Analysis of Input to State Stability for Discrete Time Nonlinear Systems via Dynamic Programming ⋆

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Abstract

The input-to-state stability (ISS) property for systems with disturbances has received considerable attention over the past decade or so, with many applications and characterizations reported in the literature. The main purpose of this paper is to present analysis results for ISS that utilize dynamic programming techniques to characterize minimal ISS gains and transient bounds. These characterizations naturally lead to computable necessary and sufficient conditions for ISS. Our results make a connection between ISS and optimization problems in nonlinear dissipative systems theory (including L_2-gain analysis and nonlinear H_∞ theory). As such, the results presented address an obvious gap in the literature.

Key words: Nonlinear systems; Stability analysis; Disturbances; Dynamic Programming; Input-to-state stability.

1 Introduction

Among the many stability properties for systems with disturbances that have been proposed in the literature, the input-to-state stability (ISS) property proposed by Sontag (1989) deserves special attention. Indeed, ISS is fully compatible with Lyapunov stability theory (Sontag & Wang, 1995) while its other equivalent characterizations relate it to robust stability, dissipativity and input-output stability theory (Sontag & Wang, 1996, Sontag, 2000). The ISS property has found its main application in the ISS small gain theorem that was first proved by Jiang, Teel and Praly (1994). Several different versions of the ISS small gain theorem that use different (equivalent) characterizations of the ISS property and their various applications to nonlinear controller design can be found in Jiang, Mareels, and Wang (1996), Jiang and Mareels (1997), Teel (1996) and references defined therein.

The ISS property and the ISS small gain theorems naturally lead to the concept of nonlinear disturbance gain functions or simply "nonlinear gains". In this context, obtaining sharp estimates for the nonlinear gains is an important issue. Indeed, the better the nonlinear gain estimate that we can obtain, the larger the class of systems to which the ISS small gain results can be applied. Currently, the main tool for estimating the nonlinear gains are the so called ISS Lyapunov functions that typically produce rather conservative estimates (over bounds) for the ISS nonlinear gains.

It is the main purpose of this paper to present several results that provide a constructive framework based on dynamic programming for obtaining *minimum ISS nonlinear gains*. These results are related to optimization based methods in nonlinear dissipative systems theory, such as L_2-gain analysis and nonlinear H_∞ theory (see Helton & James, 1999 and references defined therein), as well as recently developed optimization based L_∞ methods (see Fialho & Georgiou, 1999, Huang & James, 2003 and references defined therein). Needless to say, the optimization approach that we take in this paper can inflict a heavy (and sometimes infeasible) computational burden on the user. This is a reflection of the intrinsic complexity of the problem that we are trying to solve. We present results only for discrete-time nonlinear sys-

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tems since many calculations and technical details are in this way simplified.

The paper is organized as follows. In Section 2 we present several equivalent definitions of the ISS property and state a result from the literature that motivates our definitions and results. A fundamental dynamic programming equation that we need to state our main results is given in Section 3. Sections 4, 5 and 6 contain results on minimum nonlinear gains for different equivalent definitions of the ISS property. Two related ISS properties are analysed in Section 7 using the techniques of Sections 5 and 6. Several illustrative examples are presented in Section 8 and the paper is closed with conclusions in Section 9.

2 Preliminaries

Sets of real numbers, integers and nonnegative integers are denoted respectively as $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Z}_+$. A function $\gamma : [0,\infty) \to [0,\infty)$ is of class $\bar{\mathcal{K}}$ if it is nondecreasing, satisfies $\gamma(0) = 0$ and is right continuous at 0. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\bar{\mathcal{K}}$ and for each fixed $s \geq 0$, $\lim_{t\to+\infty} \beta(s,t) = 0$. Denote $l_\infty = \{u : \mathbf{Z}_+ \to \mathbf{R}^m : \|u\|_\infty = \sup_{k\in\mathbf{Z}_+} |u_k| < \infty\}$ where $|\cdot|$ is the Euclidean norm.

Consider the following dynamical system

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \mathbf{R}^n, u_k \in \mathbf{R}^m$, and $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is continuous and satisfies $f(0,0) = 0$. For any $x_0 \in \mathbf{R}^n$ and any input $u : \mathbf{Z}_+ \to \mathbf{R}^m$, we denote by $x(\cdot, x_0, u)$ the solution of (1) with initial state x_0 and input u.

The following definitions are taken from ISS related literature. It was shown in Jiang and Wang (2001) that these definitions of ISS are qualitatively equivalent. However, the gains in different definitions are not the same and since we are interested in minimum disturbance gains for different characterizations, we find it useful to introduce different notation for each of the different characterizations. In all the definitions below we assume that $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

Definition 1 *(Input-to-state stability with + formulation) System (1) is ISS$_+$ (with (β,γ)) if*

$$|x(k, x_0, u)| \leq \beta(|x_0|, k) + \gamma(\|u\|_\infty), \tag{2}$$

for all $x_0 \in \mathbf{R}^n$, all $u \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Definition 2 *(Asymptotic gain property) System (1) is AG (with gain γ) if*

$$\limsup_{k\to+\infty} |x(k, x_0, u)| \leq \gamma(\|u\|_\infty), \tag{3}$$

for all $x_0 \in \mathbf{R}^n$ and all $u \in l_\infty$.

Remark 3 *Using arguments as in Lemma II.1 of Sontag and Wang (1996), we can show that the above definition is equivalent to the following: for all $x_0 \in \mathbf{R}^n$ and all $u \in l_\infty$,*

$$\limsup_{k\to+\infty} |x(k, x_0, u)| \leq \gamma(\limsup_{k\to+\infty} |u_k|), \tag{4}$$

which is the definition of asymptotic gain property in Jiang and Wang (2001).

Definition 4 *(Zero global asymptotic stability property) System (1) is 0-GAS (with β) if the state trajectories with $u \equiv 0$ satisfy*

$$|x(k, x_0, 0)| \leq \beta(|x_0|, k). \tag{5}$$

for all $x_0 \in \mathbf{R}^n$ and all $k \in \mathbf{Z}_+$.

Definition 5 *(Input-to-state stability with asymptotic gain formulation) System (1) is ISS$_{AG}$ (with (β,γ)) if it is AG (with gain γ) and 0-GAS (with β).*

Remark 6 *The above definition is motivated by the result proved in Sontag and Wang (1996) which shows for continuous-time systems that $ISS_+ \Leftrightarrow AG + 0\text{-}GAS$. A similar result for discrete-time systems was proved in Gao and Lin (2000), Jiang and Wang (2001). This result is restated below in Theorem 9 for convenience.*

Definition 7 *(Input-to-state stability with* max *formulation) System (1) is ISS$_{\max}$ (with (β,γ)) if*

$$|x(k, x_0, u)| \leq \max\{\beta(|x_0|, k), \gamma(\|u\|_\infty)\} \tag{6}$$

for all $x_0 \in \mathbf{R}^n$, all $u \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Remark 8 *It is more common in the literature to use the classes of functions $\mathcal{K}$ and $\mathcal{KL}$ when defining ISS and related properties. A function $\gamma : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$ $\beta(s, \cdot)$ decreases to zero.*

It is not hard to see that the stability definitions that we use are qualitatively equivalent to the stability definitions when the classes of functions $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$ are replaced respectively by $\mathcal{K}$ and $\mathcal{KL}$. This follows from the following three facts: (i) $\mathcal{K} \subset \bar{\mathcal{K}}$ and $\mathcal{KL} \subset \bar{\mathcal{K}}\bar{\mathcal{L}}$; (ii) given any $\gamma \in \bar{\mathcal{K}}$, there exists $\gamma_1 \in \mathcal{K}$ such that $\gamma(s) \leq \gamma_1(s), \forall s \geq 0$; (iii) given any $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, there exists $\beta_1 \in \mathcal{KL}$ such that $\beta(s,k) \leq \beta_1(s,k), \forall s \geq 0, \forall k \in \mathbf{Z}_+$. Consequently, most results that were proved in the literature for classes of functions $\mathcal{K}$ and $\mathcal{KL}$ are still true when stated with function classes $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$.

