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System theoretic properties of a class of spatially invariant systems[☆]

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ABSTRACT

In this paper we develop new readily testable criteria for system theoretic properties such as stability, controllability, observability, stabilizability and detectability for a class of spatially invariant systems. Our approach uses the well-established theory developed to solve infinite-dimensional systems. The theoretical results are illustrated by several examples.

1. Introduction

In Freedman, Falb, and Zames (1969) a stability theory was developed for a very general class of continuous-time systems defined on a locally compact abelian group $\mathbb{G}$ and taking values in a separable Hilbert space. It relied on a generalization of the known transform theory in Loomis (1953) and Rudin (1962) to Hilbert space valued functions on $\mathbb{G}$ in Falb and Freedman (1969). Three decades later, motivated by technological progress in microelectromechanical systems (MEMS) and possible applications to platoons of vehicles (see Chu (1974), Jovanović and Bamieh (2005), Levine and Athans (1966) and Melzer and Kuo (1971a,b)), Bamieh, Paganini, and Dahleh (2002) reconsidered this idea under the name of *spatially invariant systems*. This is to apply the Fourier transform to the spatially invariant system, thus obtaining a mathematically simpler system defined on the character group $\hat{\mathbb{G}}$ of $\mathbb{G}$. Green and Kamen (1985) and Kamen and Green (1980) examined stabilizability concepts for discrete-time systems with $\mathbb{G} = \mathbb{Z}$ and the character group $\hat{G} = \partial\mathbb{D}$, where $\mathbb{Z}$ is the set of integer numbers and $\partial\mathbb{D}$ is the unit circle. In this paper we consider continuous-time systems with $\mathbb{G} = \mathbb{Z}$. More precisely, the class of infinite-dimensional systems (with spatially invariant dynamics (Bamieh et al., 2002)) under consideration is

$$\dot{z}_r(t) = \sum_{l=-\infty}^{\infty} A_l z_{r-l}(t) + \sum_{l=-\infty}^{\infty} B_l u_{r-l}(t), \tag{1}$$

$$y_r(t) = \sum_{l=-\infty}^{\infty} C_l z_{r-l}(t) + \sum_{l=-\infty}^{\infty} D_l u_{r-l}(t), \tag{2}$$

where $r \in \mathbb{Z}$, $A_l \in \mathbb{C}^{n\times n}$, $B_l \in \mathbb{C}^{n\times m}$, $C_l \in \mathbb{C}^{p\times n}$, $D_l \in \mathbb{C}^{p\times m}$ and $z_r(t) \in \mathbb{C}^n$, $u_r(t) \in \mathbb{C}^m$ and $y_r(t) \in \mathbb{C}^p$ are the state, the input and the output vectors, respectively, at time $t \geq 0$ and spatial point $r \in \mathbb{Z}$. Using the terminology and formalism of Curtain and Zwart (1995) we can formulate (1) and (2) as a standard state linear system $\Sigma(A, B, C, D)$

$$\begin{aligned} \dot{z}(t) &= (Az)(t) + (Bu)(t), \\ y(t) &= (Cz)(t) + (Du)(t), \quad t \geq 0, \end{aligned} \tag{3}$$

with the state space $Z = \ell_2(\mathbb{C}^n)$, the input space $U = \ell_2(\mathbb{C}^m)$ and the output space $Y = \ell_2(\mathbb{C}^p)$ (defined in Appendix). Note that Z, U, Y are all infinite dimensional and so $z(t) = (z_r(t))_{r=-\infty}^{\infty} \in \ell_2(\mathbb{C}^n)$, $u(t) = (u_r(t))_{r=-\infty}^{\infty} \in \ell_2(\mathbb{C}^m)$, $y(t) = (y_r(t))_{r=-\infty}^{\infty} \in \ell_2(\mathbb{C}^p)$ and A, B, C, D are convolution operators. We denote the signals and the convolution operators generically by $x(t)$ and T, respectively. Then

$$((Tx)(t))_r = \sum_{l=-\infty}^{\infty} T_l x_{r-l}(t) = \sum_{l=-\infty}^{\infty} T_{r-l} x_l(t).$$

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To derive conditions for these operators to be bounded we take Fourier transforms (see Definition A.4)

$$\begin{aligned}((\check{Tx})(t))(\mathrm{e}^{\mathrm{j}\theta}) &= \sum_{r=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} T_l x_{r-l}(t)\mathrm{e}^{-\mathrm{j}r\theta} \\ &= \sum_{l=-\infty}^{\infty} T_l \mathrm{e}^{-\mathrm{j}l\theta} \sum_{r=-\infty}^{\infty} x_{r-l}(t)\mathrm{e}^{-\mathrm{j}(r-l)\theta} \\ &:= \check{T}(\mathrm{e}^{\mathrm{j}\theta}) \sum_{r=-\infty}^{\infty} x_r(t)\mathrm{e}^{-\mathrm{j}r\theta} \\ &= \check{T}(\mathrm{e}^{\mathrm{j}\theta})\check{x}(\mathrm{e}^{\mathrm{j}\theta}, t),\end{aligned}$$

where $\check{T}(\mathrm{e}^{\mathrm{j}\theta}) := \sum_{l=-\infty}^{\infty} T_l \mathrm{e}^{-\mathrm{j}l\theta}$. According to Property A.2 this will define a bounded operator from $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^\bullet)$ to $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^\bullet)$ provided that $\check{T} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{\bullet\times\bullet})$, i.e., provided that (see Definition A.1) $\|\check{T}\|_\infty < \infty$ ("$\bullet$" denotes the appropriate dimension). Note that the Fourier transform of the convolution product $\check{CA} = \check{C}\check{A}$ is just matrix multiplication and the product of two operators is also a convolution operation provided, of course, that this is well defined.

