Identifiability of discrete-time nonlinear systems: the local state isomorphism approach

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Abstract

A new theorem is provided to test the identifiability of discrete-time systems with polynomial nonlinearities. That extends to discrete-time systems the local state isomorphism approach for continuous-time systems. Two examples are provided to illustrate the approach.

Key words: Identifiability; discrete-time systems; polynomial nonlinearities.

1 Problem statement

Very often, the equations of a model involve unknown parameters that must be estimated from experimental data. A fundamental problem is whether the value of these parameters can be uniquely determined from input/output measurements. This is known as the parameter identifiability problem.

Several approaches for testing the parameter identifiability have been proposed in the literature for continuous and discrete-time controlled systems. An overview for continuous-time nonlinear systems can be found in [16][17] including the input/output relation approach, the output equality approach and the local state isomorphism approach. The input/output relation approach, based on algebra, leads, under some conditions, to a necessary and sufficient condition of identifiability. In the continuous-time case [6][7], the system is transformed, by eliminating the state variables considered as unknowns, into a system depending only on the input, the output, their derivatives and the parameters. If, from the resulting system, the parameters can be rewritten as a unique expression depending only on the input, the output and their derivatives, they are identifiable. For instance, for polynomial systems, the state elimination can be achieved with the Gröbner bases approach [2], the characteristic set approach [11] or the resultant approach [19]. In the discrete-time case, the derivatives are replaced by the iterates [1]. In general, in the input/output relation approach, the initial conditions on the state are not considered since they are eliminated. However, in [4][5][9], it is shown that the input/output relation approach can fail when the system starts at specific initial condition and, in this particular case, another procedure is provided.
The output equality approach leads only, in general, to a sufficient condition of identifiability. It consists in testing whether the equality of two output trajectories from the same initial condition, depending respectively on two parameter values, implies the equality of both these parameter values. If so, the parameters are identifiable. In the continuous-time case, this is formulated as the Taylor series expansion approach [10]. In the discrete-time case, the equality of the output trajectories is tested directly, sample by sample [8].
The local state isomorphism approach takes into account the initial conditions on the state and leads to a necessary and sufficient condition of identifiability for controlled systems. This approach, only proposed for continuous-time systems [14][15], is based on the isomorphism theorem [13]. Basically, this theorem states that if the system is locally reduced (locally observable and controllable) and is conjugated to another system, up to an isomorphism, the system is identifiable if this isomorphism is unique and is the identity.
In the literature, there is no attempt to extend the local state isomorphism approach to the discrete-time case. And yet, having several approaches at hand can be useful, because it is difficult to determine a priori the best suited approach for a particular case.

In this paper, the local state isomorphism approach is extended to discrete-time systems. Our study is restricted to polynomial systems. However, this restriction is not severe

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as explained in [7].

The paper is organized as follows. In Section 2, a formal definition of identifiability for discrete-time nonlinear systems is recalled. Then, in Section 3, a new condition for testing the identifiability of discrete-time systems with polynomial nonlinearities is derived. Finally, in Section 4, two examples illustrate the proposed approach.

2 Identifiability

Consider the discrete-time system of the general form:

$$\Sigma_\theta \begin{cases} x_{k+1} = f_\theta(x_k, u_k) \\ y_k \quad = h_\theta(x_k, u_k) \end{cases} \tag{1}$$

where $x_k \in \mathscr{X} \subset \mathbb{R}^n$ is the state vector, $y_k \in \mathscr{Y} \subset \mathbb{R}^p$ the measurable and so available output vector, $u_k \in \mathscr{U} \subset \mathbb{R}^m$ the input vector, f_θ and h_θ are functions in x_k and u_k, both parametrized by $\theta \in \Theta \subset \mathbb{R}^l$, the parameter vector.

A lot of definitions of parameter identifiability are given in the literature [8][14][15][18]. Local and global identifiability can be distinguished. Local identifiability [1] ensures a finite number of solutions for $\theta \in \Theta$ and thus holds for $\theta \in v(\theta) \subset \Theta$, where $v(\theta)$ is a neighborhood of θ. Local identifiability is a necessary condition for global identifiability which ensures the uniqueness of the solution for $\theta \in \Theta$. Besides, identifiability is said structural if it holds for almost every (a.e.) $\theta \in \Theta$, except possibly for a subset of zero measure in Θ (atypical values) which leads to singularities and where no conclusion about identifiability is possible.
The following definitions, borrowed from [8], will be considered.

Definition 1 *An input sequence over a window of iterations* $[0,N]$, *denoted by* $\{u_k\}_0^N$, *is called an* admissible input on $[0,N]$ *if the difference equation (1) admits a unique local solution.*

For any positive N, $\mathscr{U}^N$ denotes hereafter the space of all sequences of admissible inputs $\{u_k\}_0^N$.

