H-infinity functional filtering for stochastic bilinear systems with multiplicative noises

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$\mathcal{H}_\infty$ functional filtering for stochastic bilinear systems with multiplicative noises

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Abstract

This paper deals with the design of a reduced-order $\mathcal{H}_\infty$ filter for a stochastic bilinear system with a prescribed $\mathcal{H}_\infty$ norm criterion. The considered system is bilinear in control and with multiplicative noises in the dynamics and in the measurement equations. The problem is transformed into the search of a unique gain matrix by using Sylvester-like constraints. The approach is based on the resolution of LMI and is then easily implementable.

Keywords : Reduced-order $\mathcal{H}_\infty$ filter; Stochastic differential equation; Multiplicative noise; Wiener process; Bilinear system; Quadratic Lyapunov function; LMI.

1 Introduction

This paper is devoted to the synthesis of functional filter for continuous-time systems bilinear in the control inputs and corrupted by multiplicative noises both in the state and in the measurement equations. Bilinear systems can represent an efficient tool to describe physical processes when linear models are not sufficiently efficient (see (Mohler, 1991) and references therein). For example, an activated sludge process can be modelled by a bilinear differential equations as shown in (Ekman, 2005); a semi-active suspension can be also described by mean of a bilinear model (Hać, 1992; Saif, 1994). Stochastic differential equations may be used to represent physical systems when their models are not exactly known or are corrupted by noises. In such situation, deterministic models are not suitable. Stochastic systems have been used in various areas of application, for example, systems with human operators, mathematical models of finance which represent some kind of uncertainties as stochastically varying lags, mechanical systems subject to random vibrations (e.g. earthquakes), ... (see J.L. Willems and J.C. Willems (1976) and B. Øksendal (2003)).
One of the motivations to study continuous-time systems bilinear in the control inputs with multiplicative noises is the robustness point of view. Ugrinovskii *et al.* (Ugrinovskii, 1998; Ugrinovskii and Petersen, 1999) show that parameter uncertainties can be efficiently modelled by Wiener processes. This leads to a stochastic differential equation with multiplicative noises. In these two papers, an example of mechanical system with an uncertainty on the spring stiffness is given. This case can be extended to the bilinear model of a semi-active suspension (see (Hać, 1992; Saif, 1994)) with uncertainties both in the tire and suspension stiffnesses, where, in addition, the road elevation at the point of contact with the tire can be considered as a brownian motion.
Stability and control of stochastic systems have been studied in numerous references (Willems and Willems, 1976; Kozin, 1969; Has'minskii, 1980; Mao, 1997; Hinrichsen and Pritchard, 1998). The filtering of stochastic systems with multiplicative noises has been treated in many papers (Carravetta and Germani, 1998; Carravetta *et al.*, 2000; Germani *et al.*, 2002). In (Carravetta *et al.*, 2000), a suboptimal filter is proposed using polynomial approximations. In (Germani *et al.*, 2002), the filtering problem with unknown inputs is solved using a linear filter based on suitable decomposition of the state. The full and

the reduced-order $\mathcal{H}_\infty$ filtering for stochastic systems with multiplicative noises has been studied in (Xu and Chen, 2003; Gershon *et al.*, 2001; Xu and Chen, 2002; Stoica, 2002; Halabi *et al.*, 2006*a*; Halabi *et al.*, 2006*b*). Notice that a stochastic system in which the measurement equation is corrupted by a multiplicative noise is considered in (Carravetta and Germani, 1998; Carravetta *et al.*, 2000; Gershon *et al.*, 2001; Halabi *et al.*, 2006*a*; Halabi *et al.*, 2006*b*).

In this paper, the problem of reduced-order $\mathcal{H}_\infty$ filtering for a larger class of stochastic systems than those cited above is considered since the studied systems are with multiplicative noises in the state and measurement equations and with multiplicative control inputs. The stochastic differential equations considered in this paper will be of Itô type (Has'minskii, 1980).

The filtering approach proposed in this paper leads to Sylvester-like constraints on the drift of the estimation error and a change of variable on the control inputs. All solutions of the Sylvester-like constraints are parametrized by a unique gain matrix which is computed through a LMI ensuring both the mean-square stability and a given $\mathcal{H}_\infty$ norm criterion. Then the problem is reduced to the search of this unique gain matrix and the reduced-order stochastic filter matrices are computed using this gain. The $\mathcal{H}_\infty$ optimization method is used to attenuate the effect of additional exogenous disturbances with finite energy.

The paper is organized as follows. Section 2 states the reduced-order filtering problem for a stochastic bilinear system with multiplicative noises both in the state and in the measurement equations. The synthesis of the reduced-order filter is treated in section 3. First the filter matrices are parametrized through a unique gain matrix, second the mean-square stability of the estimation error is established using an augmented system, and third the $\mathcal{H}_\infty$ performance is guaranteed by computing the gain matrix. In section 4, a numerical example is given to illustrate the proposed approach.

Notations Throughout the paper, $\mathbf{E}$ represents expectation operator with respect to some probability measure $\mathcal{P}$. $L_2\left(\Omega, \mathbb{R}^k\right)$ is the space of square-integrable $\mathbb{R}^k$-valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of subsets of the sample space called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. $(\mathcal{F}_t)_{t\geqslant 0}$ denote an increasing family of σ-algebras $(\mathcal{F}_t) \in \mathcal{F}$. We denote by $\widehat{L}_2\left([0,\infty);\mathbb{R}^k\right)$ the space of non-anticipatory square-integrable stochastic process $f(.) = (f(t))_{t\in[0,\infty)}$ in $\mathbb{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying

$$\|f\|^2_{\widehat{L}_2} = \mathbf{E}\left\{\int_0^\infty \|f(t)\|^2 \,\mathrm{d}\, t\right\} < \infty$$

where $\|.\|$ is the well-known Euclidean norm. $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and the largest eigenvalues of a symmetric square matrix, respectively. $\langle .,. \rangle$ denotes the usual inner product associated with $\mathbb{R}^k$.