Finally, we note that our relaxed function class definitions are necessitated by the fact that the minimal ISS gain for some systems can be of class $\bar{\mathcal{K}} \setminus \mathcal{K}$, as is demonstrated in Section 8.1, Example 1.

The following theorem has been proved in the context of function classes $\mathcal{K}$ and $\mathcal{KL}$ for continuous-time systems in Sontag and Wang (1996) and for discrete-time systems in Gao and Lin (2000), Jiang and Wang (2001). However, this result remains valid for function classes $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$.

Theorem 9 *The following statements are equivalent:*

(1) There exist $\beta_{AG} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_{AG} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{AG}$ with $(\beta_{AG}, \gamma_{AG})$;
(2) There exist $\beta_{+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_{+} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{+}$ with (β_{+}, γ_{+});
(3) There exist $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_{\max} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$.

In the sequel we use the non-standard notation from Theorem 9 since it is important to distinguish between different characterizations and the related functions. Indeed, the functions $\beta_{AG}, \beta_{+}, \beta_{\max}$ (respectively functions $\gamma_{AG}, \gamma_{+}, \gamma_{\max}$) in the above theorem are all different in general. Note that although notation β_{AG} characterizing 0-GAS seems counterintuitive, it is consistent with the definition of ISS$_{AG}$ in Definition 5.

Remark 10 *We note that each of the properties ISS$_{AG}$, ISS$_{+}$ and ISS$_{\max}$ has been used in the literature. In particular, there exist small gain theorems that use each of these different characterizations (see, for instance, Jiang & Wang, 2001, Jiang, Teel & Praly, 1994, Jiang, Mareels, & Wang, 1996, Jiang & Mareels, 1997, Teel, 1996). Computing the smallest possible functions β, γ (or their estimates) in each of these properties is an important problem for the following reasons: (i) the smaller the estimates of gains functions, the larger the class of systems to which the small gain theorem can be applied; (ii) better estimates of the functions β, γ for subsystems produce (via the small gain theorems) sharper bounds on solutions of the composite system; (iii) the smallest functions will be different in general for each of the properties ISS$_{AG}$, ISS$_{+}$ and ISS$_{\max}$ (this further motivates our notation). In the sequel, we provide a framework for the computation of minimum functions $\beta_{AG}, \beta_{+}, \beta_{\max}$ and $\gamma_{AG}, \gamma_{+}, \gamma_{\max}$ via dynamic programming.*

3 Dynamic Programming

In this section we define a value function that is used in the derivation of our subsequent results, and present a dynamic programming equation to compute it. The dynamic programming equation can be used in developing numerical algorithms for testing each of the characterizations of the ISS property that were defined in the previous section. In particular, we can obtain minimum disturbance gains and/or the minimal bounds on the transients by using this technique.

For $x \in \mathbf{R}^n$, $\delta \geq 0$, integer $k \in \mathbf{Z}_{+}$, denote

$$V^{\delta}(x,k) := \sup_{\|u\|_{\infty} \leq \delta} \{|x(k,x_0,u)| : x_0 = x\} \,. \tag{7}$$

The value function $V^{\delta}(x,k)$ satisfies the Dynamic Programming Equation (DPE)

$$V^{\delta}(x,k) = \sup_{|u| \leq \delta} V^{\delta}(f(x,u), k-1) \tag{8}$$

with the initial condition

$$V^{\delta}(x,0) = |x| \,. \tag{9}$$

In subsequent sections, we show how $V^{\delta}(x,k)$ can be used to compute the functions β, γ needed in different characterizations of ISS.

4 Necessary and sufficient conditions for ISS$_{AG}$

The main results of this section are necessary and sufficient conditions for ISS$_{AG}$. The results do not require a Lyapunov function but rather use the value function $V^{\delta}(x,k)$ to generate γ_{AG} and β_{AG} directly. More importantly, we show that the computed functions are minimal. This type of result is impossible to obtain via Lyapunov techniques since they involve a certain conservatism in estimating γ_{AG} and β_{AG}.

Using $V^{\delta}(x,k)$ we introduce

$$V^{\delta}_a(x) := \limsup_{k \to +\infty} V^{\delta}(x,k) \tag{10}$$

and define

$$\gamma_{\infty}(\delta) := \sup_{x \in \mathbf{R}^n} V^{\delta}_a(x), \quad \beta_a(s,k) := \sup_{|x| \leq s} V^{0}(x,k). \tag{11}$$

Using the above definitions, we can state the main result of this section:

Theorem 11 *If the system (1) is ISS$_{AG}$ with $(\beta_{AG}, \gamma_{AG})$ then $\gamma_{\infty} \in \bar{\mathcal{K}}$, $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\begin{aligned} \gamma_{\infty}(s) &\leq \gamma_{AG}(s), && \forall s \geq 0 \\ \beta_a(s,k) &\leq \beta_{AG}(s,k), && \forall s \geq 0, \forall k \in \mathbf{Z}_{+} \,. \end{aligned}$$

If, on the other hand, $\gamma_{\infty} \in \bar{\mathcal{K}}$ and $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system (1) is ISS$_{AG}$ with $(\beta_a, \gamma_{\infty})$.

Proof. Suppose the system (1) is ISS$_{AG}$ with $(\beta_{AG}, \gamma_{AG})$. Then, the system is AG with $\gamma_{AG} \in \bar{\mathcal{K}}$.

Since f is continous, by Lemma 10 in Gao and Lin (2000), $\forall\delta \geq 0, \forall x_0 \in \mathbf{R}^n$, we can prove the following property: $\forall\varepsilon > 0, \exists K$ (depend only on x_0 and ε), such that

$$|x(k, x_0, u)| \leq \gamma_{AG}(\delta) + \varepsilon, \quad \forall\, k \geq K, \forall \|u\|_\infty \leq \delta,$$

which implies $V^\delta(x_0, k) \leq \gamma_{AG}(\delta) + \varepsilon, \;\; \forall k \geq K$. Since ε is arbitrary, we have $V_a^\delta(x_0) \leq \gamma_{AG}(\delta)$. Hence by (11),

$$0 \leq \gamma_\infty(\delta) \leq \gamma_{AG}(\delta) < +\infty, \;\forall\delta \geq 0.$$

Since $\gamma_{AG}(0) = 0$ and γ_{AG} is right continuous at 0, we have $\gamma_\infty(0) = 0$ and γ_∞ is right continuous at 0. Thus $\gamma_\infty \in \bar{\mathcal{K}}$.

