Affine and Predictive Control Policies for a Class of Nonlinear Systems

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Abstract

The input-state linear horizon (ISLH) for a nonlinear discrete-time system is defined as the smallest number of time steps it takes the system input to appear nonlinearly in the state variable. In this paper, we employ the latter concept and show that the class of constraint admissible N-step affine *state-feedback* policies is equivalent to the associated class of constraint admissible *disturbance-feedback* policies, provided that N is less than the system's ISLH. The result generalizes a recent result in Goulart, Kerrigan, and Maciejowski (2006) and is significant because it enables one: (i) to determine a constraint admissible state-feedback policy by employing well-known convex optimization techniques; and (ii) to guarantee robust recursive feasibility of a class of model predictive control (MPC) policies by imposing a suitable terminal constraint. In particular, we propose an input-to-state stabilizing MPC policy for a class of nonlinear systems with bounded disturbance inputs and mixed polytopic constraints on the state and the control input. At each time step, the proposed MPC policy requires the solution of a single convex quadratic programme parameterized by the current system state.

Key words: Constrained control; Predictive control; Nonlinear systems

1 Introduction

Since an appropriate optimization problem needs to be solved on-line for the current system state, it can be impractical to apply model predictive control (MPC) to relatively fast, nonlinear or high-dimensional systems. An interesting area of research has thus been the task of reducing online computation times by means of, for example, explicit solutions (Bemporad et al., 2002) or "reformulations" that enable the optimal control problem to be solved efficiently as a standard convex problem.

In this context, Mare, De Doná, Seron, Haimovich, and Ramagge (2008); Mare, Lazar, and De Doná (2007) recently made the observation that the optimal control sequence for a nonlinear system with a quadratic stage cost and linear constraints may be computed exactly via a quadratic programme (QP), provided the number of stages in the open-loop optimal control problem is less than some critical integer. To characterize the latter integer, the notions of input-output linear horizon (IOLH) and input-state linear horizon (ISLH) were introduced by Mare et al. (2008). In particular, for the control problem considered in Mare et al. (2007), the critical integer was shown to be the system's ISLH.

Another line of research which rely on a reformulation in order to pose and solve the optimization problem of interest efficiently is the work of Goulart, Kerrigan, and Maciejowski (2006) on feedback control of linear systems with disturbance inputs and (polytopic) constraints. Here a relevant problem is to minimize a quadratic cost over a class of N-step affine state-feedback policies. In Goulart et al. (2006), the authors established equivalence between the class of N-step affine *state-feedback* policies and an associated class of *disturbance-feedback* polices (van Hessem and Bosgra, 2002; Löfberg, 2003; Kerrigan and Alamo, 2004). The equivalence result was subsequently used to pose the optimization problem as a convex QP and to prove properties (e.g., robust recursive feasibility) of the associated class of MPC policies.

In this paper, we combine and generalize some of the results of Goulart et al. (2006) and Mare et al. (2007). In particular, we generalize the equivalence result of Goulart et al. (2006) to nonlinear dynamics by showing that the class of constraint admissible N-step affine state-feedback policies is equivalent to the associated class of disturbance-feedback policies, provided that $1 \leq N < \ell_F$, where ℓ_F denotes the system's ISLH. We then use the latter result to construct an input-to-state stabilizing MPC policy based on a convex QP. The region of attraction obtained using the proposed MPC policy equals the set of states that can be steered to a suitable target set in $N < \ell_F$ steps using any constraint admissible N-step affine state-feedback policy. Whilst the restriction $N < \ell_F$ is the key device used to obtain our results, we acknowledge that it may lead to poor performance and a small region of attraction, unless ℓ_F is relatively large. However, from a theoretical point of view, the results of the present paper are interesting in their own right, since they are more general than existing results for linear systems (in which case $\ell_F = \infty$).

The paper outline is as follows: Section 2 describes the dynamical system under consideration. Section 3 presents results on N-step affine feedback policies. Section 4 uses the results of Section 3 to construct a robust MPC policy based on a convex QP. Notation and terminology: A polyhedral set is a set of the form $\{x \in \mathbb{R}^n | \, Ax \leq b\}$. A polytope is a bounded polyhedral set. A matrix $B \in \mathbb{R}^{Nn_x \times Nn_u}$, with block entries $B_{ij} \in \mathbb{R}^{n_x \times n_u}$, is said to be (strictly) lower block triangular if $B_{ij} = 0$ when $i \leq j$ ($i < j$). The notation $\mathbb{Z}_{[m,l]}$ denotes the integers on the interval $\{m, m+1, \cdots, l\}$ and the notation $\mathbb{Z}_{[m,\infty)}$ denotes the integers i satisfying $i \geq m$. We also use $\mathbb{Z}_+ \triangleq \mathbb{Z}_{[0,\infty)}$ (and $\mathbb{R}_+$) to denote the non-negative integers (and non-negative real numbers).

2 System Description

2.1 State Space Model

We consider the following discrete-time system

$$x_{k+1} = F(x_k, \mu_k), \quad x_0 = x, \tag{1}$$

where $x_k \in \mathbb{M} \subseteq \mathbb{R}^{n_x}$ and $\mu_k \in \mathbb{R}^m$, respectively, are the system state and the system input at time $k \in \mathbb{Z}_+$, and where the function $F : \mathbb{M} \times \mathbb{R}^m \to \mathbb{M}$ is analytic on its domain. The input $\mu_k = (u_k, w_k)$ of the system comprises a control input $u_k \in \mathbb{R}^{n_u}$ and a disturbance input $w_k \in \mathbb{R}^{n_w}$, and we assume that the dynamics (1) may be expressed alternatively in terms of u_k and w_k as follows:

$$x_{k+1} = a(x_k) + b(x_k)u_k + b_w(x_k)w_k. \tag{2}$$

Here, the matrix $b_w(x)$ is assumed to have full column rank for all $x \in \mathbb{M}$. The disturbance input satisfies

$$w_k \in \mathbb{W}, \quad \forall k \in \mathbb{Z}_+, \tag{3}$$

whereas the state and control input are subject to the following mixed constraints:

$$(x_k, u_k) \in \mathbb{C}, \quad \forall k \in \mathbb{Z}_+, \tag{4}$$

where $\mathbb{C} \subseteq \mathbb{M} \times \mathbb{R}^m$. In the MPC design of Section 4, the sets $\mathbb{W}$ and $\mathbb{C}$ are assumed to be polytopes with the origin contained in the interior, but this assumption is not required in Section 3.

2.2 Input-State Linear Horizon

Next we define the *input-state linear horizon* (ISLH) for the system (1). To this end, we also define the *input-output linear horizon* (IOLH) for the system when the system is assigned an output of the form $y_k = H(x_k)$. For a more extensive treatment of the ISLH and IOLH of a nonlinear system, the reader is referred to Mare et al. (2008); Mare (2007).

Denoting by $\mathcal{U}_k \triangleq (\mu_0, \ldots, \mu_k)$, we use the following notation:

$$\begin{aligned} F_0(x, \mathcal{U}_0) &\triangleq F(x, \mu_0), \\ F_{k+1}(x, \mathcal{U}_{k+1}) &\triangleq F(F_k(x, \mathcal{U}_k), \mu_{k+1}), \quad k \in \mathbb{Z}_+, \end{aligned}$$

where $\mathcal{U}_{k+1} = (\mathcal{U}_k, \mu_{k+1})$. Also, to denote the i^{th} component of the input vector μ_k at time k, we use $\mu_{i,k}$, $i \in \mathbb{Z}_{[1,m]}$, and similarly for the state x_k, $x_{i,k}$, $i \in \mathbb{Z}_{[1,n_x]}$.

The IOLH of the system (1) with output $y_k = H(x_k)$ is the smallest number of time steps it takes *any of the components of the input* to appear on the output in a *nonlinear* form.

Definition 1 (Input-Output Linear Horizon) *Consider any "output function" $H : \mathbb{M} \to \mathbb{R}$ analytic on its domain. The system* (1) *with output $y_k = H(x_k)$ is said to have input-output linear horizon (IOLH) $\ell \in \mathbb{Z}_{[1,\infty)}$ in the set $\mathbb{M}$ if*

(i) $\dfrac{\partial^2}{\partial\mu_{p,0}\partial\mu_{q,i}} H \circ F_k(x, \mathcal{U}_k) = 0,$

$\forall p, q \in \mathbb{Z}_{[1,m]}, \qquad \forall i \in \mathbb{Z}_{[0,k]},$

$\forall [x, \mathcal{U}_k] \in \mathbb{M} \times \mathbb{R}^{m(k+1)}, \quad \forall k \in \mathbb{Z}_{[0,\,\ell-2]}.$

(ii) $\dfrac{\partial^2}{\partial\mu_{p,0}\partial\mu_{q,i}} H \circ F_{\ell-1}(x, \mathcal{U}_{\ell-1}) \neq 0$ *a.e. in* $\mathbb{M} \times \mathbb{R}^{m\ell}$,

for some $p, q \in \mathbb{Z}_{[1,m]}$ and some $i \in \mathbb{Z}_{[0,\,\ell-1]}$.