2 Problem statement

Consider the following stochastic bilinear system

$$\left\{\begin{array}{rcl} \mathrm{d}\, x(t) &=& \left(A_{t0} + \sum_{i=1}^m u_i(t) A_{ti}\right) x(t)\,\mathrm{d}\, t + Bv(t)\,\mathrm{d}\, t + A_{w0}x(t)\,\mathrm{d}\, w_0(t) \\ \mathrm{d}\, y(t) &=& Cx(t)\,\mathrm{d}\, t + Jx(t)\,\mathrm{d}\, w_1(t) \\ z(t) &=& Lx(t) \end{array}\right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output, $u(t) \in \mathbb{R}^m$ is the control input vector (with $u_i(t)$ the i^{th} component of $u(t)$), $z(t) \in \mathbb{R}^r$ is a functional to be estimated with $r < n$ and $v(t) \in \mathbb{R}^q$ is the perturbation signal with bounded energy. Without loss of generality L is assumed to be a full row rank matrix. $w_i(t)$ is a Wiener process verifying (Has'minskii, 1980)

$$\begin{aligned} \mathbf{E}\left\{\mathrm{d}\, w_j(t)\right\} &= 0, \quad \mathbf{E}\left\{\mathrm{d}\, w_j(t)^2\right\} = \mathrm{d}\, t, \quad \text{for } j = 0, 1, && \text{(2a)} \\ \mathbf{E}\left\{\mathrm{d}\, w_0(t)\,\mathrm{d}\, w_1(t)\right\} &= \mathbf{E}\left\{\mathrm{d}\, w_1(t)\,\mathrm{d}\, w_0(t)\right\} = \varphi\,\mathrm{d}\, t, \quad \text{with } |\varphi| < 1. && \text{(2b)} \end{aligned}$$

As in the most cases for physical processes, we assume that the stochastic bilinear system (1) has known bounded control inputs, i.e. $u(t) \in \Gamma \subset \mathbb{R}^m$, where

$$\Gamma = \left\{u(t) \in \mathbb{R}^m \mid u_{i\,\min} \leqslant u_i(t) \leqslant u_{i\,\max}, \quad \text{for } i = 1, \ldots, m\right\}. \tag{3}$$

First of all, let us give the following definition and assumption.

Definition 1. *(Kozin, 1969; Has'minskii, 1980) The stochastic bilinear system (1) with* $v(t) \equiv 0$ *is said to be mean-square stable if all initial states* $x(0)$ *yield*

$$\lim_{t\to\infty} \mathbf{E}\left\{\|x(t)\|^2\right\} = 0, \quad \forall u(t) \in \Gamma. \tag{4}$$

Assumption 1. *The stochastic bilinear system (1) is assumed to be mean-square stable.*

In this paper, we consider a reduced-order filter in the following form

$$\mathrm{d}\,\widehat{z}(t) = \left(M_0 + \sum_{i=1}^{m} u_i(t)M_i\right)\widehat{z}(t)\,\mathrm{d}\,t + \left(N_0 + \sum_{i=1}^{m} u_i(t)N_i\right)\mathrm{d}\,y(t) \tag{5}$$

where $\widehat{z}(t) \in \mathbb{R}^r$ is the filter state with $r < n$ and the matrices M_i and N_i (for $i = 0, \ldots, m$) are to be determined. Let $e(t) = z(t) - \widehat{z}(t)$ be the filtering error, then the design of a $\mathcal{H}_\infty$ reduced-order filter for system (1) can be formulated as follows.

Problem 1. *The goal of this paper is to design a reduced-order* $\mathcal{H}_\infty$ *filter (5) such that the filtering error* $e(t)$ *is mean-square stable and the following* $\mathcal{H}_\infty$ *performance*

$$\|e\|^2_{\widehat{L}_2} \leqslant \gamma^2\|v\|^2_{\widehat{L}_2} \tag{6}$$

is achieved from the disturbance $v(t)$ *to the filtering error* $e(t)$*, where the positive scalar* γ *denotes the* $\mathcal{H}_\infty$ *performance of the filter.*

3 Reduced-order filter synthesis

In this section, the filter matrices are parametrized through a unique gain matrix, then we show the mean-square stability for an augmented system (composed of the state and the error) and finally we give a theorem to ensure the $\mathcal{H}_\infty$ performance and to compute this gain matrix and thus the filter matrices are easily derived.

3.1 Parametrization of the filter matrices through a unique gain matrix Z

Let $e(t) = z(t) - \widehat{z}(t) = Lx(t) - \widehat{z}(t)$ be the estimation error, then its dynamics is given by the following stochastic differential equation

$$\begin{aligned}\mathrm{d}\,e(t) = &\left(M_0 + \sum_{i=1}^{m} u_i(t)M_i\right)e(t)\,\mathrm{d}\,t + LBv(t)\,\mathrm{d}\,t \\ &+ \left((LA_{t0} - M_0L - N_0C) + \sum_{i=1}^{m} u_i(t)(LA_{ti} - M_iL - N_iC)\right)x(t)\,\mathrm{d}\,t \\ &+ LA_{w0}x(t)\,\mathrm{d}\,w_0(t) - \left(N_0 + \sum_{i=1}^{m} u_i(t)N_i\right)Jx(t)\,\mathrm{d}\,w_1(t).\end{aligned} \tag{7}$$

Consider the following Sylvester-like constraints

$$LA_{ti} - M_iL - N_iC = 0, \quad \text{for } i = 0, \ldots, m. \tag{8}$$

The solution of these constraints is based on the approach proposed in (Darouach, 2000; Souley Ali *et al.*, 2006) such that the filter matrices can be expressed through a unique gain matrix.
In fact, since L is a full row rank matrix, relations (8) are equivalent to

$$(LA_{ti} - M_iL - N_iC)\begin{bmatrix}L^\dagger & (I_n - L^\dagger L)\end{bmatrix} = 0, \quad \text{for } i = 0, \ldots, m. \tag{9}$$

where $L^\dagger$ is a generalized inverse of matrix L satisfying $L = LL^\dagger L$ (since $\operatorname{rank} L = r$, we have $LL^\dagger = I_r$). Using the approach given in (Darouach, 2000; Souley Ali *et al.*, 2006), relation (9) can be equivalently rewritten as an expression of filter matrices M_i

$$M_i = \overline{\overline{A}}_i - N_i\overline{\overline{C}}, \quad \text{for } i = 0, \dots, m, \tag{10}$$

coupled with an algebraic constraint

$$N\Sigma = L\overline{A}, \tag{11}$$

where

$$N = \begin{bmatrix} N_0 & N_1 & \dots & N_m \end{bmatrix}, \qquad \overline{A} = \begin{bmatrix} \overline{A}_0 & \overline{A}_1 & \dots & \overline{A}_m \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \overline{C} & 0 & \dots & 0 \\ 0 & \overline{C} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \overline{C} \end{bmatrix},$$

$$\begin{aligned} \overline{A}_i &= A_{ti}(I_n - L^\dagger L), & \overline{\overline{A}}_i &= LA_{ti}L^\dagger, & &\text{for } i = 0, \dots, m, \\ \overline{C} &= C(I_n - L^\dagger L), & \overline{\overline{C}} &= CL^\dagger, \end{aligned}$$