Definition 2 *The system Σ_θ is locally x_0-identifiable at θ, through the admissible input sequence $\{u_k\}_0^N$ and for a given initial condition x_0, if there exists an open neighborhood of θ, $v(\theta) \subset \Theta$, such that for any $\hat{\theta} \in v(\theta)$ and for any $\theta \in v(\theta)$:*

$$\hat{\theta} \neq \theta \Rightarrow \{y_k(x_0, u_k, \hat{\theta})\}_0^N \neq \{y_k(x_0, u_k, \theta)\}_0^N \tag{2}$$

where $\{y_k(x_0, u_k, \theta)\}_0^N$ represents the input/output behavior of the system Σ_θ (1), depending on the parameter vector θ, i.e. the output sequence, from the initial condition x_0, for the input u_k, over the time interval $[0,N]$.

[1] Local identifiability defined in [7] is called algebraic identifiability in [6].

Definition 2 is the direct counterpart of the definition given in [14] for continuous-time nonlinear systems.

Definition 3 *The system Σ_θ is structurally identifiable if there exist $N > 0$, an open subset $\mathscr{X}_0 \subset \mathscr{X}$ and some dense subsets Θ and $\mathscr{U}_0^N \subset \mathscr{U}^N$, such that, $\forall\, x_0 \in \mathscr{X}_0$, a.e. $\theta \in \Theta$ and $\forall\, \{u_k\}_0^N \in \mathscr{U}_0^N$, the system Σ_θ is locally x_0-identifiable at θ through the admissible input sequence $\{u_k\}_0^N$.*

Hereafter, the identifiability will be considered in the sense of Def. 3. In the next section, the local state isomorphism approach for testing the identifiability is presented.

3 Approach based on the isomorphism theorem

The proposed approach is based on the isomorphism theorem for discrete-time polynomial systems [12], recalled in the following. The definitions required to render this theorem self-consistent are given in Appendix A.

3.1 Isomorphism theorem

Consider the discrete-time polynomial system Σ of the general form:

$$\Sigma \begin{cases} x_{k+1} = f(x_k, u_k) \\ y_k \quad = h(x_k, u_k) \end{cases} \tag{3}$$

where $x_k \in \mathscr{X} \subset \mathbb{R}^n$, $u_k \in \mathscr{U} \subset \mathbb{R}^m$ and $y_k \in \mathscr{Y} \subset \mathbb{R}^p$.

System (3) is called a K-system (Appendix A, Def. 7) since the functions f and h are polynomials in x_k and u_k. The following assumptions are required to state the isomorphism theorem.

- Each state x_k of Σ can be rewritten as a polynomial in y_k, u_k and their iterates, i.e. Σ is algebraically observable (Appendix A, Def. 8).
- It is possible to find an input sequence allowing, from a given initial condition $x_0 \in \mathscr{X}$, to reach a final state x_f, for almost every $x_f \in \mathscr{X}$, except a set of zero measure, i.e. Σ is quasi-reachable (Appendix A, Def. 9 and 10). Since Σ is algebraically observable and quasi-reachable, it is canonical (Appendix A, Def. 11).
- The system Σ admits, or *realizes*, a response map defined as follows.

Definition 4 *The response map $\mathscr{C}_\Sigma$ of Σ is the function that maps, for a given x_0, a finite sequence of non zero inputs, $\{u_k\}_0^N$, to the output y_N:*

$$\mathscr{C}_\Sigma(u_0, \dots, u_N) = h(f^N(x_0, u_0), u_N) = y_N$$
$$\textit{with:} \begin{cases} f^N(x_0, u_0) = x_0 & \textit{if } N = 0 \\ f^N(x_0, u_0) = f(f^{N-1}(x_0, u_0), u_{N-1}) & \forall N \geq 1 \end{cases} \tag{4}$$

Rather than working with the input/output map (Appendix A, Def. 12) of the system as in Def. 2, we can alternatively consider the response map (4) because there is an one-to-one correspondence between the input/output map and the response map (see [12], page 55).

The following definition of a K-system isomorphism is needed for the isomorphism theorem, given hereafter.

