Convergence analysis of instrumental variable recursive subspace identification algorithms

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Abstract

The convergence properties of recently developed recursive subspace identification methods are investigated in this paper. The algorithms operate on the basis of instrumental variable (IV) versions of the propagator method for signal subspace estimation. It is proved that, under suitable conditions on the input signal and the system, the considered recursive subspace identification algorithms converge to a consistent estimate of the propagator and, by extension, to the state-space system matrices.

Key words: Subspace methods, recursive algorithms, linear systems, instrumental variable, convergence analysis

1 Introduction

The problem of recursive subspace model identification (RSMI) has been an active area of research in recent years (see, e.g., [16, 1, 4, 9, 14, 10]). Most RSMI algorithms are inspired by offline versions of subspace model identification (SMI) techniques and therefore rely on the availability of efficient updating methods for the numerical linear algebra algorithms used in batch SMI.

So far, two main approaches to the RSMI problem have been considered. First, some works have proposed adaptations of SMI algorithms in order to update the singular value decomposition (SVD) [16, 1]. Unfortunately, these techniques have the drawback of requiring the disturbances acting on the system output to be spatially and temporally white, which is restrictive in practice. The other approach [4, 9, 14, 10] relies on the strong analogies between RSMI and signal processing techniques dedicated to direction of arrival (DOA) estimation [6]. More precisely, two points of view are suggested to find alternatives to the SVD in a recursive framework:

- The first one consists in adapting the so-called Yang's criterion [18] to the recursive update of the observability matrix [9, 14]. In particular, DOA estimation algorithms have been adjusted in order to deal with more general types of perturbations thanks to the use of instrumental variables (IV).

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- The second one rests on the adaptation of another array signal processing technique: the propagator method [13]. The advantage of this approach over the previous conception lies in the use of a linear operator and quadratic criteria which lead to recursive least squares implementations for the algorithms [11, 10, 12].

While a significant level of maturity has now been reached on the algorithmic side, very limited attention has been dedicated to the analysis of the convergence properties of the developed RSMI techniques. In [14], the proposed gradient-based RSMI method was analysed and conditions on the gain of the gradient iteration were derived. However, the convergence study is based on assumptions on the signal-to-noise ratio which limit the validity of the results.

In the light of the above discussion, the aim of this paper is to analyse the convergence properties of two recursive implementations of the `MOESP` class [15] of subspace identification algorithms. More precisely, the Instrumental Variable Propagator Method (`IVPM`) and the Extended Instrumental Variable Propagator Method (`EIVPM`) RSMI techniques, which operate on the basis of the IV version of the propagator method for signal subspace estimation, have been considered and the following results have been derived:

- for the `EIVPM` algorithm it can be shown that asymptotic convergence can be characterised in terms of the so-called *critical relation* for the consistency of IV subspace identification algorithms, first derived in the classical paper [5]. As a consequence, convergence can be guaranteed
  - under suitable persistency of excitation conditions in the absence of process noise;
  - in a number of special cases (e.g., single input systems, white noise or `ARMA` input signal) whenever process noise is present.
- for the `IVPM` algorithm, which can be considered as a special case of `EIVPM`, a detailed analysis is proposed for single input systems. More precisely, it is possible to show that in this case `IVPM` converges to consistent estimates under less restrictive persistency of excitation conditions than for `EIVPM`.

The paper is organised as follows. In Section 2, the system model is introduced, the main notations are defined and the general assumptions are stated. In Section 3 an overview of the considered RSMI algorithms is provided. In particular, the stages necessary to the recursive estimation of the state-space matrices are developed. The analysis of the convergence properties of the considered algorithms is presented in Section 4. Conditions on the input signal and the identified system are more precisely given, which ensure the consistency of the propagator estimates and, by extension, of the extended observability matrix. These convergence properties of the propagator are completed by the expressions of the asymptotic distribution of the estimates. Some simulations are presented in Section 5 to illustrate these theoretical results. Finally, concluding remarks are provided in Section 6.

2 Problem formulation and notations

Assume that the true system can be described by the discrete-time linear time-invariant state-space model in innovation form

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}\mathbf{e}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{e}(t) \end{aligned} \tag{1}$$

with n_y outputs $\mathbf{y}$, n_u inputs $\mathbf{u}$, n_x states $\mathbf{x}$. $\mathbf{e}$ is a white process noise. Furthermore, the following assumptions hold throughout the paper:

1. the innovation process $\mathbf{e}$ is a stationary zero mean white process noise with second moments
$$\mathbb{E}\left[\mathbf{e}(t)\mathbf{e}^T(\tau)\right] = \mathbf{R}_{\mathbf{e}}\delta_{t\tau} \tag{2}$$
where $\delta_{t\tau}$ is the Kronecker delta;
2. the system (1) is asymptotically stable;

3. the pair $\{\mathbf{A}, \mathbf{C}\}$ is observable and the pair $\{\mathbf{A}, [\mathbf{B}\ \mathbf{K}]\}$ is reachable;
4. the input $\mathbf{u}$ is a quasi stationary deterministic sequence uncorrelated with the noise $\mathbf{e}$.

To deal with deterministic and stochastic signals in a compact manner, the following operator is defined

$$\bar{\mathbb{E}}[\cdot] = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathbb{E}[\cdot] \tag{3}$$

where $\mathbb{E}$ is the expectation operator. For two signals $\mathbf{a}(t)$ and $\mathbf{b}(t)$, the cross covariance matrix will be denoted as $\mathbf{R_{ab}} = \bar{\mathbb{E}}\left[\mathbf{a}(t)\mathbf{b}^T(t)\right]$ while estimates of signal correlations will be denoted by $\hat{\mathbf{R}}_{\mathbf{ab}}(t) = \sum_{k=1}^{t} \lambda^{t-k}\mathbf{a}(k)\mathbf{b}^T(k)$, where $0 < \lambda \leq 1$ is a forgetting factor.

3 Overview of RSMI algorithms

The algorithms considered in this paper recursively estimate the $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ state space matrices at each new data acquisition. The proposed methods are based on the estimation of a basis for the observability subspace from the input-output (I/O) relation [10]

$$\begin{aligned}\mathbf{y}_f(t) &= \mathbf{\Gamma}_f\mathbf{x}(t) + \mathbf{H}_f\mathbf{u}_f(t) + \mathbf{G}_f\mathbf{e}_f(t) = \\ &= \mathbf{\Gamma}_f\mathbf{x}(t) + \mathbf{H}_f\mathbf{u}_f(t) + \mathbf{b}_f(t)\end{aligned} \tag{4}$$

where the stacked vectors $\mathbf{y}_f$, $\mathbf{u}_f$ and $\mathbf{e}_f$ are defined as

$$\mathbf{y}_f(t) = \begin{bmatrix}\mathbf{y}^T(t) & \cdots & \mathbf{y}^T(t+f-1)\end{bmatrix}^T \in \mathbb{R}^{n_y f\times 1} \tag{5}$$

with $f > n_x$, $\mathbf{\Gamma}_f$ is the observability matrix

$$\mathbf{\Gamma}_f = \begin{bmatrix}\mathbf{C}^T & (\mathbf{CA})^T & \cdots & \left(\mathbf{CA}^{f-1}\right)^T\end{bmatrix}^T, \tag{6}$$