The system (1) *with output $y_k = H(x_k)$ is said to have input-output linear horizon ∞ in $\mathbb{M}$ if (i) holds for all $\ell \in \mathbb{Z}_{[1,\infty)}$. We use $\ell_{H,F}$ to denote the IOLH.*

The following definitions and properties of the IOLH of a dynamical system, stated in Mare et al. (2007) for the SISO case, also hold in the present case of multiple-input systems (e.g., the proofs in Mare et al. (2007) generalize straightforwardly).

Definition 2 (Coordinate Transformation) *A mapping $z = \psi(x)$, where $\psi : \mathbb{M} \to \mathbb{M}$ is an analytic invertible function, defines a coordinate transformation.*

Definition 3 (Regular Feedback) *A mapping $\mu = \gamma(x, v)$, where $\gamma : \mathbb{M} \times \mathbb{R}^{n_v} \to \mathbb{R}^m$ is an analytic function such that $J(x, v) \neq 0_{m \times n_v}$ a.e. in $\mathbb{M} \times \mathbb{R}^{n_v}$, (where $J(x, v)$ denotes the Jacobian of $\gamma(x, v)$ w.r.t. v), defines a regular feedback law for system* (1).

Lemma 4 *Given a discrete-time system* (1) *with output $y_k = H(x_k)$, its IOLH $\ell_{H,F}$ is invariant under coordinate transformation but, in general, not under feedback.*

The ISLH is the smallest number of time steps it takes *any component of the input* to appear in the *state-variable* in a *nonlinear* form.

Definition 5 (Input-State Linear Horizon) *Consider the discrete-time system* (1) *and output functions $H_i(x) \triangleq c_i\, x$, where c_i, $i \in \mathbb{Z}_{[1,n_x]}$, are row vectors given by the i^{th} row of the $n_x \times n_x$ identity matrix. The input-state linear horizon (ISLH) $\ell_F \in \mathbb{Z}_+$ in the set $\mathbb{M}$ is defined as*

$$\ell_F \triangleq \min_{i \in \mathbb{Z}_{[1,n_x]}} \{\ell_{H_i,F}\}. \tag{5}$$

Lemma 6 *Given the discrete-time system* (1)*, its ISLH ℓ_F is invariant under coordinate transformation but, in general, not under feedback.*

Note that, for a linear system, we have $\ell_F = \infty$, whereas, for a nonlinear system, we have $\ell_F \leq n_x + 1$ (c.f. Mare (2007)). The following example illustrates a nonlinear system structure with $\ell_F = n_x + 1$ (i.e., the system order plus one).

Example 1 (The Flexible Joint Robot) *Consider the following nonlinear model of a flexible joint robot, taken* *from Sira-Ramírez and Castro-Linares (2000), where we have introduced an additive disturbance w_k:*

$$x_{1,k+1} = x_{1,k} + T_s x_{2,k}, \tag{6a}$$
$$x_{2,k+1} = x_{2,k} + \frac{MgLT_s}{I}\sin(x_{1,k}) + \frac{K_a T_s}{I}(x_{1,k} - x_{3,k}), \tag{6b}$$
$$x_{3,k+1} = x_{3,k} + T_s x_{4,k}, \tag{6c}$$
$$x_{4,k+1} = x_{4,k} + \frac{K_a T_s}{J}(x_{1,k} - x_{3,k}) + \frac{T_s}{J}u_k + w_k, \tag{6d}$$

and where $x_{1,k}$ is the link angular position, $x_{2,k}$ is the link angular velocity, $x_{3,k}$ is the motor axis angular position, $x_{4,k}$ is the motor axis angular velocity, and u_k is the motor applied torque. The parameters I, J, mgL and K_a represent the inertia of the link, the motor inertia, the nominal load in the link, and the flexible joint stiffness coefficient, respectively, and T_s is the sampling period. A vector function F can be found such that the above system can be written as $x_{k+1} = F(x_k, \mu_k)$, where $\mu_k = (u_k, w_k)$. It follows by direct computation that, when applying the input sequence $(\mu_0, \mu_1, \mu_2, \dots)$, the state variable $x_{2,5}$ is the first sample of the state that shows a nonlinear dependency on the input μ_0. Thus, $\ell_F = 5$.

Remark 1 *Example 1 exhibits no dependency of the state in the system matrices $b(x)$, $b_w(x)$ in (2). As an example of a system structure with $\frac{\partial b_w(x)}{\partial x} \neq 0$, consider the system (6) but with the last term of (6d) replaced by $\sigma(x_{1,k})w_k$, where $\frac{\partial \sigma(y)}{\partial y} \neq 0$. It can be verified that the ISLH of such a system is, as above, given by $\ell_F = 5$. Further examples can be found in Mare et al. (2008).*

2.3 "N-step Lifted" System Model

For systems of the present form (1)-(2) the ISLH is always greater than, or equal to, two. Hence, for any prediction horizon length N, satisfying $N \in \mathbb{Z}_{[1,\ell_F-1]}$ (i.e., $1 \leq N < \ell_F$), we have that the sequence of future states can be expressed as an affine function of the control- and disturbance sequences. That is, for some functions, $A : \mathbb{M} \to \mathbb{R}^{Nn_x}$, $B : \mathbb{M} \to \mathbb{R}^{Nn_x \times Nn_u}$ and $B_w : \mathbb{M} \to \mathbb{R}^{Nn_x \times Nn_w}$, we have

$$\mathbf{x} = A(x) + B(x)\mathbf{u} + B_w(x)\mathbf{w}, \tag{7}$$

where $x_0 = x$ and

$$\mathbf{x} \triangleq \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{u} \triangleq \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}, \quad \mathbf{w} \triangleq \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix}. \tag{8}$$

3 N-step Affine Feedback Policies

In this section we generalize recent results on N-step affine feedback policies for linear systems to the case of nonlinear systems. We treat the case $N \in \mathbb{Z}_{[1,\ell_F-1]}$ when the affine relation (7)-(8) holds.

3.1 State-Feedback Policies

For a given initial state x and prediction horizon $N \in \mathbb{Z}_{[1,\ell_F-1]}$, consider an N-step affine *state-feedback policy* of the following form:

$$u_i = g_i + \sum_{j=1}^{i} K_{i,j}x_j, \ \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{9}$$

where g_i and $K_{i,j}$ are design parameters that may depend on the initial state x. In view of (8), we may express the state-feedback policy (9) in matrix form as follows:

$$\mathbf{u} = \mathbf{g} + \mathbf{K}\mathbf{x}, \tag{10}$$

where

$$\mathbf{g} \triangleq \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}, \quad \mathbf{K} \triangleq \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ K_{1,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ K_{N-1,1} & \cdots & K_{N-1,N-1} & 0 \end{bmatrix}. \tag{11}$$

Remark 2 *To relate the present state-feedback parameterization* (9)-(10) *to the state-feedback parameterization considered in Goulart et al. (2006), that is,*

$$u_i = c_i + \sum_{j=0}^{i} L_{i,j} x_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{12}$$

note that, to realize any given policy of the form (12) *using parameterization* (9)*, we simply need to assign the design parameters as follows:*

$$g_i = c_i + L_{i,0} x_0, \; \forall i \in \mathbb{Z}_{[0,N-1]}, \quad K_{i,j} = L_{i,j}, \; \forall i,j \in \mathbb{Z}_{[1,N-1]}. \tag{13}$$

Thus, since each g_i may depend on the initial state x_0, parameterization (9) *is as general as the parameterization* (12) *considered in Goulart et al. (2006).*

Since $N \in \mathbb{Z}_{[1,\ell_F-1]}$, we have by combining (7) and (10) that the closed-loop state and control sequence under the state-feedback policy (10) can be expressed as

$$\mathbf{x} = (I - B(x)\mathbf{K})^{-1} (A(x) + B(x)\mathbf{g} + B_w(x)\mathbf{w}), \tag{14}$$

$$\mathbf{u} = (I - \mathbf{K}B(x))^{-1} (\mathbf{g} + \mathbf{K}A(x) + \mathbf{K}B_w(x)\mathbf{w}). \tag{15}$$

Here, we note that both the indicated matrix inverses exist, since the matrices $B(x)\mathbf{K}$ and $\mathbf{K}B(x)$ are strictly lower triangular for all $x \in \mathbb{M}$ (c.f. strict causality (1)). In the sequel, we use the notation $x_0^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w}) = x$ and $x_i^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w})$, $i \in \mathbb{Z}_{[1,N]}$, to refer, respectively, to the initial state and the i^{th} block component of $\mathbf{x}$ in (14) (i.e., x_i is the state at time i). Similarly, we use the notation $u_i^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w})$, $i \in \mathbb{Z}_{[0,N-1]}$, to refer to the $(i+1)^{th}$ block component of $\mathbf{u}$ in (15). We say that the policy (10) is *constraint admissible* for the initial state x, if: (i) the constraints (4) are satisfied for the first N steps, that is,

$$\begin{aligned} &\left(x_i^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w}), u_i^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w})\right) \in \mathbb{C}, \\ &\forall i \in \mathbb{Z}_{[0,N-1]}, \; \forall \mathbf{w} \in \mathbb{W}^N \triangleq \underbrace{\mathbb{W} \times \cdots \times \mathbb{W}}_{N \text{ times}}, \end{aligned} \tag{16}$$

and (ii) the terminal state satisfies

$$x_N^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w}) \in \mathbb{X}_f, \quad \forall \mathbf{w} \in \mathbb{W}^N, \tag{17}$$

where $\mathbb{X}_f$ is some given "terminal constraint" set.