Definition 5 *[12] A* K-system isomorphism $T : \Sigma \to \hat{\Sigma}$ *is a map* $T : \mathscr{X} \to \hat{\mathscr{X}}$ *satisfying* $T(x_k) = \hat{x}_k$ *and,* $\forall x_k \in \mathscr{X}$, $\forall u_k \in \mathscr{U}$*:*

$$\begin{aligned} &(i) \quad \hat{x}_0 = T(x_0) \\ &(ii) \quad \hat{f}(T(x_k),u_k) = T(f(x_k,u_k)) \\ &(iii) \quad \hat{h}(T(x_k),u_k) = h(x_k,u_k) \end{aligned} \tag{5}$$

with f and $\hat{f}$ the dynamic functions and h and $\hat{h}$ the output functions of the K-systems Σ and $\hat{\Sigma}$ respectively.

Theorem 1 *[12] Let $\mathscr{C}$ be a polynomial response map. Then there exists a canonical K-system Σ realizing $\mathscr{C}$. If $\hat{\Sigma}$ is another canonical K-system realizing $\mathscr{C}$, there is a unique K-system isomorphism $T : \Sigma \to \hat{\Sigma}$.*

A condition for testing the identifiability in the discrete-time case, according to Def. 3, is formulated in the next section.

3.2 Main result

Consider the system Σ_θ of the form (1) with f_θ and h_θ being polynomials in x_k and u_k and depending on the parameter vector θ. The response map of Σ_θ, denoted $\mathscr{C}_{\Sigma_\theta}$, is given by $\mathscr{C}_{\Sigma_\theta}(u_0,\dots,u_N) = h_\theta(f_\theta^N(x_0(\theta),u_0),u_N)$. The notation $x_0(\theta)$ means that one or several components of the initial condition x_0 can be considered as parameter.
Consider also $\Sigma_{\hat{\theta}}$, the same system as Σ_θ, except that it depends on the parameter vector $\hat{\theta}$. The response map of $\Sigma_{\hat{\theta}}$ is $\mathscr{C}_{\Sigma_{\hat{\theta}}}(u_0,\dots,u_N) = h_{\hat{\theta}}(f_{\hat{\theta}}^N(x_0(\hat{\theta}),u_0),u_N)$.

Theorem 2 *If Σ_θ (1) is a canonical K-system, then it is structurally identifiable if and only if there exist $N > 0$, an open subset $\mathscr{X}_0 \subset \mathscr{X}$, some dense subsets Θ and $\mathscr{U}_0^N \subset \mathscr{U}^N$, such that, $\forall x_0 \in \mathscr{X}_0$, $\forall x_k \in \mathscr{X}$, $\forall \{u_k\}_0^N \in \mathscr{U}_0^N$, a.e. $\theta \in \Theta$, a.e. $\hat{\theta} \in \Theta$, for any K-system isomorphism $T : \Sigma_\theta \to \Sigma_{\hat{\theta}}$,*

$$\begin{aligned} &(i) \quad x_0(\hat{\theta}) = T(x_0(\theta)) \\ &(ii) \quad f_{\hat{\theta}}(T(x_k),u_k) = T(f_\theta(x_k,u_k)) \\ &(iii) \quad h_{\hat{\theta}}(T(x_k),u_k) = h_\theta(x_k,u_k) \end{aligned} \tag{6}$$

implies that $\hat{\theta} = \theta$.

Proof *Necessity*
For necessity, the following implication is shown.
The system Σ_θ is structurally identifiable implies that there exist $N > 0$, an open subset $\mathscr{X}_0 \subset \mathscr{X}$, some dense subsets Θ and $\mathscr{U}_0^N \subset \mathscr{U}^N$, such that, $\forall x_0 \in \mathscr{X}_0$, $\forall x_k \in \mathscr{X}$, $\forall \{u_k\}_0^N \in \mathscr{U}_0^N$, a.e. $\theta \in \Theta$, a.e. $\hat{\theta} \in \Theta$, for any K-system isomorphism $T : \Sigma_\theta \to \Sigma_{\hat{\theta}}$, (6) $\Rightarrow \theta = \hat{\theta}$.
The proof is made by the contrapositive.
$\forall N > 0$, $\forall \mathscr{X}_0 \subset \mathscr{X}$ an open subset, $\forall \Theta$ and $\forall \mathscr{U}_0^N \subset \mathscr{U}^N$ some dense subsets, there exist $x_0 \in \mathscr{X}_0$, $x_k \in \mathscr{X}$, $\{u_k\}_0^N \in \mathscr{U}_0^N$, $\theta \in \Theta$, $\hat{\theta} \in \Theta$, a K-system isomorphism $T : \Sigma_\theta \to \Sigma_{\hat{\theta}}$ such that the relations (6) are fulfilled and $\theta \neq \hat{\theta}$ imply that the system Σ_θ is not structurally identifiable.
Assume that, $\forall N > 0$, $\forall \mathscr{X}_0 \subset \mathscr{X}$ an open subset, $\forall \Theta$ and $\forall \mathscr{U}_0^N \subset \mathscr{U}^N$ some dense subsets, there exist $x_0 \in \mathscr{X}_0$, $x_k \in \mathscr{X}$, $\{u_k\}_0^N \in \mathscr{U}_0^N$, $\theta \in \Theta$, $\hat{\theta} \in \Theta$, a K-system isomorphism $T : \Sigma_\theta \to \Sigma_{\hat{\theta}}$ such that the relations (6) are fulfilled and $\theta \neq \hat{\theta}$. According to Def. 4, the response map of Σ_θ is:

$$\mathscr{C}_{\Sigma_\theta}(u_0,\dots,u_N) = h_\theta(f_\theta^N(x_0(\theta),u_0)),u_N) \tag{7}$$

By (6(iii)), (7) can be rewritten as:

$$\mathscr{C}_{\Sigma_\theta}(u_0,\dots,u_N) = h_{\hat{\theta}}(T(f_\theta^N(x_0(\theta),u_0)),u_N) \tag{8}$$

By compositions of (6(ii)), (8) can be rewritten as:

$$\mathscr{C}_{\Sigma_\theta}(u_0,\dots,u_N) = h_{\hat{\theta}}(f_{\hat{\theta}}^N(T(x_0(\theta)),u_0),u_N) \tag{9}$$

By (6(i)), (9) can be rewritten as:

$$\mathscr{C}_{\Sigma_\theta}(u_0,\dots,u_N) = h_{\hat{\theta}}(f_{\hat{\theta}}^N(x_0(\hat{\theta}),u_0),u_N) = \mathscr{C}_{\Sigma_{\hat{\theta}}}(u_0,\dots,u_N) \tag{10}$$

According to (10), the systems Σ_θ and $\Sigma_{\hat{\theta}}$ have the same response map and thus, the same input/output behavior (there is an one-to-one correspondence between the input/output map and the response map), with different parameters θ and $\hat{\theta}$. By Def. 3, the system Σ_θ is not structurally identifiable, which proves the contrapositive.

Sufficiency
For sufficiency, the following implication is shown.
There exist $N > 0$, an open subset $\mathscr{X}_0 \subset \mathscr{X}$, some dense subsets Θ and $\mathscr{U}_0^N \subset \mathscr{U}^N$, such that, $\forall x_0 \in \mathscr{X}_0$, $\forall x_k \in \mathscr{X}$, $\forall \{u_k\}_0^N \in \mathscr{U}_0^N$, a.e. $\theta \in \Theta$, a.e. $\hat{\theta} \in \Theta$, for any K-system isomorphism $T : \Sigma_\theta \to \Sigma_{\hat{\theta}}$, (6) $\Rightarrow \theta = \hat{\theta}$ imply that the system Σ_θ is structurally identifiable.
The proof is made by the contrapositive.
The system Σ_θ is not structurally identifiable implies that, $\forall N > 0$, $\forall \mathscr{X}_0 \subset \mathscr{X}$ an open subset, $\forall \Theta$ and $\forall \mathscr{U}_0^N \subset \mathscr{U}^N$ some dense subsets, there exist $x_0 \in \mathscr{X}_0$, $x_k \in \mathscr{X}$, $\{u_k\}_0^N \in \mathscr{U}_0^N$, $\theta \in \Theta$, $\hat{\theta} \in \Theta$, a K-system isomorphism $T : \Sigma_\theta \to \Sigma_{\hat{\theta}}$ such that the relations (6) are fulfilled and $\theta \neq \hat{\theta}$.
Assume that the system Σ_θ is canonical and is not structurally identifiable. It means that, $\forall N > 0$, $\forall \mathscr{X}_0 \subset \mathscr{X}$ an

open subset, $\forall\Theta$ and $\forall\mathscr{U}_0^N \subset \mathscr{U}^N$ some dense subsets, there exist $x_0 \in \mathscr{X}_0$, $x_k \in \mathscr{X}$, $\{u_k\}_0^N \in \mathscr{U}_0^N$, $\theta \in \Theta$, $\hat{\theta} \in \Theta$ such that the systems Σ_θ and $\Sigma_{\hat{\theta}}$ have the same input/output behavior, and so the same response map $\mathscr{C}_{\Sigma_\theta} = \mathscr{C}_{\Sigma_{\hat{\theta}}}$ (there is a one-to-one correspondence between the input/output map and the response map), with different parameters θ and $\hat{\theta}$. On the other hand, if, $\forall N > 0$, $\forall\mathscr{X}_0 \subset \mathscr{X}$, $\forall\Theta$ and $\forall\mathscr{U}_0^N \subset \mathscr{U}^N$, there exist $x_0 \in \mathscr{X}_0$, $x_k \in \mathscr{X}$, $\{u_k\}_0^N \in \mathscr{U}_0^N$, $\theta \in \Theta$, $\hat{\theta} \in \Theta$ such that $\mathscr{C}_{\Sigma_\theta} = \mathscr{C}_{\Sigma_{\hat{\theta}}}$, according to Theorem 1 with $\hat{\Sigma} = \Sigma_{\hat{\theta}}$, $\hat{f} = f_{\hat{\theta}}$ and $\hat{h} = h_{\hat{\theta}}$, there exists a unique K-system isomorphism $T : \Sigma_\theta \to \Sigma_{\hat{\theta}}$ fulfilling (6). This completes the proof.