$\mathbf{H}_f$ is the block Toeplitz matrix of the impulse responses from $\mathbf{u}$ to $\mathbf{y}$ [17] and $\mathbf{b}_f = \mathbf{G}_f\mathbf{e}_f$ with $\mathbf{G}_f$ the block Toeplitz matrix of the impulse responses from $\mathbf{e}$ to $\mathbf{y}$. The class of techniques considered herein is based on the application of the so-called propagator method [13] to the recursive estimation of $\mathbf{\Gamma}_f$. To this purpose, note that letting $\mathbf{Y}_f \in \mathbb{R}^{n_y f\times N}$, $\mathbf{U}_f \in \mathbb{R}^{n_u f\times N}$ and $\mathbf{B}_f \in \mathbb{R}^{n_y f\times N}$ be the Hankel I/O data matrices defined as

$$\mathbf{Y}_f(\bar{t}) = \begin{bmatrix}\mathbf{y}_f(t) & \cdots & \mathbf{y}_f(t+N-1)\end{bmatrix}, \tag{7}$$

with $N >> f > n$ and $\bar{t} = t + N - 1$, equation (4) can be written in matrix form as

$$\mathbf{Y}_f(\bar{t}) = \mathbf{\Gamma}_f\mathbf{X}(\bar{t}) + \mathbf{H}_f\mathbf{U}_f(\bar{t}) + \mathbf{B}_f(\bar{t}), \tag{8}$$

where $\mathbf{X}(\bar{t}) = \begin{bmatrix}\mathbf{x}(\bar{t}) & \cdots & \mathbf{x}(\bar{t}+N-1)\end{bmatrix}$. As is well known from the offline subspace identification literature, a quantity directly related with the observability subspace can be obtained by computing the projection $\mathbf{Z}_f$ of $\mathbf{Y}_f$ on the kernel of $\mathbf{U}_f$

$$\begin{aligned}\mathbf{Z}_f(\bar{t}) &= \mathbf{Y}_f(\bar{t})\mathbf{\Pi}_{\mathbf{U}_f^\perp}(\bar{t}) = \\ &= \left(\mathbf{\Gamma}_f\mathbf{X}(\bar{t}) + \mathbf{B}_f(\bar{t})\right)\mathbf{\Pi}_{\mathbf{U}_f^\perp}(\bar{t}).\end{aligned} \tag{9}$$

Considering now the time update of $\mathbf{Z}_f$

$$\mathbf{Z}_f(\bar{t}) = \begin{bmatrix}\mathbf{Z}_f(\bar{t}-1) & \mathbf{z}_f(\bar{t})\end{bmatrix}, \tag{10}$$

it is clear that the *observation vector* $\mathbf{z}_f(\bar{t})$ will carry all the relevant information for the estimation of the observability subspace contained in the data at time $\bar{t}$. Therefore, a two-step procedure for the recursive estimation of the system matrices can be devised:

1. the update of the observation vector $\mathbf{z}_f$ from the I/O measurements by considering a recursive formulation of the orthogonal projection performed in equation (9) (see Subsection 3.1);
2. the estimation of a basis of $\mathbf{\Gamma}_f$ from this observation vector by adapting the propagator method [13] (see Subsection 3.2).

Both stages are considered in the following subsections.

3.1 Recursive estimation of the observation vector

The problem of updating the observation vector can be solved by recursively updating the projection (9) at each iteration. Several techniques for the computation of this update have been developed in the literature (see, e.g., [9, 14]). In this paper, an approach based on the matrix inversion lemma [3] is used. It has indeed the advantage of providing an explicit expression of the observation vector in terms of the I/O data. The idea is to recursively update the quantity $\mathbf{Z}_f(\bar{t}) = \mathbf{Y}_f(\bar{t})\mathbf{\Pi}_{\mathbf{U}_f^\perp}(\bar{t})$

$$\mathbf{Z}_f(\bar{t}) = \mathbf{Y}_f(\bar{t})\left\{\mathbf{I} - \mathbf{U}_f^T(\bar{t})\left(\mathbf{U}_f(\bar{t})\mathbf{U}_f^T(\bar{t})\right)^{-1}\mathbf{U}_f(\bar{t})\right\}$$

at each new data acquisition, knowing that

$$\mathbf{U}_f(\bar{t}) = \begin{bmatrix}\mathbf{U}_f(\bar{t}-1) & \mathbf{u}_f(\bar{t})\end{bmatrix}, \mathbf{Y}_f(\bar{t}) = \begin{bmatrix}\mathbf{Y}_f(\bar{t}-1) & \mathbf{y}_f(\bar{t})\end{bmatrix}$$

by applying the matrix inversion lemma to $\left(\mathbf{U}_f\mathbf{U}_f^T\right)^{-1}$. It can be shown that the observation vector $\mathbf{z}_f(\bar{t})$ can be recursively computed with the following algorithm:

Algorithm 1 *Assume that, at time* $\bar{t}-1$, $\mathbf{W}_f(\bar{t}-1) = \left(\mathbf{U}_f(\bar{t}-1)\mathbf{U}_f^T(\bar{t}-1)\right)^{-1}$ *and* $\mathbf{V}_f(\bar{t}-1) = \mathbf{Y}_f(\bar{t}-1)\mathbf{U}_f^T(\bar{t}-1)$ *have been estimated. Then, when a new I/O data sequence* $\{\mathbf{u}_f(\bar{t}), \mathbf{y}_f(\bar{t})\}$ *is acquired, the observation vector is updated by means of the recursion*

$$\boldsymbol{\beta}_f(\bar{t}) = \mathbf{W}_f(\bar{t}-1)\mathbf{u}_f(\bar{t}) \tag{11a}$$
$$\delta_f(\bar{t}) = \mathbf{u}_f^T(\bar{t})\boldsymbol{\beta}_f(\bar{t}) \tag{11b}$$
$$\alpha_f(\bar{t}) = \frac{1}{1+\delta_f(\bar{t})} \tag{11c}$$
$$\mathbf{z}_f(\bar{t}) = \alpha_f(\bar{t})\left(\mathbf{y}_f(\bar{t}) - \mathbf{V}_f(\bar{t}-1)\boldsymbol{\beta}_f(\bar{t})\right) \tag{11d}$$
$$\mathbf{V}_f(\bar{t}) = \mathbf{V}_f(\bar{t}-1) + \mathbf{y}_f(\bar{t})\mathbf{u}_f^T(\bar{t}) \tag{11e}$$
$$\mathbf{W}_f(\bar{t}) = \mathbf{W}_f(\bar{t}-1) - \alpha_f(\bar{t})\boldsymbol{\beta}_f(\bar{t})\boldsymbol{\beta}_f^T(\bar{t}). \tag{11f}$$

Remark 2 *It is easy to show that the observation vector* $\mathbf{z}_f(\bar{t})$ *can be equivalently written as*

$$\mathbf{z}_f(\bar{t}) = \alpha_f(\bar{t})\left(\mathbf{\Gamma}_f\tilde{\mathbf{x}}(\bar{t}) + \tilde{\mathbf{b}}_f(\bar{t})\right), \tag{12}$$

where

$$\tilde{\mathbf{b}}_f(\bar{t}) = \mathbf{b}_f(\bar{t}) - \mathbf{B}_f(\bar{t}-1)\mathbf{U}_f^T(\bar{t}-1)\boldsymbol{\beta}_f(\bar{t}) \tag{13}$$
$$\tilde{\mathbf{x}}(\bar{t}) = \mathbf{x}(\bar{t}) - \mathbf{X}(\bar{t}-1)\mathbf{U}_f^T(\bar{t}-1)\boldsymbol{\beta}_f(\bar{t}). \tag{14}$$

Therefore, it is possible to apply the propagator method to the subspace identification problem by exploiting the analogy between (12) *and the data generation model used in array signal processing problems [6]*

$$\mathbf{z}(t) = \mathbf{\Gamma}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{b}(t) \tag{15}$$

where $\mathbf{z}$ *is the output of the* n_z *sensors of the antenna array,* $\mathbf{\Gamma}(\boldsymbol{\theta})$ *the steering matrix,* $\mathbf{s}$ *the vector of the* n_s *signal waveforms and* $\mathbf{b}$ *the additive noise.*

Remark 3 *In the following, the scaling factor* α_f *(see* (12)*) will be neglected for simplicity*

$$\mathbf{z}_f(\bar{t}) = \mathbf{\Gamma}_f\tilde{\mathbf{x}}(\bar{t}) + \tilde{\mathbf{b}}_f(\bar{t}). \tag{16}$$

Note that this simplification does not affect the properties of the algorithm since (16) *provides all the information needed to estimate* $\mathrm{span}_{\mathrm{col}}\{\mathbf{\Gamma}_f\}$.