Now, $\ell_2(\mathbb{C}^q)$ is isometrically isomorphic to $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$ under the Fourier transform $\mathfrak{F}$. Hence

$$\check{x} = \mathfrak{F}x, \qquad x = \mathfrak{F}^{-1}\check{x}, \qquad \mathfrak{F}(Tx) = \mathfrak{F}T\mathfrak{F}^{-1}\mathfrak{F}x, \qquad T = \mathfrak{F}^{-1}\check{T}\mathfrak{F},$$

and T is bounded if and only if $\check{T}$ is. They have the same norms, since

$$\begin{aligned}\|T\| &= \sup_{\|x\|_{\ell_2}=1} \|Tx\|_{\ell_2} = \sup_{\|x\|_{\ell_2}=1} \|\mathfrak{F}^{-1}\check{T}\check{x}\|_{\ell_2} \\ &= \sup_{\|\check{x}\|_{\mathbf{L}_2(\partial\mathbb{D})}=1} \|\check{T}\check{x}\|_{\mathbf{L}_2(\partial\mathbb{D})} = \|\check{T}\|_\infty,\end{aligned}$$

where we have used the fact that $\|x\|_{\ell_2(\mathbb{C}^q)} = \|\check{x}\|_{\mathbf{L}_2(\partial\mathbb{D},\mathbb{C}^q)}$. The Fourier transformed operators form the space $\mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times q})$.

Taking Fourier transforms of the system equations (3), we obtain

$$\begin{aligned}\dot{\check{z}}(t) &= \mathfrak{F}\dot{z}(t) = \check{A}\check{z}(t) + \check{B}\check{u}(t), \\ \check{y}(t) &= \mathfrak{F}y(t) = \check{C}\check{z}(t) + \check{D}\check{u}(t),\end{aligned} \tag{4}$$

where $\check{A} = \mathfrak{F}A\mathfrak{F}^{-1}$, $\check{B} = \mathfrak{F}B\mathfrak{F}^{-1}$, $\check{C} = \mathfrak{F}C\mathfrak{F}^{-1}$ and $\check{D} = \mathfrak{F}D\mathfrak{F}^{-1}$ are multiplicative operators. In our case they are all bounded operators. The state linear system $\Sigma(A, B, C, D)$ is isometrically isomorphic to the state linear system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ on the state space $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$ with input and output spaces $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^m)$ and $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^p)$ respectively. Their system theoretic properties are identical (see Curtain and Zwart (1995) Exercise 2.5) and so it suffices to apply the standard theory from Curtain and Zwart (1995) to the particular class of spatially invariant infinite-dimensional systems. For almost all $\theta \in [0, 2\pi]$ the system (4) can be written as

$$\begin{aligned}\frac{\partial}{\partial t}\check{z}(\mathrm{e}^{\mathrm{j}\theta}, t) &= \check{A}(\mathrm{e}^{\mathrm{j}\theta})\check{z}(\mathrm{e}^{\mathrm{j}\theta}, t) + \check{B}(\mathrm{e}^{\mathrm{j}\theta})\check{u}(\mathrm{e}^{\mathrm{j}\theta}, t) \\ \check{y}(\mathrm{e}^{\mathrm{j}\theta}, t) &= \check{C}(\mathrm{e}^{\mathrm{j}\theta})\check{z}(\mathrm{e}^{\mathrm{j}\theta}, t) + \check{D}(\mathrm{e}^{\mathrm{j}\theta})\check{u}(\mathrm{e}^{\mathrm{j}\theta}, t), \quad t \geq 0.\end{aligned} \tag{5}$$

where $\check{A}$, $\check{B}$, $\check{C}$ and $\check{D}$ need not be defined for all $\theta \in [0, 2\pi]$. The motivation for studying this special class of system stems from the interest shown in the literature for controlling infinite platoons of vehicles over the years (see Bamieh et al. (2002), Chu (1974), Jovanović and Bamieh (2005), Levine and Athans (1966) and Melzer and Kuo (1971a,b)). The models obtained for these configurations have the spatially invariant form (5).

In what follows we shall sometimes assume for the convolution operators T the stronger condition $T \in \ell_1(\mathbb{C}^{k\times q})$, i.e.,

$$\|T\|_1 = \|\check{T}\|_1 = \sum_{l=-\infty}^{\infty} |T_l| < \infty,$$

where $|\cdot|$ denotes the matrix spectral norm (i.e., the operator norm). In the case that $k = q$, the space $\ell_1(\mathbb{C}^{k\times k})$ is a Banach subalgebra of $\mathcal{L}(\ell_2(\mathbb{C}^k))$ with convolution as the product operation.

Definition 1.1. The operator $\check{T} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times q})$ is in the *Wiener class* if it has the well-defined expansion $\check{T}(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{l=-\infty}^{\infty} T_l \mathrm{e}^{\mathrm{j}l\theta}$ and $\|\check{T}\|_1 < \infty$. We denote the space of operators with this property by $\mathbf{W}(\partial\mathbb{D}; \mathbb{C}^{k\times q})$.

