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Abstract

A more general class of stochastic nonlinear systems with irreducible homogenous Markovian switching are considered in this paper. As preliminaries, the stability criteria and the existence theorem of strong solution are firstly presented by using the inequality of mathematic expectation of Lyapunov function. The state-feedback controller is designed by regarding Markovian switching as constant such that the closed-loop system has a unique solution, and the equilibrium is asymptotically stable in probability in the large. The output-feedback controller is designed based on a quadratic-plus-quartic-form Lyapunov function such that the closed-loop system has a unique solution with the equilibrium being asymptotically stable in probability in the large in unbiased case and has a unique bounded-in-probability solution in biased case.

Key words: Nonlinear stochastic systems, backstepping, Markovian switching.

1 Introduction

Stability of stochastic differential equations(SDE) has been one of the most important research topics not only for pure mathematics but also for other subjects such as cybernetics. For some representative work on this general topic, to name a few, we refer readers to (Arnold 1972, Friedman 1976, Khas'minskii 1980, Skorohod 1989), and the references therein. Recently, stability of SDE with Markovian switching has received a lot of attention. Ji & Chizeck (1990) studied the stability of a jump linear equation; Basak, Bisi & Ghosh (1996) discussed the stability of a semi-linear SDE with Markovian switching; Mao (1999) discussed the exponential stability of general nonlinear differential equations with Markovian switching. As a direct application and also an origination, controller design for this type of hybrid SDE has been an important issue (Yuan & Mao 2004, Shi, Xia, Liu & Rees 2006, Shi, Boukas & Agarwal 1999, Shi & Boukas 1997). It should be noted that these design methods are in linear case, which means much strict conditions imposed on the practical controlled system such as global Lipschitz condition or linear growth condition. For nonlinear control a breakthrough came in 1990s: backstepping, a recursive design for systems with nonlinearities non constrained by linear bound (Krstić, Kanellakopoulos & Kokotović 1995, Marino & Tomei 1995, Tong & Li 2007). Backstepping designs for systems with stochastic disturbance were firstly proposed by (Krstić & Deng 1998, Pan & Başar 1999) and were further developed by the recent work of (Liu & Zhang 2006, Liu, Zhang & Jiang 2007, Tian & Xie 2006, Xie & Tian 2007, Wu, Xie & Zhang 2007).

The purpose of this paper is to design nonlinear controllers for stochastic strict-feedback systems with Markovian switching. The main work consists of the following aspects:

- By adopting the backstepping control design used in

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(Krstić & Deng 1998, Wu et al. 2007), Wiener process can be well dealt with. While in the infinitesimal generator of Lyapunov function there appear the "interconnected" items caused by Markovian switching (see (22)). They can not be suppressed by summing up all the single Lyapunov functions as in (Wu, Xie & Zhang 2004) because the infinitesimal generator is not a linear operator about Lyapunov function.

- Under the standard assumption that the finite homogeneous Markov process is irreducible, we can suppress the "interconnections" in the expectation of the infinitesimal generator and then obtain the result that the expectation of Lyapunov function can be bounded by exponential functions of time (see (25) and (28)).
- This leads to two new problems—how to guarantee the existence and uniqueness of strong solution to closed-loop system and how to prove the stochastic stability of this solution based on these inequalities. As mathematic preliminaries, Lemma 1 and Theorems 1,2 are presented, which are different from the corresponding results in (Yuan & Mao 2004).
- State-feedback and output-feedback backstepping controllers are designed for stochastic nonlinear systems (17) and (30), respectively, which are proved to be robust against the irreducible homogenous Markovian switchings.

The paper is organized as follows: Section 2 begins with the mathematical preliminaries. In section 3 and 4 the state-feedback and the output-feedback backstepping controller are designed, respectively. Finally, the paper is concluded in Section 5.

Notations: The following notations are used throughout the paper. For a vector x, $|x|$ denotes its usual Euclidean norm, x^T denotes its transpose and $\bar{x}_i = (x_1, \cdots, x_i)^T$. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}^{n\times r}$ denotes the real $n \times r$ matrix space. C^i denotes the set of all functions with continuous ith partial derivative; $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ denotes the family of all nonnegative functions $V(x(t), t, r(t))$ on $\mathbb{R}^n \times \mathbb{R}_+ \times S$ which are C^2 in x and C^1 in t.

2 Mathematical preliminaries

In this section, we will extend the existence and uniqueness theorem of strong solution, the criteria on asymptotic stability in probability and boundedness in probability to hybrid stochastic nonlinear systems, which is required for backstepping control design.

Consider the following stochastic nonlinear system with Markovian switching

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dW(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state of system. $W(t)$ is an m-dimensional independent standard Wiener process (or Brownian motion). The underlying complete probability space is taken to be the quartet $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $r(t)$ be a right-continuous homogeneous Markov process on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{pq})_{N\times N}$ given by

$$\begin{aligned} P_{pq}(t) &= P\{r(t+s) = q | r(s) = p\} \\ &= \begin{cases} \gamma_{pq} t + o(t) & \text{if } p \neq q \\ 1 + \gamma_{pp} t + o(t) & \text{if } p = q \end{cases} \end{aligned} \quad (2)$$

for any $s, t \geq 0$. Here $\gamma_{pq} > 0$ is the transition rate from p to q if $p \neq q$ while

$$\gamma_{pp} = -\sum_{q=1, q\neq p}^{N} \gamma_{pq}.$$

We assume that the Markov process $r(t)$ is independent of the Brownian motion $W(t)$. For nonlinear controller design, other standard assumptions on $r(t)$ will be given in the end of this section. The following hypothesis is imposed on the the Borel measurable functions $f : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times p}$.

H: Both f and g are locally Lipschitz in $x \in \mathbb{R}^n$ for all $t \geq 0$, namely, for any $R > 0$, there exists a constant $C_R \geq 0$ such that

$$\begin{aligned} &|f(x_1, t, p) - f(x_2, t, p)| \\ &+ |g(x_1, t, p) - g(x_2, t, p)| \leq C_R |x_1 - x_2| \end{aligned}$$

for any $(t, p) \in \mathbb{R}_+ \times S$ and $(x_1, x_2) \in U_R = \{\xi : |\xi| \leq R\}$. Moreover, $f(0, t, p) = g(0, t, p) = 0$.