(with $\Sigma \in \mathbb{R}^{(m+1)p \times (m+1)n}$) and a general solution to equation (11), if it exists, is given by

$$N = L\overline{A}\,\Sigma^\dagger + Z(I_{(m+1)p} - \Sigma\,\Sigma^\dagger), \tag{12}$$

where

$$Z = \begin{bmatrix} Z_0 & Z_1 & \dots & Z_m \end{bmatrix}, \tag{13}$$

is an arbitrary matrix of appropriate dimensions. Z can be seen as the "observer gain matrix". The solution of the constraint (11) exists if and only if the following condition is verified

$$\operatorname{rank} \begin{bmatrix} LA_{t0} & LA_{t1} & \dots & LA_{tm} \\ C & 0 & \dots & 0 \\ 0 & C & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C \\ L & 0 & \dots & 0 \\ 0 & L & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & L \end{bmatrix} = \operatorname{rank} \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C \\ L & 0 & \dots & 0 \\ 0 & L & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & L \end{bmatrix}. \tag{14}$$

To reduce the conservatism in the study of the stability conditions inherent to the definition of the set Γ in relation (3), let us introduce the following change of variable on the control inputs $u_i(t)$

$$u_i(t) = \alpha_i + \sigma_i\varepsilon_i(t), \quad \text{for } i = 1, \dots, m, \tag{15}$$

where $\alpha_i \in \mathbb{R}$ and $\sigma_i \in \mathbb{R}$ are given by

$$\alpha_i = \frac{1}{2}(u_{i\min} + u_{i\max}),\ \sigma_i = \frac{1}{2}(u_{i\max} - u_{i\min}), \quad \text{for } i = 1, \dots, m, \tag{16}$$

and $\alpha_0 = 1$ and $\sigma_0 = 0$. The new variable $\varepsilon(t)$ belongs to the polytope $\overline{\Gamma}$ defined by

$$\overline{\Gamma} = \left\{ \varepsilon(t) \in \mathbb{R}^m \mid \varepsilon_{i\min} = -1 \leqslant \varepsilon_i(t) \leqslant \varepsilon_{i\max} = 1, \quad \text{for } i = 1, \dots, m \right\}. \tag{17}$$

Using (15), the state equation (1) becomes

$$\mathrm{d}\,x(t) = (A_\alpha + A_\sigma\Delta_x(\varepsilon(t))H_x)\,x(t)\,\mathrm{d}\,t + Bv(t)\,\mathrm{d}\,t + A_{w0}x(t)\,\mathrm{d}\,w_0(t) \tag{18}$$

where

$$A_\alpha = \sum_{i=0}^{m} \alpha_i A_{ti}, \qquad A_\sigma = \begin{bmatrix} \sigma_1 A_{t1} & \dots & \sigma_m A_{tm} \end{bmatrix}$$

$$H_x = \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} \in \mathbb{R}^{mn\times n}, \qquad \Delta_x(\varepsilon(t)) = \begin{bmatrix} \varepsilon_1(t)I_n & 0 & \dots & 0 \\ 0 & \varepsilon_2(t)I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \varepsilon_m(t)I_n \end{bmatrix},$$

and then, under the algebraic constraint (11), the error dynamics (7) can be rewritten as

$$\begin{aligned} \mathrm{d}\, e(t) = \Big(\mathbb{A}_t - Z\mathbb{C}_t + (\widetilde{\mathbb{A}}_t - Z\widetilde{\mathbb{C}}_t)\Delta_e(\varepsilon(t))H_e\Big) e(t)\mathrm{d}\, t + \mathbb{B}v(t)\mathrm{d}\, t + \mathbb{A}_{w0}x(t)\mathrm{d}\, w_0(t) \\ + \Big(\mathbb{A}_{w1} - Z\mathbb{C}_{w1} + (\widetilde{\mathbb{A}}_{w1} - Z\widetilde{\mathbb{C}}_{w1})\Delta_x(\varepsilon(t))H_x\Big)\, x(t)\,\mathrm{d}\, w_1(t) \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\mathbb{A}_t = \sum_{i=0}^{m} \alpha_i \overline{\overline{A}}_i - L\overline{A}\ \Sigma^\dagger \Upsilon_\alpha, && \mathbb{C}_t = (I_{(m+1)p} - \Sigma\ \Sigma^\dagger)\Upsilon_\alpha, && \mathbb{B} = LB, \\ &\widetilde{\mathbb{A}}_t = \begin{bmatrix} \sigma_1 \overline{\overline{A}}_1 & \dots & \sigma_m \overline{\overline{A}}_m \end{bmatrix} - L\overline{A}\ \Sigma^\dagger \Upsilon_\sigma, && \widetilde{\mathbb{C}}_t = (I_{(m+1)p} - \Sigma\ \Sigma^\dagger)\Upsilon_\sigma, && \mathbb{A}_{w1} = L\overline{A}\ \Sigma^\dagger \Psi_\alpha, \\ &\mathbb{C}_{w1} = (I_{(m+1)p} - \Sigma\ \Sigma^\dagger)\Psi_\alpha, && \widetilde{\mathbb{A}}_{w1} = L\overline{A}\ \Sigma^\dagger \Psi_\sigma, && \mathbb{A}_{w0} = LA_{w0}, \\ &\widetilde{\mathbb{C}}_{w1} = (I_{(m+1)p} - \Sigma\ \Sigma^\dagger)\Psi_\sigma, \end{aligned}$$