■

Remark 1 *When Σ_θ is structurally identifiable, according to Theorem 2, the only value $\hat{\theta}$ that satisfies (6) is $\hat{\theta} = \theta$. From (6), $T(x_k) = x_k$, i.e. the identity map on $\mathbb{R}^n$, is an obvious solution. Since the isomorphism T is unique, the identity map is the unique solution. Conversely, if $\hat{\theta} \neq \theta$ then $T(x_k) \neq x_k$.*

3.3 Testing the identifiability

The proposed approach for testing the identifiability of the system Σ_θ of the form (1) with f_θ and h_θ being polynomials in x_k and u_k is summed up by the steps below.

(1) Check whether Σ_θ is algebraically observable.
(2) Check whether Σ_θ is quasi-reachable. If Σ_θ is algebraically observable and quasi-reachable, it is canonical.
(3) Solve the three equations (6) in the variables $(\hat{\theta}, T)$. If the only solution is $(\hat{\theta}, T) = (\theta, \mathbf{1}_n)$, where $\mathbf{1}_n$ is the identity map on $\mathbb{R}^n$ (n is the dimension of the system), Σ_θ is structurally identifiable.

The steps (1) and (2) consist in checking whether Σ_θ is canonical, a necessary property required to apply Theorem 2. Only step (3) represents actually the test of identifiability.

It is worth noting that, for the computation, one can resort to computer algebra system, for instance the software Maxima, available for free at `http://maxima.sourceforge.net`. In the next section, two examples illustrate the proposed approach.

4 Examples

4.1 Example 1

Consider the following system:

$$\begin{cases} x_{k+1}^{(1)} = (1+\theta^{(1)})x_k^{(1)} + x_k^{(2)} & \text{(a)} \\ x_{k+1}^{(2)} = (1-\theta^{(2)})x_k^{(1)} + (x_k^{(2)})^2 + u_k & \text{(b)} \\ y_k = x_k^{(1)} & \text{(c)} \end{cases} \tag{11}$$

where $x_k = [x_k^{(1)} \quad x_k^{(2)}]^T \in \mathscr{X} \subset \mathbb{R}^2$. The initial condition, $x_0 = [x_0^{(1)} \quad x_0^{(2)}]^T$, is independent of the parameters. The structural identifiability of the parameter vector $\theta = [\theta^{(1)} \quad \theta^{(2)}]^T$ is tested with the proposed approach.

4.1.1 Algebraic observability

By using (11c) and (11a) successively, it can be shown that the two states can be rewritten as polynomials of the observable quantities:

$$\begin{aligned} x_k^{(1)} &= y_k \\ x_k^{(2)} &= y_{k+1} - (1+\theta^{(1)})y_k \end{aligned} \tag{12}$$

Consequently, the system (11) is algebraically observable. For the implementation, the aim is to express $x_k^{(1)}$ and $x_k^{(2)}$ in function of y_k and its iterates. To this end, (11) is iterated up to its observability indice in order to get as much equations as unknowns, the unknowns being the iterates of $x_k^{(1)}$ and $x_k^{(2)}$. The elimination of the unknowns can be achieved with the function `eliminate` in Maxima. It results a set of equations in $x_k^{(1)}$ and $x_k^{(2)}$ to be solved (function `solve`).