The set $\Pi_N^{sf}(x)$ of all pairs $(\mathbf{g},\mathbf{K})$ that define a constraint admissible state-feedback policy for the initial state x is as follows:

$$\Pi_N^{sf}(x) \triangleq \left\{ (\mathbf{g},\mathbf{K}) \;\middle|\; \begin{aligned} &(\mathbf{g},\mathbf{K}) \text{ satisfies (11)}, \\ &x_i = x_i^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w}), \\ &u_i = u_i^{\{\mathbf{g},\mathbf{K}\}}(x,\mathbf{w}), \\ &(x_i,u_i) \in \mathbb{C}, \; x_N \in \mathbb{X}_f, \\ &\forall i \in \mathbb{Z}_{[0,N-1]}, \; \forall \mathbf{w} \in \mathbb{W}^N \end{aligned} \right\}. \tag{18}$$

Also of interest below is the set of states for which there exists at least one constraint admissible state-feedback policy of the form (10), that is,

$$X_N^{sf} \triangleq \{x \in \mathbb{M} \,|\, \Pi_N^{sf}(x) \neq \emptyset\}. \tag{19}$$

Unfortunately, even in the case of linear dynamics, the set $\Pi_N^{sf}(x)$ may be non-convex for a given initial state $x \in X_N^{sf}$ (Goulart et al., 2006), and the present state-feedback parameterization is thus, in general, not suited to numerical optimization. However, following Goulart et al. (2006), we next introduce an associated class of disturbance-feedback policies, which, via a nonlinear transformation, is equivalent to the present class of state-feedback policies. Furthermore, as shown subsequently in Section 3.4, computing a constraint admissible disturbance-feedback policy for a given initial state $x \in X_N^{sf}$ is a linear programming problem, provided that the sets $\mathbb{C}$, $\mathbb{W}$ and $\mathbb{X}_f$ are polyhedral.

3.2 Disturbance-Feedback Policies

For a given initial state x and prediction horizon $N \in \mathbb{Z}_{[1,\ell_F-1]}$, consider an N-step affine *disturbance-feedback policy* of the following form:

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{20}$$

where v_i and $M_{i,j}$ are design parameters that may depend on the initial state x. In view of (8), we may express the control policy (20) in matrix form as follows:

$$\mathbf{u} = \mathbf{v} + \mathbf{M}\mathbf{w}, \tag{21}$$

where

$$\mathbf{v} \triangleq \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad \mathbf{M} \triangleq \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}. \tag{22}$$

Since $N \in \mathbb{Z}_{[1,\ell_F-1]}$, we have from (7) that the state sequence under the policy (21) can be expressed as

$$\mathbf{x} = A(x) + B(x)\mathbf{v} + [B(x)\mathbf{M} + B_w(x)]\,\mathbf{w}. \tag{23}$$

In the sequel, we use the notation $x_0^{\{\mathbf{v},\mathbf{M}\}}(x,\mathbf{w}) = x$ and $x_i^{\{\mathbf{v},\mathbf{M}\}}(x,\mathbf{w})$, $i \in \mathbb{Z}_{[1,N]}$, to refer, respectively, to the initial state and the i^{th} block component of $\mathbf{x}$ in (23). Similarly, we use the notation $u_i^{\{\mathbf{v},\mathbf{M}\}}(\mathbf{w})$, $i \in \mathbb{Z}_{[0,N-1]}$, to refer to the $(i+1)^{th}$ block component of $\mathbf{u}$ in (21). The set $\Pi_N^{df}(x)$ of all pairs $(\mathbf{v},\mathbf{M})$ that define a constraint admissible disturbance-feedback policy for the initial state x is as follows:

$$\Pi_N^{df}(x) \triangleq \left\{ (\mathbf{v},\mathbf{M}) \;\middle|\; \begin{array}{l} (\mathbf{v},\mathbf{M}) \text{ satisfies (22)}, \\ x_i = x_i^{\{\mathbf{v},\mathbf{M}\}}(x,\mathbf{w}), \\ u_i = u_i^{\{\mathbf{v},\mathbf{M}\}}(\mathbf{w}), \\ (x_i,u_i) \in \mathbb{C},\ x_N \in \mathbb{X}_f, \\ \forall i \in \mathbb{Z}_{[0,N-1]},\ \forall \mathbf{w} \in \mathbb{W}^N \end{array} \right\}. \tag{24}$$

The associated set of states for which at least one constraint admissible disturbance-feedback policy exists is

$$X_N^{df} \triangleq \{x \in \mathbb{M} \,|\, \Pi_N^{df}(x) \neq \emptyset\}. \tag{25}$$

3.3 Equivalence Result

The following theorem is the main result of this section; it provides a generalization to nonlinear dynamics of the equivalence result stated as Theorem 9 in Goulart et al. (2006).

Theorem 7 *Suppose that $N \in \mathbb{Z}_{[1,\ell_F-1]}$. (i) Given any $x \in X_N^{sf}$ and any state-feedback policy* (10) *with $(\mathbf{g},\mathbf{K}) \in \Pi_N^{sf}(x)$, one can find a corresponding disturbance-feedback policy* (21) *with $(\mathbf{v},\mathbf{M}) \in \Pi_N^{df}(x)$ that results in the same sequences of states and control inputs for all $\mathbf{w} \in \mathbb{W}^N$. Conversely, (ii) given any $x \in X_N^{df}$ and any disturbance-feedback policy* (21) *with $(\mathbf{v},\mathbf{M}) \in \Pi_N^{df}(x)$, one can find a corresponding state-feedback policy* (10) *with $(\mathbf{g},\mathbf{K}) \in \Pi_N^{sf}(x)$ that results in the same sequences of states and control inputs for all $\mathbf{w} \in \mathbb{W}^N$. Hence, it follows that (iii) $X_N^{sf} = X_N^{df}$.*

Proof *As in Goulart et al. (2006), we prove statement (i) and the set inclusion $X_N^{sf} \subseteq X_N^{df}$ by constructing a pair $(\mathbf{v},\mathbf{M}) \in \Pi_N^{df}(x)$ from any given pair $(\mathbf{g},\mathbf{K}) \in \Pi_N^{sf}(x)$, $x \in X_N^{sf}$, in such a way that the associated control input sequences coincide for all disturbance input sequences $\mathbf{w} \in \mathbb{W}^N$. Indeed, from* (15) *and the fact that both $B(x)$ and $B_w(x)$ are lower block triangular, we have that such a construction is obtained using:*

$$\mathbf{v} \triangleq (I - \mathbf{K}B(x))^{-1}(\mathbf{g} + \mathbf{K}A(x)), \tag{26}$$

$$\mathbf{M} \triangleq (I - \mathbf{K}B(x))^{-1}\mathbf{K}B_w(x). \tag{27}$$

To prove statement (ii) and the set inclusion $X_N^{df} \subseteq X_N^{sf}$ in a similar fashion, we firstly note from the appendix that the matrix $B_w(x)$ has a lower block triangular left inverse $B_w^\dagger(x)$ satisfying $B_w^\dagger(x)B_w(x) = I, \forall x \in \mathbb{M}$. Left multiplying (7) *by $B_w^\dagger(x)$ and rearranging yields*

$$\mathbf{w} = B_w^\dagger(x)\mathbf{x} - B_w^\dagger(x)A(x) - B_w^\dagger(x)B(x)\mathbf{u}. \tag{28}$$

Under any disturbance-feedback policy (21), *we thus have*

$$\begin{aligned}&\mathbf{u} = \mathbf{v} + \mathbf{M}\mathbf{w} = \\ &\mathbf{v} - \mathbf{M}B_w^\dagger(x)B(x)\mathbf{u} + \mathbf{M}\left(B_w^\dagger(x)\mathbf{x} - B_w^\dagger(x)A(x)\right).\end{aligned}$$

Alternatively, since both $B_w^\dagger(x)$ and $B(x)$ are lower block triangular, we may express the control policy in the required state-feedback form (10) *using*

$$\mathbf{g} \triangleq \left(I + \mathbf{M}B_w^\dagger(x)B(x)\right)^{-1}\left(\mathbf{v} - \mathbf{M}B_w^\dagger(x)A(x)\right), \tag{29}$$

$$\mathbf{K} \triangleq \left(I + \mathbf{M}B_w^\dagger(x)B(x)\right)^{-1}\mathbf{M}B_w^\dagger(x). \tag{30}$$

This completes the proof of (i)-(iii).

