Note that an output injection does not change the achievable H_2 norm for the error dynamics and hence we can equally well study the system

$$\begin{aligned}\dot{\tilde{x}}_{au}&=\begin{pmatrix}A+(\mu-\lambda)\tilde{p}\tilde{m} & BC_a\\ 0 & S\end{pmatrix}\tilde{x}_{au}+\begin{pmatrix}\tilde{p}\\ \tilde{q}\end{pmatrix}\omega,\\ \tilde{y}_{au}&=\begin{pmatrix}\tilde{m} & 0\\ C & D_2C_a\end{pmatrix}\tilde{x}_{au}+\begin{pmatrix}\tilde{n}\\ 0\end{pmatrix}\omega,\\ \tilde{z}_{au}&=(E\ \ F_2C_a)\tilde{x}_{au}.\end{aligned}$$

It is easy to see that the state of this system (given zero initial conditions) will satisfy $\tilde{x}_{au}(t)=(\tilde{p}'\tilde{q}'))'r(t)$ for some scalar valued function r. Next, we derive a differential equation for r and express the whole system in terms of the function r,

$$\begin{aligned}\dot{r}&=\mu r+\omega,\\ \tilde{y}_{au}&=(1\ 0)'r+((\mu-\lambda)^{-1}\ 0)'\omega,\\ \tilde{z}_{au}&=(E\tilde{p}\quad F_2C_a\tilde{q})r.\end{aligned}$$

However, for this scalar system, the achievable performance measure can very easily be computed. For both strictly proper and proper filters we obtain as the optimal performance measure,

$$2\|(E\tilde{p}\ \ F_2C_a\tilde{q})\|^2\frac{\operatorname{Re}\lambda}{\|\mu-\lambda\|^2}. \tag{19}$$

The expression given in (19) is a lower bound for the average transient performance measure $\tilde{J}^{*g}_{\mathrm{sp}}$ as well as $\tilde{J}^{*g}_{\mathrm{p}}$. We clearly see that $\tilde{J}^{*g}_{\mathrm{sp}}$ as well as $\tilde{J}^{*g}_{\mathrm{p}}$ is inversely related to the distance between the poles of an exosystem and the non-minimum phase invariant zeros. That is, when λ gets close to an aligned eigenvalue of S then the achievable performance measure goes to infinity. However, there is one exception to this unbounded behavior. That is, when $E\tilde{p}$ and $F_2C_a\tilde{q}$ converge asymptotically to zero, $\tilde{J}^{*g}_{\mathrm{sp}}$ as well as $\tilde{J}^{*g}_{\mathrm{p}}$ can be bounded. Under this circumstance, the effect of the invariant zero is asymptotically invisible in the to-be-estimated output. Note that, in the special cases we considered above, poles of exosystem and non-minimum phase invariant zeros are always aligned.

We consider next an example in which the poles of the exosystem and the non-minimum phase invariant zeros are misaligned. As seen in this example, $\tilde{J}_{\rm sp}^{*g}$ as well as $\tilde{J}_{\rm p}^{*g}$ need not be unbounded as the distance between the poles of an exosystem and the non-minimum phase invariant zeros goes to zero. Thus, the alignment of poles of the exosystem and the non-minimum phase invariant zeros as mentioned in the beginning of this section plays a crucial role.

Example 13. Consider system (4) where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1-\varepsilon & 0 \\ 0 & 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

while the exosystem is given by

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad C_a = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

The system has an invariant zero in ε which, when $\varepsilon \to 0$, converges to a pole 0 of the exosystem without exhibiting the alignment property. In contrast to the aligned case, we see here that the average transient performance measure $\tilde{J}_{\rm sp}^{*g}$ as well as $\tilde{J}_{\rm p}^{*g}$ does not go to ∞. This can be seen by realizing that removing the first part of the measurement removes the invariant zero, while relying simply only on the second measurement the system is still detectable. Therefore, an observer design based on the second measurement only, will not have a transient performance measure that converges to ∞. Although due to space limitations we are not showing here all the details, the reader can easily work them out.

Remark 14. The observations we made above regarding the impact of non-minimum-phase dynamics on the transient performance measure of generalized H_2 optimal filtering might give the impression of being a dual of a property of the output regulation problem where we also try to reject sinusoidal signals generated by an exosystem but in a control context. It is indeed not so. There exist two crucial differences. First of all, the exosystem represents uncontrollable dynamics in the output regulation problem and hence we cannot stabilize the dynamics in the standard sense. Here in the context of estimation, this problem does not arise at all since the exosystem in most cases is detectable. Secondly and most importantly, in the case of output regulation the rejection of sinusoidal signals is only related to the location of the invariant zeros (see Qiu and Davison, 1993) while, in the case of estimation, in addition to the locations of invariant zeros, their directionality (input zero directions) as well as the directionality (the eigenvectors) of the modes of the exosystem play significant roles. Thus, the issues here in the context of estimation are vastly more complex than those in the output regulation problem.

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