It is readily seen that $\ell_1(\mathbb{C}^{k\times q})$ is isometrically isomorphic to $\mathbf{W}(\partial\mathbb{D}; \mathbb{C}^{k\times q})$. So they are Banach spaces under the norm $\|\cdot\|_1$. Clearly, $\|\check{T}\|_1 \geq \|\check{T}\|_\infty$, and $\check{T}(\mathrm{e}^{\mathrm{j}\theta})$ is continuous in θ on $[0, 2\pi]$. In the case that $k = q$ we have that $\mathbf{W}(\partial\mathbb{D}; \mathbb{C}^{k\times k})$ is a Banach algebra. In the subsequent sections we consider the particular class of spatially invariant infinite-dimensional systems (5) and arrive at simple readily checkable tests for system theoretic properties such as stability, stabilizability, detectability, observability and controllability. The same tests are applicable to the isometrically isomorphic class of spatially invariant systems (1) and (2). The theory is illustrated by simple examples.

2. Stability properties

In this section we are concerned with exponential stability and strong stability of the autonomous differential equation

$$\frac{\partial}{\partial t}\check{z}(t) = \check{A}\check{z}(t), \quad t \geq 0, \tag{6}$$

where $\check{A}$ is the Fourier transform of the convolution operator A, and $\check{A} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$. Denote by $\mathrm{e}^{\check{A}t}$ and, for a given $\theta \in [0, 2\pi]$, $\mathrm{e}^{\check{A}(\mathrm{e}^{\mathrm{j}\theta})t}$ the strongly continuous semigroups generated by $\check{A}$ and $\check{A}(\mathrm{e}^{\mathrm{j}\theta})$, respectively (Curtain & Zwart, 1995). The semigroup $\mathrm{e}^{\check{A}t}$ is exponentially stable if there exist positive constants M and α such that $\|\mathrm{e}^{\check{A}t}\|_\infty \leq M\mathrm{e}^{-\alpha t}$ for all $t \geq 0$. We also say that the system (6) is *exponentially stable*. In particular, the system (6) is exponentially stable if and only if there exist positive constants M and α such that $\operatorname{ess\,sup}_{0\leq\theta\leq 2\pi} \|\mathrm{e}^{\check{A}(\mathrm{e}^{\mathrm{j}\theta})t}\| \leq M\mathrm{e}^{-\alpha t}$ for all $t \geq 0$.

Since $\check{A}$ is a bounded operator, the semigroup $\mathrm{e}^{\check{A}t}$ satisfies the spectrum determined growth assumption (see Curtain and Pritchard (1978, p. 74)), i.e.,

$$\sup\{\mathrm{Re}(\lambda), \lambda \in \sigma(\check{A})\} = \lim_{t\to\infty} \frac{\log \|\mathrm{e}^{\check{A}t}\|_\infty}{t} = \omega_0.$$

Thus the equivalence stated in the following remark holds.

Remark 2.1. The system (6) is exponentially stable if and only if

$$\sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(\check{A})\} < 0. \tag{7}$$

Conditions for the spectrum of $\check{A}$ are given in Lemma A.3. In particular, if $\check{A}(\mathrm{e}^{\mathrm{j}\theta})$ is continuous, $\lambda \in \sigma(\check{A})$ if and only if $\det(\lambda I - \check{A}(\mathrm{e}^{\mathrm{j}\theta})) = 0$ for some value of θ. More precisely

$$\sigma(\check{A}) = \bigcup_{\theta\in[0,2\pi]} \sigma(\check{A}(\mathrm{e}^{\mathrm{j}\theta})). \tag{8}$$

This allows us to obtain a succinct proof of a sharper result than the one in Bamieh et al. (2002, Corollary 3).

Theorem 2.2. *Consider $\check{A} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$ and let $\check{A}(\mathrm{e}^{\mathrm{j}\theta})$ be continuous in θ on $[0, 2\pi]$. Then $\check{A}$ is exponentially stable if and only if for all $\theta \in [0, 2\pi]$ the matrix $A(\mathrm{e}^{\mathrm{j}\theta})$ is exponentially stable and this is true if and only if*

$$\sup\{\mathrm{Re}(\lambda) \mid \exists\theta \in [0, 2\pi]\ \text{s.t.}\ \det(\lambda I - \check{A}(\mathrm{e}^{\mathrm{j}\theta})) = 0\} < 0. \tag{9}$$

Proof. Since the semigroup $e^{\check{A}t}$ satisfies the spectrum determined growth assumption, it will be exponentially stable if and only if (7) holds. From the continuity assumption of $\check{A} \in \mathbf{L}_\infty(\partial\mathbb{D};\mathbb{C}^{n\times n})$ it follows that we have equality (8). So (7) holds if and only if (9) holds. ■

Remark 2.3. Let $\check{A} \in \mathbf{W}(\partial\mathbb{D};\mathbb{C}^{n\times n})$. It is known that $\lambda I - \check{A}$ is invertible in $\mathbf{W}(\partial\mathbb{D};\mathbb{C}^{n\times n})$ if and only if

$$\det(\lambda I - \check{A}(e^{j\theta})) \neq 0 \quad \forall \theta \in [0, 2\pi].$$

So the spectrum of $\check{A}$ with respect to $\mathbf{W}(\partial\mathbb{D};\mathbb{C}^{n\times n})$ is also given by (8). Then (7) holds if and only if (9) holds. So (9) is also a necessary and sufficient condition for exponential stability of $\check{A} \in \mathbf{W}(\partial\mathbb{D};\mathbb{C}^{n\times n})$.