An important consequence of the equivalence result established above is Corollary 8 below, which requires the following assumption on the terminal constraint set $\mathbb{X}_f$:

Assumption 1 *The set $\mathbb{X}_f$ introduced in* (17) *is robustly invariant for the closed-loop system obtained using a linear feedback gain law $u_k = -K_f x_k$, that is, for some $K_f \in \mathbb{R}^{n_u \times n_x}$,*

$$a(x) - b(x)K_f x + b_w(x)w \in \mathbb{X}_f, \ \forall x \in \mathbb{X}_f, \ \forall w \in \mathbb{W}.$$

Moreover, the system constraints are satisfied along the associated trajectories, that is,

$$(x, -K_f x) \in \mathbb{C}, \ \forall x \in \mathbb{X}_f,$$

where the set $\mathbb{C}$ is as in (4).

Corollary 8 *Suppose that $N \in \mathbb{Z}_{[1,\ell_F-1]}$ and that Assumption 1 holds. Then, the following set inclusions hold:*

$$\mathbb{X}_f \subseteq X_1^{sf} \subseteq \cdots \subseteq X_N^{sf}, \tag{31}$$

$$\mathbb{X}_f \subseteq X_1^{df} \subseteq \cdots \subseteq X_N^{df}. \tag{32}$$

Proof *It suffices to prove* (31) *since* (32) *then follows from point (iii) of Theorem 7. Proving* (31) *may be accomplished as in Goulart et al. (2006). For example, to prove the set inclusion $X_f \subseteq X_1^{sf}$, note from Assumption 1 that $x \in \mathbb{X}_f$ implies*

$$(-K_f x, 0) \in \Pi_1^{sf}(x) \;\Rightarrow x \in X_1^{sf}. \tag{33}$$

Similarly, to prove the set inclusions $X_1^{sf} \subseteq \cdots \subseteq X_N^{sf}$, consider an integer $L \in \mathbb{Z}_{[2,N]}$, and note from Assumption 1 that $x \in X_{L-1}^{sf}$ implies

$$(\mathbf{g}_+, \mathbf{K}_+) \in \Pi_L^{sf}(x) \;\Rightarrow x \in X_L^{sf}, \tag{34}$$

where

$$\mathbf{g}_+ \triangleq \begin{bmatrix} \mathbf{g}_{L-1} \\ 0 \end{bmatrix}, \quad K_+ \triangleq \begin{bmatrix} \mathbf{K}_{L-1} & 0 \\ \begin{bmatrix} 0 & \cdots & 0 & -K_f \end{bmatrix} & 0 \end{bmatrix}, \tag{35}$$

and where $(\mathbf{g}_{L-1}, \mathbf{K}_{L-1}) \in \Pi_{L-1}^{sf}(x)$ is some pair whose existence follows from the assumption $x \in X_{L-1}^{sf}$.

Corollary 8 can be used to establish recursive feasibility and thereby robust constraint satisfaction of MPC policies that are based on optimization over the class of affine feedback policies. See, for example, Goulart et al. (2006) and Corollary 10 below.

3.4 Convex Parameterization of Admissible Policies

When $N \in \mathbb{Z}_{[1,\ell_F-1]}$ and the sets $\mathbb{C}$, $\mathbb{W}$ and $\mathbb{X}_f$ are polyhedral, it is straightforward to find matrix functions $H_x(x)$, $H_u(x)$, $H_w(x)$ and matrices S, s and h such that the set $\Pi_N^{df}(x)$ in (24) can be expressed as follows:

$$\Pi_N^{df}(x) = \left\{ (\mathbf{v}, \mathbf{M}) \;\middle|\; \begin{array}{l} (\mathbf{v}, \mathbf{M}) \text{ satisfies (22)} \\ \max_{\mathbf{w} \in \mathbb{W}^N} \{[H_u(x)\mathbf{M} + H_w(x)]\,\mathbf{w}\} \\ +H_x(x) + H_u(x)\mathbf{v} \le h \end{array} \right\}, \tag{36}$$

where $\mathbb{W}^N = \left\{ \mathbf{w} \in \mathbb{R}^{Nn_w} \;\middle|\; S\mathbf{w} \le s \right\}$. The following corollary may be proven as in Ben-Tal et al. (2004) or Goulart et al. (2006) and shows that the set $\Pi_N^{df}(x)$ is convex and, in fact, characterized by linear inequalities.

Corollary 9 *The set $\Pi_N^{df}(x)$ in* (36) *can be expressed as*

$$\Pi_N^{df}(x) = \left\{ (\mathbf{v}, \mathbf{M}) \;\middle|\; \begin{array}{l} (\mathbf{v}, \mathbf{M}) \text{ satisfies (22)}, \\ \exists\, \mathbf{Z} \ge 0 \textit{ such that} \\ \mathbf{Z}s + H_x(x) + H_u(x)\mathbf{v} \le h \\ \mathbf{Z}S = [H_u(x)\mathbf{M} + H_w(x)] \end{array} \right\},$$

where $\mathbf{Z}$ is a matrix of appropriate dimensions, and where all inequalities are componentwise.

Thanks to Corollary 9, in the polyhedral case (36), we are able to compute a disturbance-feedback policy given any initial state $x \in X_N^{df} = X_N^{sf}$, by formulating and solving, for example, a linear programme. As explored further in the following section, it is also possible to use the results of this section to design robust MPC policies that are based on a QP parameterized by the current system state.

Remark 3 *By employing the results in Goulart (2006), it is possible to treat the case when the set $\mathbb{W}^N$ is ellipsoidal (as opposed to polytopic) using a second-order cone (as opposed to a linear or quadratic) programme.*

4 Robust Nonlinear Model Predictive Control

In this section we propose an input-to-state stabilizing MPC policy for the system (1).

4.1 Nonlinear MPC via QP

Consider the following quadratic cost function which penalizes the "nominal trajectory" obtained using $\mathbf{u} = \mathbf{v}$ and $\mathbf{w} = 0$ [see (21)]:

$$V(x, \mathbf{v}) = \|x_N\|_{P_f}^2 + \sum_{i=0}^{N-1} \left(\|x_i\|_Q^2 + \|u_i\|_R^2\right), \tag{37}$$

where $Q > 0$, $R > 0$, $P_f > 0$, $x_i = x_i^{\{\mathbf{v},\mathbf{M}\}}(x, 0) = x_i^{\{\mathbf{v},0\}}(x, 0)$ and $u_i = u_i^{\{\mathbf{v},\mathbf{M}\}}(0) = u_i^{\{\mathbf{v},0\}}(0)$, and where $x_i^{\{\mathbf{v},\mathbf{M}\}}(x, \mathbf{w})$ and $u_i^{\{\mathbf{v},\mathbf{M}\}}(\mathbf{w})$ are as in (24). The MPC policy we propose is obtained by minimizing $V(x, \mathbf{v})$ over the set $(\mathbf{v}, \mathbf{M}) \in \Pi_N^{df}(x)$ using the current system state $x = x_k$ as it evolves. More precisely, the MPC policy is as follows:

Algorithm 1 *For the current system state $x = x_k \in X_N^{df}$, $k \in \mathbb{Z}_+$, solve the following optimization problem*

$$(\mathbf{v}^*(x), \mathbf{M}^*(x)) \triangleq \arg \min_{(\mathbf{v},\, \mathbf{M}) \,\in\, \Pi_N^{df}(x)} V(x, \mathbf{v}) \tag{38}$$

and apply $u_k = \mathbf{v}_0^*(x_k)$ to the system (1), where $\mathbf{v}_0^*(x_k)$ denotes the n_u first components of $\mathbf{v}^*(x_k)$.

To facilitate our stability analysis below and ensure that the optimization problem (38) can be posed as a standard convex QP (c.f. Lemma 9), we require the following assumption:

Assumption 2 *The prediction horizon N satisfies $N \in \mathbb{Z}_{[1,\ell_F-1]}$ and the constraint sets $\mathbb{C}$, $\mathbb{W}$ and $\mathbb{X}_f$ are polytopes with the origin contained in the interior.*

In the sequel, we write the closed-loop dynamics under Algorithm 1 as follows:

$$x_{k+1} = f(x_k, w_k), \tag{39}$$

where $f(x, w) \triangleq a(x) + b(x)\mathbf{v}_0^*(x) + b_w(x)w$. We also refer to the MPC value function $V^* : X_N^{df} \to \mathbb{R}$, defined as

$$V^*(x) \triangleq V(x, \mathbf{v}^*(x)). \tag{40}$$

Remark 4 *Note that there is a unique solution $\mathbf{v}^*(x)$ for each $x \in X_N^{df}$, since the cost function $V(x, \mathbf{v})$ is strictly convex in $\mathbf{v}$ by our choice of $R > 0$. On the other hand, the mapping $\mathbf{M}^*(x)$ may be set-valued (i.e., $\mathbf{M}^*(x) \subset \mathbb{R}^{Nn_u \times Nn_w}$), since the cost function $V(x, \mathbf{v})$ does not depend on $\mathbf{M}$. To prove the main theorem below, we make an arbitrary but single-valued selection $\mathbf{M}^{**}(x) \in \mathbf{M}^*(x)$. For example, we take $\mathbf{M}^{**}(x)$ to be the unique $\mathbf{M}$ that minimizes* Trace $\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)$ *subject to $(\mathbf{v}^*(x), \mathbf{M}) \in \Pi_N^{df}(x)$. We also make use of the following block partition:*

$$\mathbf{v}^*(x) = \begin{bmatrix} \mathbf{v}_0^*(x) \\ \bar{v}(x) \end{bmatrix}, \quad \mathbf{M}^{**}(x) = \begin{bmatrix} 0 \\ \begin{bmatrix} \bar{M}(x) & \mathbf{M}_+^{N-1}(x) \end{bmatrix} \end{bmatrix}, \tag{41}$$

where $\bar{v}(x) \in \mathbb{R}^{(N-1)n_u}$, $\bar{M}(x) \in \mathbb{R}^{n_u(N-1)\times n_w}$ and $\mathbf{M}_{+}^{N-1}(x) \in \mathbb{R}^{n_u(N-1)\times(N-1)n_w}$, and where $\mathbf{v}_0^*(x)$ is the model predictive control law.