3.2 Recursive update of the observability matrix

Once the observation vector is estimated, the second step of the recursive subspace identification procedure consists in the online update of the observability matrix. A number of methods have been developed in the literature in order to avoid the computation of a complete SVD in this step of recursive subspace identification [9, 14, 10]. In this paper, the focus is on algorithms based on the propagator concept [13].

Under assumption 3, since $\mathbf{\Gamma}_f \in \mathbb{R}^{n_y f \times n_x}$ with $n_y f > n_x$, the extended observability matrix has at least n_x linearly independent rows, which can be gathered in a submatrix $\mathbf{\Gamma}_{f_1}$. Then, the complement $\mathbf{\Gamma}_{f_2}$ of $\mathbf{\Gamma}_{f_1}$ can be expressed as a linear combination of these n_x rows. So, there is a unique linear operator $\mathbf{P}_f \in \mathbb{R}^{n_x \times (n_y f - n_x)}$, named *propagator* [13], such that

$$\mathbf{\Gamma}_{f_2} = \mathbf{P}_f^T \mathbf{\Gamma}_{f_1}. \tag{17}$$

Furthermore, it is easy to verify that

$$\mathbf{\Gamma}_f = \begin{bmatrix} \mathbf{\Gamma}_{f_1} \\ \mathbf{\Gamma}_{f_2} \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_{f_1} \\ \mathbf{P}_f^T \mathbf{\Gamma}_{f_1} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P}_f^T \end{bmatrix} \mathbf{\Gamma}_{f_1}. \tag{18}$$

Thus, since $\operatorname{rank}\{\mathbf{\Gamma}_{f_1}\} = n_x$,

$$\operatorname{span}_{\mathrm{col}}\{\mathbf{\Gamma}_f\} = \operatorname{span}_{\mathrm{col}}\left\{ \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P}_f^T \end{bmatrix} \right\}. \tag{19}$$

Equation (19) implies that it is possible to estimate the observability matrix (in a particular basis) by estimating the propagator. This operator can be estimated from (16). Indeed, applying a data reorganization to the observation vector so that the first n_x rows of $\mathbf{\Gamma}_f$ are linearly independent, (16) can be partitioned as

$$\mathbf{z}_f(t) = \begin{bmatrix} \mathbf{z}_{f_1}(t) \\ \mathbf{z}_{f_2}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P}_f^T \end{bmatrix} \mathbf{\Gamma}_{f_1} \tilde{\mathbf{x}}(t) + \begin{bmatrix} \tilde{\mathbf{b}}_{f_1}(t) \\ \tilde{\mathbf{b}}_{f_2}(t) \end{bmatrix} \tag{20}$$

where $\mathbf{z}_{f_1} \in \mathbb{R}^{n_x \times 1}$ and $\mathbf{z}_{f_2} \in \mathbb{R}^{(n_y f - n_x) \times 1}$ are the components of $\mathbf{z}_f$ respectively corresponding to $\mathbf{\Gamma}_{f_1}$ and $\mathbf{\Gamma}_{f_2}$. In the noise free case, it is easy to show that

$$\mathbf{z}_{f_2} = \mathbf{P}_f^T \mathbf{z}_{f_1}. \tag{21}$$

In the presence of noise, this relation no longer holds. However, by assuming we have collected (or are about to collect) N I/O measurements, an unbiased estimate of the propagator $\mathbf{P}_f$ can be obtained by introducing an instrumental variable $\boldsymbol{\xi} \in \mathbb{R}^{n_\xi \times 1}$, assumed to be uncorrelated with the noise but sufficiently correlated with the state vector $\mathbf{x}$, in the sense that

$$\mathbf{R}_{\mathbf{e}_f \boldsymbol{\xi}} = \mathbf{0} \text{ and } \operatorname{rank}\{\mathbf{R}_{\mathbf{x}\boldsymbol{\xi}}\} = n_x, \tag{22}$$

and by defining the cost function (see [4])

$$J_{IV}(\mathbf{P}_f) = \left\| \hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(N) - \mathbf{P}_f^T \hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(N) \right\|_F^2. \tag{23}$$

Four algorithms (`IVPM`, `EIVPM`, `EIVsqrtPM` and `COIVPM` [11, 12]) have been developed to minimise this criterion according to the number of instruments in $\boldsymbol{\xi}$. In the following the `IVPM` and `EIVPM` algorithms will be described in detail. However it should be noticed that, along the same lines, the asymptotic properties of the other two algorithms can be worked out (the details are omitted for brevity). In fact, the `EIVsqrtPM` and the `COIVPM` algorithms differ from the `IVPM` and `EIVPM` only in an algorithmic way and are based on the same least squares estimate [12].

3.3 The `IVPM` algorithm

Assuming that it is possible to build an instrumental variable such that $n_\xi = n_x$, the least squares estimate of the propagator is given by

$$\hat{\mathbf{P}}_f^T(N) = \hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(N) \hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}^{-1}(N) \tag{24}$$

provided the indicated inverse exists (see Section 4 for invertibility conditions). Then, an RLS recursive version of (24), named `IVPM` [10], can be obtained by applying the matrix inversion lemma [3] to $\mathbf{R}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}$

$$\mathbf{K}_f(t) = \frac{\boldsymbol{\xi}^T(t)\mathbf{L}_f(t-1)}{\lambda + \boldsymbol{\xi}^T(t)\mathbf{L}_f(t-1)\mathbf{z}_{f_1}(t)} \tag{25a}$$

$$\mathbf{L}_f(t) = \frac{1}{\lambda}\left(\mathbf{L}_f(t-1) - \mathbf{L}_f(t-1)\mathbf{z}_{f_1}(t)\mathbf{K}_f(t)\right) \tag{25b}$$

$$\begin{aligned}\hat{\mathbf{P}}_f^T(t) = {}& \hat{\mathbf{P}}_f^T(t-1) + \\ &+ \left(\mathbf{z}_{f_2}(t) - \hat{\mathbf{P}}_f^T(t-1)\mathbf{z}_{f_1}(t)\right)\mathbf{K}_f(t),\end{aligned} \tag{25c}$$

where $\mathbf{L}_f(t) = \hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}^{-1}(t)$ and $0 < \lambda \leq 1$ is a forgetting factor.