For $V(x, t, r(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, introduce the infinitesimal generator by

$$\begin{aligned} &\mathcal{L}V(x, t, p) = V_t(x, t, p) + V_x(x, t, p) f(x, t, p) \\ &+ \tfrac{1}{2} \mathrm{Tr}\left[g^T(x, t, p) V_{xx}(x, t, p) g(x, t, p)\right] \\ &+ \textstyle\sum_{q=1}^{N} \gamma_{pq} V(x, t, q), \end{aligned} \quad (3)$$

where $V_t(x, t, p) = \frac{\partial V(x,t,p)}{\partial t}$, $V_x(x, t, p) = (\frac{\partial V(x,t,p)}{\partial x_1}, \ldots, \frac{\partial V(x,t,p)}{\partial x_n})$, $V_{xx}(x, t, p) = \left(\frac{\partial^2 V(x,t,p)}{\partial x_p \partial x_q}\right)_{n\times n}$. Just for the convenience of the reader we cite the following useful property proposed by (Mao & Yuan 2006, Lemma 1.9): Let $V(x, t, r(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ and τ_1, τ_2 be bounded stopping times such that $0 \leq \tau_1 \leq \tau_2$ *a.s.* If $V(x, t, r(t))$ and $\mathcal{L}V(x, t, r(t))$ are bounded on

$t \in [\tau_1,\ \tau_2]$ a.s., then

$$
\begin{aligned}
&E[V(x,\tau_2,r(\tau_2)) - V(x,\tau_1,r(\tau_1))] \\
&= E\int_{\tau_1}^{\tau_2} \mathcal{L}V(x,t,r(t))dt.
\end{aligned} \tag{4}
$$

The following statement about the existence and uniqueness of strong solution to a jump stochastic differential equation and the line of its proof are originated from (Mao & Yuan 2006, Theorem 3.19).

Lemma 1 *Let H holds for system (1). For any $l > 0$, define the first exit time η_l as*

$$
\eta_l = \inf\{t : t \geq t_0,\ |x(t)| \geq l\}.
$$

Assume that there exist a positive function $V(x,t,r(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ and parameters d and $D \geq 0$ such that

$$
EV(x, \eta_l \wedge t, r(\eta_l \wedge t)) \leq De^{d(\eta_l \wedge t - t_0)}, \tag{5a}
$$

$$
R \to \infty \Longrightarrow V_R = \inf_{t \geq t_0, |x| > R} V(x,t,r(t)) \to \infty. \tag{5b}
$$

Then for every $x(t_0) = x_0 \in \mathbb{R}^n$ and $r(t_0) = i_0 \in S$, there exists a solution $x(t) = x(x_0, i_0; t, r(t))$, unique up to equivalence, of system (1).

Proof. By (Mao & Yuan 2006, Theorem 3.15), the locally Lipschitz condition guarantees that there exists a unique maximal solution $x(t)$ on $[t_0, \eta_\infty)$, where η_∞ is the explosion time. Following the same line as in (Mao & Yuan 2006, Theorem 3.19), from (5a) and (5b) we can obtain

$$
\eta_\infty = \infty, a.s., \tag{6}
$$

which implies the result of this theorem. □

The following definitions about stability that were proposed by (Khas'minskii 1980), are represented now for the research on stochastic systems with Markovian switching.

Definition 1 *The equilibrium $x(t) = 0$ of (1) is said to be*

•*(weakly) stable in probability if, for every $\varepsilon > 0$ and $\delta > 0$, there exists an r such that if $t > t_0$, $|x_0| < r$ and $i_0 \in S$, then*

$$
P\{|x(t)| > \varepsilon\} < \delta. \tag{7}
$$

•*asymptotically stable in probability in the large if it is stable in probability and, for each $\varepsilon > 0$, $x_0 \in \mathbb{R}^n$ and $i_0 \in S$ there is*

$$
\lim_{t \to \infty} P\{|x(t)| > \varepsilon\} = 0. \tag{8}
$$

Definition 2 *A stochastic process $x(t)$ is said to be bounded in probability if the random variables $|x(t)|$ are bounded in probability uniformly in t, i.e.,*

$$
\lim_{R \to \infty} \sup_{t > t_0} P\{|x(t)| > R\} = 0. \tag{9}
$$

The asymptotic stability criterion is given as follows.

Theorem 1 *Assume that system (1) has a uniqueness solution in almost surely sense in $t \in [t_0,\ \infty)$ and that there exist a positive function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ and parameters $D > 0$ and $c > 0$ such that*

$$
EV(x,t,r(t)) \leq De^{-c(t-t_0)}, \tag{10a}
$$

$$
\bar{V}_R = \sup_{t \geq t_0, |x| < R} V(x,t,r(t)) \to 0 \Longleftrightarrow R \to 0. \tag{10b}
$$

Then for any $x_0 \in \mathbb{R}^n$ and $i_0 \in S$, the equilibrium $x(t) = 0$ is asymptotically stable in probability in the large.

Proof. From (10b), it can be inferred that for each $\varepsilon > 0$ there exists a $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon)$ such that $|x(t)| \leq \varepsilon$ when $V(x(t),t,r(t)) \leq \bar{\varepsilon}$, while for any $\varepsilon, \delta > 0$ there exists an $r = r(\varepsilon, \delta)$ such that $V(x_0, t_0, i_0) < \frac{1}{D}\varepsilon\delta$ when $|x_0| < r$. Then by Chebyshev's inequality, from (10a), it follows that

$$
\begin{aligned}
&P\{|x(t)| > \varepsilon\} \leq P\{V(x,t,r(t)) > \bar{\varepsilon}\} \\
&\leq \frac{DEV(x_0,t_0,i_0)e^{-c(t-t_0)}}{\bar{\varepsilon}} < \delta,
\end{aligned} \tag{11}
$$

which means that $x(t) = 0$ is stable in probability. By (11), it is obvious that (8) is valid for each $\varepsilon > 0$, $x_0 \in \mathbb{R}^n$ and $i_0 \in S$, which implies that the equilibrium is asymptotically stable in probability in the large. □

The criterion for boundedness in probability is given as follows.

Theorem 2 *Assume that system (1) has a uniqueness solution in almost surely sense in $t \in [t_0,\ \infty)$ and that there exist a positive function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ and parameters $d_c > 0$ such that*

$$
EV(x,t,r(t)) \leq d_c \tag{12}
$$

and (5b) hold. Then for any $x_0 \in \mathbb{R}^n$ and $i_0 \in S$, the solution of system (1) is bounded in probability .