Since the system is ISS_{AG} with $(\beta_{AG}, \gamma_{AG})$, it is 0-GAS with $\beta_{AG} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$. Hence, when $u \equiv 0$, the trajectories satisfy

$$|x(k, x_0, 0)| \leq \beta_{AG}(|x_0|, k), \;\forall x_0 \in \mathbf{R}^n, \forall k \in \mathbf{Z}_+.$$

Consequently, $\forall s \geq 0, \forall k \in \mathbf{Z}_+$, for any initial state x_0 such that $|x_0| \leq s$, we have

$$|x(k, x_0, 0)| \leq \beta_{AG}(|x_0|, k) \leq \beta_{AG}(s, k).$$

By (7) and (11),

$$\beta_a(s, k) \leq \beta_{AG}(s, k) < +\infty, \quad \forall s \geq 0, \forall k \in \mathbf{Z}_+.$$

For fixed $k \in \mathbf{Z}_+$, since $0 \leq \beta_a(s, k) \leq \beta_{AG}(s, k)$ and $\beta_{AG}(s, k)$ is right continuous at 0 with $\beta_{AG}(0, k) = 0$, $\beta_a(s, k)$ must be right continuous at 0 with $\beta_a(0, k) = 0$. So $\beta_a(\cdot, k) \in \bar{\mathcal{K}}$. Moreover, for fixed $s \geq 0$, since $0 \leq \beta_a(s, k) \leq \beta_{AG}(s, k)$ and $\beta_{AG}(s, k)$ tends to zero as $k \to \infty$, $\beta_a(s, k)$ also tends to zero as $k \to \infty$. Thus, we have proved that $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

The sufficiency part of the proof follows directly from the definitions of ISS_{AG}, AG, 0-GAS, the gain γ_∞ and the function β_a. $\square$

Remark 12 *It is clear from the above proof that system (1) is AG if and only if $\gamma_\infty \in \bar{\mathcal{K}}$. Moreover, system (1) is 0-GAS if and only if $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.*

5 Necessary and sufficient conditions for ISS_+

In this section we show how the value function $V^\delta(x, k)$ can be used in analysing the ISS_+ property. Results of this section are slightly weaker than the results of the previous section since they do not produce minimal β_+ and γ_+ simultaneously. Instead, we show that given a fixed γ_+ it is possible to compute a minimal β_+ corresponding to the given γ_+ and vice versa. Consequently, results of this section are divided into two subsections addressing respectively the case when β_+ is fixed and the case when γ_+ is fixed.

We note that the gain γ_∞ (defined in (11)) which was used in characterizing the ISS_{AG} property is not appropriate for results in this section. For this reason, we introduce a new function γ_a. Define

$$\gamma_a(\delta) := \max\{\gamma_\infty(\delta), \sup_{k\geq 0} V^\delta(0, k)\}. \tag{12}$$

We first show that β_a (defined in (11)) and γ_a are respectively lower bounds for β_+ and γ_+.

Lemma 13 *If the system (1) is ISS_+ with (β_+, γ_+), then $\gamma_a \in \bar{\mathcal{K}}, \beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\gamma_a(\delta) \leq \gamma_+(\delta), \qquad \forall\delta \geq 0$$
$$\beta_a(s, k) \leq \beta_+(s, k), \qquad \forall s \geq 0, \forall k \in \mathbf{Z}_+.$$

Proof. Since system (1) is ISS_+ with (β_+, γ_+), it is AG with γ_+ and 0-GAS with β_+. From Theorem 11, we only need to prove that $\sup_{k\geq 0} V^\delta(0, k) \leq \gamma_+(\delta), \forall\delta \geq 0$. Choosing $x_0 = 0$, the ISS_+ property implies that $\forall k \in \mathbf{Z}_+$,

$$\sup_{\|u\|_\infty \leq \delta} |x(k, 0, u)| \leq \gamma_+(\delta).$$

Hence $V^\delta(0, k) \leq \gamma_+(\delta), \forall k \in \mathbf{Z}_+$ and hence $\sup_{k\geq 0} V^\delta(0, k) \leq \gamma_+(\delta)$. $\square$

5.1 Minimal β_+ for fixed γ_+

For a fixed $\gamma_+ \in \bar{\mathcal{K}}$, we define

$$\beta^{\gamma_+}(\delta, s, k) := \max\left\{\sup_{|x|\leq s} V^\delta(x, k) - \gamma_+(\delta), 0\right\} \tag{13}$$

and

$$\beta_a^{\gamma_+}(s, k) := \sup_{\delta\geq 0} \beta^{\gamma_+}(\delta, s, k). \tag{14}$$

The main result of the subsection is presented below.

Theorem 14 *For fixed $\gamma_+ \in \bar{\mathcal{K}}$, if there exists $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that system (1) is ISS_+ with (β_+, γ_+), then $\beta_a^{\gamma_+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\beta_a^{\gamma_+}(s, k) \leq \beta_+(s, k), \qquad \forall s \geq 0, k \in \mathbf{Z}_+\ . \tag{15}$$

Conversely, if $\beta_a^{\gamma_+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system (1) is ISS_+ with $(\beta_a^{\gamma_+}, \gamma_+)$.

PROOF. Let $\gamma_+ \in \bar{\mathcal{K}}$ be fixed, if there exists $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that system (1) is ISS$_+$ with (β_+, γ_+), then $\forall \delta \geq 0$,

$$|x(k, x_0, u)| \leq \beta_+(|x_0|, k) + \gamma_+(\delta),$$

$\forall x_0 \in \mathbf{R}^n, \forall \|u\|_\infty \leq \delta, \forall k \in \mathbf{Z}_+$. Hence

$$V^\delta(x,k) - \gamma_+(\delta) \leq \beta_+(|x|, k),\ \forall x \in \mathbf{R}^n, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Since $\beta_+(s,k)$ is nondecreasing in s (for fixed k), we have

$$\sup_{|x|\leq s} V^\delta(x,k) - \gamma_+(\delta) \leq \beta_+(s,k),\ \forall s \geq 0, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Noting that $\beta_+(s,k)$ is nonnegative, by (13) we have

$$0 \leq \beta^{\gamma_+}(\delta, s, k) \leq \beta_+(s,k),\ \forall s \geq 0, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Since δ is arbitrary, we have

$$0 \leq \beta_a^{\gamma_+}(s,k) \leq \beta_+(s,k),\ \forall s \geq 0, \forall k \in \mathbf{Z}_+.$$

It is easy to see that $\beta_a^{\gamma_+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, as $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

The sufficiency part of the proof follows from the definitions of $\beta_a^{\gamma_+}$ and ISS$_+$. $\square$

5.2 Minimal γ_+ for fixed β_+

For a fixed β_+, we define

$$\gamma_a^{\beta_+}(\delta) := \sup_{x\in\mathbf{R}^n} \sup_{k\in\mathbf{Z}_+} \max\left\{V^\delta(x,k) - \beta_+(|x|,k), 0\right\}. \quad (16)$$

The main result of the subsection is presented below. The proof is similar to that of Theorem 14 and is omitted.