4.1.2 Quasi-reachability

Consider $x_f = [x_f^{(1)} \quad x_f^{(2)}]^T \in \mathscr{X}$, an arbitrary final state to be reached from a given initial condition $x_0 \in \mathscr{X}$. By two compositions of (11a) and (11b) successively, it is possible to find an input sequence that allows reaching x_f, for all $x_f \in \mathscr{X}$:

$$\begin{cases} u_0 = x_f^{(1)} - A_1(x_0^{(1)}, x_0^{(2)}) \\ u_1 = x_f^{(2)} - A_2(x_0^{(1)}, x_0^{(2)}) - (A_3(x_0^{(1)}, x_0^{(2)}) \\ \qquad + x_f^{(1)} - A_1(x_0^{(1)}, x_0^{(2)}))^2 \end{cases} \tag{13}$$

with

$$\begin{aligned} A_1(x_0^{(1)}, x_0^{(2)}) &= (1+\theta^{(1)})((1+\theta^{(1)})x_0^{(1)} + x_0^{(2)}) \\ &\quad + (1-\theta^{(2)})x_0^{(1)} + (x_0^{(2)})^2 \\ A_2(x_0^{(1)}, x_0^{(2)}) &= (1-\theta^{(2)})((1+\theta^{(1)})x_0^{(1)} + x_0^{(2)}) \\ A_3(x_0^{(1)}, x_0^{(2)}) &= (1-\theta^{(2)})x_0^{(1)} + (x_0^{(2)})^2 \end{aligned} \tag{14}$$

As a result, x_f, $\forall x_f \in \mathscr{X}$, can be reached with input sequence of length two $\{u_0, u_1\}$. Thus, the set of reachable states $\mathscr{X}_R$ is $\mathscr{X}$. Hence, the system (11) is reachable and consequently, quasi-reachable. Since the system (11) is algebraically observable and quasi-reachable, it is canonical. For the implementation purposes, the aim is to express u_0 and u_1 as a function of the final states $x_f^{(1)}$ and $x_f^{(2)}$ and the initial conditions $x_0^{(1)}$ and $x_0^{(2)}$. Thus, $x_k^{(1)}$, $x_k^{(2)}$ and their iterates must be eliminated in (11) (function `eliminate`). It results a set of equations to solve in u_0 and u_1 (function `solve`).

4.1.3 Solving (6) in $(\hat{\theta}, T)$

Let define the map $T : \mathscr{X} \to \mathscr{X}$:

$$T(x_k) = \begin{bmatrix} T_1(x_k^{(1)}, x_k^{(2)}) \\ T_2(x_k^{(1)}, x_k^{(2)}) \end{bmatrix} \tag{15}$$

with T_1 and T_2 two maps. Condition (6)(iii) implies:

$$T_1(x_k^{(1)}, x_k^{(2)}) = x_k^{(1)} \tag{16}$$

Condition (6)(ii) implies, for the equation (11a):

$$(\hat{\theta}^{(1)} - \theta^{(1)})x_k^{(1)} + T_2(x_k^{(1)}, x_k^{(2)}) - x_k^{(2)} = 0 \tag{17}$$

This leads to $\hat{\theta}^{(1)} = \theta^{(1)}$, $\forall x_k^{(1)} \neq 0$, and $T_2(x_k^{(1)}, x_k^{(2)}) = x_k^{(2)}$, $\forall x_k^{(2)}$. Consequently, T is reduced to the identity map on $\mathbb{R}^2$. Condition (6)(ii) implies, for the equation (11b):

$$(\hat{\theta}^{(2)} - \theta^{(2)})x_k^{(1)} = 0 \tag{18}$$

and then, (18) implies $\hat{\theta}^{(2)} = \theta^{(2)}$, $\forall x_k^{(1)} \neq 0$, $\forall x_k^{(2)}$. Hence, Theorem 2 is fulfilled. Thus, the system (11) is structurally identifiable. For the implementation, it leads to solve (16), (17) and (18) in $\hat{\theta}^{(1)}$, $\hat{\theta}^{(2)}$, $T_1(x_k^{(1)}, x_k^{(2)})$ and $T_2(x_k^{(1)}, x_k^{(2)})$ (function `solve` in Maxima).

4.2 Example 2

Consider the following system:

$$\begin{cases} x_{k+1}^{(1)} = \theta^{(1)}(x_k^{(1)})^2 + \theta^{(2)}x_k^{(2)} + u_k & \text{(a)} \\ x_{k+1}^{(2)} = \theta^{(3)}x_k^{(1)} & \text{(b)} \\ y_k \quad = x_k^{(1)} & \text{(c)} \end{cases} \tag{19}$$

where $x_k = [x_k^{(1)} \quad x_k^{(2)}]^T \in \mathscr{X} \subset \mathbb{R}^2$. The initial condition, $x_0 = [x_0^{(1)} \quad x_0^{(2)}]^T$, is independent of the parameters. The structural identifiability of the parameter vector $\theta = [\theta^{(1)} \quad \theta^{(2)} \quad \theta^{(3)}]^T$ is tested with the proposed approach.