Proof. By (Khas'minskii 1980, Lemma 1.4.1), from (12), it follows that

$$
\begin{aligned}
&P\{|x(x_0,i_0;t,r(t))| > R\} \\
&\leq \frac{EV(x,t,r(t))}{\inf_{|x|>R, t \geq t_0} V(x,t,r(t))} \leq \frac{d_c}{V_R},
\end{aligned} \tag{13}
$$

which, together with (5b), means that (9) holds. □

For controller design by using backstepping techniques, we further assume, as a standard hypothesis, that $r(t)$ is irreducible (Please see (Mao 1999, P.183)). The algebraic interpretation of irreducibility is rank $(\Gamma) = N-1$. This means that the Markov process admits a unique stationary distribution $\pi = (\pi_1, \cdots, \pi_N)$, which is also a limit distribution and can be obtained by solving

$$\pi\Gamma = 0, \ \sum_{p=1}^{N} \pi_p = 1, \text{ and } \pi_p > 0, \ \forall p \in S. \tag{14}$$

Consider how to generate a stationary Markov process based on stationary (limit) distribution π (Ross 1996, P.259). Since $r(t)$ is a homogenous finite irreducible Markov process, which means that it is ergodic, then one can suppose that it started at the moment $t = -\infty$. Such a process will be stationary, i.e, it satisfies

$$P(r(t) = p) = \sum_{l=1}^{N} \pi_l P_{lp}(t) = \pi_p, \ \forall t \geq 0, \forall p \in S. \tag{15}$$

Another method is to choose the initial state according to stationary distribution, that is, if $r(0)$ satisfies π, then (15) holds, which is exactly what we pursue in this paper. In fact, these two approaches are consistent with each other from the viewpoint that the initial state of a new Markov process is the final state of an old one with the same distribution. For $V(x, t, r(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, according to (Ross 1996, P.21), we have

$$\begin{aligned} &EV(x,t,r(t)) = \textstyle\sum_{p=1}^{N} EV(x,t,p)\pi_p, \\ &E\mathcal{L}(V(x,t,r(t))) = \textstyle\sum_{p=1}^{N} E(\mathcal{L}V(x,t,p))\pi_p, \end{aligned} \tag{16}$$

as long as the expectations involved exist and are finite.

Remark 1 *For the same Markov process, transition probability (2) and unconditional probability (15) are introduced. The former is used to define infinitesimal generator (3), and the latter is used in the calculus of expatiations such as (16).*

3 State-feedback backstepping controller design

Consider the stochastic nonlinear systems with Markovian switching as follows

$$\begin{aligned} &dx_i = x_{i+1}dt + \varphi_i(\bar{x}_i, r(t))dW(t), \\ &\ \ i = 1, \cdots, n-1, \\ &dx_n = udt + \varphi_n(\bar{x}_n, r(t))dW(t). \end{aligned} \tag{17}$$

where $x = \bar{x}_n = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ are the state, the input of system, respectively. The known function φ_i is smooth and $\varphi_i(0, r(t)) = 0$. Let $r(t)$ be a right-continuous homogeneous irreducible Markov process in a state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{pq})_{N\times N}$ and assume that initial state $r(0) = i_0$ satisfying stationary distribution π given by (14). The explanations about Wiener process and probability space are as same as those for system (1).

The objective of this section is to design a state-feedback controller such that the equilibrium of the closed-loop system is asymptotically stable in probability in the large.

Remark 2 *When $r(t)$ equals to a constant, system (17) will reduce to system (3.28),(3.29) of (Krstić & Deng 1998). For a smooth controller $u = u(x, r(t))$ with $u(0, r(t)) = 0$, the right side of system (1) satisfies local Lipschitz condition H.*

3.1 Controller design

Introducing the following transformation

$$\begin{aligned} &z_1 = x_1, \\ &z_i(\bar{x}_i, r(t)) = x_i - \alpha_{i-1}(\bar{x}_{i-1}, r(t)), \ i = 2, \ldots, n, \end{aligned} \tag{18}$$

where smooth function α_{i-1} will be designed later.

Choose the Lyapunov function candidate

$$V(x, r(t)) = \sum_{i=1}^{n} \frac{1}{4} z_i^4(\bar{x}_i, r(t)). \tag{19}$$

For the simplicity, we introduce the notions

$$\begin{aligned} &\varphi_{ip} = \varphi_i(\bar{x}_i, p), \quad z_{ip} = z_i(\bar{x}_i, p), \\ &\alpha_{ip} = \alpha_i(\bar{x}_i, p), \quad u_p = u(x, p), \quad \forall p \in S. \end{aligned}$$

According to (3), we have

$$\begin{aligned} &\mathcal{L}V(x,p) = \textstyle\sum_{i=1}^{n-1} z_{ip}^3 z_{i+1,p} \\ &+ z_{np}^3(u_p - \textstyle\sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1,p}}{\partial x_j} x_{j+1} \\ &- \frac{1}{2}\textstyle\sum_{j,k=1}^{n-1} \varphi_{jp}^T \frac{\partial^2 \alpha_{n-1,p}}{\partial x_j \partial x_k}(\varphi_{kp}) \\ &+ \textstyle\sum_{i=1}^{n-1} z_{ip}^3(\alpha_{ip} - \textstyle\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1,p}}{\partial x_j} x_{j+1} \\ &- \frac{1}{2}\textstyle\sum_{j,k=1}^{i-1} \varphi_{jp}^T \frac{\partial^2 \alpha_{i-1,p}}{\partial x_j \partial x_k}\varphi_{kp}) \\ &+ \frac{3}{2}\textstyle\sum_{i=1}^{n} z_{ip}^2(\varphi_{ip} - \textstyle\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1,p}}{\partial x_j} x_{j+1})^T \\ &\times(\varphi_{ip} - \textstyle\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1,p}}{\partial x_j} x_{j+1}) + \textstyle\sum_{q=1}^{N} \gamma_{pq} V(x, q). \end{aligned} \tag{20}$$

Compared (20) with (3.40) of (Krstić & Deng 1998), the only difference is $\sum_{q=1}^{N} \gamma_{pq} V(x, q)$ caused by Markovian switching. By ignoring this "interconnected" item and following the same line of this work by M. Krstić *et al.*, we obtain the controller

$$\begin{aligned}
\alpha_1 &= -c_1 z_1 - \tfrac{3}{4} d_1^{\frac{4}{3}} z_1 - \tfrac{3}{2} z_1 \beta_{11}^T \beta_{11} \\
&\quad - \tfrac{3m}{4} z_1 \textstyle\sum_{k=2}^{n} \sum_{l=1}^{k-1} d_{k1l}, \\
\alpha_i &= -c_i z_i + \textstyle\sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l} x_{l+1} \\
&\quad + \tfrac{1}{2} \textstyle\sum_{j,k=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_j \partial x_k} \varphi_j^T \varphi_k - \tfrac{3}{4} d_i^{\frac{4}{3}} z_i \\
&\quad - \tfrac{1}{4 d_{i-1}^4} z_i - \tfrac{3}{2} z_i \beta_{ii}^T \beta_{ii} - 3 \beta_{ii}^T \textstyle\sum_{k=1}^{i-1} z_k \beta_{ik} \\
&\quad - \tfrac{3}{4} z_i \textstyle\sum_{j=1}^{r} \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} \frac{1}{d_{ikl}^2} \beta_{ikj}^2 \beta_{ilj}^2 \\
&\quad - \tfrac{3m}{4} \textstyle\sum_{k=i+1}^{n} \sum_{l=1}^{k-1} d_{kil}, \\
u &= -c_n z_n + \textstyle\sum_{l=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_l} x_{l+1} \\
&\quad + \tfrac{1}{2} \textstyle\sum_{j,k=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial x_k} \varphi_j^T \varphi_k - \tfrac{1}{4 d_{n-1}^4} z_n \\
&\quad - \tfrac{3}{2} z_n \beta_{nn}^T \beta_{nn} - 3 \beta_{nn}^T \textstyle\sum_{k=1}^{i-1} z_k \beta_{nk} \\
&\quad - \tfrac{3}{4} z_n \textstyle\sum_{j=1}^{r} \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \frac{1}{d_{nkl}^2} \beta_{nkj}^2 \beta_{nlj}^2,
\end{aligned} \tag{21}$$