4.2 Robust Constraint Satisfaction

The following corollary is a direct consequence of Corollary 8 and shows that Algorithm 1 guarantees constraint satisfaction in the presence of the disturbance input $w_k \in \mathbb{W}$.

Corollary 10 (Robust Constraint Satisfaction) *Suppose that Assumptions 1 and 2 hold. Then, the set X_N^{df} is robustly invariant for the closed-loop system* (39)*, that is,*

$$f(x, w) \in X_N^{df}, \quad \forall x \in X_N^{df}, \forall w \in \mathbb{W}. \tag{42}$$

Consequently, the MPC policy ensures that the system constraints are satisfied for all admissible disturbance sequences provided that $x_0 \in X_N^{df}$, that is, provided that the optimization problem (38) *is solvable at initial time $k = 0$.*

Proof *It can be verified that, for $N = 1$, $f(x, w) \in X_f, \forall x \in X_N^{df}, \forall w \in \mathbb{W}$, whereas, for $2 \le N < \ell_F$, $f(x, w) \in X_{N-1}^{df}, \forall x \in X_N^{df}, \forall w \in \mathbb{W}$. Thus,* (42) *follows from the set inclusions* (32) *of Corollary 8.*

4.3 Input-To-State Stability

Here we show that the closed-loop system is input-to-state stable in the set X_N^{df} provided that Assumptions 1, 2 and the following additional assumption hold:

Assumption 3 *Consider the matrices $Q > 0$, $R > 0$ and $P_f > 0$ of the cost function and the feedback gain K_f of Assumption 1. For all $x \in \mathbb{X}_f$, we have*

$$V_f\left(a\left(x\right) - b\left(x\right) K_f x\right) - V_f\left(x\right) \le -\|x\|^2_{Q + K_f^{\mathrm{T}} R K_f},$$

where $V_f\left(x\right) \triangleq \|x\|^2_{P_f}$.

Remark 5 *Note that we require the "terminal controller" $u = -K_f x$ in Assumptions 1 and 3 to be linear. Also note that, disregarding the linearity of the terminal controller, Assumption 3 is typical for MPC of systems without disturbances (Mayne, Rawlings, Rao, and Scokaert, 2000), whereas Assumption 1 is typical for MPC of systems with disturbances (Mayne, 2001).*

Following Jiang and Wang (2001), we next define the notion of input-to-state stability. To this end, we also introduce standard definitions of $\mathcal{K}$-, $\mathcal{K}_\infty$- and $\mathcal{KL}$-functions.

Definition 11 (Comparison Functions) *A function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\mathcal{K}$-function if it is continuous, strictly increasing and $\sigma(0) = 0$. A $\mathcal{K}$-function $\sigma(r)$ which is radially unbounded (i.e., $\sigma(r) \to \infty$ as $r \to \infty$) is said to be a $\mathcal{K}_\infty$-function. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\mathcal{KL}$-function if for each fixed $k \in \mathbb{R}_+$, the function $\beta(\cdot, k)$ is a $\mathcal{K}$-function, and for each fixed $r \in \mathbb{R}_+$, the function $\beta(r, \cdot)$ is non-increasing with $\lim_{k\to\infty} \beta(r, k) = 0$.*

Definition 12 (Input-To-State Stability) *Consider the discrete-time system*

$$x_{k+1} = f(x_k, w_k), \tag{43}$$

where $x_k \in X$, $0 \in \mathrm{int}(X)$, $w_k \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ and $f(0, 0) = 0$, and let $\phi\left(k, x, \{w_t\}_{t=0}^{k-1}\right)$ denote the state at time k when the initial state is $x_0 = x$ and the disturbance sequence is $\{w_t\}_{t=0}^{k-1}$. The system (43) *is input-to-state stable (ISS) in* *the set X, if there exist a $\mathcal{KL}$-function $\beta(\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$, such that, for all integers $k \geq 0$, for all $x \in X$ and for all sequences $\{w_t\}_{t=0}^{k-1}$ taking values in $\mathbb{W}$, we have $\phi\left(k, x, \{w_t\}_{t=0}^{k-1}\right) \in X$ and*

$$\left\|\phi\left(k, x, \{w_t\}_{t=0}^{k-1}\right)\right\| \leq \beta\left(\|x\|, k\right) + \gamma\left(\max_{t \in \mathbb{Z}_{[0,k-1]}} \|w_t\|\right).$$

We are then ready to state the main result of this section.

Theorem 13 *The closed-loop system* (39) *is ISS in the set $X_N^{df} = X_N^{sf}$ provided that Assumptions 1, 2, 3 hold.*

Proof *By Corollary 10 above and Lemma 3.5 in Jiang and Wang (2001) it suffices to show that the MPC value function $V^*(x)$ in* (40) *satisfies the following condition:*[1] *There exist three $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ and a $\mathcal{K}$-function $\sigma(\cdot)$, such that*

$$\alpha_1(\|x\|) \leq V^*\left(x\right) \leq \alpha_2(\|x\|), \quad \forall x \in X_N^{df}. \tag{44}$$

$$V^*\left(f\left(x,w\right)\right) - V^*\left(x\right) \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \quad \forall x \in X_N^{df},\ \forall w \in \mathbb{W}. \tag{45}$$

Since $Q > 0$, we may choose the $\mathcal{K}_\infty$-function $\alpha_1(r)$ in (44) *as $\alpha_1(r) \triangleq \lambda_{\min}(Q)r^2$, where $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of Q. Furthermore, by exploiting the terminal controller $u = -K_f x$ and Assumptions 1, 2 and 3, it can be shown that $V^*\left(x\right) \leq V_f\left(x\right) \leq \lambda_{\max}(P_f)\|x\|^2$, $\forall x \in \mathbb{X}_f$. Hence, since $V^*\left(x\right)$ is bounded on X_N^{df} and $0 \in \operatorname{int}(\mathbb{X}_f)$, there exists a sufficiently large scalar c such that* (44) *holds using $\alpha_2(r) \triangleq cr^2$.*

It remains to find a $\mathcal{K}_\infty$-function $\alpha_3(r)$ and a $\mathcal{K}$-function $\sigma(r)$ satisfying (45)*. To this end, we define two functions, $\mathbf{v}_+\left(x,w\right)$ and $\mathbf{M}_+\left(x,w\right)$, satisfying*

$$\left(\mathbf{v}_+\left(x,w\right), \mathbf{M}_+\left(x,w\right)\right) \in \Pi_N^{df}\left(f\left(x,w\right)\right), \quad \forall x \in X_N^{df},\ \forall w \in \mathbb{W}. \tag{46}$$

[Note that (46) *implies that the pair $(\mathbf{v}_+(\cdot), \mathbf{M}_+(\cdot))$ is a feasible solution to the problem* (38) *when the current system state is $f(\cdot)$.] When $N = 1$, we use $\mathbf{v}_+\left(x,w\right) \triangleq -K_f f\left(x,w\right)$ and $\mathbf{M}_+\left(x,w\right) \triangleq 0$. When $2 \leq N < \ell_F$, we use*

$$\mathbf{v}_+\left(x,w\right) \triangleq \begin{bmatrix} \mathbf{v}_+^{N-1}\left(x,w\right) \\ v_+^N\left(x,w\right) \end{bmatrix}, \quad \mathbf{M}_+\left(x,w\right) \triangleq \begin{bmatrix} \mathbf{M}_+^{N-1}(x) & 0 \\ M_+^N\left(x,w\right) & 0 \end{bmatrix}, \tag{47}$$

where $\mathbf{v}_+^{N-1}\left(x,w\right) \triangleq \bar{v}(x) + \bar{M}(x)w$, and where $\mathbf{M}_+^{N-1}(x)$, $\bar{v}(x)$ and $\bar{M}(x)$ are as in (41)*. To describe the bottom entries in* (47)*, we first note that the top entries in* (47) *satisfy*

$$\left(\mathbf{v}_+^{N-1}\left(x,w\right), \mathbf{M}_+^{N-1}\left(x\right)\right) \in \Pi_{N-1}^{df}\left(f\left(x,w\right)\right), \quad \forall x \in X_N^{df},\ \forall w \in \mathbb{W}. \tag{48}$$