In the case that $\check{A}(e^{j\theta}) = A_0$, a constant matrix, we have $\sigma(\check{A}) = \sigma(A_0)$ which comprises eigenvalues with infinite multiplicity. However, in general, $\sigma(\check{A})$ also contains a continuous spectrum. If $\check{A}$ is a scalar function, the spectrum of $\check{A}$ is equal to the essential range of $\check{A}$ (Böttcher & Silberman, 1999, Theorem 1.2, p. 4). If $\check{A}$ is continuous, the essential range of $\check{A}$ is $\{\check{A}(e^{j\theta}) \mid \theta \in [0, 2\pi]\}$ (Böttcher & Silberman, 1999, Example 1.6, p. 7). The spectrum of $\check{A}$ can be very complicated, as some of the examples show. Moreover, the system can be exponentially stable or only strongly stable, a weaker form of stability.

Definition 2.4. The system (6) is *strongly stable* if and only if the semigroup $e^{\check{A}t}$ is strongly stable, i.e.,

$$\lim_{t\to\infty} \|e^{\check{A}t} z\|_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)} = 0$$

for all $z \in \mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)$.

The following result provides sufficient conditions for strong stability.

Theorem 2.5. *If the strongly continuous semigroup $e^{\check{A}t}$ is uniformly bounded in norm for $t \geq 0$, i.e.,*

$$\sup_{t\geq 0} \operatorname{ess\,sup}_{0\leq\theta\leq 2\pi} \|e^{\check{A}(e^{j\theta})t}\| < \infty, \tag{10}$$

and the finite-dimensional semigroups $\{e^{\check{A}(e^{j\theta})t} \mid \theta \in [0, 2\pi]\}$ are exponentially stable except for a set of measure zero, then the system (6) is strongly stable.

Proof. Define

$$\Theta_n := \{\theta \in [0, 2\pi] \mid \text{there exists an eigenvalue } \lambda(e^{j\theta}) \text{ of } A(e^{j\theta}) \text{ such that } \operatorname{Re}(\lambda(e^{j\theta})) \geq -1/n\}.$$

Since $\Theta_n \subset \Theta_m$ if $m \leq n$, and since the finite-dimensional semigroups $e^{\check{A}(e^{j\theta})t}$ are exponentially stable except for a set of measure zero, we find that the measure of Θ_n is converging to zero for $n \to \infty$, see Rudin (1966, Theorem 1.19).

Let $z \in L_2(\partial\mathbb{D},\mathbb{C}^n)$ and let $\varepsilon > 0$ be given. By the above, we can find an N such that

$$\frac{1}{2\pi}\int_{\Theta_N} \|z(e^{j\theta})\|^2 d\theta \leq \varepsilon^2. \tag{11}$$

Define $z_N \in L_2(\partial\mathbb{D},\mathbb{C}^n)$ as being zero on Θ_N and equals to $z(e^{j\theta})$ on the complement of this set. Then we have that for $\theta \in [0, 2\pi]$

$$\|e^{\check{A}(e^{j\theta})t} z_N(e^{j\theta})\| \leq \begin{cases} 0 & \theta \in \Theta_N \\ M_N e^{-\frac{t}{N}} \|z(e^{j\theta})\| & \theta \notin \Theta_N. \end{cases}$$

From this inequality and (11), it can be concluded as in the following

$$\begin{aligned} \|e^{\check{A}t} z\|_{L_2(\partial\mathbb{D},\mathbb{C}^n)} &\leq \|e^{\check{A}t}(z - z_N)\|_{L_2(\partial\mathbb{D},\mathbb{C}^n)} + \|e^{\check{A}t} z_N\|_{L_2(\partial\mathbb{D},\mathbb{C}^n)} \\ &\leq M\varepsilon + \|e^{\check{A}t} z_N\|_{L_2(\partial\mathbb{D},\mathbb{C}^n)} \\ &\leq M\varepsilon + M_N e^{-\frac{t}{N}} \|z\|_{L_2(\partial\mathbb{D},\mathbb{C}^n)}. \end{aligned}$$

Consequently, the limit of $\|e^{\check{A}t} z\|_{L_2(\partial\mathbb{D},\mathbb{C}^n)}$ is less than or equal to $M\varepsilon$. This holds for every $\varepsilon > 0$, and so $e^{\check{A}t} z$ converges to zero. ■

If $\check{A}(e^{j\theta})$ is continuous in θ on $[0, 2\pi]$, then the esssup in (10) can be replaced by max. We conjecture that the converse of Theorem 2.5 is also true.

The uniform boundedness condition (10) can also be written as $\operatorname{ess\,sup}_{0\leq\theta\leq 2\pi} \sup_{t\geq 0} \|e^{\check{A}(e^{j\theta})t}\| < \infty$. So, it is tempting to just check that the finite-dimensional semigroups $e^{\check{A}(e^{j\theta})t}$ are bounded for all $\theta \in [0, 2\pi]$ and $t \geq 0$ by showing that the eigenvalues of $\check{A}(e^{j\theta})$ are on the left half-plane (including the imaginary axis). The following example shows that it is not sufficient for uniform boundedness of $e^{\check{A}t}$.

Example 2.6. As $\check{A}(e^{j\theta})$ we define

$$\check{A}(e^{j\theta}) = \begin{pmatrix} -\sin^2(\theta) & \sin(\theta) \\ 0 & -\sin^2(\theta) \end{pmatrix}.$$

If θ is not a multiple of π, then the eigenvalues are negative, and so the exponential of $\check{A}(e^{j\theta})$ is bounded. If θ is a multiple of π, then $\check{A}(e^{j\theta})$ is the zero matrix, and so its exponential is bounded.