and

$$\Upsilon_\alpha = \begin{bmatrix} CL^\dagger \\ \alpha_1 CL^\dagger \\ \vdots \\ \alpha_m CL^\dagger \end{bmatrix}, \qquad \Psi_\alpha = \begin{bmatrix} J \\ \alpha_1 J \\ \vdots \\ \alpha_m J \end{bmatrix},$$

$$\Upsilon_\sigma = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \sigma_1 CL^\dagger & 0 & \dots & 0 \\ 0 & \sigma_2 CL^\dagger & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_m CL^\dagger \end{bmatrix}, \qquad \Psi_\sigma = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \sigma_1 J & 0 & \dots & 0 \\ 0 & \sigma_2 J & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_m J \end{bmatrix},$$

$$\Delta_e(\varepsilon(t)) = \begin{bmatrix} \varepsilon_1(t)I_r & 0 & \dots & 0 \\ 0 & \varepsilon_2(t)I_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \varepsilon_m(t)I_r \end{bmatrix}, \qquad H_e = \begin{bmatrix} I_r \\ \vdots \\ I_r \end{bmatrix} \in \mathbb{R}^{mr\times r}.$$

Let us consider the following augmented state vector

$$\xi(t) = \begin{bmatrix} x^T(t) & e^T(t) \end{bmatrix}^T. \tag{20}$$

Then under the constraint (11), the dynamics of the augmented system is given by

$$\begin{cases} \mathrm{d}\,\xi(t) = \Big(\widehat{A}_t + \Delta\widehat{A}_t(t)\Big)\,\xi(t)\,\mathrm{d}\,t + \widehat{B}v(t)\,\mathrm{d}\,t + \widehat{A}_{w0}\xi(t)\,\mathrm{d}\,w_0(t) + \Big(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\Big)\,\xi(t)\,\mathrm{d}\,w_1(t) \\ e(t) = \widehat{C}\xi(t) \end{cases} \tag{21}$$

where $\widehat{C} = [\,0 \ \ I_r\,]$,

$$\widehat{A}_t = \begin{bmatrix} A_\alpha & 0 \\ 0 & \mathbb{A}_t - Z\mathbb{C}_t \end{bmatrix}, \qquad \Delta\widehat{A}_t(t) = H_1\Delta_\xi(\varepsilon(t))H, \qquad \widehat{B} = \begin{bmatrix} B \\ \mathbb{B} \end{bmatrix},$$

$$\widehat{A}_{w1} = \begin{bmatrix} 0 & 0_{n\times r} \\ \mathbb{A}_{w1} - Z\mathbb{C}_{w1} & 0 \end{bmatrix}, \qquad \Delta\widehat{A}_{w1}(t) = H_2\Delta_\xi(\varepsilon(t))H, \qquad \widehat{A}_{w0} = \begin{bmatrix} A_{w0} & 0 \\ \mathbb{A}_{w0} & 0_{r\times r} \end{bmatrix},$$

and

$$H_1 = \begin{bmatrix} A_\sigma & 0 \\ 0 & \widetilde{\mathbb{A}}_t - Z\widetilde{\mathbb{C}}_t \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 0_{n\times rm} \\ \widetilde{\mathbb{A}}_{w1} - Z\widetilde{\mathbb{C}}_{w1} & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} H_x & 0 \\ 0 & H_e \end{bmatrix}, \qquad \Delta_\xi(\varepsilon(t)) = \begin{bmatrix} \Delta_x(\varepsilon(t)) & 0 \\ 0 & \Delta_e(\varepsilon(t)) \end{bmatrix}.$$

Notice that from (17), $\Delta_\xi(\varepsilon(t))$ satisfies

$$\|\Delta_\xi(\varepsilon(t))\| \leqslant 1. \tag{22}$$

3.2 Mean-square stability of the estimation error

In this section, the mean-square stability conditions of the augmented system (21) are established. We first give the following lemma which will be used in the proof of the mean-square stability conditions.

Lemma 1. *(Wang* et al.*, 1992) Let $\mathcal{A}$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{W}$ and F be real matrices of appropriate dimensions such that $\mathcal{W} > 0$ and $F^TF \leqslant I$. Then the following inequalities hold :*

(i) For any scalar $\eta > 0$ and vectors x and $y \in \mathbb{R}^n$,

$$2x^T\mathcal{D}F\mathcal{S}y \leqslant \eta^{-1}x^T\mathcal{D}\mathcal{D}^Tx + \eta y^T\mathcal{S}^T\mathcal{S}y.$$

(ii) For any scalar $\eta > 0$ such that $\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}^T \geqslant 0$,

$$(\mathcal{A} + \mathcal{D}F\mathcal{S})^T\,\mathcal{W}^{-1}\,(\mathcal{A} + \mathcal{D}F\mathcal{S}) \leqslant \mathcal{A}^T\left(\mathcal{W} - \eta\mathcal{D}\mathcal{D}^T\right)^{-1}\mathcal{A} + \eta^{-1}\mathcal{S}^T\mathcal{S}.$$

Then the following theorem ensures the mean-square stability of the augmented system (21).

Theorem 1. *The system (21), with $v(t) \equiv 0$, is mean-square stable if there exist a matrix $\mathcal{P} = \mathcal{P}^T > 0$ and some given positive reals μ_1, μ_2 and μ_3, such that the following inequality holds*

$$\begin{bmatrix} (1,1) & \mathcal{P}H_1 & \widehat{A}_{w0}^T\mathcal{P} & \varphi^{\frac{1}{2}}\widehat{A}_{w0}^T\mathcal{P}H_2 & \widehat{A}_{w1}^T\mathcal{P} & 0 \\ H_1^T\mathcal{P} & -\mu_1 I_{(n+r)m} & 0 & 0 & 0 & 0 \\ \mathcal{P}\widehat{A}_{w0} & 0 & -\mathcal{P} & 0 & 0 & 0 \\ \varphi^{\frac{1}{2}}H_2^T\mathcal{P}\widehat{A}_{w0} & 0 & 0 & -\mu_3 I_{(n+r)m} & 0 & 0 \\ \mathcal{P}\widehat{A}_{w1} & 0 & 0 & 0 & -\mathcal{P} & \mathcal{P}H_2 \\ 0 & 0 & 0 & 0 & H_2^T\mathcal{P} & -\mu_2 I_{(n+r)m} \end{bmatrix} < 0 \tag{23}$$

where

$$(1,1) = \mathcal{P}\widehat{A}_t + \widehat{A}_t^T\mathcal{P} + (\mu_1 + \mu_2 + \varphi\mu_3)H^TH + \varphi\left(\widehat{A}_{w0}^T\mathcal{P}\widehat{A}_{w1} + \widehat{A}_{w1}^T\mathcal{P}\widehat{A}_{w0}\right). \tag{24}$$