Hence, in view of Assumption 1, we may define the bottom entries by "adding a step" in a similar manner to the proof of Corollary 8 [see (34)*-*(35)*]. That is,*

$$v_+^N\left(x,w\right) \triangleq -K_f\left[A^{N-1}\left(f\left(x,w\right)\right) + B^{N-1}\left(f\left(x,w\right)\right)\mathbf{v}_+^{N-1}\left(x,w\right)\right], \tag{49}$$

$$M_+^N\left(x,w\right) \triangleq -K_f\left[B^{N-1}\left(f\left(x,w\right)\right)\mathbf{M}_+^{N-1}(x) + B_w^{N-1}\left(f\left(x,w\right)\right)\right], \tag{50}$$

where the notation $A^{N-1}(\cdot)$, $B^{N-1}(\cdot)$ and $B_w^{N-1}(\cdot)$, respectively, denote the last block row of the functions $A(\cdot)$, $B(\cdot)$ and $B_w(\cdot)$ in (7) *when N is reduced to $N \leftarrow N-1$ in* (8)*.*

[1] As remarked by Goulart (2006) and others, the ISS-Lyapunov function $V^*(x)$ need not be continuous in the proof of Lemma 3.5 in Jiang and Wang (2001).

A useful observation regarding the function $\mathbf{v}_+(x, w)$ defined above is that, subject to Assumption 3, the following holds:

$$V\left(f\left(x,0\right),\mathbf{v}_+\left(x,0\right)\right) - V^*(x) \leq -\|x\|_Q^2 - \|u\|_R^2, \quad \forall x \in X_N^{df}, \tag{51}$$

where $u = \mathbf{v}_0^*(x)$. Another useful property is that $\mathbf{v}_+(x, w)$ is Lipschitz continuous in w on its domain $X_N^{df} \times \mathbb{W}$, that is, there exists a scalar L (i.e., a Lipschitz constant) such that the following holds:

$$\left\|\mathbf{v}_+\left(x, w_2\right) - \mathbf{v}_+\left(x, w_1\right)\right\| \leq L\left\|w_2 - w_1\right\|, \quad \forall x \in X_N^{df},\ \forall w_1, w_2 \in \mathbb{W}. \tag{52}$$

This follows from (42), (47) and (49) since: (i) the sets X_N^{df} and $\Pi_N^{df}(x)$, $\forall x \in X_N^{df}$, are bounded; (ii) the function $F(x,\mu)$ is analytic, hence Lipschitz continuous when restricted to a bounded set; and (iii) the function $f(x,w)$ is Lipschitz continuous in w on $X_N^{df} \times \mathbb{W}$. Similarly, note that the cost function $V(x, \mathbf{v})$ is Lipschitz continuous in both its arguments when restricted to any set of the form $X_N^{df} \times \mathbb{U}^N$, where $\mathbb{U}$ is bounded. Next let us consider a bounded set $\mathbb{U}$ such that $\mathbb{C} \subseteq \mathbb{M} \times \mathbb{U}$ and note that, since $f(x,w) \in X_N^{df}$ and $\mathbf{v}_+(x,w) \in \mathbb{U}^N$, we have using some Lipschitz constants k_1, k_2, k_3 and k_4 (which, in general, depend on the sets X_N^{df}, $\mathbb{W}$ and $\mathbb{U}$) that

$$\begin{aligned}
&\left\|V\left(f\left(x,w\right),\mathbf{v}_+\left(x,w\right)\right) - V\left(f\left(x,0\right),\mathbf{v}_+\left(x,0\right)\right)\right\| \\
&\leq k_1\left\|f\left(x,w\right) - f\left(x,0\right)\right\| + k_2\left\|\mathbf{v}_+\left(x,w\right) - \mathbf{v}_+\left(x,0\right)\right\| \\
&\leq \left(k_1 k_3 + k_2 k_4\right)\|w\| \triangleq k\|w\|, \quad \forall x \in X_N^{df},\ \forall w \in \mathbb{W}.
\end{aligned} \tag{53}$$

Hence we can establish (45) by using the property (46) to upper bound the value function $V^*\left(f\left(x,w\right)\right)$ and then employing (51) and (53) as follows:

$$\begin{aligned}
&V^*\left(f\left(x,w\right)\right) - V^*(x) \leq V\left(f\left(x,w\right),\mathbf{v}_+\left(x,w\right)\right) - V^*(x) \\
&= V\left(f\left(x,0\right),\mathbf{v}_+\left(x,0\right)\right) - V^*(x) + \left[V\left(f\left(x,w\right),\mathbf{v}_+\left(x,w\right)\right) - V\left(f\left(x,0\right),\mathbf{v}_+\left(x,0\right)\right)\right] \\
&\leq -\|x\|_Q^2 - \|u\|_R^2 + k\|w\| \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \quad \forall x \in X_N^{df},\ \forall w \in \mathbb{W},
\end{aligned} \tag{54}$$

where $u = \mathbf{v}_0^*(x)$, $\sigma(r) \triangleq kr$ and $\alpha_3(r) \triangleq \lambda_{\min}(Q)r^2 = \alpha_1(r)$.

Remark 6 *As is typical in MPC analysis (see, e.g., Rawlings and Muske (1993); Mayne et al. (2000)), the proof of Theorem 13 employs the feasibility of a "candidate solution" $\mathbf{v}_+(x,w)$ in order to establish an upper bound on the value function $V^*\left(f\left(x,w\right)\right)$. In contrast to the ISS results in, for example, Kerrigan and Maciejowski (2003); Goulart et al. (2006), the proof of Theorem 13 does not rely on (local Lipschitz) continuity properties of $V^*(x)$, rather, it relies on Lipschitz continuity properties of the candidate solution $\mathbf{v}_+(x,w)$. In fact, it remains unclear to the author whether the value function $V^*(x)$ can be discontinuous at points in X_N^{df} under the stated assumptions. A related observation is that, since the cost function $V(x,\mathbf{v})$ may be non-convex in its first argument x, the value function $V^*(x)$ may also fail to be convex, even when restricted to a convex subset of its domain X_N^{df}. Finally, note that Theorem 13 does not exploit the fact that the optimization problem is a QP and thus the results generalize straightforwardly to the case of an ellipsoidal disturbance set, in which case the optimization problem is a second-order cone programme (c.f. Remark 3).*

4.4 Numerical Example

We consider the flexible joint manipulator model described in Example 1 with the sampling time set to $T_s = 0.05$ and the parameter values as described in Sira-Ramírez and Castro-Linares (2000) (i.e., $m = 0.4$, $g = 9.81$, $L = 0.185$, $J = 0.002$, $I = 0.0059$ and $K_a = 1.61$). The disturbance input is assumed to satisfy $w_k \in \mathbb{W} \triangleq \{w \mid |w| \leq 1\}$, whereas the mixed constraints on the state and the control input are given by $(x_k, u_k) \in \mathbb{C} \triangleq \{(x_k, u_k) \mid |x_{1,k}| \leq \pi/2, |x_{i,k}| \leq 100, i \in \mathbb{Z}_{[2,4]}, |u_k| \leq 5\}$. Our goal is to design an instance of Algorithm 1 using the following weighting matrices:

$$Q = \operatorname{diag}\{1, 0.1, \epsilon_0, \epsilon_0\},\ R = \epsilon_0,\ \epsilon_0 \triangleq 0.001. \tag{55}$$

4.4.1 Finding the Terminal Triplet K_f, $\mathbb{X}_f$ and P_f

We first consider the problem of finding the feedback gain K_f, the constraint set $\mathbb{X}_f$ and the weighting matrix P_f so as to satisfy Assumptions 1 and 3 using the given weighting matrices (55). To this end, we note that the system dynamics (6) may be re-written as follows:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \\ x_{4,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0.050 & 0 & 0 \\ -7.492 & 1 & 13.64 & 0 \\ 0 & 0 & 1 & 0.050 \\ 40.25 & 0 & -40.25 & 1 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{bmatrix} + \mathcal{B}u_k + \mathcal{B}_\varphi \varphi(x_{1,k}) + \mathcal{B}_w w_k, \tag{56}$$

where $\mathcal{B} \triangleq [0\ 0\ 0\ 25]^{\mathrm{T}}$, $\mathcal{B}_\varphi \triangleq [0\ -6.152\ 0\ 0]^{\mathrm{T}}$, $\mathcal{B}_w \triangleq [0\ 0\ 0\ 1]^{\mathrm{T}}$ and $\varphi(y) \triangleq y - \sin(y)$. Furthermore, we observe from Fig. 1 that the nonlinearity $\varphi(y)$ is relatively small on the interval $y \in [-0.5, 0.5]$. Hence, whenever the system