However, we can show that (10) does not hold. We have

$$e^{\check{A}(e^{j\theta})t} = \begin{pmatrix} e^{-\sin^2(\theta)t} & \sin(\theta)te^{-\sin^2(\theta)t} \\ 0 & e^{-\sin^2(\theta)t} \end{pmatrix}.$$

If we choose $\theta_m = \frac{1}{m}$ and $t_m = (\sin^2(\theta_m))^{-1}$, then

$$e^{\check{A}(e^{j\theta_m})t_m} = \begin{pmatrix} e^{-1} & (\sin(\theta_m))^{-1}e^{-1} \\ 0 & e^{-1} \end{pmatrix}$$

and it is clear that this sequence is unbounded in norm. ■

In what follows we discuss stability properties of several particular systems.

Example 2.7. Consider the autonomous part of the system (1) with $A_0 = a$, $A_1 = c$ and $A_{-1} = b$ and all other $A_l = 0$. Then

$$\check{A}(e^{j\theta}) = ce^{-j\theta} + a + be^{j\theta} = (b + c)\cos\theta + a + j(b - c)\sin\theta,$$

for positive constants b, c. If $b = c$, we have $\sigma(\check{A}) = [a-2b, a+2b]$. If $b \neq c$, we have

$$\sigma(\check{A}) = \{ce^{-j\theta} + a + be^{j\theta} \mid \theta \in [0, 2\pi]\}$$

(Fig. 1, Left). In both cases the spectrum comprises continuous spectrum. If $a + |b + c| < 0$ the semigroup $e^{\check{A}t}$ is exponentially stable. If $a = -|b + c|$ the semigroup $e^{\check{A}t}$ is not exponentially stable, since $0 \in \sigma(\check{A})$. However, it is strongly stable. That follows from Theorem 2.5 using the fact that the intersection of $\sigma(\check{A})$ with the imaginary axis contains only the origin and the uniform boundness with respect to t of the semigroup, which follows from the inequality

$$|e^{\check{A}(e^{j\theta})t}| = e^{(b+c)\cos(\theta)t}e^{-|b+c|t} \leq 1,$$

for all $\theta \in [0, 2\pi]$ and $t \geq 0$.

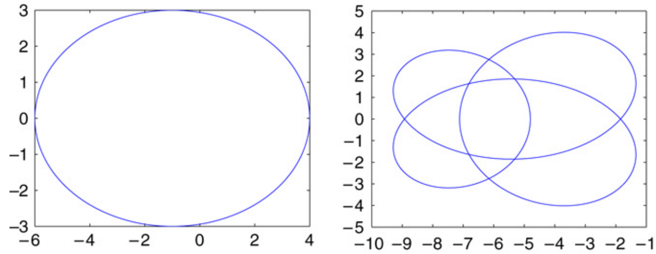

Fig. 1. Left: Spectrum of $\check{A}$ for Example 2.7 $b = 4, c = 1, a = -1$; Right: Spectrum of $\check{A}$ for Example 2.8.

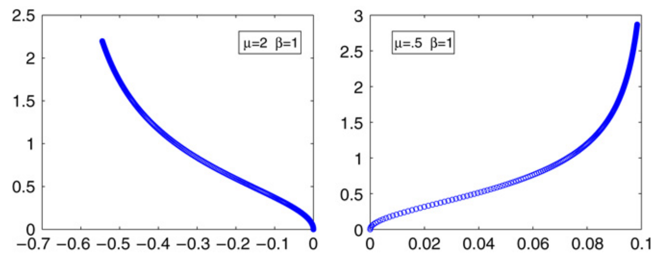

Fig. 2. Spectrum of $\check{A}$ for Example 2.9; Left: $\mu = 2, \beta = 1$ (strongly stable); Right: $\mu = 0.5, \beta = 1$ (unstable).

Example 2.8. Consider the autonomous part of the system (1) with $A_0 = -5.5, A_1 = 1.5, A_{-1} = 0.34, A_{-2} = -0.46$ and $A_{-3} = -3$ and all other $A_l = 0$. The spectrum of $\check{A}$ is $\sigma(\check{A}) = \{\check{A}(e^{j\theta}) \mid \theta \in [0, 2\pi]\}$ provided in Fig. 1 (Right) and indicates that the system is exponentially stable. The exponential stability of the semigroup follows also from the inequality

$$|e^{\check{A}(e^{j\theta})t}| < e^{-0.1t}, \quad \text{for all } \theta \in [0, 2\pi].$$

Example 2.9. Consider the autonomous part of the system (1) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -\beta & -\mu \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

and all other $A_l = 0$, where β and μ are positive numbers. The Fourier transformed system has

$$\check{A}(e^{j\theta}) = \begin{bmatrix} 0 & 1 - e^{-j\theta} \\ -\beta & -\mu \end{bmatrix}.$$

The spectrum of $\check{A}$ is

$$\sigma(\check{A}) = \left\{ \frac{1}{2}(-\mu \pm (x(e^{j\theta}) + jy(e^{j\theta}))), \theta \in [0, 2\pi] \right\},$$

where $x(e^{j\theta}), y(e^{j\theta})$ are the positive square roots of

$$\begin{aligned} 2x(e^{j\theta})^2 &= \sqrt{(8\beta \sin^2\theta/2 - \mu^2)^2 + 16\beta^2 \sin^2\theta} \\ &\quad + \mu^2 - 8\beta \sin^2\theta/2 \\ 2y(e^{j\theta})^2 &= \sqrt{(8\beta \sin^2\theta/2 - \mu^2)^2 + 16\beta^2 \sin^2\theta} \\ &\quad - \mu^2 + 8\beta \sin^2\theta/2. \end{aligned}$$