where $c_i, d_i, d_{ikj} > 0$ are design parameters with $d_{ikj} = d_{kij}$; smooth function

$$\begin{aligned}
&\beta_{ik}(\bar{x}_i, r(t)) = \psi_{ik}(\bar{x}_i, r(t)) \\
&- \textstyle\sum_{l=k}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \psi_{lk}(\bar{x}_k, r(t)), \; k = 1, \cdots, i,
\end{aligned}$$

with $\frac{\partial \alpha_{i-1,p}}{\partial x_i} = 0$ and β_{ikj} is the j-th component of the vector β_{ik}; and ψ_{ik} is smooth function defined by

$$\varphi_i(\bar{x}_i, r(t)) = \sum_{k=1}^{i} z_k \psi_{ik}(\bar{x}_i, r(t)).$$

Then the infinitesimal generator of the system satisfies:

$$\begin{aligned}
&\mathcal{L}V(x,p) \le -\textstyle\sum_{i=1}^{n} c_i z_{ip}^4 + \sum_{q=1}^{N} \gamma_{pq} V(x,q) \\
&\le -cV(x,p) - \tfrac{\bar{c}}{4} \textstyle\sum_{i=2}^{n} z_{ip}^4 + \tfrac{1}{4} \sum_{q=1}^{N} \gamma_{pq} \sum_{i=2}^{n} z_{iq}^4.
\end{aligned} \tag{22}$$

where $c = \min\{4c_1, c_2, \ldots, c_n\}$ and $\bar{c} = 3 \min_{i=2}^{n} \{c_i\}$.

3.2 Stability analysis

Theorem 3 *By choosing the design parameters $c_2, \cdots, c_n$ appropriately, the closed-loop system consist of (17) and (21) has a unique solution, and $x(t) = 0$ is asymptotically stable in probability in the large for any $i_0 \in S$ satisfying distribution π and every $x_0 \in \mathbb{R}^n$.*

Proof. From (18) and (21), step by step, we can conclude that bounded z_i implies bounded x_i, and vice versa, which results in that

$$V_R = \inf_{t \ge t_0, |x| > R} V(x, r(t)) \to \infty \iff R \to \infty. \tag{23}$$

For any $l > 0$, define the first exit time $\eta_l = \inf\{t : t \ge t_0, \; |x(t)| \ge l\}$. Let $t_l = \eta_l \wedge t$ for any $t \ge t_0$. Since $|x(\cdot)| < l$ in the interval $[t_0, t_l]$ a.s., which, together with (23), implies that $V(x, r(\cdot))$ is bounded on $[t_0, \; t_l]$ a.s. From (22), it can be obtained that $\mathcal{L}V(x, r(\cdot))$ is also bounded on $[t_0, \; t_l]$ a.s. It comes from (16) and (22) that

$$\begin{aligned}
&E\mathcal{L}(V(x, r(t_l))) = \textstyle\sum_{p=1}^{N} E(\mathcal{L}V(x,p)) \pi_p \\
&\le -c \textstyle\sum_{p=1}^{N} \pi_p EV(x,p) - E(\sum_{p=1}^{N} \frac{\bar{c}\pi_p}{4} \sum_{i=2}^{n} z_{ip}^4 \\
&+ \textstyle\sum_{p=1}^{N} \frac{1}{4} \pi_p \sum_{q=1}^{N} \gamma_{pq} \sum_{i=2}^{n} z_{iq}^4) \\
&\le -cEV(x, r(t_l)),
\end{aligned} \tag{24}$$

where $c_2, \ldots, c_n$ are chosen such that $\bar{c} \ge \bar{\bar{c}} = \frac{\max_{p=1}^{N}\{\pi_p\}}{\min_{p=1}^{N}\{\pi_p\}} \max_{q=1}^{N} \left\{ \sum_{p=1}^{N} \gamma_{pq} \right\}$. According to formula (4), it is followed from (24) that

$$EV(x, r(t_l)) \le EV(x_0, i_0). \tag{25}$$

By Lemma 1, from (25) and (23), for every $x(t_0) = x_0 \in \mathbb{R}^n$ and $r(t_0) = i_0 \in S$, there exists a solution $x(t) = x(x_0, i_0; t, r(t))$, unique up to equivalence, of the closed-loop system consist of (17) and (21). The subsequent part is originated from line of (Mao 1997, Theorem 4.4). From (6), one has $\eta_l \to \infty$ (a.s.) when $l \to \infty$. Again from (24) and (4), we obtain that

$$\begin{aligned}
&E(e^{ct_l} V(x, r(t_l))) \\
&\le e^{ct_0} E(V(x_0, i_0)) + E \textstyle\int_{t_0}^{t_l} e^{cs} \mathcal{L}(V(x, r(s))) ds \\
&+ cE \textstyle\int_{t_0}^{t_l} e^{cs} V(x, r(s)) ds,
\end{aligned} \tag{26}$$

which together with (24) implies that

$$EV(x, r(t_l)) \le e^{-c(t_l - t_0)} EV(x_0, i_0). \tag{27}$$

Letting $l \to \infty$ gives

$$EV(x, r(t)) \le e^{-c(t - t_0)} EV(x_0, i_0). \tag{28}$$

From (18) and (21), step by step, we can obtain

$$R \to 0 \iff \bar{V}_R = \sup_{t \ge t_0, |x| < R} V(x, r(t)) \to 0. \tag{29}$$

From (28), (29) and Theorem 1, we can conclude that $x(t) = 0$ is asymptotically stable in probability in the large for any $x_0 \in \mathbb{R}^n$ and $i_0 \in S$, which completes the proof. $\square$

Remark 3 *By comparison, our controller is in the same form as that in (Krstić & Deng 1998) except for that the design parameters $c_2, \cdots, c_n$ should be large enough. In other words, we can design controller for system (17) by regarding $r(t)$ as constant and tuning our parameter c_i appropriately.*