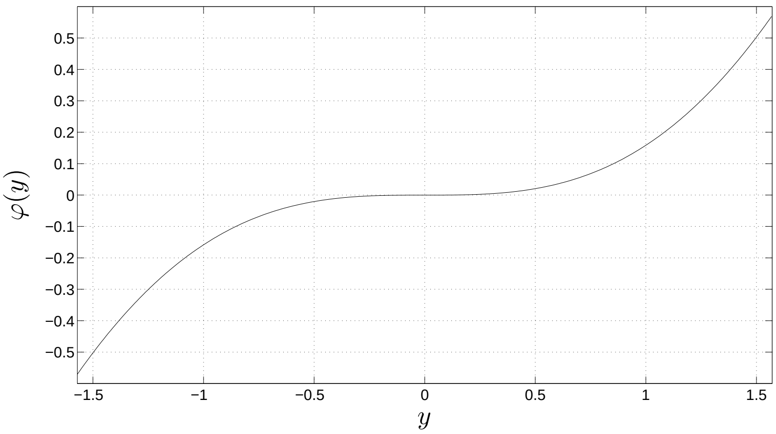

Fig. 1. The function $\varphi(y) = (y - \sin(y))$ on the interval $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

state satisfies $x_k \in \mathbb{X}_0 \triangleq \{x_k \,|\, |x_{1,k}| \le 0.5\}$, the nonlinear dynamics may be approximated relatively accurately by the linear dynamics obtained using $\mathcal{B}_\varphi = 0$ in (56). Motivated by these observations we propose to compute K_f as the LQR gain of the linear system model $x_{k+1} = \mathcal{A}x_k + \mathcal{B}u_k$ for the given weighting matrices; the result is $K_f = [0.4961\ 0.0123\ 0.7110\ 0.1223]$.

Next, to compute an associated terminal constraint set $\mathbb{X}_f$, we consider the following sufficient condition for Assumption 1 to hold with our choice of K_f:

$$A_f x + [\mathcal{B}_\varphi\ \mathcal{B}_w]\,\bar{w} \in \mathbb{X}_f, \quad \forall x \in \mathbb{X}_f, \forall \bar{w} \in \mathbb{W}_\varphi \times \mathbb{W}, \tag{57a}$$

$$[x, -K_f x] \in \mathbb{C} \cap \{\mathbb{X}_0 \times \mathbb{R}\}, \quad \forall x \in \mathbb{X}_f, \tag{57b}$$

where $A_f \triangleq \mathcal{A} - \mathcal{B}K_f$, $\mathbb{W}_\varphi \triangleq \{\varphi \in \mathbb{R} \,|\, |\varphi| \le \varphi(0.5) \approx 0.0206\}$, and where we have made use of the fact that $\varphi(x_{1,k}) \in \mathbb{W}_\varphi$, $\forall x_k \in \mathbb{X}_0$. To arrive at the sufficient condition (57), we have effectively: (i) introduced the additional constraint $[x_k, u_k] \in \mathbb{X}_0 \times \mathbb{R}$ and (ii) modeled the nonlinearity $\varphi(x_{1,k})$ as a bounded additive disturbance input $\varphi_k \in \mathbb{W}_\varphi$. These two steps are motivated by the facts that: (i) computing the maximal set $\mathbb{X}_f$ that satisfies (57) is a linear programming problem (Kolmanovsky and Gilbert, 1998); and (ii) the set $\mathbb{W}_\varphi$ can be taken to relatively small when we require $\mathbb{X}_f \subseteq \mathbb{X}_0$. Indeed, as is well known, since the relevant sets are polytopes and the set $\mathbb{W}_\varphi \times \mathbb{W}$ is sufficiently small, we may use Algorithm 6.2 in Kolmanovsky and Gilbert (1998) to compute the maximal set $\mathbb{X}_f$ that satisfies (57). The result is a polytopic constraint set $\mathbb{X}_f$ which contains the origin in the interior. To illustrate the shape of the resulting terminal constraint set $\mathbb{X}_f$, Fig. 2 shows the specific "slice" of $\mathbb{X}_f$ that intersects the surface $x_{3,k} = x_{4,k} = 0$. For an alternative approach to compute $\mathbb{X}_f$, see, for example, Cannon et al. (2003).

Towards our next goal of computing the weighting matrix P_f, we observe that the nonlinearity $\varphi(y)$ on the interval $y \in [-0.5, 0.5]$ belongs to the sector $[0, K_2]$ (see, e.g., Khalil (2002)), where $K_2 \triangleq 2\varphi(0.5) \approx 0.0411$. That is to say,

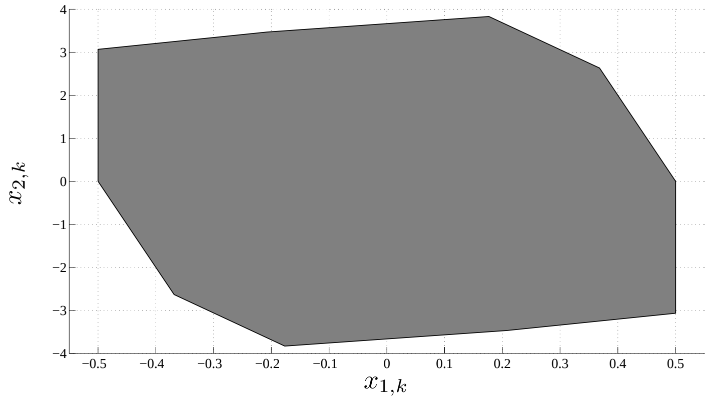

Fig. 2. Slice along $x_{3,k} = x_{4,k} = 0$ of the maximal set $\mathbb{X}_f \subset \mathbb{R}^4$ that satisfies (57).

the function $\varphi : [-0.5, 0.5] \to \mathbb{R}$ satisfies:

$$\begin{bmatrix} y \\ \varphi(y) \end{bmatrix}^{\mathrm{T}} S \begin{bmatrix} y \\ \varphi(y) \end{bmatrix} \geq 0,\ \forall y \in [-0.5, 0.5], \quad S \triangleq \begin{bmatrix} 0 & K_2 \\ K_2 & -2 \end{bmatrix}. \tag{58}$$

Hence, a sufficient condition for Assumption 3 to hold using K_f and $\mathbb{X}_f$ is as follows:

$$\begin{bmatrix} A_f & \mathcal{B}_\varphi \end{bmatrix}^{\mathrm{T}} P_f \begin{bmatrix} A_f & \mathcal{B}_\varphi \end{bmatrix} - \begin{bmatrix} I & 0 \end{bmatrix}^{\mathrm{T}} P_f \begin{bmatrix} I & 0 \end{bmatrix} \tag{59}$$

$$\leq - \begin{bmatrix} I & 0 \end{bmatrix}^{\mathrm{T}} \left(Q + K_f^{\mathrm{T}} R K_f\right) \begin{bmatrix} I & 0 \end{bmatrix} + m C_S^{\mathrm{T}} S C_S, \tag{60}$$

where $C_S \triangleq \operatorname{diag}\{[1\ 0\ 0\ 0], 1\}$ and $m \geq 0$ is a scalar "multiplier". [To see that this is the case, left- and right-multiply (60) by $[x_k\ \varphi(x_{1,k}]$ and $[x_k\ \varphi(x_{1,k}]^{\mathrm{T}}$, respectively, and then use $\mathbb{X}_f \subseteq \mathbb{X}_0$ and (58).] It follows that determining an (in some sense) optimal matrix P_f satisfying Assumption 3 may be accomplished by solving the following SDP:

$$\min_{P_f,\, m}\ \operatorname{Trace}(P_f)\ \text{ subject to (60) and } m \geq 0.$$

Specifically, by solving this SDP we obtain

$$P_f = \begin{bmatrix} 32.3713 & -1.7679 & -56.4408 & -1.1638 \\ -1.7679 & 0.4031 & 5.6113 & 0.0853 \\ -56.4408 & 5.6113 & 138.2601 & 2.7077 \\ -1.1638 & 0.0853 & 2.7077 & 0.0648 \end{bmatrix}. \tag{61}$$

4.4.2 Simulation Results

Using a prediction horizon of $N = 4 < \ell_F = 5$ we next implement Algorithm 1 by solving the quadratic programme (38) at each time step k. Note from the discussion above that Assumptions 1, 2 and 3 hold and that our MPC policy thus input-to-state stabilizes the system in the set $X_4^{df} = X_4^{sf}$ (c.f. Theorem 13). To illustrate that Algorithm 1 also leads to good nominal performance, Fig. 3 compares the closed-loop responses with the responses of the linear system $x_{k+1} = A_f x_k$ when the initial state is set to $x_0 = [0.5\ 0\ 0\ 0]^{\mathrm{T}} \in \mathbb{X}_f \subseteq X_4^{df}$ and $w_k = 0$. Note that the closed-loop responses of Algorithm 1 shown in Fig. 3 are relatively close to being optimal over an infinite horizon, since they closely approximate the responses of $x_{k+1} = A_f x_k$ and we have $\varphi(x_{1,k}) \approx 0$ and $w_k = 0$ (c.f. (56) and Fig. 1). More importantly perhaps, the MPC policy approximates the associated infinite horizon problem and input-to-state stabilizes the system in the set X_4^{df} which is significantly larger than the terminal constraint set $\mathbb{X}_f$. To illustrate this point, Fig. 4 shows the closed-loop responses obtained when the initial state $x_0 \in X_4^{df}$ is close to the boundary of the set X_4^{df} and the disturbance w_k varies randomly within its constraint set $\mathbb{W}$. Note from Fig. 2 that the initial state in Fig. 4 is far outside the terminal constraint set, and from Fig. 1 that the initial state of $x_{1,k}$