Theorem 15 *For fixed $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, if there exists $\gamma_+ \in \bar{\mathcal{K}}$ such that system (1) is ISS$_+$ with (β_+, γ_+), then $\gamma_a^{\beta_+} \in \bar{\mathcal{K}}$ and*

$$\gamma_a^{\beta_+}(\delta) \leq \gamma_+(\delta), \qquad \forall \delta \geq 0\ . \quad (17)$$

Conversely, if $\gamma_a^{\beta_+} \in \bar{\mathcal{K}}$, then system (1) is ISS$_+$ with $(\beta_+, \gamma_a^{\beta_+})$.

6 Necessary and sufficient conditions for ISS$_{\max}$

In this section we present necessary and sufficient conditions for ISS$_{\max}$ and moreover, we obtain in a similar manner as in the previous section, a minimum gain $\gamma_{\max}$ for a fixed transient bound $\beta_{\max}$ and vice versa. The constructions of the minimal functions are different from the constructions in the previous section although the ideas are the same. The following lemma follows directly from the definitions of β_a, γ_a and the property ISS$_{\max}$.

Lemma 16 *If the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$, then $\gamma_a \in \bar{\mathcal{K}}, \beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\begin{aligned} \gamma_a(\delta) &\leq \gamma_{\max}(\delta), && \forall \delta \geq 0 \\ \beta_a(s,k) &\leq \beta_{\max}(s,k), && \forall s \geq 0, \forall k \in \mathbf{Z}_+\ . \end{aligned}$$

6.1 Minimal $\beta_{\max}$ for fixed $\gamma_{\max}$

For $\gamma_{\max} \in \bar{\mathcal{K}}$, we define

$$\tilde{\beta}^{\gamma_{\max}}(\delta, s, k) := \begin{cases} \sup\limits_{|x|\leq s} V^\delta(x,k) & \text{if } \sup\limits_{|x|\leq s} V^\delta(x,k) > \gamma_{\max}(\delta), \\ 0 & \text{if } \sup\limits_{|x|\leq s} V^\delta(x,k) \leq \gamma_{\max}(\delta). \end{cases} \quad (18)$$

and

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) := \sup_{\delta\geq 0} \tilde{\beta}^{\gamma_{\max}}(\delta, s, k) \quad (19)$$

The main result of this subsection is presented next.

Theorem 17 *For a fixed $\gamma_{\max} \in \bar{\mathcal{K}}$, if there exists $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$, then $\tilde{\beta}_a^{\gamma_{\max}} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) \leq \beta_{\max}(s,k), \qquad \forall s \geq 0, k \in \mathbf{Z}_+\ . \quad (20)$$

Conversely, if $\tilde{\beta}_a^{\gamma_{\max}} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system is ISS$_{\max}$ with $(\tilde{\beta}_a^{\gamma_{\max}}, \gamma_{\max})$.

PROOF. Let $\gamma_{\max} \in \bar{\mathcal{K}}$ be fixed. If there exists $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$, then $\forall \delta \geq 0$,

$$|x(k, x_0, u)| \leq \max\{\beta_{\max}(|x_0|, k), \gamma_{\max}(\delta)\},$$

$\forall x_0 \in \mathbf{R}^n, \forall \|u\|_\infty \leq \delta, \forall k \in \mathbf{Z}_+$. Hence

$$V^\delta(x,k) \leq \max\{\beta_{\max}(|x|, k), \gamma_{\max}(\delta)\},$$

$\forall x \in \mathbf{R}^n, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0$. Since $\beta_{\max}(s,k)$ is nondecreasing in s (for fixed k), we have

$$\sup_{|x|\leq s} V^\delta(x,k) \leq \max\{\beta_{\max}(s,k), \gamma_{\max}(\delta)\}, \quad (21)$$

$\forall s \geq 0, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0$.

By (18), if $\sup_{|x|\leq s} V^\delta(x,k) > \gamma_{\max}(\delta)$, then

$$\begin{aligned} \tilde{\beta}^{\gamma_{\max}}(\delta, s, k) &= \sup_{|x|\leq s} V^\delta(x,k) \\ &\leq \max\{\beta_{\max}(s,k), \gamma_{\max}(\delta)\} = \beta_{\max}(s,k). \end{aligned}$$

If $\sup_{|x|\le s} V^\delta(x,k) \le \gamma_{\max}(\delta)$, then

$$\tilde{\beta}^{\gamma_{\max}}(\delta, s, k) = 0 \le \beta_{\max}(s,k).$$

So, in either case we have

$$0 \le \tilde{\beta}^{\gamma_{\max}}(\delta, s, k) \le \beta_{\max}(s,k).$$

Since δ is arbitrary,

$$0 \le \tilde{\beta}_a^{\gamma_{\max}}(s,k) \le \beta_{\max}(s,k), \qquad \forall s \ge 0, \forall k \in \mathbf{Z}_+.$$

It is easy to see that $\tilde{\beta}_a^{\gamma_{\max}} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, since $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

The sufficiency part of the proof follows from the definitions of $\tilde{\beta}_a^{\gamma_{\max}}$ and $\text{ISS}_{\max}$. $\square$

6.2 Minimal $\gamma_{\max}$ for fixed $\beta_{\max}$

For a fixed $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, we define

$$\tilde{\gamma}^{\beta_{\max}}(\delta, s, k) := \begin{cases} \sup_{|x|\le s} V^\delta(x,k) & \text{if } \sup_{|x|\le s} V^\delta(x,k) > \beta_{\max}(s,k), \\ 0 & \text{if } \sup_{|x|\le s} V^\delta(x,k) \le \beta_{\max}(s,k). \end{cases} \tag{22}$$

and

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) := \sup_{s\ge 0} \sup_{k\in\mathbf{Z}_+} \tilde{\gamma}^{\beta_{\max}}(\delta, s, k). \tag{23}$$

The main result of this subsection is presented next. The proof is similar to that of Theorem 17 and is omitted.

Theorem 18 *For a fixed $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, if there exists $\gamma_{\max} \in \bar{\mathcal{K}}$ such that the system (1) is $ISS_{\max}$ with $(\beta_{\max}, \gamma_{\max})$ for some $\gamma_{\max} \in \bar{\mathcal{K}}$, then $\tilde{\gamma}_a^{\beta_{\max}} \in \bar{\mathcal{K}}$ and*

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \le \gamma_{\max}(\delta), \qquad \forall \delta \ge 0. \tag{24}$$

Conversely, if $\tilde{\gamma}_a^{\beta_{\max}} \in \bar{\mathcal{K}}$, then the system is $ISS_{\max}$ with $(\beta_{\max}, \tilde{\gamma}_a^{\beta_{\max}})$.