3.4 The `EIVPM` algorithm

In many cases, it is difficult or impossible to construct an instrumental variable such as $n_\xi = n_x$. Thus, it is necessary to increase the number of instruments. By assuming that the input is sufficiently "rich" (see Section 4) so that the (rectangular) matrix $\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(N)$ is full rank, the least squares estimate of the propagator is given by

$$\hat{\mathbf{P}}_f^T(N) = \hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(N)\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}^{\dagger}(N) \tag{26}$$

with $\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}^{\dagger}(N) = \hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}^{T}(N)\left(\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(N)\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}^{T}(N)\right)^{-1}$. Then, a recursive version of (26), named `EIVPM` [10], can be obtained by adapting the overdetermined instrumental variable technique first proposed in [2]

$$\mathbf{g}_f(t) = \begin{bmatrix}\hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(t)\boldsymbol{\xi}(t) & \mathbf{z}_{f_2}(t)\end{bmatrix} \tag{27a}$$

$$\boldsymbol{\Lambda}(t) = \begin{bmatrix}-\boldsymbol{\xi}^T(t)\boldsymbol{\xi}(t) & \lambda \\ \lambda & 0\end{bmatrix} \tag{27b}$$

$$\boldsymbol{\Psi}_f(t) = \begin{bmatrix}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t-1)\boldsymbol{\xi}(t) & \mathbf{z}_{f_1}(t)\end{bmatrix} \tag{27c}$$

$$\begin{aligned}\mathbf{K}_f(t) = {}& \left(\boldsymbol{\Lambda}(t) + \boldsymbol{\Psi}_f^T(t)\mathbf{L}_f(t-1)\boldsymbol{\Psi}_f(t)\right)^{-1} \\ & \qquad\qquad\qquad \boldsymbol{\Psi}_f^T(t)\mathbf{L}_f(t-1)\end{aligned} \tag{27d}$$

$$\begin{aligned}\mathbf{P}_f^T(t) = {}& \mathbf{P}_f^T(t-1) + \\ &+ \left(\mathbf{g}_f(t) - \mathbf{P}_f^T(t-1)\boldsymbol{\Psi}_f(t)\right)\mathbf{K}_f(t)\end{aligned} \tag{27e}$$

$$\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t) = \lambda\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t-1) + \mathbf{z}_{f_1}(t)\boldsymbol{\xi}^T(t) \tag{27f}$$

$$\hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(t) = \lambda\hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(t-1) + \mathbf{z}_{f_2}(t)\boldsymbol{\xi}^T(t) \tag{27g}$$

$$\mathbf{L}_f(t) = \frac{1}{\lambda^2}\left(\mathbf{L}_f(t-1) - \mathbf{L}_f(t-1)\boldsymbol{\Psi}(t)\mathbf{K}_f(t)\right) \tag{27h}$$

with $\mathbf{L}_f(t) = \left(\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}^T(t)\right)^{-1}$.

Remark 4 *The computational complexity of `EIVPM` and `IVPM` has been analysed in [10], where a comparison with other RSMI algorithms based on `PAST` [18] and `EIVPAST` [4] has been presented. It was more precisely shown that the overall computational cost of `PM` approaches is comparable to the one of `PAST` and its by-products.*

It is interesting to note that, from the second iteration on, the update of the estimated subspace is always in the form $\hat{\boldsymbol{\Gamma}}_f = \begin{bmatrix}\mathbf{I}_{n_x} & \hat{\mathbf{P}}_f\end{bmatrix}^T$. This means that, after a short transient period, the recursive estimation is made in the same state-space basis. This loss of flexibility can be firstly considered as a drawback, more particularly from a numerical point of view. Indeed, the fact that the first n_x rows of $\hat{\boldsymbol{\Gamma}}_f$ are equal to the identity matrix could lead to conditioning problems when the proposed algorithms are applied to ill conditioned systems. However, we have not encountered any numerical problems when simulating the algorithm in MATLAB. Furthermore, contrary to the PAST approach where the signal subspace is obtained by minimising a modified function obtained after an approximation, the PM cost functions are always computed without approximations nor constraints. More particularly, in the PAST framework, the minimization of approximated criterion leads to a matrix having columns that are not exactly orthonormal. This property evolves during the recursive minimization since, under some conditions, the minimizer converges to a matrix with orthonormal columns [18]. This evolution can be interpreted as a slow change of basis, which implies that it is not possible to guarantee that $\mathbf{\Gamma}_f(t)$ and $\mathbf{\Gamma}_f(t-1)$ are expressed in the same state-space coordinates. This might represent a problem in the estimation of the state-space realization $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$, and particularly so whenever RSMI techniques are used for change detection purposesThe PM algorithms do not suffer from this drawback.

4 Convergence analysis

In this Section, the convergence conditions and the asymptotic distribution of the estimation error of the EIVPM and IVPM algorithms are derived. More precisely, conditions will be established by which the considered recursive algorithms converge to consistent estimates of the propagator $\mathbf{P}_f$.

4.1 Convergence analysis of EIVPM

Consider the extended instrumental variable estimate (26) and assume that $\lambda = 1$. Then

$$\hat{\mathbf{P}}_f^T(t)\mathbf{R}(t) = \left[\frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(t)\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)\right] \tag{28}$$

with $\mathbf{R}(t) = \left[\frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)\right]$. Note that the right hand side of (28) can be equivalently written as

$$\hat{\mathbf{P}}_f^T(t)\mathbf{R}(t) = \left[\frac{1}{t}\sum_{\tau=1}^{t}\mathbf{z}_{f_2}(\tau)\boldsymbol{\xi}^T(\tau)\right]\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}. \tag{29}$$

Now, since

$$\begin{bmatrix}\mathbf{z}_{f_1}(t)\\ \mathbf{z}_{f_2}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{n_x}\\ \mathbf{P}_f^T\end{bmatrix}\mathbf{\Gamma}_{f_1}\tilde{\mathbf{x}}(t) + \begin{bmatrix}\tilde{\mathbf{b}}_{f_1}(t)\\ \tilde{\mathbf{b}}_{f_2}(t)\end{bmatrix}, \tag{30}$$

it is possible to write $\mathbf{z}_{f_2}$ in terms of the true $\mathbf{P}_f^T$

$$\mathbf{z}_{f_2}(t) = \mathbf{P}_f^T\mathbf{z}_{f_1}(t) + \left(\tilde{\mathbf{b}}_{f_2}(t) - \mathbf{P}_f^T\tilde{\mathbf{b}}_{f_1}(t)\right). \tag{31}$$

Letting $\boldsymbol{\eta}(t) = \left(\tilde{\mathbf{b}}_{f_2}(t) - \mathbf{P}_f^T\tilde{\mathbf{b}}_{f_1}(t)\right)$ and introducing equation (31) in the relation (29), we get

$$\hat{\mathbf{P}}_f^T(t)\mathbf{R}(t) = \mathbf{P}_f^T\frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t) + \frac{1}{t}\hat{\mathbf{R}}_{\boldsymbol{\eta}\boldsymbol{\xi}}(t)\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t) \tag{32}$$

and finally, from (28) and (32) we have

$$\left(\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right)\mathbf{R}(t) = \frac{1}{t}\hat{\mathbf{R}}_{\boldsymbol{\eta}\boldsymbol{\xi}}(t)\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t). \tag{33}$$

Along the lines of [8, Chapter 4], the convergence analysis of EIVPM is based on the analysis of (33), *i.e.*,

1. proving that $\left(\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right)\mathbf{R}(t) \to \mathbf{0}$ with probability 1 (w.p.1) as $t \to \infty$;
2. deriving conditions under which $\mathbf{R}(t)$ converges to a constant matrix $\bar{\mathbf{R}}$ as $t \to \infty$, with $\bar{\mathbf{R}}$ full rank.

These two steps are considered in the following subsections.