4.2.1 Algebraic observability

By using (19c) and (19a) successively, it can be shown that the two states can be rewritten as polynomials of the observable quantities:

$$\begin{aligned} x_k^{(1)} &= y_k \\ x_k^{(2)} &= \frac{y_{k+1} - \theta^{(1)}y_k^2 - u_k}{\theta^{(2)}} \end{aligned} \tag{20}$$

assuming that $\theta^{(2)} \neq 0$. Consequently, the system (19) is algebraically observable.

4.2.2 Quasi-reachability

Consider $x_f = [x_f^{(1)} \quad x_f^{(2)}]^T \in \mathscr{X}$, an arbitrary final state to be reached from a given initial condition $x_0 \in \mathscr{X}$. By two compositions of (19a) and (19b), it is possible to find an input sequence that allows reaching x_f, for all $x_f \in \mathscr{X}$:

$$\begin{cases} u_0 = \dfrac{x_f^{(2)} - \theta^{(1)}\theta^{(3)}(x_0^{(1)})^2 - \theta^{(2)}\theta^{(3)}x_0^{(2)}}{\theta^{(3)}} \\ u_1 = x_f^{(1)} - \theta^{(2)}\theta^{(3)}x_0^{(1)} - \theta^{(1)}\Big(\theta^{(1)}(x_0^{(1)})^2 + \theta^{(2)}x_0^{(2)} \\ \qquad + \dfrac{x_f^{(2)} - \theta^{(1)}\theta^{(3)}(x_0^{(1)})^2 - \theta^{(2)}\theta^{(3)}x_0^{(2)}}{\theta^{(3)}}\Big)^2 \end{cases} \tag{21}$$

assuming that $\theta^{(3)} \neq 0$. As previously, it is shown that x_f, $\forall x_f \in \mathscr{X}$, can be reached with input sequence of length two. Thus, the set of reachable states $\mathscr{X}_R$ is $\mathscr{X}$. Hence, the system (19) is reachable and consequently, quasi-reachable. Since the system (19) is algebraically observable and quasi-reachable, it is canonical.

4.2.3 Solving (6) in $(\hat{\theta}, T)$

Let define the map $T : \mathscr{X} \to \mathscr{X}$:

$$T(x_k) = \begin{bmatrix} T_1(x_k^{(1)}, x_k^{(2)}) \\ T_2(x_k^{(1)}, x_k^{(2)}) \end{bmatrix} \tag{22}$$

with T_1 and T_2 two maps. Condition (6)(iii) implies:

$$T_1(x_k^{(1)}, x_k^{(2)}) = x_k^{(1)} \tag{23}$$

Condition (6)(ii) implies, for the equation (19a):

$$(\hat{\theta}^{(1)} - \theta^{(1)})(x_k^{(1)})^2 + \hat{\theta}^{(2)}T_2(x_k^{(1)}, x_k^{(2)}) - \theta^{(2)}x_k^{(2)} = 0 \tag{24}$$

This leads to $\hat{\theta}^{(1)} = \theta^{(1)}$, $\forall x_k^{(1)} \neq 0$, and $T_2(x_k^{(1)}, x_k^{(2)}) = \dfrac{\theta^{(2)}x_k^{(2)}}{\hat{\theta}^{(2)}}$, $\forall x_k^{(2)}$. Condition (6)(ii) implies, for the equation (19b):

$$(\hat{\theta}^{(3)} - \frac{\theta^{(2)}\theta^{(3)}}{\hat{\theta}^{(2)}})x_k^{(1)} = 0 \tag{25}$$

The equation (25) implies $\hat{\theta}^{(2)}\hat{\theta}^{(3)} = \theta^{(2)}\theta^{(3)}$, $\forall x_k^{(1)} \neq 0$, $\forall x_k^{(2)}$. Consequently, $T(x_k)$ becomes:

$$\begin{bmatrix} x_k^{(1)} \\ \dfrac{\theta^{(2)}}{\hat{\theta}^{(2)}}x_k^{(2)} \end{bmatrix} \tag{26}$$

Theorem 2 is not fulfilled. Thus, the system (19) is not structurally identifiable.

5 Conclusion

In this paper, a new condition for testing the identifiability of discrete-time polynomial systems has been established. This condition has extended to discrete-time systems the local state isomorphism approach for continuous-time systems and constitutes an alternative to the other approaches. Extending the result to other types of nonlinearity is a challenging problem.

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References

[1] F. Anstett, G. Millérioux, and G. Bloch. Chaotic cryptosystems: cryptanalysis and identifiability. *IEEE Trans. on Circuits and Systems I*, 53(12):2673–2680, 2006.

[2] B. Buchberger. *An algorithm for finding a basis for the residue class ring of zero-dimensional polynomial ideal*. PhD thesis, Math. Inst. Univ. of Innsbrück, Austria, 1965.