Remark 19 *It can be seen from Theorem 11, Lemmas 13 and 16 (see also equation (12)) that the minimal ISS_{AG} gain γ_∞ defined by (11) is a lower bound of both the minimal ISS_+ gain and the minimal $ISS_{\max}$ gain (this is also clear from the different ISS definitions). However, we do not have clear formulas for the the minimal ISS_+ gain and the minimal $ISS_{\max}$ gain. In fact, there is a tradeoff between the minimal ISS gain and the minimal transient bound for the ISS_+ and $ISS_{\max}$ cases. Moreover, our examples (see Examples 2 and 3 in Section 8) shows that the limit of some good $ISS_{\max}$ gains may not be a good $ISS_{\max}$ gain itself. Our results (see (16) and (23)) also show that for a fixed transient bound $\beta_+ = \beta_{\max}$, the minimal ISS_+ gain $\gamma_a^{\beta_+}$ is not greater than the minimal $ISS_{\max}$ gain $\tilde{\gamma}_a^{\beta_{\max}}$ if they both exist. The minimal transient bounds of different ISS definitions enjoy a similar property.*

7 Analysis of related ISS like properties

It is possible to analyse several other ISS like properties using techniques of Sections 5 and 6. In particular, we sketch below how one can analyse input-to-output stability (IOS) and incremental input-to-state stability (Δ-ISS) that were respectively considered in Sontag and Wang (2001) and Angeli (2002). Other ISS like properties can be analysed using similar techniques, but we have omitted those results for space reasons.

Consider the system (1) with the output

$$y_k = h(x_k). \tag{25}$$

We introduce the following two IOS properties:

Definition 20 *The system (1) with the output (25) is IOS_+ (with (β, γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that*

$$|h(x(k, x_0, u))| \le \beta(|x_0|, k) + \gamma(\|u\|_\infty), \tag{26}$$

$\forall x_0 \in \mathbf{R}^n, u \in l_\infty, k \in \mathbf{Z}_+$.

Definition 21 *The system (1) with the output (25) is $IOS_{\max}$ (with (β, γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that*

$$|h(x(k, x_0, u))| \le \max\{\beta(|x_0|, k), \gamma(\|u\|_\infty)\}, \tag{27}$$

$\forall x_0 \in \mathbf{R}^n, u \in l_\infty, k \in \mathbf{Z}_+$.

For $x \in \mathbf{R}^n$, $\delta \ge 0$, integer $k \in \mathbf{Z}_+$, denote

$$U^\delta(x,k) := \sup_{\|u\|_\infty \le \delta} \{|h(x(k, x_0, u))| : x_0 = x\}. \tag{28}$$

The Dynamic Programming Equation (DPE) for $U^\delta(x,k)$ is

$$U^\delta(x,k) = \sup_{|u|\le\delta} U^\delta(f(x,u), k-1) \tag{29}$$

with the initial condition $U^\delta(x,0) = |h(x)|$.

Another property that can be treated in a similar way is incremental ISS (Δ-ISS) considered in Angeli (2002). In particular, we can define the following two characterizations of Δ-ISS:

Definition 22 *The system (1) is Δ-ISS$_+$ (with (β,γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that any two solutions $x(k,x_0,u)$ and $x(k,z_0,v)$ satisfy:*

$$|x(k,x_0,u) - x(k,z_0,v)| \le \beta(|x_0 - z_0|, k) + \gamma(\|u-v\|_\infty),$$

for all $x_0, z_0 \in \mathbf{R}^n$, all $u, v \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Definition 23 *The system (1) is Δ-ISS$_{\max}$ (with (β,γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that any two solutions $x(k,x_0,u)$ and $x(k,z_0,v)$ satisfy:*

$$\begin{aligned} &|x(k,x_0,u) - x(k,z_0,v)| \\ &\le \max\{\beta(|x_0 - z_0|, k), \gamma(\|u-v\|_\infty)\}, \end{aligned}$$

for all $x_0, z_0 \in \mathbf{R}^n$, all $u, v \in l_\infty$ and all $k \in \mathbf{Z}_+$.

In order to state the appropriate dynamic programming equation for Δ-ISS, we introduce the following $2n$ dimensional auxiliary system containing system (1) and an augmented exact copy:

$$x_{k+1} = f(x_k,u_k), \quad z_{k+1} = f(z_k,v_k).$$

Here $x_k, z_k \in \mathbf{R}^n$ and $u_k, v_k \in \mathbf{R}^m$. Then, we introduce for $x, z \in \mathbf{R}^n$, $\delta \ge 0$, integer $k \in \mathbf{Z}_+$

$$\begin{aligned} &W^\delta(x,z,k) \\ :=& \sup_{\|u-v\|_\infty \le \delta} \{|x(k,x_0,u) - x(k,z_0,v)| : x_0 = x, z_0 = z\}. \end{aligned}$$

The Dynamic Programming Equation (DPE) for $W^\delta(x,z,k)$ is

$$W^\delta(x,z,k) = \sup_{|u-v|\le\delta} W^\delta(f(x,u), f(z,v), k-1) \tag{30}$$

with the initial condition $W^\delta(x,z,0) = |x-z|$.

Results similar to those in Sections 5 and 6 still hold for IOS/Δ-ISS properties defined above. It should be noted that the results in Section 4 do not hold since we do not have an appropriate asymptotic gain characterization of IOS/Δ-ISS.

8 Examples

In this section, we present three examples to which the results of Sections 3, 4, 5 and 6 are applied. The first example shows that the minimal asymptotic gain for an ISS system may be discontinuous. The second and third examples consider respectively scalar linear systems and a second order nonlinear system.

Where necessary in analysing these examples, a numerical scheme is applied to solve DPE (8) approximately. This scheme utilizes a bounded discretized input bound space Δ, state space X and input space U. In terms of notation, these spaces are denoted respectively by

$$\begin{aligned} \Delta &= \{\delta \in \mathbf{R} : \delta_{\min} \le \delta \le \delta_{\max}\}_{N_\Delta}, \\ X &= \{x \in \mathbf{R} : |x| \le x_{\max}\}_{N_X}, \\ U^\delta &= \{u \in \mathbf{R} : |u| \le \delta\}_{N_U}, \ \delta \in \Delta. \end{aligned} \tag{31}$$

Here, N_Δ, N_X and N_U respectively refer to the number of points in each of the discretized spaces Δ, X and U^δ. The result of applying DPE (8) over these discretized spaces is an approximation for V^δ. With $V^\delta(x,k)$ computed for all $\delta \in \Delta$, computation of approximations for the remaining quantities is then possible.

We acknowledge that, while straightforward in principle, these approximations can be computationally expensive to obtain. Aside from this observation, we stress that while the details of the attendant numerical scheme are important, the scheme itself is not fundamental to understanding the concepts presented in this paper. Consequently, a detailed discussion of possible numerical schemes is postponed for inclusion in a later paper.

8.1 *Example 1: A system with discontinuous minimal asymptotic gain*

Consider the one dimensional system

$$x_{k+1} = \frac{1}{2} x_k \left(1 + \phi(|x_k|)\, a(|u_k|)\right) \tag{32}$$

where

$$\phi(s) = \begin{cases} 1, & s \in [0,20), \\ 21 - s, & s \in [20,21), \\ 0, & s \in [21,\infty), \end{cases} \tag{33}$$

and

$$a(s) = \begin{cases} 0, & s \in [0,9), \\ s - 9, & s \in [9,10), \\ 1, & s \in [10,\infty), \end{cases} \tag{34}$$

are both continuous functions.