4.1.1 Convergence of $\left(\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right)\mathbf{R}(t)$

From (33), it is easy to establish the following proposition:

Proposition 5 *Under assumptions 1-4, consider algorithm* (27) *and further assume that* $\boldsymbol{\xi} \in \mathbb{R}^{n_\xi \times 1}$ *($n_\xi \geq n_x$) is uncorrelated with the noise but sufficiently correlated with the state vector* $\mathbf{x}$, i.e., (22) *holds. Then* $\left(\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right)\mathbf{R}(t) \to \mathbf{0}$ *w.p.1 as* $t \to \infty$.

Proof 6 *This result can be proved by showing that the quantity* $\frac{1}{t}\hat{\mathbf{R}}_{\boldsymbol{\eta}\boldsymbol{\xi}}(t)$, *which appears in the right hand side of equation* (33), *converges to* $\mathbf{0}$. *To this purpose, consider the definition of* $\tilde{\mathbf{b}}_f$ *in equation* (13) *and compute the correlation*

$$\frac{1}{t}\sum_{\tau=1}^{t}\tilde{\mathbf{b}}_f(\tau)\boldsymbol{\xi}^T(\tau) = \frac{1}{t}\sum_{\tau=1}^{t}\left[\mathbf{G}_f\left(\mathbf{e}_f(\tau) - \mathbf{E}_f(\tau-1)\mathbf{U}_f^T(\tau-1)\boldsymbol{\beta}_f(\tau)\right)\boldsymbol{\xi}^T(\tau)\right] \tag{34}$$

and note that the above quantity tends to (see, e.g., [7])

$$\mathbf{G}_f\left(\mathbf{R}_{\mathbf{e}_f\boldsymbol{\xi}} - \mathbf{R}_{\mathbf{e}_f\mathbf{u}_f}\mathbf{R}_{\mathbf{u}_f}^{-1}\mathbf{R}_{\mathbf{u}_f\boldsymbol{\xi}}\right) \tag{35}$$

w.p.1 when $t \to \infty$. *Furthermore, according to the assumptions on* $\boldsymbol{\xi}$ *and* $\mathbf{u}$, *it is straightforward that* $\mathbf{R}_{\mathbf{e}_f\mathbf{u}_f} = \mathbf{0}$ *and* $\mathbf{R}_{\mathbf{e}_f\boldsymbol{\xi}} = \mathbf{0}$. *Thus, we have*

$$\frac{1}{t}\sum_{\tau=1}^{t}\tilde{\mathbf{b}}_f(\tau)\boldsymbol{\xi}^T(\tau) \to \mathbf{0} \text{ w.p.1 as } t \to \infty \tag{36}$$

and recalling that $\boldsymbol{\eta}(t) = \tilde{\mathbf{b}}_{f_2}(t) - \mathbf{P}_f^T\tilde{\mathbf{b}}_{f_1}(t)$ *we have*

$$\frac{1}{t}\sum_{\tau=1}^{t}\boldsymbol{\eta}(\tau)\boldsymbol{\xi}^T(\tau) \to \mathbf{0} \text{ w.p.1 as } t \to \infty, \tag{37}$$

from which, in view of (33), *the thesis follows.*

4.1.2 Convergence of $\mathbf{R}(t)$

In order to complete the convergence analysis, we need to study under which conditions $\mathbf{R}(t)$ converges to a full rank matrix in order to conclude that $\hat{\mathbf{P}}_f^T(t)$ tends to $\mathbf{P}_f^T$ w.p.1 as t tends to infinity. To this purpose, it is sufficient to analyse the rank of $\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)$ as t tends to infinity by construction of $\mathbf{R}(t)$. Indeed, if conditions are fulfilled such as $\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)$ is full rank, $\mathbf{R}(t)$ will satisfy the same property and will be invertible. For that, note first of all that in the propagator basis (see (30)) the first n_x rows of (16) (*i.e.*, the $\mathbf{z}_{f_1}$ part) are given by

$$\mathbf{z}_{f_1}(t) = \tilde{\mathbf{x}}(t) + \tilde{\mathbf{b}}_{f_1}(t). \tag{38}$$

Then, recalling the definition of $\tilde{\mathbf{x}}$ given in (14), we have that

$$\frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t) = \frac{1}{t}\sum_{\tau=1}^{t}\tilde{\mathbf{b}}_{f_1}(\tau)\boldsymbol{\xi}^T(\tau) + \frac{1}{t}\sum_{\tau=1}^{t}\left[\left(\mathbf{x}(\tau) - \mathbf{X}(\tau-1)\mathbf{U}_f^T(\tau-1)\boldsymbol{\beta}_f(\tau)\right)\boldsymbol{\xi}^T(\tau)\right]. \tag{39}$$

Since (36) holds, the first term of the right hand side of (39) can be neglected. Therefore, we have that

$$\frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t) \to \bar{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}} = \mathbf{R}_{\mathbf{x}\boldsymbol{\xi}} - \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\mathbf{R}_{\mathbf{u}_f}^{-1}\mathbf{R}_{\mathbf{u}_f\boldsymbol{\xi}} \tag{40}$$

w.p.1 when t tends to infinity. Noting that matrix $\mathbf{R}_{\mathbf{x}\boldsymbol{\xi}} - \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\mathbf{R}_{\mathbf{u}_f}^{-1}\mathbf{R}_{\mathbf{u}_f\boldsymbol{\xi}}$ is the Schur complement of the block $\mathbf{R}_{\mathbf{u}_f}$ in matrix

$$\begin{bmatrix}\mathbf{R}_{\mathbf{x}\boldsymbol{\xi}} & \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\\ \mathbf{R}_{\mathbf{u}_f\boldsymbol{\xi}} & \mathbf{R}_{\mathbf{u}_f}\end{bmatrix},$$

a sufficient condition that guarantees that $\bar{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}$ is full rank is given by

$$\operatorname{rank}\left\{\bar{\mathbb{E}}\left[\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}_f(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \mathbf{u}_f(t)\end{bmatrix}^T\right]\right\} = n_x + fn_u. \tag{41}$$

Note that the above condition does not depend on the choice of the instrumental variable. Thus, the invertibility of $\bar{\mathbf{R}}$ is ensured for any choice of instrumental variable such that (41) holds.

Therefore, (41) is a sufficient condition that guarantees that $\mathbf{R}(t)$ converges to a constant, full rank matrix $\bar{\mathbf{R}}$, as $t \to \infty$.

If the instrumental variable vector is constructed as

$$\boldsymbol{\xi}(t) = \begin{bmatrix}\mathbf{y}_p^T(t) & \mathbf{u}_p^T(t)\end{bmatrix}^T = \begin{bmatrix}\mathbf{y}^T(t-p) \cdots \mathbf{y}^T(t-1)\ \mathbf{u}^T(t-p) \cdots \mathbf{u}^T(t-1)\end{bmatrix}^T, \tag{42}$$

where parameter p defines the horizon over which past input and output data are taken to form the IV vector, then equation (41) corresponds to the so-called *critical relation* for the consistency of IV subspace identification algorithms, first derived in the classical paper [5]. In particular, conditions under which (41) holds have been derived in the cited paper and lead to the following result for the convergence of the `EIVPM` algorithm.