Note that for $\theta = 0\, x(e^{j0}) = \mu$ and so $0 \in \sigma(\check{A})$, which shows that $e^{\check{A}t}$ does not generate an exponentially stable semigroup. Since $x(e^{j\theta}) > 0$ for every $\theta \in [0, 2\pi]$, $\check{A}(e^{j\theta})$ has two distinct eigenvalues $\lambda_{1,2}(e^{j\theta}) = \frac{1}{2}(-\mu \pm (x(e^{j\theta}) + jy(e^{j\theta})))$. Then

$$\check{A}(e^{j\theta}) = L^{-1} \begin{bmatrix} \lambda_1(e^{j\theta}) & 0 \\ 0 & \lambda_2(e^{j\theta}) \end{bmatrix} L,$$

where L and its inverse are bounded, and

$$\|e^{\check{A}(e^{j\theta})t}\| \le \|L\|\|L^{-1}\| \left\| \begin{bmatrix} e^{\lambda_1(e^{j\theta})t} & 0 \\ 0 & e^{\lambda_2(e^{j\theta})t} \end{bmatrix} \right\|.$$

If $\mu^2 - 2\beta > 0$ then the inequality $x(e^{j\theta}) \le \mu$ is satisfied for all $\theta \in [0, 2\pi]$, which implies that $e^{\lambda_{1,2}(e^{j\theta})t}$ are uniformly bounded. Then $e^{\check{A}t}$ is uniformly bounded (with respect to t). The intersection of $\sigma(\check{A})$ with the imaginary axis contains only the origin. From Theorem 2.5, it follows that the semigroup $e^{\check{A}t}$ is strongly stable provided that $\mu^2 > 2\beta$ and $\beta > 0$ (Fig. 2).

Example 2.10. Consider the autonomous part of the system (1) with

$$A_0 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and all other $A_l = 0$. The Fourier transformed system has

$$\check{A}(e^{j\theta}) = \begin{bmatrix} -1 + e^{-j\theta} & 1 \\ 0 & -1 + e^{-j\theta} \end{bmatrix}.$$

The system is not exponentially stable because $0 \in \sigma(\check{A})$. Moreover, it is not strongly stable because

$$\begin{aligned} \|e^{\check{A}(e^{j\theta})t}\| &= |e^{(\cos\theta - 1 + j\sin(\theta))t}| \left\| \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\| \\ &= |e^{(\cos\theta - 1)t}| \sqrt{\frac{2 + t^2 + \sqrt{t^4 + 4t^2}}{2}}, \end{aligned}$$

which shows that $\|e^{\check{A}t}\|_\infty$ tends to infinity as $t \to \infty$.

A Lyapunov-type condition follows from Curtain and Zwart (1995, Theorem 5.1.3), (see also in Bamieh et al. (2002, Theorem 1) for a similar claim).

Theorem 2.11. *An operator $\check{A} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$ generates an exponentially stable semigroup if and only if there exists a unique positive operator $\check{P} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$ such that*

$$\check{A}^*\check{P} + \check{P}^*\check{A} = -I.$$

If $\check{A}(e^{j\theta})$ is continuous in θ on $[0, 2\pi]$ then $\check{P}(e^{j\theta})$ is also continuous. If $A \in \ell_1(\mathbb{C}^{n\times n})$, then $P \in \ell_1(\mathbb{C}^{n\times n})$ and so $\check{P} \in \mathbf{W}(\partial\mathbb{D}; \mathbb{C}^{n\times n})$.

Proof. We only need to prove the continuity. If $\check{A}(e^{j\theta})$ is continuous in θ, one can apply (Lancaster & Rodman, 1995, Theorem 11.2.1) to obtain the continuity of $\check{P}(e^{j\theta})$.

Consider the case $A \in \ell_1(\mathbb{C}^{n\times n})$. The solution to the isometrically isomorphic Lyapunov equation $A^*P + P^*A = -I$ is $P = \int_0^\infty e^{A^*t}e^{At}\,dt$. The space $\ell_1(\mathbb{C}^{n\times n})$ is a Banach algebra and so closed under limits. Hence $e^{At} \in \ell_1(\mathbb{C}^{n\times n})$ and so is the integrand. The semigroup e^{At} is exponentially stable and so, for some constants $M, \alpha > 0$, we have the inequality

$$\|e^{A^*t}e^{At}\|_\infty \le M^2 e^{-2\alpha t}.$$

By the Lebesgue lemma we conclude that $P \in \ell_1(\mathbb{C}^{n\times n})$, which implies that $\check{P} \in \mathbf{W}(\partial\mathbb{D}; \mathbb{C}^{n\times n})$. ■

For Example 2.7 the Lyapunov equation has the solution

$$\check{P}(e^{j\theta}) = -\frac{1}{2[(b + c)\cos\theta + a]},$$

which shows that $e^{\check{A}t}$ will generate an exponentially stable semigroup if and only if $-a > |b + c|$.

The Lyapunov equation associated to the system considered in Example 2.9 does not have a positive solution which confirms our earlier conclusion that $e^{\check{A}t}$ does not generate an exponentially stable semigroup.