(i) AG property: It is not difficult to prove the following two facts:

$$\begin{aligned} \|u\|_\infty \in [0,10) &\Rightarrow \sup_{x_0 \in \mathbf{R}} \limsup_{k\to\infty} |x(k,x_0,u)| = 0. \\ \|u\|_\infty \in [10,\infty) &\Rightarrow \sup_{x_0 \in \mathbf{R}} \limsup_{k\to\infty} |x(k,x_0,u)| \in [20,21]. \end{aligned} \tag{35}$$

This implies that system (32) satisfies the AG property and a continuous asymptotic gain can be chosen as

$$\gamma_1(s) \;=\; \begin{cases} 0, & s \in [0,9), \\ 21(s-9), & s \in [9,10), \\ 21, & s \in [10,\infty). \end{cases}$$

(ii) Minimal asymptotic gain: It also follows from (35) that any candidate asymptotic gain $\gamma_{AG} \in \mathcal{K}$ for system (32) must satisfy the inequality $\gamma_{AG}(s) \geq \gamma_0(s)$ for all $s \geq 0$, where

$$\gamma_0(s) \;=\; \begin{cases} 0, & s \in [0,10), \\ 20, & s \in [10,\infty). \end{cases}$$

Hence, the minimal asymptotic gain γ_∞ defined by (11) must satisfy $\gamma_0(s) \leq \gamma_\infty(s) \leq \gamma_1(s)$ for all $s \geq 0$, which implies a jump discontinuity in γ_∞ at $s = 10$. Using the dynamic programming technique provided in Section 4, we obtain an approximation of γ_∞ which is shown in Figure 1. Although computed on a finite grid, this approximation clearly demonstrates the jump discontinuity (at $s = 10$) in γ_∞.

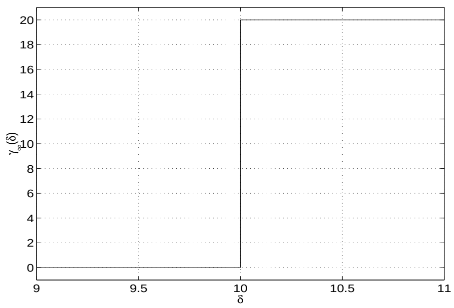

Fig. 1. Approximation of γ_∞ obtained by dynamic programming (Example 1, $\delta_{\min} = 9, \delta_{\max} = 11,\ x_{\max} = 20$).

Remark 24 *This example demonstrates that for some systems, the minimal asymptotic gain γ_∞ can be of class $\bar{\mathcal{K}} \setminus \mathcal{K}$.*

8.2 Example 2: A class of scalar linear systems

Consider the class of scalar linear systems given by

$$x_{k+1} \;=\; ax_k + bu_k, \tag{36}$$

where $0 < a < 1$ and $b \geq 0$.

By direct calculation using (8) and (9), we obtain

$$V^\delta(x,k) = a^k|x| + \left(\frac{1-a^k}{1-a}\right) b\delta. \tag{37}$$

ISS$_{AG}$ property: Applying definitions (10) and (11), we have

$$\begin{aligned} V_a^\delta(x) &= \limsup_{k\to\infty} V^\delta(x,k) \;=\; \left(\frac{b}{1-a}\right)\delta, \\ \gamma_\infty(\delta) &= \sup_{x\in\mathbf{R}^n} V_a^\delta(x) \;=\; \left(\frac{b}{1-a}\right)\delta, \\ \beta_a(s,k) &= \sup_{|x|\leq s} V^0(x,k) \;=\; sa^k. \end{aligned}$$

Since $\gamma_\infty \in \bar{\mathcal{K}}$ and $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, Theorem 11 implies that system (36) is ISS$_{AG}$ with (β_a, γ_∞).

ISS$_+$ property: Applying definition (12) of $\gamma_a(\delta)$,

$$\gamma_a(\delta) = \max\left\{\gamma_\infty(\delta), \left(\frac{b}{1-a}\right)\delta\right\} \;=\; \gamma_\infty(\delta). \tag{38}$$

(i) Minimal β_+ for fixed γ_+: Using γ_a as a candidate (fixed) gain in testing ISS$_+$ (i.e. $\gamma_+ = \gamma_a$), the definition (14) of the minimal corresponding transient bound yields

$$\beta_a^{\gamma+}(s,k) = sa^k = \beta_a(s,k).$$

(ii) Minimal γ_+ for fixed β_+: Using β_a as a candidate (fixed) transient bound in testing ISS$_+$ (i.e. $\beta_+ = \beta_a$), the definition (16) of the minimal corresponding gain yields

$$\gamma_a^{\beta+}(\delta) = \left(\frac{b}{1-a}\right)\delta \;=\; \gamma_a(\delta).$$

Both Theorems 14 and 15 imply that system (36) is ISS$_+$ with (β_a, γ_a), indeed, this is the minimal possible pair.

Remark 25 *Calculations (i) and (ii) above highlight an important property of scalar linear systems. In particular, (i) shows that the minimal ISS$_+$ transient bound $\beta_a^{\gamma+}$ determined using the minimal candidate ISS$_+$ gain $\gamma_+ = \gamma_a$ is exactly the minimal candidate ISS$_+$ transient bound β_a. Similarly, (ii) shows that the minimal candidate ISS$_+$ gain bound γ_a is recovered as the minimal ISS$_+$ gain. That is, both approaches yield that the ISS$_+$ property holds with the transient bound / gain* pair *defined by the minimal candidate transient bound β_a and the minimal candidate gain γ_a. We note that this is not in general the case, either for other classes of systems or other equivalent ISS properties. This is illustrated below in the ISS$_{\max}$ case.*

ISS$_{\max}$ property: Unlike the ISS$_+$ property however, we find that (for this example) the ISS$_{\max}$ property does not hold for the *pair* defined by the minimal candidate transient bound β_a and the minimal candidate gain γ_a.

(i) Minimal $\beta_{\max}$ for fixed $\gamma_{\max}$: Using γ_a as a candidate (fixed) gain in testing ISS$_{\max}$ (i.e. $\gamma_{\max} = \gamma_a$), by definition (19) we obtain

$\tilde{\beta}_a^{\gamma_{\max}}(s,k) = s,$

which is not of class $\bar{\mathcal{KL}}$. Hence, the gain γ_a is too small to be a gain candidate for computing the minimal transient bound. To illustrate this point further, suppose a slightly larger candidate gain is chosen, namely

$$\gamma_{\max}(\delta) \;=\; (1+\varepsilon)\gamma_a(\delta)$$

where $\varepsilon > 0$ is fixed and small. Using (19) again yields that

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) = \left(\frac{1+\varepsilon}{a^k+\varepsilon}\right) s a^k \;=\; \left(\frac{1+\varepsilon}{a^k+\varepsilon}\right)\beta_a(s,k),$$

which is of class $\bar{\mathcal{KL}}$ for any $\varepsilon > 0$. Hence, by Theorem 17, system (36) is $\mathrm{ISS}_{\max}$ with $(\tilde{\beta}_a^{\gamma_{\max}}, \gamma_{\max})$, $\gamma_{\max} = (1+\varepsilon)\gamma_a$.