Proof. Consider the system (21) with $v(t) \equiv 0$ and the following Lyapunov function candidate

$$V(\xi(t)) = \xi^T(t)\mathcal{P}\xi(t) \quad \text{with} \quad \mathcal{P} = \mathcal{P}^T > 0. \tag{25}$$

Applying Itô formula (Has'minskii, 1980; Mao, 1997) to the system (21), we get

$$\mathrm{d}\,V(\xi(t)) = \mathcal{L}V(\xi(t))\,\mathrm{d}\,t + 2\xi^T(t)\mathcal{P}\Phi(t)\xi(t) \tag{26}$$

where

$$\Phi(t) = \widehat{A}_{w0}\,\mathrm{d}\,w_0(t) + \left(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\right)\mathrm{d}\,w_1(t) \tag{27}$$

and

$$\mathcal{L}V(\xi(t))\,\mathrm{d}\,t = 2\xi^T(t)\left(\widehat{A}_t + \Delta\widehat{A}_t(t)\right)\xi(t)\,\mathrm{d}\,t + \langle\Phi(t)\xi(t), \mathcal{P}\Phi(t)\xi(t)\rangle. \tag{28}$$

With the above notations, relation (26) can be written as

$$\mathrm{d}\,V(\xi(t)) = 2\xi^T(t)\mathcal{P}\left(\widehat{A}_t + \Delta\widehat{A}_t(t)\right)\xi(t)\,\mathrm{d}\,t + \xi^T(t)\widehat{A}_{w0}^T\mathcal{P}\widehat{A}_{w0}\xi(t)\,\mathrm{d}\,t$$
$$+\xi^T(t)\left(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\right)^T\mathcal{P}\left(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\right)\xi(t)\,\mathrm{d}\,t + 2\xi^T(t)\widehat{A}_{w0}^T\mathcal{P}\left(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\right)\xi(t)\varphi\,\mathrm{d}\,t + \Psi(t) \tag{29}$$

where

$$\Psi(t) = 2\xi^T(t)\mathcal{P}\widehat{A}_{w0}\xi(t)\,\mathrm{d}\,w_0(t) + 2\xi^T(t)\mathcal{P}\left(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\right)\xi(t)\,\mathrm{d}\,w_1(t). \tag{30}$$

Using the majoration lemma 1, it can be shown that, there exist $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_3 > 0$ such that the following inequalities hold

$$2\xi^T(t)\mathcal{P}\Delta\widehat{A}_t(t)\xi(t) \leqslant \xi^T(t)\left(\mu_1^{-1}\mathcal{P}H_1H_1^T\mathcal{P} + \mu_1H^TH\right)\xi(t), \tag{31}$$

$$\left(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\right)^T\mathcal{P}\left(\widehat{A}_{w1} + \Delta\widehat{A}_{w1}(t)\right) \leqslant \widehat{A}_{w1}^T\left(\mathcal{P}^{-1} - \mu_2^{-1}H_2H_2^T\right)^{-1}\widehat{A}_{w1} + \mu_2H^TH, \tag{32}$$

where μ_2 verifies $\mathcal{P}^{-1} - \mu_2^{-1}H_2H_2^T > 0$, and

$$2\xi^T(t)\widehat{A}_{w0}^T\mathcal{P}\Delta\widehat{A}_{w1}(t)\xi(t) \leqslant \xi^T(t)\left(\mu_3^{-1}\widehat{A}_{w0}^T\mathcal{P}H_2H_2^T\mathcal{P}\widehat{A}_{w0} + \mu_3H^TH\right)\xi(t). \tag{33}$$

Using the Schur lemma, inequality (23) becomes

$$\mathcal{P}\widehat{A}_t + \widehat{A}_t^T\mathcal{P} + \mu_1^{-1}\mathcal{P}H_1H_1^T\mathcal{P} + \mu_1H^TH + \widehat{A}_{w0}^T\mathcal{P}\widehat{A}_{w0} + \widehat{A}_{w1}^T\left(\mathcal{P}^{-1} - \mu_2^{-1}H_2H_2^T\right)^{-1}\widehat{A}_{w1} + \mu_2H^TH$$
$$+\varphi\left(\widehat{A}_{w0}^T\mathcal{P}\widehat{A}_{w1} + \widehat{A}_{w1}^T\mathcal{P}\widehat{A}_{w0}\right) + \varphi\left(\mu_3^{-1}\widehat{A}_{w0}^T\mathcal{P}H_2H_2^T\mathcal{P}\widehat{A}_{w0} + \mu_3H^TH\right) = -\Theta < 0. \tag{34}$$

Now, taking the expectation of (29) (see (Mao, 1997)) and using relations (2) and inequalities (31)−(34), $\mathbf{E}\{\mathrm{d}\,V(\xi(t))\}$ can be bounded as

$$\mathbf{E}\left\{\mathrm{d}\,V(\xi(t))\right\} \leqslant \mathbf{E}\left\{\xi^T(t)(-\Theta)\xi(t)\,\mathrm{d}\,t\right\} + \underbrace{\mathbf{E}\left\{\Psi(t)\right\}}_{=0}. \tag{35}$$

Then the inequality (35) yields to

$$\mathbf{E}\left\{\mathrm{d}\,V(\xi(t))\right\} \leqslant -\lambda_{\min}(\Theta)\,\mathbf{E}\left\{\|\xi(t)\|^2\,\mathrm{d}\,t\right\} \tag{36}$$

with $\lambda_{\min}(\Theta) > 0$ if inequality (23) is verified.
Now let $c_1 > 0$ and $c_2 > 0$ be given by

$$c_1 = \frac{\lambda_{\min}(\Theta)}{\lambda_{\max}(\mathcal{P})} \text{ and } c_2 = \mathbf{E}\left\{\xi^T(0)\mathcal{P}\xi(0)\right\} \tag{37}$$

and with some elementary developments the following inequality is obtained

$$\mathbf{E}\left\{\|\xi(t)\|^2\right\} \leqslant \frac{c_2}{\lambda_{\min}(\mathcal{P})}e^{-c_1t} \tag{38}$$

and we have

$$\lim_{t\to\infty}\mathbf{E}\left\{\|\xi(t)\|^2\right\} = 0. \tag{39}$$

So the mean-square stability of the augmented system (21) is proved. □ □

3.3 Filter gain computation for $\mathcal{H}_\infty$ performance

In this section, the synthesis of the reduced-order $\mathcal{H}_\infty$ filter stated in problem 1 is given.