Proposition 7 *Under assumptions 1-4 (see Section 2), consider the* `EIVPM` *algorithm* (27) *and further assume that:*

- $\mathbf{K} = \mathbf{0}$ *(*i.e.*, no process noise);*
- *the instruments are composed by past input and output data with* $p \geq n_x$ *(see* (42)*);*
- *the input* $\mathbf{u}$ *is persistently exciting of order* $f + p + n_x$*;*
- *the forgetting factor* λ *is chosen equal to 1.*

Then $\hat{\mathbf{P}}_f^T(t) \to \mathbf{P}_f^T$ *w.p.1 as* $t \to \infty$

In addition, convergence can also be guaranteed if $\mathbf{K} \neq \mathbf{0}$, for example[1]

Proposition 8 *Under assumptions 1-4 (see Section 2), consider the* `EIVPM` *algorithm* (27) *and further assume that:*

- *the instruments are composed by past input and output samples with* $p \geq n_x$ *(see* (42)*);*
- *the input* $\mathbf{u}$ *is a zero mean white sequence;*
- $\operatorname{rank}\{\mathcal{C}_p\{\mathbf{A},\mathbf{B}\}\ \mathcal{C}_p\{\mathbf{A},\mathbf{G}\}\} = n_x$ *with*

$$\mathcal{C}_p\{\mathbf{L},\mathbf{M}\} = \begin{bmatrix}\mathbf{L}^{p-1}\mathbf{M} & \cdots \mathbf{LM} & \mathbf{M}\end{bmatrix} \tag{43}$$

$$\mathbf{G} = \bar{\mathbb{E}}\left\{\mathbf{x}(t+1)\mathbf{y}^T(t)\right\}; \tag{44}$$

- *the forgetting factor* λ *is chosen equal to 1.*

Then $\hat{\mathbf{P}}_f^T(t) \to \mathbf{P}_f^T$ *w.p.1 as* $t \to \infty$

Other special cases (e.g., single input systems, `ARMA` input signal) can be analysed by exploiting the results in [5], and are omitted for brevity.

[1]This particular case will be considered in the numerical experiments.

4.2 Convergence analysis of IVPM

In the special case of the IVPM algorithm, the previous analysis can be further developed in order to derive more stringent convergence conditions. For that, note first of all that, assuming $\lambda = 1$, the quadratic criterion (23) is minimised by $\hat{\mathbf{P}}_f^T(t)$ such that

$$\hat{\mathbf{P}}_f^T(t)\mathbf{R}(t) = \frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_2}\boldsymbol{\xi}}(t) \tag{45}$$

where now $\mathbf{R}(t) = \frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)$. By following the same steps as in paragraph 4.1.1, it is straightforward to prove that

$$\left(\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right)\mathbf{R}(t) = \frac{1}{t}\hat{\mathbf{R}}_{\boldsymbol{\eta}\boldsymbol{\xi}}(t) \tag{46}$$

and that $\frac{1}{t}\hat{\mathbf{R}}_{\boldsymbol{\eta}\boldsymbol{\xi}}(t) \to \mathbf{0}$ w.p.1 as $t \to \infty$ when the input $\mathbf{u}$ is uncorrelated with the innovation $\mathbf{e}$ (system in open loop) and when the instrumental variable $\boldsymbol{\xi} \in \mathbb{R}^{n_x \times 1}$ is uncorrelated with the noise but sufficiently correlated with the state vector $\mathbf{x}$. Thus, the following proposition is verified:

Proposition 9 *Under assumptions 1-4 (see Section 2), consider algorithm* (25) *and further assume that* $\boldsymbol{\xi} \in \mathbb{R}^{n_x \times 1}$ *is uncorrelated with the noise but sufficiently correlated with the state vector* $\mathbf{x}$*. Then* $\left(\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right)\mathbf{R}(t) \to \mathbf{0}$ *w.p.1 as* $t \to \infty$.

The second step of the convergence analysis of IVPM is, just as before, the derivation of conditions under which $\mathbf{R}(t)$ converges to a full rank matrix. More precisely, it is possible to exploit the conclusions obtained in analysing EIVPM in order to conclude that

$$\mathbf{R}(t) \to \bar{\mathbf{R}} = \mathbf{R}_{\mathbf{x}\boldsymbol{\xi}} - \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\mathbf{R}_{\mathbf{u}_f}^{-1}\mathbf{R}_{\mathbf{u}_f\boldsymbol{\xi}} \tag{47}$$

w.p.1 when t tends to infinity. Indeed, the sufficient condition (41) insures, as previously, that $\mathbf{R}(t)$ converges to a full rank matrix. However, in the IVPM case, a constraint has to be satisfied: $n_\xi = n_x$. To study the influence of this condition in a widespread situation, consider that the instruments are chosen as past inputs, *i.e.*,

$$\boldsymbol{\xi}(t) = \mathbf{u}_p(t) = \begin{bmatrix}\mathbf{u}^T(t-p) & \cdots & \mathbf{u}^T(t-1)\end{bmatrix}^T \tag{48}$$

with $n_u p = n_x$. Thus, we get

$$\mathbf{R}(t) \to \bar{\mathbf{R}} = \mathbf{R}_{\mathbf{x}\mathbf{u}_p} - \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\mathbf{R}_{\mathbf{u}_f}^{-1}\mathbf{R}_{\mathbf{u}_f\mathbf{u}_p} \tag{49}$$

w.p.1 when t tends to infinity. As in paragraph 4.1.2, the application of the Schur complement leads to a sufficient condition for the invertibility of $\bar{\mathbf{R}}$:

$$\begin{bmatrix}\mathbf{R}_{\mathbf{x}\mathbf{u}_p} & \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\\ \mathbf{R}_{\mathbf{u}_f\mathbf{u}_p} & \mathbf{R}_{\mathbf{u}_f}\end{bmatrix} = \bar{\mathbb{E}}\left[\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}_f(t)\end{bmatrix}\begin{bmatrix}\mathbf{u}_p(t)\\ \mathbf{u}_f(t)\end{bmatrix}^T\right] > \mathbf{0}. \tag{50}$$

In order to check if (50) holds, we proceed along the lines of the analysis of the IV-4SID algorithm performed in [5]. To this purpose, the state vector is decomposed into its deterministic and its stochastic parts

$$\mathbf{x}(t) = \mathbf{x}^d(t) + \mathbf{x}^s(t) \tag{51}$$

where $\mathbf{x}^d$ is due to the observed inputs $\mathbf{u}$

$$\mathbf{x}^d(t) = \mathbf{F}^{\mathbf{u}}(q^{-1})\mathbf{u}(t) \tag{52}$$

and $\mathbf{x}^s$ is caused by the innovation $\mathbf{e}$

$$\mathbf{x}^s(t) = \mathbf{F}^{\mathbf{e}}(q^{-1})\mathbf{e}(t). \tag{53}$$

Substituting (51)-(53) in (50) we have

$$\bar{\mathbb{E}}\left[\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}_f(t)\end{bmatrix}\begin{bmatrix}\mathbf{u}_p(t)\\ \mathbf{u}_f(t)\end{bmatrix}^T\right] = \bar{\mathbb{E}}\left[\begin{bmatrix}\mathbf{x}^d(t)\\ \mathbf{u}_f(t)\end{bmatrix}\begin{bmatrix}\mathbf{u}_p(t)\\ \mathbf{u}_f(t)\end{bmatrix}^T\right] \tag{54}$$

since $\mathbf{u}$ is assumed to be uncorrelated with the innovation. Now, let

$$\mathbf{F}^{\mathbf{u}}(q^{-1}) = \frac{N_{\mathbf{F}^{\mathbf{u}}}(q^{-1})}{D_{\mathbf{F}^{\mathbf{u}}}(q^{-1})} \tag{55}$$

with $N_{\mathbf{F}^{\mathbf{u}}}(q^{-1}) = N_{\mathbf{F}^{\mathbf{u}}_1}q^{-1} + \cdots + N_{\mathbf{F}^{\mathbf{u}}_{n_x}}q^{-n_x}$ and $D_{\mathbf{F}^{\mathbf{u}}}(q^{-1}) = D_{\mathbf{F}^{\mathbf{u}}_0} + \cdots + D_{\mathbf{F}^{\mathbf{u}}_{n_x}}q^{-n_x}$. Then, the deterministic state $\mathbf{x}^d$ can be easily related to the input and the state-space matrices as follows

$$\mathbf{x}^d(t) = \frac{1}{D_{\mathbf{F}^{\mathbf{u}}}(q^{-1})}\begin{bmatrix} N_{\mathbf{F}^{\mathbf{u}}_{n_x}} & \cdots & N_{\mathbf{F}^{\mathbf{u}}_1}\end{bmatrix}\begin{bmatrix}\mathbf{u}(t-n_x)\\ \vdots \\ \mathbf{u}(t-1)\end{bmatrix}. \tag{56}$$