(ii) Minimal $\gamma_{\max}$ for fixed $\beta_{\max}$: Using β_a as a candidate (fixed) transient bound in testing $\mathrm{ISS}_{\max}$ (i.e. $\beta_{\max} = \beta_a$), by definition (23) we obtain

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) = \infty, \tag{39}$$

for all $\delta > 0$, which is clearly not of class $\bar{\mathcal{K}}$. This implies that the transient bound β_a is too small to be a candidate transient bound for $\mathrm{ISS}_{\max}$. To illustrate that this system is $\mathrm{ISS}_{\max}$, choose the slightly larger transient bound

$$\beta_{\max}(s,k) \;=\; (1+\varepsilon)\beta_a(s,k)$$

where $\varepsilon > 0$. By (23),

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \;=\; \left(1+\frac{1}{\varepsilon}\right)\frac{b\delta}{1-a} \;=\; \left(1+\frac{1}{\varepsilon}\right)\gamma_a(\delta),$$

which is of class $\bar{\mathcal{K}}$. Theorem 18 then implies that system (36) is $\mathrm{ISS}_{\max}$ with $(\beta_{\max}, \tilde{\gamma}_a^{\beta_{\max}})$, $\beta_{\max} = (1+\varepsilon)\beta_a$.

8.3 Example 3: A two dimensional nonlinear system

Consider the 0-GAS closed loop system

$$\begin{aligned} x_{1,k+1} &= x_{2,k} + \sqrt[3]{x_{1,k}} \\ x_{2,k+1} &= x_{2,k} - \tfrac{1}{4}x_{1,k} + \sqrt[3]{x_{1,k}} - \sqrt[3]{x_{2,k} + \sqrt[3]{x_{1,k}}} + w_k, \end{aligned} \tag{40}$$

obtained via backstepping from the corresponding open loop system with $x_{2,k+1} = u_k + w_k$. Here, $u_k \in \mathbf{R}$ and $w_k \in \mathbf{R}$ represent respectively control and disturbance inputs at time $k \in \mathbf{Z}_+$. The Lyapunov function utilized in the backstepping procedure was

$$V(x_1,x_2) = \frac{1}{2}|x_1| + \frac{3}{2}\left|x_2 - \frac{x_1}{2} + \sqrt[3]{x_1}\right|. \tag{41}$$

The aim is to determine the minimal asymptotic gain and transient bound for which the ISS_{AG} property holds (from disturbance to state) for this closed loop system.

ISS Lyapunov characterizations: We present Lyapunov characterizations of ISS for two state space representations of the closed loop system (40). Using the Lyapunov function V given by (41), we find (in both characterizations) the functions α_1, α_2, α_3, $\sigma \in \mathcal{K}_\infty$ such that for all $x \in \mathbf{R}^2$,

$$\begin{aligned} \alpha_1(|x|) \;&\le\; V(x) \le \alpha_2(|x|) \\ V(f(x,w)) - V(x) \;&\le\; -\alpha_3(|x|) + \sigma(|w|). \end{aligned} \tag{42}$$

(i) Characterization 1: (State vector $x = [x_1\ x_2]^T$.) System (40) satisfies the Lyapunov characterization (42) of ISS with Lyapunov function (41), where

$$\begin{aligned} \alpha_1(s) \;&=\; \min\left(\tfrac{1}{10}s,\ \tfrac{1}{16}s^3\right),\ \alpha_2(s) \;=\; \tfrac{5\sqrt{2}}{4}s + \tfrac{3\sqrt[6]{2}}{2}\sqrt[3]{s}, \\ \alpha_3(s) \;&=\; \min\left(\tfrac{1}{20}s,\ \tfrac{1}{32}s^3\right),\ \ \sigma(s) \;=\; \tfrac{3}{2}s. \end{aligned} \tag{43}$$

(ii) Characterization 2: (State vector $\xi = [x_1\ y_2]^T$.) System (40) can be expressed in the coordinates $\xi := [x_1\ y_2]^T$, where $y_2 := x_2 - \frac{x_1}{2} + \sqrt[3]{x_1}$. Then, the closed loop dynamics are linear, with

$$\xi_{k+1} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}\xi_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k. \tag{44}$$

Consequently, system (44) satisfies the Lyapunov characterization (42) of ISS (in the ξ coordinates) with Lyapunov function (41), $V(x_1,y_2) = \frac{1}{2}|x_1| + \frac{3}{2}|y_2|$, and

$$\begin{aligned} \alpha_1^{lin}(s) \;&=\; \tfrac{1}{4}s,\ \alpha_2^{lin}(s) \;=\; \tfrac{3}{\sqrt{2}}s, \\ \alpha_3^{lin}(s) \;&=\; \tfrac{1}{4}s,\ \sigma^{lin}(s) \;=\; \tfrac{3}{2}s. \end{aligned} \tag{45}$$

Applying results from Nešić and Teel (2001b), Nešić and Teel (2002), there exists κ_1,$\kappa_2 \in \mathcal{K}_\infty$ such that

$$\kappa_1(|x|) \;\le\; |\xi| \;\le\; \kappa_2(|x|). \tag{46}$$

In this case,

$$\kappa_1(s) = \sqrt{\tilde{\kappa}_1\left(\frac{1}{\sqrt{2}}s\right)},\ \kappa_2(s) = \sqrt{3}s + 2\sqrt[3]{s}, \tag{47}$$

where $\tilde{\kappa}_1(s) = \min\left(\frac{1}{4}s^2,\ \left(\tilde{\varphi}_1^{-1}\left(\frac{1}{2}s\right)\right)^2\right)$ and $\tilde{\varphi}_1(s) := s + \sqrt[3]{s}$. Here, it can be shown that $\tilde{\varphi}_1^{-1} \in \mathcal{K}_\infty$. Combining (42), (45), (46) and (47) implies that the closed loop system (40) satisfies the Lyapunov characterization of ISS in the original x coordinates with bounds

$$\begin{aligned} \alpha_1(s) &= \alpha_1^{lin} \circ \kappa_1(s) = \tfrac{1}{4}\kappa_1(s), \\ \alpha_2(s) &= \alpha_2^{lin} \circ \kappa_2(s) = \tfrac{3}{\sqrt{2}}\kappa_2(s), \\ \alpha_3(s) &= \alpha_3^{lin} \circ \kappa_1(s) = \tfrac{1}{4}\kappa_1(s), \\ \sigma(s) &= \sigma^{lin}(s) = \tfrac{3}{2}s. \end{aligned} \tag{48}$$

ISS$_{AG}$ gains: Applying results from Jiang and Wang (2001), the Lyapunov characterization (42) of ISS implies that the ISS$_{\max}$ property holds with gain

$$\gamma_{\max}(s) = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\lambda\sigma(s)), \qquad \lambda > 1. \tag{49}$$

As the ISS$_{\max}$ property implies the ISS$_{AG}$ property (with identical gains), (49) is an upper bound for the minimal ISS$_{AG}$ gain γ_∞. That is,

$$\gamma_\infty(s) \le \gamma_{AG}(s) := \gamma_{\max}(s). \tag{50}$$

(i) Upper bound 1: By direct calculation using Characterization 1 and (49), we obtain an upper bound for the minimal asymptotic gain γ_∞:

$$\gamma_\infty(s) \le \gamma_{\max,1}(s) = \max \left(2^{\frac{4}{3}} \left(\frac{3\max(30\,\lambda\, s, 2\,(6\lambda\, s)^{\frac{1}{3}})^{\frac{1}{3}}}{2^{\frac{5}{6}}} + \frac{5\,\max(30\,\lambda\, s, 2\,(6\lambda\, s)^{\frac{1}{3}})}{2\sqrt{2}} \right)^{\frac{1}{3}}, \; 10 \left(\frac{3\max(30\,\lambda\, s, 2\,(6\lambda\, s)^{\frac{1}{3}})^{\frac{1}{3}}}{2^{\frac{5}{6}}} + \frac{5\,\max(30\,\lambda\, s, 2\,(6\lambda\, s)^{\frac{1}{3}})}{2\sqrt{2}} \right) \right),$$

where $\lambda > 1$.