Theorem 2. *If the rank condition (14) is verified, the reduced-order $\mathcal{H}_\infty$ filtering problem 1 is solved for the system (1) with the filter (5) if, for some reals $\mu_1 > 0, \mu_2 > 0, \mu_3 > 0$ and $\beta > 0$, there exist matrices $P_1 = P_1^T > 0 \in \mathbb{R}^{n\times n}$, $P_2 = P_2^T > 0 \in \mathbb{R}^{r\times r}$ and $Y \in \mathbb{R}^{r\times(m+1)p}$ such that $\gamma > 0$ is minimized and the following LMI holds*

$$\left[\begin{array}{ccccccccccc}
(1,1) & A_\alpha^T L^T P_2 + L^T P_2 \mathbb{A}_t - L^T Y \mathbb{C}_t & P_1 B + L^T P_2 \mathbb{B} & P_1 A_\sigma & L^T P_2 \widetilde{\mathbb{A}}_t - L^T Y \widetilde{\mathbb{C}}_t & (1,6) & (1,7) & (1,8) & (1,9) & (1,10) & 0 \\
P_2 L A_\alpha + \mathbb{A}_t^T P_2 L - \mathbb{C}_t^T Y^T L & (2,2) & P_2 \mathbb{B} + \beta P_2 \mathbb{B} & P_2 L A_\sigma & \beta\left(P_2 \widetilde{\mathbb{A}}_t - Y \widetilde{\mathbb{C}}_t\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
B^T P_1 + \mathbb{B}^T P_2 L & B^T L^T P_2 + \beta \mathbb{B}^T P_2 & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_\sigma^T P_1 & A_\sigma^T L^T P_2 & 0 & -\mu_1 I_{mn} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\widetilde{\mathbb{A}}_t^T P_2 L - \widetilde{\mathbb{C}}_t^T Y^T L & \beta\left(\widetilde{\mathbb{A}}_t^T P_2 - \widetilde{\mathbb{C}}_t^T Y^T\right) & 0 & 0 & -\mu_1 I_{mr} & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,6)^T & 0 & 0 & 0 & 0 & -P_1 & -L^T P_2 & 0 & 0 & 0 & 0 \\
(1,7)^T & 0 & 0 & 0 & 0 & -P_2 L & -\beta P_2 & 0 & 0 & 0 & 0 \\
(1,8)^T & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_3 I_{mn} & 0 & 0 & 0 \\
(1,9)^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -P_1 & -L^T P_2 & (9,11) \\
(1,10)^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -P_2 L & -\beta P_2 & (10,11) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (9,11)^T & (10,11)^T & -\mu_2 I_{mn}
\end{array}\right] < 0 \quad (40)$$

where

$$\begin{aligned}
(1,1) &= P_1 A_\alpha + A_\alpha^T P_1 + \varphi\left(\mathbb{A}_{w0}^T\left(P_2 \mathbb{A}_{w1} - Y \mathbb{C}_{w1}\right) + \left(\mathbb{A}_{w1}^T P_2 - \mathbb{C}_{w1}^T Y^T\right)\mathbb{A}_{w0}\right) + 2(\mu_1 + \mu_2 + \varphi\mu_3) I_n, \\
(2,2) &= \beta\left(P_2 \mathbb{A}_t + \mathbb{A}_t^T P_2 - Y \mathbb{C}_t - \mathbb{C}_t^T Y^T\right) + 2(\mu_1 + \mu_2 + \varphi\mu_3) I_r, \\
(1,6) &= A_{w0}^T P_1 + \mathbb{A}_{w0}^T P_2 L, \\
(1,7) &= (1+\beta)\mathbb{A}_{w0}^T P_2, \\
(1,8) &= \varphi^{\frac{1}{2}}(1+\beta)\mathbb{A}_{w0}^T\left(P_2 \widetilde{\mathbb{A}}_{w1} - Y \widetilde{\mathbb{C}}_{w1}\right), \\
(1,9) &= \mathbb{A}_{w1}^T P_2 L - \mathbb{C}_{w1}^T Y^T L, \\
(1,10) &= \beta\left(\mathbb{A}_{w1}^T P_2 - \mathbb{C}_{w1}^T Y^T\right), \\
(9,11) &= L^T\left(P_2 \widetilde{\mathbb{A}}_{w1} - Y \widetilde{\mathbb{C}}_{w1}\right), \\
(10,11) &= \beta\left(P_2 \widetilde{\mathbb{A}}_{w1} - Y \widetilde{\mathbb{C}}_{w1}\right).
\end{aligned}$$

Then the gain matrix Z is given by

$$Z = P_2^{-1} Y. \quad (41)$$

Proof. Consider the following performance index with $\xi(0) = 0$

$$J_{\xi v} = \int_0^\infty \mathbf{E}\left\{e^T(t)e(t) - \gamma^2 v^T(t)v(t)\right\} \mathrm{d}\,t$$

$$= \int_0^\infty \left(\mathbf{E}\left\{ \left(\xi^T(t)\widehat{C}^T\widehat{C}\xi(t) - \gamma^2 v^T(t)v(t) \right) \mathrm{d}\,t + \mathrm{d}\,V(\xi(t)) \right\} \right) + \mathbf{E}\left\{V(\xi(t))\right\}_{t=0} - \mathbf{E}\left\{V(\xi(t))\right\}_{t=\infty}. \tag{42}$$

where $V(\xi)$ is the Lyapunov function defined in (25).
Using the facts that $\mathbf{E}\left\{V(\xi(t))\right\}_{t=\infty} \geqslant 0$ and $\mathbf{E}\left\{V(\xi(t))\right\}_{t=0} = 0$, we obtain

$$J_{\xi v} \leqslant \int_0^\infty \mathbf{E}\left\{ \left(\xi^T(t)\widehat{C}^T\widehat{C}\xi(t) - \gamma^2 v^T(t)v(t) \right) \mathrm{d}\,t + \mathrm{d}\,V(\xi(t)) \right\}. \tag{43}$$