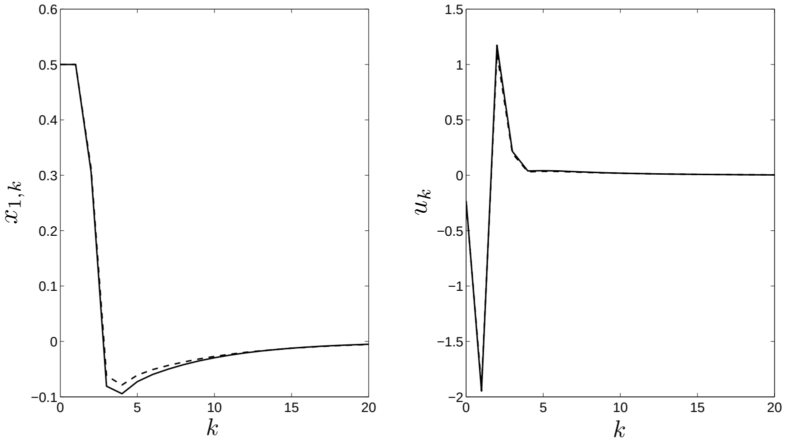

Fig. 3. Closed-loop responses under Algorithm 1 when $w_k = 0$ (solid) and the corresponding responses of the linear system $x_{k+1} = A_f x_k$, $u_k = -K_f x_k$ (dashed).

is far outside the "flat" region of the nonlinearity $\varphi(x_{1,k})$. Also note that the control input u_k hits its constraints at $-5, 5$, and that the responses are stable and converge to a neighborhood around zero in the presence of the persistent disturbance $w_k \in \mathbb{W}$, as established in Theorem 13.

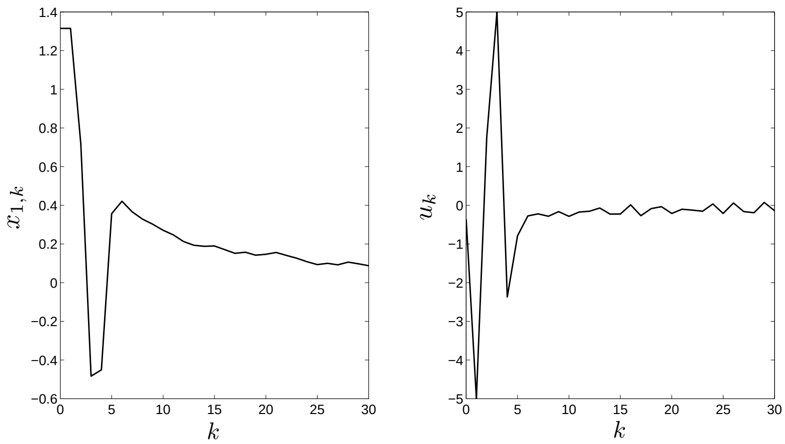

Fig. 4. Closed-loop responses under Algorithm 1 when $w_k \in \mathbb{W}$ and $x_0 = [1.32\ 0\ 0\ 0]^{\mathrm{T}}$.

5 Conclusions

We have presented results on N-step affine feedback policies that enable the systematic design of affine and predictive control policies for a class of nonlinear systems with bounded disturbance inputs. Our results generalize recent results for linear systems to the case when N is chosen to be less than the nonlinear system's input-state linear horizon ℓ_F. As a particular application, we have presented an input-to-state stabilizing MPC policy based on a convex QP parameterized by the current system state. The MPC policy has been applied to control a nonlinear discrete-time model of a flexible joint manipulator, and simulations results illustrating this have been reported.

The authors would like to stress that the results presented here are well-known in the linear case when $\ell_F = \infty$ (see Goulart et al. (2006)). We also acknowledge that the results seem to be restricted to a relatively small class of nonlinear systems due to the restriction $N < \ell_F$. We hope, however, that our approach will motivate further developments that exploit structure in order to reduce computational complexity and better understand nonlinear MPC.

A promising extension of our approach is to attempt to relax the restriction $N < \ell_F$, at the expense of introducing an approximation error, by making use of an *approximate* N-step prediction model of the form (7)-(8); whilst the QP of the present MPC policy yields the *exact solution* to the problem of optimizing a quadratic criterion over the class N-step affine state-feedback policies whenever $N < \ell_F$, such an approach would produce an *approximate solution* for the cases when $N \geq \ell_F$. It is also of significant practical interest to attempt to extend our results to cases with imperfect state measurements.

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Appendix: Lower Block Triangular Left Inverse

Lemma 14 *Suppose that $N \in \mathbb{Z}_{[1,\ell_F-1]}$ and that $b_w(x)$ in (2) has full column rank for all $x \in \mathbb{M}$. Then, it follows that $B_w(x)$ in (7) has a lower block triangular left inverse $B_w^\dagger(x) \in \mathbb{R}^{Nn_w \times Nn_x}$ for all $x \in \mathbb{M}$.*

Proof *Firstly note that the matrix $B_w(x)$ is lower block triangular with $\operatorname{rank}(B_w(x)) = Nn_w$ for all $N \in \mathbb{Z}_{[0,\ell_F-1]}$ and all $x \in \mathbb{M}$. [Conversely, if $B_w(x)$ is rank deficient for some $N \in \mathbb{Z}_{[0,\ell_F-1]}$ and some $x \in \mathbb{M}$, then there exist two different disturbance sequences, say $\mathbf{w}_1$ and $\mathbf{w}_2$, $\mathbf{w}_1 \neq \mathbf{w}_2$, which result in the same state sequence under the open-loop control policy $\mathbf{u} = 0$ when $x_0 = x$. However, this, in turn, implies that the matrix $b_w(x)$ in (2) is rank deficient for some $x \in \mathbb{M}$, which contradicts our initial assumption.] Using the latter fact, which is necessary and sufficient for $B_w(x)$ to have some left inverse, we proceed with an induction argument to show that there exists, as claimed, a specific left inverse $B_w^\dagger(x)$ which also is lower block triangular. When $N = 1$, there is only one block, that* *is, we have* $B_w(x) = b_w(x)$ *and we may thus choose* $B_w^{\dagger}(x) = b_w^{\dagger,\mathrm{full}}(x)$ *where* $b_w^{\dagger,\mathrm{full}}(x)$ *is any left inverse of* $b_w(x)$*. To complete the induction argument, we next show that, if the lemma statement holds for* $N \in \mathbb{Z}_{[1,\ell_F-2]}$*, then it also holds when* N *is incremented to* $N \leftarrow N+1$*. To this end, let* $B_{w,L}(x)$ *and* $B_{w,L}^{\dagger}(x)$ *denote* $B_w(x)$ *and* $B_w^{\dagger}(x)$ *when* $N = L$*,* $L \in \mathbb{Z}_{[1,\ell_F-2]}$*, and note that* $B_{w,L+1}(x)$ *is of the form*

$$B_{w,L+1}(x) = \begin{bmatrix} \begin{bmatrix} B_{w,L}(x) & 0 \end{bmatrix} \\ B_{w,+}(x) \end{bmatrix}, \tag{62}$$

for some matrix $B_{w,+}(x)$*. As a consequence of* (62) *and the induction hypothesis that* $B_{w,L}^{\dagger}(x)\,B_{w,L}(x) = I$ *where* $B_{w,L}^{\dagger}(x)$ *is lower block triangular, it follows by direct computation that the following matrix is a lower block triangular left inverse of* $B_{w,L+1}(x)$*:*

$$B_{w,L+1}^{\dagger}(x) = \begin{bmatrix} \begin{bmatrix} B_{w,L}^{\dagger}(x) & 0 \end{bmatrix} \\ e_{L+1} B_{w,L+1}^{\dagger,\mathrm{full}}(x) \end{bmatrix}. \tag{63}$$

Here, $B_{w,L+1}^{\dagger,\mathrm{full}}(x)$ *denotes any left inverse of* $B_{w,L+1}(x)$ *and* e_{L+1} *denotes the* n_w *last rows of the identity matrix of dimension* $(L+1)n_w$*. Since the required left inverse* $B_{w,L+1}^{\dagger,\mathrm{full}}(x)$ *exists for all* $L \in \mathbb{Z}_{[1,\ell_F-2]}$ *and all* $x \in \mathbb{M}$*, this completes the proof.*