[3] M. J. Chappell, K. R. Godfrey, and S. Vajda. Global identifiability of the parameters of nonlinear systems with specified inputs: a comparison method. *Mathematical Biosciences*, 102(1):41–73, 1990.

[4] L. Denis-Vidal, G. Joly-Blanchard, and C. Noiret. Some effective approaches to check the identifiability of uncontrolled nonlinear systems. *Mathematics and Computers in Simulation*, 57(1-2):35–44, 2001.

[5] L. Denis-Vidal, G. Joly-Blanchard, and C. Noiret. System identifiability (symbolic computation) and parameter estimation (numerical computation). *Numerical Algorithms*, 34(2-4):283–292, 2003.

[6] S. Diop and M. Fliess. Nonlinear observability, identifiability and persistent trajectories. In *Proc. of the 30th Conference on Decision and Control*, pages 714–718, Brighton, England. December 11-13, 1991.

[7] L. Ljung and T. Glad. On global identifiability for arbitrary model parametrizations. *Automatica*, 30(2):265–276, 1994.

[8] S. Nõmm and C. H. Moog. Identifiability of discrete-time nonlinear systems. In *Proc. of the 6th IFAC Symposium on Nonlinear Control Systems, NOLCOS*, pages 477–489, Stuttgart, Germany. September 1-3, 2004.

[9] M. P. Saccomani, S. Audoly, and L. D'Angio. Parameter identifiability of nonlinear systems: the role of initial conditions. *Automatica*, 39(4):619-632, 2003.

[10] H. Pohjanpalo. System identifiability based on the power series expansion of the solution. *Mathematical Biosciences*, 41:21–33, 1978.

[11] J. F. Ritt. *Differential algebra*. American Mathematical Society, Providence, RI, 1950.

[12] E. D. Sontag. Polynomial response maps. In A. V. Balakrishnan and M. Thoma, editors, *Lecture Notes in Control and Information Sciences*, volume 13. Springer Verlag, 1979.

[13] H. J. Sussmann. Existence and uniqueness of minimal realizations of nonlinear systems. *Mathematical Systems Theory*, 10(3):263–284, 1977.

[14] E. T. Tunali and T.J. Tarn. New results for identifiability of nonlinear systems. *IEEE Trans. on Automatic Control*, 32(2):146–154, February 1987.

[15] S. Vajda and H. Rabitz. State isomorphism approach to global identifiability of nonlinear systems. *IEEE Trans. on Automatic Control*, 34(2):220–223, 1989.

[16] E. Walter and L. Pronzato. *Nonlinear Systems*. Chapmann and Hall, London, England, 1993.

[17] E. Walter and L. Pronzato. On the identifiability and distinguishability of nonlinear parametric models. *Mathematics and Computers in Simulation*, 42(2-3):125–134, October 1996.

[18] E. Walter and L. Pronzato. *Identification of parametric models from experimental data*. Springer-Verlag, 1997.

[19] D. Wang. Elimination theory, methods and practice. In D. Lin, W. Li, and Y. Yu, editors, *Mathematics and Mathematics-Mechanization*, pages 91–137, Jinan, 2001. Shandong Education Publishing House. Available at `http://www-calfor.lip6.fr/~wang/papers`.

A Appendix

Consider the system Σ (3).

Definition 6 *A K-space is a topological space X together with an algebra of polynomial functions on X.*

Definition 7 *The system Σ is a K-system if and only if the state space $\mathscr{X}$ is a K-space and the dynamic map f and the output map h are polynomial maps.*

Definition 8 *The system Σ is algebraically observable if and only if each state is polynomial in the observable quantities (input u_k and output y_k).*

Definition 9 *The set of reachable states $\mathscr{X}_R$ from the initial condition x_0 is the set of states which can be reached through a finite non zero input sequence $\{u_k\}_0^N$, where N represents a finite positive integer.*

Definition 10 *The system Σ is reachable if and only if $\mathscr{X}_R = \mathscr{X}$. The system Σ is quasi-reachable if the closure of $\mathscr{X}_R$ is $\mathscr{X}$.*

Definition 11 *The system Σ is canonical if and only if it is algebraically observable and quasi-reachable.*

Definition 12 *The input/output map of Σ is a function that maps, for a given initial condition x_0, a finite sequence of non zero input, $\{u_k\}_0^N$, to a finite sequence of output, denoted $\{y_k(x_0, u_k)\}_0^N$.*

Definition 13 *Given a response map $\mathscr{C}$, the system* realizes *$\mathscr{C}$ if and only if $\mathscr{C} = \mathscr{C}_\Sigma$.*