Now if the LMI (40) holds, then applying Schur lemma on the previous inequality yields

$$\underbrace{\begin{bmatrix} -\Theta & \mathcal{P}\widehat{B} \\ \widehat{B}^T\mathcal{P} & 0 \end{bmatrix}}_{\Pi} + \begin{bmatrix} \widehat{C}^T\widehat{C} & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} < 0 \tag{44}$$

with matrix Θ given in (34). If the matrix $\mathcal{P}$ is taken with the general form $\mathcal{P} = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix}$, so inequality (44) is not convex. To overcome this non convexity and to be able to use LMI method to obtain the gain Z, we choose the following structure of the Lyapunov matrix $\mathcal{P}$ with two matrices P_1 and P_2 to be determined

$$\mathcal{P} = \begin{bmatrix} P_1 & L^T P_2 \\ P_2 L & \beta P_2 \end{bmatrix}, \tag{45}$$

where $\beta > 0$ is a tuning parameter, and matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{r\times r}$ verify the following constraint

$$P_1 - \beta^{-1} L^T P_2 L > 0. \tag{46}$$

The inequality (44) implies $-\Theta < 0$. Applying the Schur lemma to Θ (see (34)), we obtain the inequality (23) of theorem 1. Inserting (44) in (43) yieds

$$J_{\xi v} \leqslant \int_0^\infty \mathbf{E}\left\{ \begin{bmatrix} \xi(t)^T & v(t)^T \end{bmatrix} \Pi \begin{bmatrix} \xi(t) \\ v(t) \end{bmatrix} \mathrm{d}\,t + \begin{bmatrix} \xi(t)^T & v(t)^T \end{bmatrix} \begin{bmatrix} \widehat{C}^T\widehat{C} & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \begin{bmatrix} \xi(t) \\ v(t) \end{bmatrix} \mathrm{d}\,t \right\} < 0. \tag{47}$$

Then the mean-square stability and the $\mathcal{H}_\infty$ performance of the augmented system (21) are satisfied if the LMI (40) holds. □ □

4 Numerical example

Consider the stochastic bilinear system (1) with one control input and two disturbance signals. The numerical values of the system matrices are

$$A_{t0} = \begin{bmatrix} -1.5 & 1 & -1 \\ 0.5 & -2.5 & 1 \\ 0 & -0.6 & -3.5 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.1 & 0.3 \\ 0.5 & -0.2 \\ -0.6 & -0.5 \end{bmatrix},$$

$$A_{t1} = \begin{bmatrix} -0.01 & 0.1 & 0 \\ 0 & -0.05 & 0 \\ 0.15 & 0 & -0.02 \end{bmatrix}, \qquad A_{w0} = \begin{bmatrix} -1 & 0 & 0.2 \\ 0.5 & -0.3 & -0.1 \\ -0.2 & 0 & 0.2 \end{bmatrix},$$

$$J = \begin{bmatrix} -0.03 & 0 & -0.03 \\ 0 & -0.01 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The control input is $u_1(t) = 5.5\sin(3t) + 0.5$ and is bounded as follows

$$u_{1\,\min} = -5 \leqslant u_1(t) \leqslant u_{1\,\max} = 6.$$

The covariance factor between the Wiener processes $w_1(t)$ and $w_2(t)$, defined in equation (2b), is $\varphi = 0.7$. For the simulation, the initial conditions are

$$x(0) = \begin{bmatrix} -1.5 & -0.5 & 1 \end{bmatrix}^T \text{ and } \hat{z}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T.$$

The LMI (40) is verified for $\mu_1 = 3, \mu_2 = 0.095, \mu_3 = 1$ and a tuning parameter $\beta = 100$. The solutions γ, $\mathcal{P}$ and Y are given by $\gamma = 1.6$ and

$$\mathcal{P} = \begin{bmatrix} P_1 & L^T P_2 \\ P_2 L & \beta P_2 \end{bmatrix} = \left[\begin{array}{ccc|cc} 5.3355 & 4.2131 & 0.2826 & -0.0090 & 0.0089 \\ 4.2131 & 8.2636 & 3.7260 & -0.0089 & 0.0202 \\ 0.2826 & 3.7260 & 4.6140 & -0.0001 & -0.0113 \\ \hline -0.0090 & -0.0089 & -0.0001 & 0.9002 & -0.8886 \\ 0.0089 & 0.0202 & -0.0113 & -0.8886 & 2.0197 \end{array}\right], Y = \begin{bmatrix} 6636 & -6636 & 13.2 & -13.2 \\ 3079.2 & -3079.2 & 5.9 & -5.9 \end{bmatrix}.$$

Using (41), the gain matrix Z is

$$Z = \begin{bmatrix} Z_0 & Z_1 \end{bmatrix} = 10^6 \times \left[\begin{array}{cc|cc} 1.5691 & -1.5691 & 0.0031 & -0.0031 \\ 0.8428 & -0.8428 & 0.0017 & -0.0017 \end{array}\right].$$

Finally, (10) and (12) give the matrices of the reduced-order filter (5)

$$M_0 = \begin{bmatrix} -4.8618 & 0.8618 \\ -1.8775 & -2.1225 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0.1368 & -0.1568 \\ 0.2237 & -0.2037 \end{bmatrix},$$

$$N_0 = \begin{bmatrix} -3.3618 & -1.2618 \\ -1.3775 & 0.2225 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} -0.0032 & 0.0568 \\ 0.1137 & 0.1137 \end{bmatrix}.$$

The estimation error $e(t)$ is plotted in figure 1. The disturbance signals $v_1(t)$ and $v_2(t)$ are depicted in figure 2. The efficiency of the approach is then shown by numerical simulation.