In the single input case, equation (54) can be equivalently written as

$$\bar{\mathbb{E}}\left[\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}_f(t)\end{bmatrix}\begin{bmatrix}\mathbf{u}_p(t)\\ \mathbf{u}_f(t)\end{bmatrix}^T\right] = \mathcal{M}\frac{1}{D_{\mathbf{F}^u}(q^{-1})}\bar{\mathbb{E}}\left[\bar{\mathbf{u}}(t)\bar{\mathbf{u}}^T(t)\right] \tag{57}$$

with $\bar{\mathbf{u}}(t) = \begin{bmatrix}u(t-n_x) & \cdots & u(t+f-1)\end{bmatrix}^T$ and

$$\mathcal{M} = \begin{bmatrix} N_{\mathbf{F}^u_{n_x}} & \cdots & N_{\mathbf{F}^u_1} & 0 & \cdots & 0 \\ D_{\mathbf{F}^u_{n_x}} & \cdots & D_{\mathbf{F}^u_1} & D_{\mathbf{F}^u_0} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & D_{\mathbf{F}^u_{n_x}} & \cdots & D_{\mathbf{F}^u_1} & D_{\mathbf{F}^u_0}\end{bmatrix}. \tag{58}$$

It is possible to show that $\mathcal{M}$ is full rank if the system is reachable (see [5, Appendix A] for details). By assuming this condition is fulfilled, the matrix in (57) is positive definite if u is persistently exciting of order $f + n_x$. Therefore $\bar{\mathbf{R}}$ in (49) is also nonsingular and $\hat{\mathbf{P}}_f^T(t) \to \mathbf{P}_f^T$ w.p.1 as $t \to \infty$. This result is summed up in the following proposition:

Proposition 10 *Under assumptions 1-4 (see Section 2), consider the* `IVPM` *algorithm* (25) *and further assume that:*

- *the system is single input;*
- *the input is persistently exciting of order $f + n_x$;*
- *the forgetting factor λ is chosen equal to 1.*

Then $\hat{\mathbf{P}}_f^T(t) \to \mathbf{P}_f^T$ w.p.1 as $t \to \infty$.

4.3 Asymptotic distribution

Once it is proved that the `IVPM` and `EIVPM` algorithms lead to consistent estimates, the following phase consists in expressing the asymptotic distribution of the projector estimates, which can provide a useful tool for the assessment of the accuracy of the computed estimates. For that, we first analyse the distribution of the `EIVPM` estimate. `IVPM` will follow as a special case.

The probability distribution of the `EIVPM` estimate is characterised by the following proposition:

Proposition 11 *Consider the* `EIVPM` *algorithm* (27)*. Then, under the assumptions of Proposition 7, the propagator estimate is asymptotically normal:*

$$\sqrt{t}\text{vec}\left\{\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right\} \in \text{AsN}\left(0, \left(\bar{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}\bar{\mathbf{R}}^{-1} \otimes \mathbf{I}\right)\begin{bmatrix}\mathbf{Q}_1 & & \\ \mathbf{0} & \ddots & \mathbf{0} \\ & & \mathbf{Q}_{n_\xi}\end{bmatrix}\right) \tag{59}$$

with $\bar{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}} = \mathbf{R}_{\mathbf{x}\boldsymbol{\xi}} - \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\mathbf{R}^{-1}_{\mathbf{u}_f}\mathbf{R}_{\mathbf{u}_f\boldsymbol{\xi}}$, $\bar{\mathbf{R}} = \bar{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}\bar{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}$ *and*

$$\mathbf{Q}_i = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\sum_{j=1}^{t}\boldsymbol{\eta}(j)\boldsymbol{\xi}_i^T(j)\sum_{k=1}^{t}\boldsymbol{\xi}_i(k)\boldsymbol{\eta}^T(k)\right]. \tag{60}$$

Proof 12 *From* (33)*, it follows that*

$$\left(\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right)\left[\frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t)\right] = \frac{1}{t}\hat{\mathbf{R}}_{\boldsymbol{\eta}\boldsymbol{\xi}}(t)\frac{1}{t}\hat{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t). \tag{61}$$

According to paragraph 4.1.2, we have (see (40)*)*

$$\frac{1}{t}\hat{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}}(t) \to \bar{\mathbf{R}}_{\mathbf{z}_{f_1}\boldsymbol{\xi}} = \mathbf{R}_{\mathbf{x}\boldsymbol{\xi}} - \mathbf{R}_{\mathbf{x}\mathbf{u}_f}\mathbf{R}^{-1}_{\mathbf{u}_f}\mathbf{R}_{\mathbf{u}_f\boldsymbol{\xi}} \text{ w.p.1 as } t \to \infty \tag{62}$$

Thus, by vectorizing $\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T$*, knowing that [3]* $\operatorname{vec}\{\mathbf{ABC}\} = \left(\mathbf{C}^T \otimes \mathbf{A}\right)\operatorname{vec}\{\mathbf{B}\}$ *where* $\otimes$ *is the Kronecker product, we get*

$$\sqrt{t}\operatorname{vec}\left\{\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right\} = \left(\bar{\mathbf{R}}^T_{\mathbf{z}_{f_1}\boldsymbol{\xi}}\bar{\mathbf{R}}^{-1} \otimes \mathbf{I}\right)\operatorname{vec}\left\{\frac{1}{\sqrt{t}}\sum_{\tau=1}^{t}\boldsymbol{\eta}(\tau)\boldsymbol{\xi}^T(\tau)\right\} \tag{63}$$

Now, from the central limit theorem we know that, by writing the $i-th$ *component of* $\boldsymbol{\xi}$ *as* $\boldsymbol{\xi}_i$*, for all* i

$$\frac{1}{\sqrt{t}}\sum_{\tau=1}^{t}\boldsymbol{\eta}(\tau)\boldsymbol{\xi}_i^T(\tau) \in \operatorname{AsN}(0, \mathbf{Q}_i), \tag{64}$$

with $\mathbf{Q}_i$ *defined as in* (60)*. Thus, it follows that*

$$\operatorname{vec}\left\{\frac{1}{\sqrt{t}}\sum_{\tau=1}^{t}\boldsymbol{\eta}(\tau)\boldsymbol{\xi}^T(\tau)\right\} \in \operatorname{AsN}\left(0, \begin{bmatrix}\mathbf{Q}_1 & & \\ \mathbf{0} & \ddots & \mathbf{0} \\ & & \mathbf{Q}_{n_\xi}\end{bmatrix}\right). \tag{65}$$

(59) *follows by combining* (63) *and* (65)*.*

In the same way, it is straightforward to prove that:

Proposition 13 *Consider the* `IVPM` *algorithm* (25)*; then, under the assumptions of Proposition 10, the propagator estimate is asymptotically normal:*

$$\sqrt{t}\operatorname{vec}\left\{\hat{\mathbf{P}}_f^T(t) - \mathbf{P}_f^T\right\} \in \operatorname{AsN}\left(0, \left(\bar{\mathbf{R}}^{-1} \otimes \mathbf{I}\right)\begin{bmatrix}\mathbf{Q}_1 & & \\ \mathbf{0} & \ddots & \mathbf{0} \\ & & \mathbf{Q}_{n_x}\end{bmatrix}\right) \tag{66}$$

with $\bar{\mathbf{R}} = \mathbf{R}_{\mathbf{x}u_p} - \mathbf{R}_{\mathbf{x}u_f}\mathbf{R}^{-1}_{u_f}\mathbf{R}_{u_f u_p}$*.*

5 Simulation example

Consider the following fourth-order system:

$$\mathbf{x}(t+1) = \begin{bmatrix} 0.67 & 0.67 & 0 & 0 \\ -0.67 & 0.67 & 0 & 0 \\ 0 & 0 & -0.67 & -0.67 \\ 0 & 0 & 0.67 & -0.67 \end{bmatrix}\mathbf{x}(t) + + \begin{bmatrix} 0.6598 \\ 1.9698 \\ 4.3171 \\ -2.6436 \end{bmatrix}u(t) + \begin{bmatrix} -0.1027 \\ 0.5501 \\ 0.3545 \\ -0.5133 \end{bmatrix}e(t) \tag{67}$$

$$y(t) = \begin{bmatrix} -0.5749 & 1.0751 & -0.5225 & 0.183 \end{bmatrix}\mathbf{x}(t) \tag{68}$$

$$-0.7139\, u(t) + 0.9706\, e(t). \tag{69}$$

This SISO process has been chosen in order to apply at the same time the `IVPM` and `EIVPM` algorithms. The input u and the innovation e are white Gaussian noises with zero mean and variance 1 and 9 respectively. This leads to a signal to noise ratio (of variances) at the output ($\operatorname{cov}[y]/\operatorname{cov}[e]$) of 4.4. The initial estimates of the system matrices are randomly generated under the constraint that the absolute value of the maximum eigenvalue of $\hat{\mathbf{A}}(0)$ is less than 1 (stability requirement). The forgetting factor is fixed at 1 in order to meet the assumptions of the convergence study. The instrumental variable vector is constructed using past values of the input u. All these conditions allow to satisfy the constraints

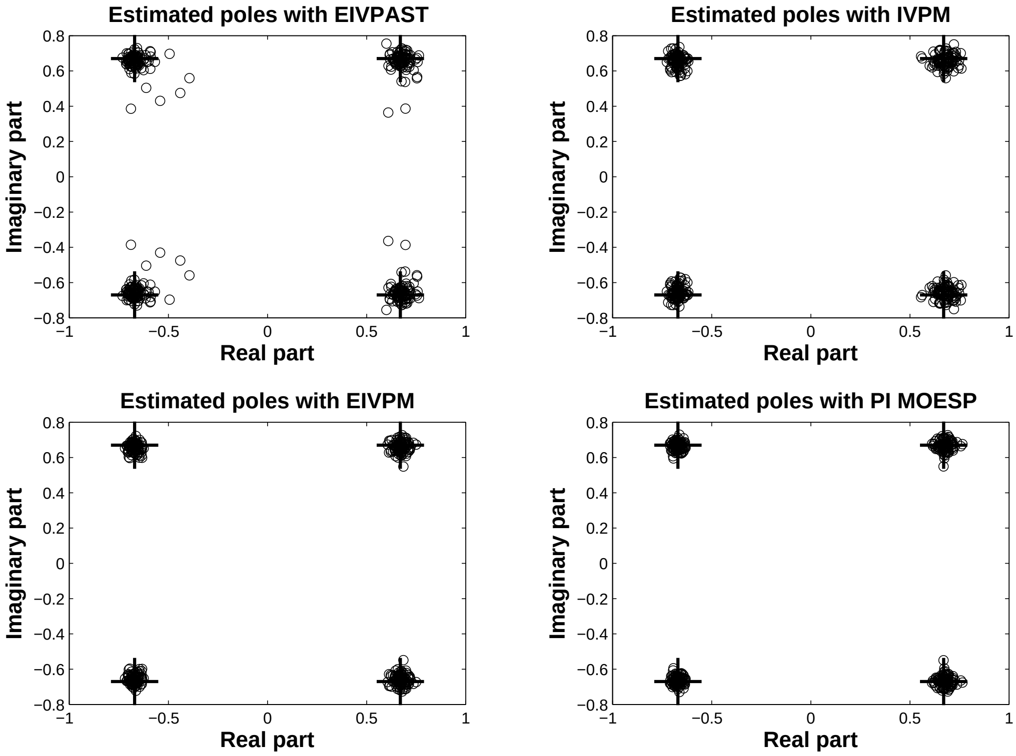

Figure 1: Estimated poles obtained with `IVPM`, `EIVPM`, `EIVPAST` and `PI MOESP` algorithms for a Monte Carlo simulation of size 100.

of Propositions 8 and 10. The results presented in this Section have been obtained by repeating the identification procedure 100 times using different noise realizations for e.

The first result to be verified is the asymptotic convergence of the matrix $\mathbf{R}(t)$ to a full rank matrix; for the considered example, $\mathbf{R}(t)$ converges to a matrix with a condition number of about 3 ($\sigma_{max} \simeq 30$ and $\sigma_{min} \simeq 3$).

In order to underline the efficiency of the instrumental variable propagator based methods, it is next proposed to compare the accuracy of the estimated poles obtained with `IVPM` and `EIVPM` and two other subspace identification techniques: the recursive `EIVPAST` algorithm [4, 9] and the batch `PI MOESP` algorithm [15]. More precisely, the eigenvalues of the estimated $\mathbf{A}$ matrices during the Monte Carlo simulation are plotted and compared with the true ones (*i.e.*, $\pm 0.67 \pm 0.67\jmath$). 300 samples are used for each realization. The final model, *i.e.*, $\left\{\hat{\mathbf{A}}(300), \hat{\mathbf{B}}(300), \hat{\mathbf{C}}(300), \hat{\mathbf{D}}(300)\right\}$ is considered for the recursive methods. The parameters are chosen as follows: $f = p = 4$ for `IVPM` and $f = p = 8$ for the other algorithms. Figure 1 shows that `IVPM` and `EIVPM` produce eigenvalue estimates which are equally accurate as the ones obtained with the batch `PI MOESP` algorithm. On the contrary, the `EIVPAST` method yields several poorly accurate estimates. It can be noted that the variance of the eigenvalue estimates of `IVPM` is slightly larger than the `EIVPM` one. Furthermore, in practice, it seems to be more convenient to use `EIVPM` instead of `IVPM` since the first one only needs an upper bound of the system order for the instrumental variable construction, which is not the case with `IVPM`.

Finally, the averaged time histories of the real parts of the estimated eigenvalues obtained with `EIVPM` and `EIVPAST` techniques have been analysed. These averaging results (omitted for brevity) illustrate the fact that the transient behaviour of `EIVPAST` method is relatively more chaotic than the `EIVPM` one. This could be explained by the fact that the projection approximation which characterises PAST-based algorithms might lead to a loss of accuracy during transients.

6 Concluding remarks

In this paper, the convergence properties of a class of recursive subspace identification algorithms have been investigated. More precisely, online implementations of the `MOESP` methods based on instrumental variable versions of the propagator technique for signal subspace estimation have been analysed, convergence proofs for the `IVPM` and `EIVPM` algorithms have been derived and conditions on the input signal have been given which guarantee consistent estimates of the observability matrix. A simulation example has been used to illustrate the validity of the underlying assumptions and to assess the performance of the algorithms.

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