3.3 A simulation example

Consider system (17) ($n = 2$) with $\varphi_1 = x_1^2(t) + x_1(t)r(t), \varphi_2 = x_1^2(t) + x_2^2(t) + x_2(t)r(t)$ where $W(t)$ being a scaler Wiener process. The state-feedback control law is given by (18) and (21) with $\beta_{11} = x_1 + r(t)$, $\beta_{21} = x_1 - \frac{\partial \alpha_{1p}}{\partial x_1}$ and $\beta_{22} = x_2 + r(t)$. $r(t)$ is a homogenous irreducible Markov process which belongs to the space $S = \{1, 2\}$ with generater $\Gamma = (\gamma_{pq})_{2\times 2}$ given by $\gamma_{11} = -4, \gamma_{12} = 4, \gamma_{21} = 3$ and $\gamma_{22} = -3$, which means that $\pi_1 = \frac{3}{7}, \pi_2 = \frac{4}{7}$. Choose the initial values $x_1(0) = -1, x_2(0) = 1$, $r(0) = 1$, the design parameters $c_1 = 1.1, c_2 = 1.1$ (satisfying $3.3 = \bar{c} > \bar{\bar{c}} = \frac{4}{3}$), $d_1 = 0.4$ and $d_2 = 10$. Figure 1 demonstrates that the state of

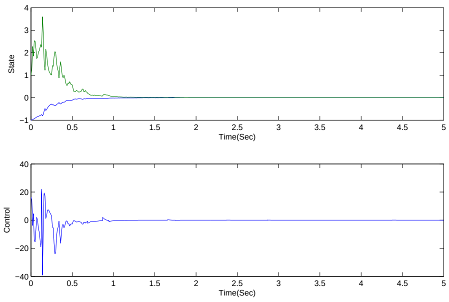

Fig.1. The responses of closed-loop system with Markovian switching.

nonlinear system with Markovian switching can be regulated to the origin asymptotically by the same controller designed for non switching case.

4 Output feedback backstepping controller design

Consider the stochastic nonlinear systems with Markovian switching as follows

$$\begin{aligned} dx_i &= x_{i+1}dt + \Delta_i(x, r(t))dt + \Omega_i(x, r(t))dW_i(t), \\ &\quad i = 1, \cdots, n-1, \\ dx_n &= udt + \Delta_n(x, r(t))dt + \Omega_n(x, r(t))dW_n(t), \\ y &= x_1, \end{aligned} \tag{30}$$

where $y \in \mathbb{R}$ is the output; the uncertain functions Δ_i and Ω_i are locally Lipschitz; other explications are as same as that for system (17).

The objective of this section is to design an output-feedback controller such that the solution of the closed-loop system is bounded in probability in biased case and the equilibrium is asymptotically stable in probability in the large in unbiased case.

The following assumption is made on system (30).

A1: For each $1 \le i \le n$, there exist unknown constants $\check{d}_i, \hat{d}_i \ge 0$ such that

$$\begin{aligned} |\Delta_i(x, r(t))| &\le \phi_i(y, r(t)) + \check{d}_i, \\ |\Omega_i(x, r(t))| &\le \varphi_i(y, r(t)) + \hat{d}_i, \end{aligned}$$

where ϕ_i, φ_i are known nonnegative smooth functions with $\phi_i(0, r(t)) = \varphi_i(0, r(t)) = 0$, which means that there exist smooth functions $\bar{\phi}_i, \bar{\varphi}_i$ such that

$$\phi_i(y, r(t)) = y\bar{\phi}_i(y, r(t)), \;\; \varphi_i(y, r(t)) = y\bar{\varphi}_i(y, r(t)).$$

Remark 4 *When Δ_i and Ω_i are known and all the states x_i are measurable, system (30) is reduced to (17). From A1 the static uncertainties are nonlinear parameterized, which is considered by (Jiang & Praly 1998, Jiang 1999) in the deterministic case and by (Wu et al. 2007) in the stochastic case without Markovian switching. A detailed example satisfying A1 can be found in Subsection 4.3.*

4.1 Controller design

Since $x_i (i = 2, \cdots, n)$ is unavailable, as in (Jiang 1999), the following reduced-order observer is introduced

$$\begin{aligned} d\hat{x}_i &= (\hat{x}_{i+1} + k_{i+1}y - k_i(\hat{x}_1 + k_1 y))dt, \\ &1 \le i \le n-2, \\ d\hat{x}_{n-1} &= (u - k_{n-1}(\hat{x}_1 + k_1 y))dt, \end{aligned} \tag{31}$$

where $k = (k_1, \cdots, k_{n-1})^T$ is chosen such that $A_0 = \begin{pmatrix} -k & \begin{matrix} I_{n-2} \\ 0 \cdots 0 \end{matrix} \end{pmatrix}$ is asymptotically stable. By (30) and (31), the observer error ε defined by

$$\begin{aligned} \varepsilon &= (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1})^T, \\ \varepsilon_i &= x_{i+1} - \hat{x}_i - k_i x_1, \end{aligned} \tag{32}$$

satisfies

$$d\varepsilon = A_0 \varepsilon dt + \Delta(x, r(t))dt + \bar{A}_0 \Omega(x, r(t))dW(t), \tag{33}$$

where $\Delta = (\Delta_1, \cdots, \Delta_{n-1})^T, W(t) = (W_1(t), \cdots, W_n(t))^T$, $\bar{A}_0 = (A_0 \quad e_{n-1})$, $e_{n-1} = (0, \cdots, 0, 1)^T \in \mathbb{R}^{n-1}$, $\Delta_i = \Delta_{i+1} - k_i \Delta_1, \Omega = \mathrm{diag}(\Omega_1, \cdots, \Omega_n)$. From assumption A1, we have

$$|\Delta_i(x, r(t))| \le \psi_i(y, r(t)) + \bar{d}_i, \tag{34}$$

where $\psi_i = \phi_{i+1} + k_i \phi_1$, and $\bar{d}_i = \check{d}_{i+1} + k_i \check{d}_1$.

From (30) and (32), one gets

$$dy = (\hat{x}_1 + \varepsilon_1 + k_1 y)dt + \Delta_1 dt + \Omega_1 dW_1(t), \tag{35}$$

which together with (30), (33) and (35) consist of the following interconnected system

$$\begin{aligned} d\varepsilon &= A_0\varepsilon dt + \Delta dt + \bar{A}_0\Omega dW(t), \\ dy &= (\hat{x}_1 + \varepsilon_1 + k_1 y)dt + \Delta_1 dt + \Omega_1 dW_1(t), \\ d\hat{x}_i &= (\hat{x}_{i+1} + k_{i+1}y - k_i(\hat{x}_1 + k_1 y))dt, \\ & 1 \le i \le n-2, \\ d\hat{x}_{n-1} &= (u - k_{n-1}(\hat{x}_1 + k_1 y))dt, \end{aligned} \tag{36}$$

Next, we will develop an output-feedback controller along the $(y, \hat{x}_1, \cdots, \hat{x}_{n-1})$-system of (36) step by step using backstepping techniques.