3. Controllability and observability

The necessary and sufficient conditions for *approximate controllability* from Curtain and Zwart (1995, Definition 4.1.17) applied to $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ yield

$$\check{B}^* e^{\check{A}^* t}\check{z} = 0 \quad \text{for } t \geq 0 \Longrightarrow \check{z} = 0,$$

and for the dual concept of *approximate observability* they yield

$$\check{C} e^{\check{A}t}\check{z} = 0 \quad \text{for } t \geq 0 \Longrightarrow \check{z} = 0.$$

Since $\check{A}, \check{B}, \check{C}$ are bounded operators, we obtain the following necessary and sufficient condition for approximate observability from the dual of Curtain and Pritchard (1978, Theorem 3.16)

$$\ker \begin{bmatrix} \check{C} \\ \check{C}\check{A} \\ \dots \\ \check{C}\check{A}^r \\ \dots \end{bmatrix} = \{0\}. \tag{12}$$

Moreover, since $\check{A}$ and $\check{C}$ have matrix values, this condition reduces to

$$\ker \begin{bmatrix} \check{C}(e^{j\theta}) \\ \check{C}(e^{j\theta})\check{A}(e^{j\theta}) \\ \dots \\ \check{C}(e^{j\theta})\check{A}(e^{j\theta})^{n-1} \end{bmatrix} = \{0\} \tag{13}$$

for almost all $\theta \in [0, 2\pi]$.

Similarly, we obtain the following necessary and sufficient condition for approximate controllability

$$\text{rank}[\check{B}(e^{j\theta}) : \check{A}(e^{j\theta})\check{B}(e^{j\theta}) : \dots \check{A}(e^{j\theta})^{n-1}\check{B}(e^{j\theta})] = n \tag{14}$$

for almost all $\theta \in [0, 2\pi]$.

As in the finite-dimensional case, this leads to the following Hautus test.

Theorem 3.1. *$\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is approximately controllable if and only if*

$$\text{rank}\left[(\lambda I - \check{A}(e^{j\theta})) : \check{B}(e^{j\theta})\right] = n$$

for almost all $\theta \in [0, 2\pi]$ and for all $\lambda \in \mathbb{C}$.

$\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is approximately observable if and only if

$$\text{rank}\begin{bmatrix} \lambda I - \check{A}(e^{j\theta}) \\ \check{C}(e^{j\theta}) \end{bmatrix} = n$$

for almost all $\theta \in [0, 2\pi]$ and for all $\lambda \in \mathbb{C}$.

One might expect that, in the case that $\check{A}(e^{j\theta}), \check{B}(e^{j\theta}), \check{C}(e^{j\theta})$ are continuous in θ on $[0, 2\pi]$, necessary and sufficient conditions for approximate observability and controllability should be that (13) and (14) hold for all $\theta \in [0, 2\pi]$. The following example shows that there are approximately controllable systems with $\check{A}(e^{j\theta}), \check{B}(e^{j\theta}), \check{C}(e^{j\theta})$ continuous in θ on $[0, 2\pi]$ for which the rank condition does not hold for all $\theta \in [0, 2\pi]$.

Example 3.2. Consider the system (1) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and all other A_l, B_l zero. To examine the approximate controllability of this system we examine the Fourier transformed system which has the operators

$$\check{A}(e^{j\theta}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \check{B}(e^{j\theta}) = \begin{bmatrix} 0 \\ 1 - e^{-j\theta} \end{bmatrix}.$$

We have

$$\check{B}(e^{j\theta})^* e^{\check{A}(e^{j\theta})^* t}\begin{bmatrix} \xi \\ \rho \end{bmatrix} = [\,0 \quad 1 - e^{-j\theta}\,]\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \rho \end{bmatrix} = (1 - e^{-j\theta})(t\xi + \rho).$$

If this equals zero almost everywhere in $[0, 2\pi]$ for $t \geq 0$, we must have $\xi = 0 = \rho$. Consequently it is approximately controllable, even though it does not satisfy the rank condition (14) in $\theta = 0$.

It turns out that, in the case that $\check{A}(e^{j\theta})$ and $\check{B}(e^{j\theta})$ are continuous in θ on $[0, 2\pi]$, if (14) hold for all $\theta \in [0, 2\pi]$, then the system is exactly controllable.

Let $T < \infty$ be a positive real constant. We recall that $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is *exactly controllable* on $[0, T]$ if and only if there exists a positive β such that

$$\int_0^T \|\check{B}^* e^{\check{A}^* t}\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^m)}\,dt \geq \beta\|\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)}, \tag{15}$$

for any $\check{z} \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$. If there exists a $\beta_0 > 0$ such that

$$\int_0^\infty \|\check{B}^* e^{\check{A}^* t}\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^m)}\,dt \leq \beta_0\|\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)}, \tag{16}$$

for any $\check{z} \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$ and the inequality (15) holds for $T = \infty$, then we say that $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is *exactly controllable in infinite time*. $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is *exactly observable* on $[0, T]$ if and only if there exists a positive γ such that

$$\int_0^T \|\check{C} e^{\check{A}t}\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^p)}\,dt \geq \gamma\|\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)}, \tag{17}$$

for any $\check{z} \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$. If there exists a $\gamma_0 > 0$ such that

$$\int_0^\infty \|\check{C} e^{\check{A}t}\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^p)}\,dt \leq \gamma_0\|\check{z}\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)}, \tag{18}$$

for any $\check{z} \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$ and the inequality (17) holds for $T = \infty$, then we say that $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is *exactly observable in infinite time*.

The following statement shows that exact observability of $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ on $[0, T]$ is independent of T and is equivalent with the observability of all finite-dimensional systems $\Sigma(\check{A}(e^{j\theta}), \check{B}(e^{j\theta}), \check{C}(e^{j\theta}), \check{D}(e^{j\theta}))$.