(ii) Upper bound 2: We repeat the calculation of (49) using the Characterization 2 and obtain another upper bound for the minimal asymptotic gain γ_∞:

$$\begin{aligned} &\gamma_\infty(s) \le \gamma_{\max,2}(s) \\ &= 4\left(6\sqrt{2}\left(6\,\lambda\, s + (6\lambda\, s)^{\frac{1}{3}}\right)^{\frac{1}{3}} + 6\sqrt{6}\left(6\,\lambda\, s + (6\lambda\, s)^{\frac{1}{3}}\right)\right)^{\frac{1}{3}} \\ &+24\left(2\sqrt{2}\left(6\,\lambda\, s + (6\lambda\, s)^{\frac{1}{3}}\right)^{\frac{1}{3}} + 2\sqrt{6}\left(6\,\lambda\, s + (6\lambda\, s)^{\frac{1}{3}}\right)\right) \end{aligned}$$

where $\lambda > 1$.

(iii) Upper bound 3: Suppose that $V^\delta(\xi, k)$ is defined for system (44). Then, from (7) and (46),

$$\begin{aligned} V^\delta(x,k) &\le \sup_{\|u\|_\infty \le \delta} \left\{ \kappa_1^{-1}\left(|\xi(k,\xi_0,u)|\right) : \xi_0 = \xi \right\} \\ &= \kappa_1^{-1}\left(V^\delta(\xi,k)\right). \end{aligned} \tag{51}$$

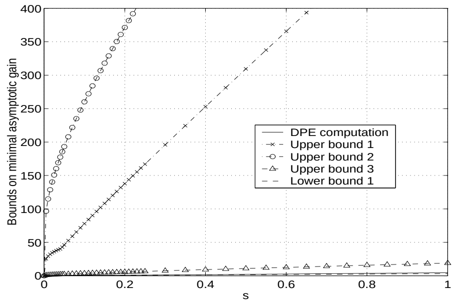

Fig. 2. Comparison of γ_∞ obtained by dynamic programming, upper bounds 1-3, and lower bound 1 (Example 3).

The minimal asymptotic gain γ_∞ for system (40) is then

$$\begin{aligned} \gamma_\infty(\delta) &= \sup_{x\in\mathbf{R}^2} \limsup_{k\to\infty} V^\delta(x,k) \\ &\le \kappa_1^{-1}\left(\sup_{\xi\in\mathbf{R}^2} \limsup_{k\to\infty} V^\delta(\xi,k)\right) = \kappa_1^{-1} \circ \gamma_\infty^{lin}(\delta), \end{aligned} \tag{52}$$

where γ_∞^{lin} is the minimal asymptotic gain for system (44). It can be shown that a candidate asymptotic gain γ_{AG}^{lin} for system (44) is $\gamma_{AG}^{lin}(s) = 5s$. Hence, an upper bound (52) for minimal asymptotic gain γ_∞ is

$$\gamma_\infty(s) \le \kappa_1^{-1}(5s). \tag{53}$$

(iv) Lower bound 1: A lower bound γ_- for the minimal ISS$_{AG}$ gain γ_∞ follows from (11). In particular, for any $\tilde{u}$ satisfying $\|\tilde{u}\|_\infty \le s$, we can define

$$\gamma_-(s) := \sup_{x\in\mathbf{R}^n} \limsup_{k\to\infty} \left\{ |x(k,x_0,\tilde{u})| : x_0 = x \right\} \le \gamma_\infty(s).$$

For this example, $\tilde{u}$ was chosen (arbitrarily) to be a square wave of amplitude s and period 10 samples.

(v) Minimal asymptotic gain via dynamic programming: An approximation to the minimal asymptotic gain $\gamma_\infty(\delta)$ for the nonlinear system (40) was computed over three overlapping intervals $\delta \in [0.00, 0.05]$, $\delta \in [0.05, 0.25]$, $\delta \in [0.25, 1.00]$ and combined. Figures 2 and 3 clearly demonstrate that asymptotic gains obtained from the Lyapunov characterization of ISS can be very conservative when transformed to other characterizations. This highlights a distinct advantage of the dynamic programming approach presented, particularly in (for example) small gain applications.

(vi) Minimal transient bound for ISS$_{AG}$: The computation outlined in (v) above also enables approximation of the minimal transient bound β_a for which the ISS$_{AG}$ property holds. This approximation is illustrated in Figure 4, which show qualitatively that $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$. Theorem 11 then implies that system (40) satisfies the ISS$_{AG}$ property with (β_a, γ_∞).

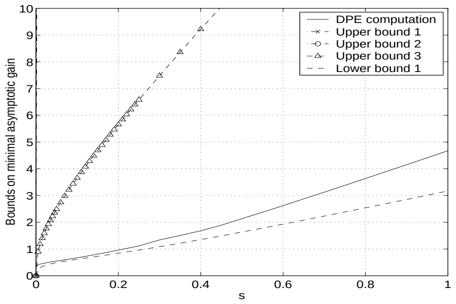

Fig. 3. Enlargement of Figure 2 illustrating the DPE computation and lower bound (Example 3).

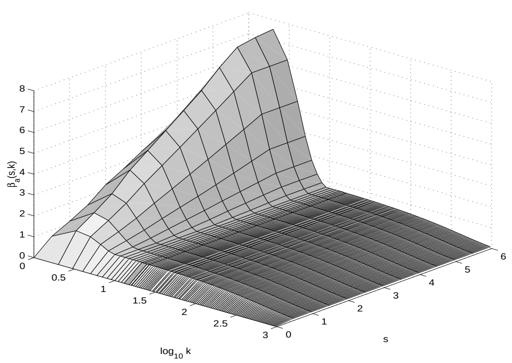

Fig. 4. Approximation of β_a (Example 3).

9 Conclusions

We have presented results for verifying different characterizations of ISS via dynamic programming. Formulas for minimum nonlinear gains and bounds on transients for different characterizations are presented. A discussion on how these results can be used to analyze input-to-output stability and incremental input-to-state stability is also given. We illustrated our approach by three examples.

The aim of this paper is to present a constructive formulation for finding minimal ISS gains and transient bounds. The results of this paper provide a framework for generating numerical algorithms for calculating ISS gains and transient bounds. The detailed development and analysis of numerical methods is an important topic for future investigation and is outside the scope of the present paper. Our example, however, indicates the potential benefits of this numerical approach and motivates careful investigation of numerical issues.

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