Let us introduce the following notions for the use of the recursive design

$$\begin{aligned} &\Xi_1 = (y, \varepsilon)^T,\ \Xi_i = (y, \hat{x}_1, \ldots, \hat{x}_{i-1}, \varepsilon)^T, \\ &X_1 = y,\ X_i = (y, \hat{x}_1, \ldots, \hat{x}_{i-1})^T,\ \ i = 1, \cdots, n \end{aligned}$$

(with $\Xi = \Xi_n, X = X_n$). Introducing the following transformations

$$\begin{aligned} &z_1 = y,\ \ z_i(X_i, r(t)) = \hat{x}_{i-1} \\ &-\alpha_{i-1}(X_{i-1}, r(t)),\ \ i = 2, \ldots, n, \end{aligned} \tag{37}$$

where the smooth function α_i will be designed later. Just for the simplify, let us further introduce the notions

$$\begin{aligned} &\Delta_{1p} = \Delta_1(x,p),\ \Omega_{1p} = \Omega_1(x,p), \\ &\alpha_{ip} = \alpha_i(X_i, p),\ z_{ip} = z_i(X_i, p). \end{aligned}$$

Step 1. Let us consider the Lyapunov function candidate

$$V_1(\Xi_1, r(t)) = \frac{1}{2}y^2 + r_0\varepsilon^T P\varepsilon, \tag{38}$$

where P is the solution of $PA_0 + A_0^T P = -I_{n-1}$, $r_0 > 0$ is design parameter. In view of (36) and (37), the infinitesimal generator of V_1 satisfies

$$\begin{aligned} \mathcal{L}V_1(\Xi_1, p) \ =\ & y(z_{2p} + \alpha_{1p} + k_1 y + \varepsilon_1 + \Delta_{1p}) \\ & + \tfrac{1}{2}\Omega_{1p}^2 - r_0|\varepsilon|^2 + 2r_0\varepsilon^T P\Delta(x,p) \\ & + \mathrm{Tr}[(\bar{A}_0\Omega(x,p))^T P\bar{A}_0\Omega(x,p)]. \end{aligned} \tag{39}$$

From assumption A1, one has

$$\begin{aligned} & 2r_0\varepsilon^T P\Delta + y(\varepsilon_1 + \Delta_1) \\ & + \tfrac{1}{2}\Omega_1^2 + \mathrm{Tr}[(\bar{A}_0\Omega)^T P\bar{A}_0\Omega] \\ & \le \tfrac{1}{4}r_0|\varepsilon|^2 + 8r_0|P|^2\sum_{i=1}^{n-1}\bar{\psi}_i^2 y^2 \\ & + 8r_0|P|^2\sum_{i=1}^{n-1}\bar{d}_i^2 + \tfrac{1}{4}r_0|\varepsilon|^2 + \left(\bar{\psi}_1 + \tfrac{1}{r_0}\right)y^2 \\ & + d_1 y^2 + \tfrac{1}{4d_1}\check{d}_1^2 + (\bar{\varphi}_1^2 + 2|P||\bar{A}_0|^2\sum_{i=1}^{n}\bar{\varphi}_i^2)y^2 \\ & + \hat{d}_1^2 + 2|P||\bar{A}_0|^2\sum_{i=1}^{n}\hat{d}_i^2 \\ & = \tfrac{1}{2}r_0|\varepsilon|^2 + z_1\Psi_1 + d_1 y^2 + \lambda_1, \end{aligned} \tag{40}$$

where $d_1 > 0$ is any design parameter, $\lambda_1 = \hat{d}_1^2 + \frac{1}{4d_1}\check{d}_1^2 + 2|P||\bar{A}_0|^2\sum_{i=1}^{n}\hat{d}_i^2 + 8r_0|P|^2\sum_{i=1}^{n-1}\bar{d}_i^2$ and $\Psi_1(y, r(t))$ is given by

$$\begin{aligned} \Psi_1 \ =\ & 8r_0|P|^2\sum_{i=1}^{n-1}\bar{\psi}_i^2 y + \left(\bar{\psi}_1 + \tfrac{1}{r_0}\right)y \\ & + \bar{\varphi}_1^2 y + 2|P||\bar{A}_0|^2\sum_{i=1}^{n}\bar{\varphi}_i^2 y, \end{aligned}$$

where $\bar{\psi}_i = \bar{\phi}_{i+1} + k_i\bar{\phi}_1$. Substituting $z_1 z_2 \le \frac{3}{4}z_1\sqrt[3]{z_1^2+1} + \frac{1}{4}z_2^4$ and (40) into (39) gives

$$\begin{aligned} \mathcal{L}V_1(\Xi_1, p) \le\ & \tfrac{1}{4}z_{2p}^4 + y(\tfrac{3}{4}z_1\sqrt[3]{z_1^2+1} + \alpha_{1p} \\ & + k_1 y + \Psi_{1p}) + d_1 y^2 - \tfrac{1}{2}r_0|\varepsilon|^2 + \lambda_1. \end{aligned} \tag{41}$$

where $\Psi_{1p} = \Psi_1(y,p)$. The stabilizing function $\alpha_1(X_1, r(t))$ is designed as

$$\alpha_1 = -\tfrac{3}{4}z_1\sqrt[3]{z_1^2+1} - k_1 y - c_1 y - \Psi_1. \tag{42}$$

From (41) and (42), it follows that

$$\mathcal{L}V_1 \le -c_1 z_1^2 + \tfrac{1}{4}z_2^4 - \tfrac{1}{4}r_0|\varepsilon|^2 + d_1 y^2 + \lambda_1.$$