5 Conclusion

In this paper, the problem of reduced-order filtering for stochastic bilinear systems with multiplicative noises both in the state and in the measurement equations and subject to finite energy perturbations, has been studied. The mean-square stability of the observation error and the $\mathcal{H}_\infty$ performance from the disturbances to the filtering error are proved using a quadratic Lyapunov function. A particular choice of the Lyapunov matrix allows to convert the filter design into a convex problem. The use of Sylvester-like constraints allows to parametrize the filter matrices through a unique gain matrix. Finally, a LMI method which ensures the mean-square stability and the $\mathcal{H}_\infty$ performance is used to compute the gain matrix. A numerical example is given to illustrate the effectiveness of the proposed approach.

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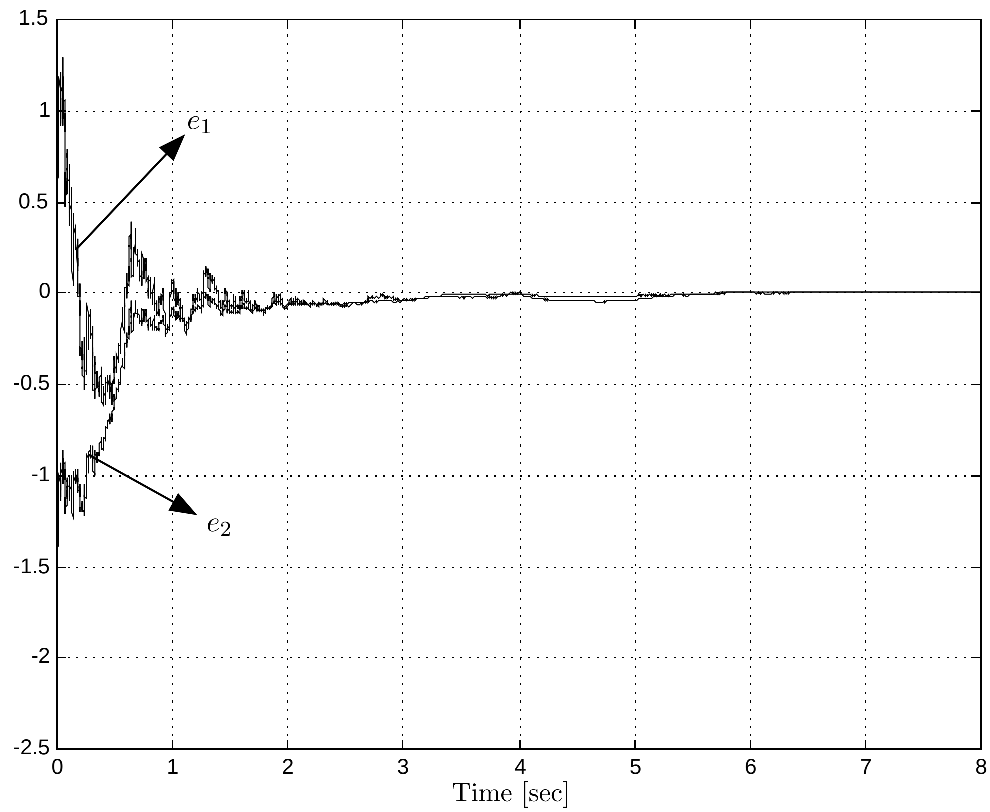

Figure 1: Filtering error $e(t)$.

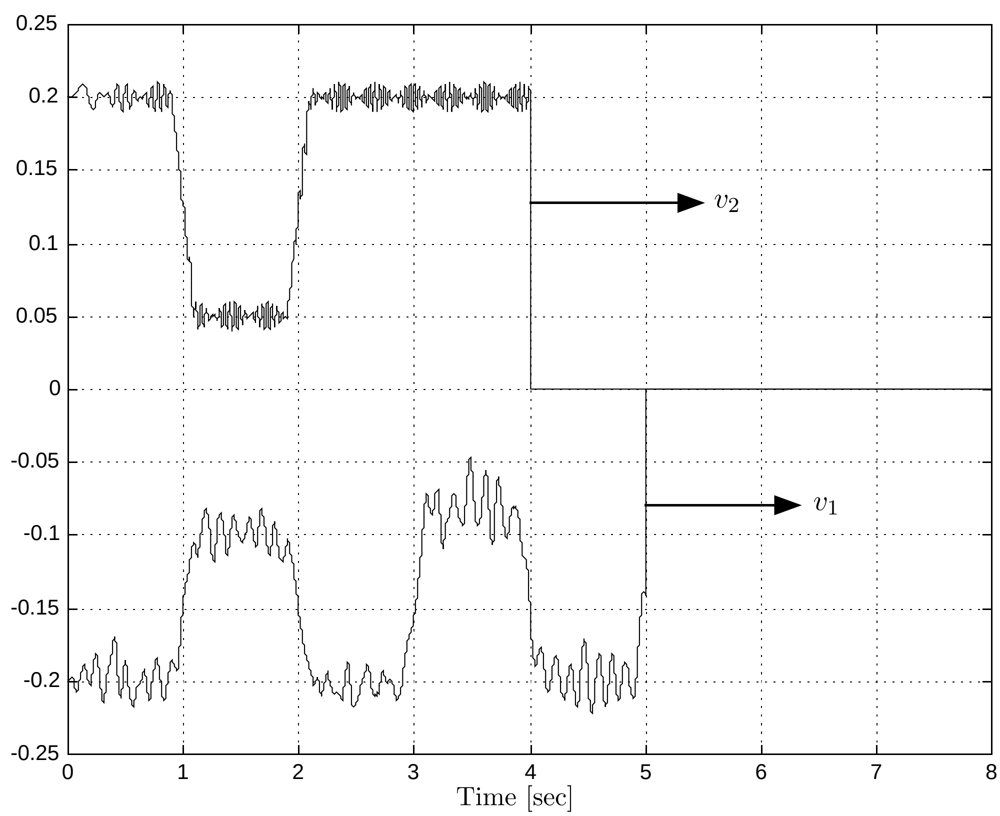

Figure 2: Disturbance signals $v_1(t)$ and $v_2(t)$.