Step $i = 2, \cdots, n$**.** Assume that one has designed smooth function $\alpha_j (2 \le j \le i-1)$ such that the infinitesimal generator of $V_{i-1} = V_{i-2} + \frac{1}{4}z_{i-1}^4$ satisfies

$$\begin{aligned} \mathcal{L}V_{i-1}(\Xi_{i-1}, p) \le\ & -c_1 z_1^2 - \textstyle\sum_{j=2}^{i-1} c_j z_{jp}^4 \\ & - \tfrac{1}{2^{i-1}}r_0|\varepsilon|^2 + \tfrac{1}{4}z_{ip}^4 + \textstyle\sum_{j=1}^{i-1} d_j y^2 \\ & + \textstyle\sum_{j=1}^{i-1}\lambda_j + \tfrac{1}{4}\sum_{q=1}^{N}\sum_{j=2}^{i-1}\gamma_{pq}z_{jq}^4,\ \forall p \in S, \end{aligned} \tag{43}$$

where $d_i, c_i > 0$ are design parameters. In the sequel, we will prove that (43) holds for the i-th Lyapunov function candidate $V_i = V_{i-1} + \frac{1}{4}z_i^4$. The infinitesimal generator of V_i satisfies

$$\begin{aligned}
&\mathcal{L}V_i(\Xi_i, p) \leq \mathcal{L}V_{i-1}(\Xi_{i-1}, p) \\
&+ \tfrac{3}{2}z_{ip}^2(\tfrac{\partial\alpha_{i-1,p}}{\partial y})^2\Omega_{1p}^2 - \tfrac{1}{2}z_{ip}^3\tfrac{\partial^2\alpha_{i-1,p}}{\partial y^2}\Omega_{1p}^2 \\
&+ z_{ip}^3\Big(z_{i+1,p} + \alpha_{ip} + \eta_{ip} - \tfrac{\partial\alpha_{i-1,p}}{\partial y}(\varepsilon_1 + \Delta_{1p})\Big) \\
&+ \tfrac{1}{4}\textstyle\sum_{q=1}^N \gamma_{pq}z_{iq}^4, \quad \forall p \in S,
\end{aligned} \tag{44}$$

where $\eta_{ip} = \eta_i(X_i, p), \eta_i(X_i, r(t)) = k_i y - k_{i-1}(\hat{x}_1 + k_1 y) - \frac{\partial\alpha_{i-1,p}}{\partial y}(\hat{x}_1 + k_1 y) - \sum_{j=1}^{i-2}\frac{\partial\alpha_{i-1,p}}{\partial\hat{x}_j}(\hat{x}_{j+1} + k_{j+1}y - k_j(\hat{x}_1 + k_1 y))$. Following the similar procedure as in the initial step, one has

$$\begin{aligned}
&-z_i^3\tfrac{\partial\alpha_i}{\partial y}(\varepsilon_1 + \Delta_1) \\
&+\tfrac{3}{2}z_i^2\left(\tfrac{\partial\alpha_{i-1}}{\partial y}\right)^2\Omega_1^2 - \tfrac{1}{2}z_i^3\tfrac{\partial^2\alpha_{i-1}}{\partial y^2}\Omega_1^2 \\
&\leq \tfrac{1}{2^i}r_0|\varepsilon|^2 + z_i^6\left(\tfrac{\partial\alpha_{i-1}}{\partial y}\right)^2\left(\tfrac{\bar{\psi}_1^2}{2d_{i1}} + \tfrac{1}{d_{i2}} + \tfrac{2^{i-2}}{r_0}\right) \\
&+\tfrac{d_{i1}}{2}y^2 + d_{i2}\check{d}_1^2 + \tfrac{d_{i3}}{2}y^2 + \tfrac{1}{d_{i3}}(9z_i^4(\tfrac{\partial\alpha_{i-1}}{\partial y})^4 \\
&+z_i^6(\tfrac{\partial^2\alpha_{i-1}}{\partial y^2})^2)(\bar{\varphi}_1)^4y^2 + \tfrac{1}{d_{i4}}(9z_i^4(\tfrac{\partial\alpha_{i-1}}{\partial y})^4 \\
&+z_i^6(\tfrac{\partial^2\alpha_{i-1}}{\partial y^2})^2) + \tfrac{1}{2}d_{i4}\hat{d}_1^4 \\
&= \tfrac{1}{2^i}r_0|\varepsilon|^2 + \Psi_i z_i^3 + d_i y^2 + \lambda_i,
\end{aligned} \tag{45}$$

where $d_i = \frac{1}{2}(d_{i1} + d_{i3})$, $\lambda_i = \frac{1}{2}d_i\hat{d}_1^4 + d_i\check{d}_1^2$ and $\Psi_i(X_i, r(t))$ is defined as

$$\begin{aligned}
\Psi_i = {}& z_i^3\left(\tfrac{\partial\alpha_{i-1}}{\partial y}\right)^2\left(\tfrac{\bar{\psi}_1^2}{2d_{i1}} + \tfrac{1}{d_{i2}} + \tfrac{2^{i-2}}{r_0}\right) \\
&+ \tfrac{1}{d_{i3}}(9z_i(\tfrac{\partial\alpha_{i-1}}{\partial y})^4 + z_i^3(\tfrac{\partial^2\alpha_{i-1}}{\partial y^2})^2)(\bar{\varphi}_1)^4y^2 \\
&+ \tfrac{1}{d_{i4}}(9z_i(\tfrac{\partial\alpha_{i-1}}{\partial y})^4 + z_i^3(\tfrac{\partial^2\alpha_{i-1}}{\partial y^2})^2).
\end{aligned}$$

From (43)-(45) and the fact $z_i^3 z_{i+1} \leq \frac{3}{4}z_i^4 + \frac{1}{4}z_{i+1}^4$, it is easy to show that

$$\begin{aligned}
&\mathcal{L}V_i(\Xi_i, p) \leq -c_1 z_1^2 - \sum_{j=2}^{i-1}c_j z_{jp}^4 - \tfrac{1}{2^i}r_0|\varepsilon|^2 \\
&+\tfrac{1}{4}z_{i+1,p}^4 + z_{ip}^3\Big(z_{ip} + \alpha_{ip} + \eta_{ip} + \Psi_{ip}\Big) \\
&+\Sigma_{j=1}^i d_j y^2 + \Sigma_{j=1}^i\lambda_j + \tfrac{1}{4}\textstyle\sum_{q=1}^N\sum_{j=2}^i\gamma_{pq}z_{jq}^4,
\end{aligned} \tag{46}$$

where $\Psi_{ip} = \Psi_i(X_i, p)$. By choosing the virtue control $\alpha_i(X_i, r(t))$ as

$$\alpha_i = -z_i - c_i z_i - \eta_i - \Psi_i, \tag{47}$$

it follows from (46) that (43) holds for V_i. At the end of the recursive procedure, we obtain the control law

$$u = \alpha_n(X, r(t)). \tag{48}$$

From (43) and (48), by selecting $c_1 \geq 2\sum_{j=1}^n d_j$, one gets

$$\begin{aligned}
&\mathcal{L}V_n(\Xi, p) \leq -cV_n(\Xi, p) - \tfrac{\bar{c}}{4}\textstyle\sum_{i=2}^n z_{ip}^4 \\
&+\tfrac{1}{4}\textstyle\sum_{q=1}^N\sum_{j=2}^n\gamma_{pq}z_{jq}^4 + d_a, \quad \forall p \in S,
\end{aligned} \tag{49}$$

where $d_a = \Sigma_{i=1}^n d_i\check{d}_1^2 + \Sigma_{i=1}^n\lambda_i$, $c = \min\{c_1, c_2, \ldots, c_n, \frac{1}{2^n}\}$ and $\bar{c} = 3\min_{i=2}^n\{c_i\}$.

4.2 Stability analysis

Theorem 4 *For any $i_0 \in S$ satisfying distribution π and every $x_0 \in \mathbb{R}^n$, by choosing the design parameters $c_1, \cdots, c_n$ such that $c_1 \geq 2\sum_{j=1}^n d_j$ and $\bar{c} \geq \bar{\bar{c}} = \frac{\max_{p=1}^N\{\pi_p\}}{\min_{p=1}^N\{\pi_p\}}\max_{q=1}^N\{\sum_{p=1}^N\gamma_{pq}\}$, the closed-loop system consist of (30) and (48) has a unique solution, which is bounded in probability. Furthermore, if $\check{d}_i = \hat{d}_i = 0$ is given, the zero solution of the closed-loop system is asymptotically stable in probability in the large.*

Proof. From (37) and (48), step by step, we can conclude that

$$R \to \infty \Longleftrightarrow V_R = \inf_{t\geq t_0, |\Xi|>R} V_n(\Xi, r(t)) \to \infty. \tag{50}$$

As in the proof of Theorem 3, from (16) and (49), one has

$$E(\mathcal{L}(V_n(\Xi, r(t_l))) \leq -cEV_n(\Xi, r(t_l)) + Nd_a, \tag{51}$$

then the existence and uniqueness of solution to the closed-loop system consist of (30) and (48) can be obtained. Continuing following the line of Theorem 3, it comes from (51) that

$$EV_n(\Xi, r(t)) \leq EV_n(\Xi_0, i_0)e^{-c(t-t_0)} + Nd_a/c, \tag{52}$$

which implies that

$$EV_n(\Xi, r(t)) \leq d_c \tag{53}$$

where $d_c = EV_n(\Xi_0, i_0) + Nd_a/c$. From (53), (50) and Theorem 2, we can prove that the solution of the closed-loop system is bounded in probability for any initial conditions, which implies the first result of this theorem.

When $\check{d}_i = \hat{d}_i = 0$, (52) reduces to

$$EV_n(\Xi, r(t)) \leq EV_n(\Xi_0, i_0)e^{-c(t-t_0)}. \tag{54}$$

From (37) and (48), step by step, we can obtain (50) and

$$\bar{V}_R = \sup_{t\geq t_0, |\Xi|<R} V_n(\Xi, r(t)) \to 0 \Longleftrightarrow R \to 0. \tag{55}$$

From (54), (55) and Theorem 1, we obtain the second result of this theorem. □

Remark 5 *To overcome the observation errors and adaptive errors, the quadratic form Lyapunov functions are used in the initial step of backstepping design. To deal with the effect of Hessian items, the quartic Lyapunov functions are remained in the subsequent steps.*

4.3 A simulation example

Let $n = 2$ in system (30). Chose $\Delta_1 = (1+r(t))x_1^2 \sin x_2 + \check{d}_1, \Omega_1 = x_1^2 \sin x_2 + \hat{d}_1$, $\Delta_2 = x_1^2 \cos x_2 + \check{d}_2, \Omega_2 = (r(t)-1)x_1^2 \cos x_2 + \hat{d}_2$ with $\check{d}_1, \hat{d}_1, \check{d}_2$ and $\hat{d}_2$ being unknown parameters. The following observer is needed

$$\dot{\hat{x}} = u - k(\hat{x} + ky),$$

where $k > 0$. Perform the transformation (37) and output feedback control (48). By verifying assumption A1, it is easy to obtain that

$$\phi_1 = (1+r(t))x_1^2, \phi_2 = x_1^2,$$
$$\varphi_1 = x_1^2, \varphi_2 = (r(t)-1)x_1^2.$$

To check the main results in Theorem 4, the following two cases are considered. Biased case: $\check{d}_1 = \check{d}_2 = \hat{d}_1 = \hat{d}_2 = 1$. Unbiased case: $\check{d}_1 = \check{d}_2 = \hat{d}_1 = \hat{d}_2 = 0$. The homogenous irreducible Markov process $r(t)$ belongs to the space $S = \{1, 2\}$ with generater with generater $\Gamma = (\gamma_{pq})_{2\times 2}$ given by $\gamma_{11} = -4, \gamma_{12} = 4, \gamma_{21} = 3, \gamma_{22} = -3$. Choose the initial values $x_1(0) = 0.5, x_2(0) = -4$, $r(0) = 1$, the design parameters $k = 1.6$, $d_1 = 0.1$, $d_{21} = d_{23} = 1, d_{22} = d_{24} = 10$, $c_1 = 2.5, c_2 = 2$ (satisfying $c_1 > 2(d_1 + d_2)$ and $6 = \bar{c} > \bar{\bar{c}} = \frac{4}{3}$), $r_0 = 0.5$ and $\hat{x}(0) = -3.3$. Figures 2 shows the responses of the closed-loop system in biased cases; Figures 3 shows the responses of the closed-loop system in unbiased cases. From Figures 2 we can see that the residual errors remain in all the solutions of closed-loop system because the non-vanishing biased parameters prevent the equilibrium at zero. Figures 3 demonstrates that the solutions of the closed-loop system can be regulated to the origin asymptotically in unbiased case.

5 Conclusion

Two important issues: backstepping and hybrid diffusion are well combined so that the stochastic nonlinear control has been improved to a new level, in which the mode studied has the capability to describe complex systems not only with exterior random perturbations but also with inner jumping parameters. Meanwhile, the novel existence theorem and stability criterion of solution are proposed to SDE with Markovian switching for nonlinear controller design without linear growth condition.

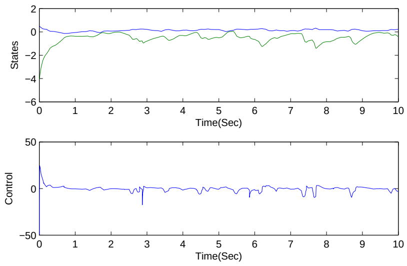

Fig.2. The responses of closed-loop system in biased case.

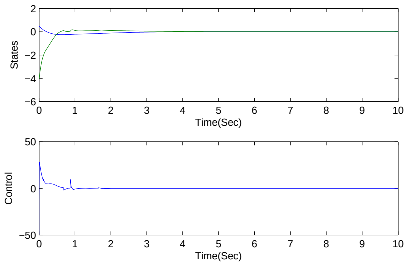

Fig.3. The responses of closed-loop system in unbiased case.

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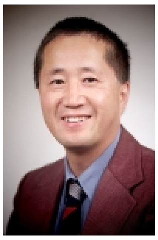

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