Theorem 3.3. *Suppose that $\check{A}(e^{j\theta})$ and $\check{C}(e^{j\theta})$ are continuous in θ on $[0, 2\pi]$. Then the following statements are equivalent:*

(1) *There exists a $T > 0$ such that the system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly observable on $[0, T]$.*
(2) *The system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly observable on $[0, T]$ for every $T > 0$.*
(3) *The systems $\Sigma(\check{A}(e^{j\theta}), \check{B}(e^{j\theta}), \check{C}(e^{j\theta}), \check{D}(e^{j\theta}))$ are observable for all $\theta \in [0, 2\pi]$, i.e.,*

$$\text{rank}\begin{bmatrix} \lambda I - \check{A}(e^{j\theta}) \\ \check{C}(e^{j\theta}) \end{bmatrix} = n \tag{19}$$

holds for all $\theta \in [0, 2\pi]$ and for all $\lambda \in \mathbb{C}$.

Proof. We prove (2) $\to$ (1) $\to$ (3) $\to$ (2). The implication (2) $\to$ (1) is trivial.

(3) $\to$ (2): Let T be a strictly positive real number. If condition (19) holds, then we know that for every $\theta \in [0, 2\pi]$ there exists an $m_\theta(T)$ such that for every $v \in \mathbb{C}^n$

$$\int_0^T \|\check{C}(e^{j\theta})e^{\check{A}(e^{j\theta})t}v\|^2 dt \geq m_\theta(T)\|v\|^2. \tag{20}$$

Note that this holds for all $T > 0$ since we are looking at a finite-dimensional system. For simplicity, write m_θ instead of $m_\theta(T)$. Now the question is whether the m_θ can become arbitrarily small. To prove the contrary we suppose that there is a sequence $\theta_k, k \in \mathbb{N}$ such that for some sequence $v_k \in \mathbb{C}^n$ of norm one, we have

$$\int_0^T \|\check{C}(\mathrm{e}^{\mathrm{j}\theta_k})\mathrm{e}^{\check{A}(\mathrm{e}^{\mathrm{j}\theta_k})t} v_k\|^2 \mathrm{d}t \to 0. \tag{21}$$

Since $\theta_k \in [0, 2\pi]$ and v_k is a sequence of norm one in a finite-dimensional space, we can find a subsequence such that θ_k and v_k converge along this subsequence. To keep the notation simple we denote this subsequence by θ_k and v_k and the limits by $\theta_k \to \theta_\infty$ and $v_k \to v_\infty$. From (21), the continuity of $\check{C}$ and $\check{A}$, and the Lebesgue Dominated Convergence theorem, we conclude that

$$\int_0^T \|\check{C}(\mathrm{e}^{\mathrm{j}\theta_\infty})\mathrm{e}^{\check{A}(\mathrm{e}^{\mathrm{j}\theta_\infty})t} v_\infty\|^2 \mathrm{d}t = 0. \tag{22}$$

This is in contradiction with (20).
So we conclude that if (19) holds, then there exists a constant $m(T) > 0$ such that for every $\theta \in [0, 2\pi]$ and every $v \in \mathbb{C}^n$

$$\int_0^T \|\check{C}(\mathrm{e}^{\mathrm{j}\theta})\mathrm{e}^{\check{A}(\mathrm{e}^{\mathrm{j}\theta})t} v\|^2 \mathrm{d}t \geq m(T)\|v\|^2.$$

Now it is easy to conclude exact observability on $[0, T]$ for every $T > 0$ as follows.

$$\begin{aligned}
\int_0^T \|\check{C}\mathrm{e}^{\check{A}t}z\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)}\mathrm{d}t &= \int_0^T \frac{1}{2\pi}\int_0^{2\pi} \|\check{C}(\mathrm{e}^{\mathrm{j}\theta})\mathrm{e}^{\check{A}(\mathrm{e}^{\mathrm{j}\theta})t}z(\mathrm{e}^{\mathrm{j}\theta})\|^2 \mathrm{d}\theta \mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^T \|\check{C}(\mathrm{e}^{\mathrm{j}\theta})\mathrm{e}^{\check{A}(\mathrm{e}^{\mathrm{j}\theta})t}z(\mathrm{e}^{\mathrm{j}\theta})\|^2 \mathrm{d}t \mathrm{d}\theta \\
&\geq \frac{1}{2\pi}\int_0^{2\pi} m(T)\|z(\mathrm{e}^{\mathrm{j}\theta})\|^2 \mathrm{d}\theta \\
&= m(T)\|z\|^2_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^n)}.
\end{aligned}$$

Concluding, we see that if (19) holds, then our system is exactly observable on $[0, T]$ for every $T > 0$.

(1) $\to$ (3): Assume that $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly observable on $[0, T]$. Now $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly observable on $[0, T]$ if and only if $\Sigma(\check{A} + \alpha I, \check{B}, \check{C}, \check{D})$ is exactly observable on $[0, T]$ for any real α (see Curtain and Zwart (1995, Lemma 4.1.6)). Hence we can assume without loss of generality that the system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exponentially stable and exactly observable on $[0, T]$. Since $\mathrm{e}^{\check{A}t}$ is exponentially stable, $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly observable in infinite time (see Russell and Weiss (1994)). From Grabowski (1990), the exact observability in infinite time of the system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is equivalent to the existence of a coercive solution $\check{Q}$ of the Lyapunov equation

$$\check{A}^*\check{Q} + \check{Q}\check{A} = -\check{C}^*\check{C}. \tag{23}$$

Suppose now that (19) does not hold, i.e., there exist $\theta_0 \in [0, 2\pi]$, a nonzero $v_0 \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ such that

$$\check{A}(\mathrm{e}^{\mathrm{j}\theta_0})v_0 = \lambda v_0, \qquad \check{C}(\mathrm{e}^{\mathrm{j}\theta_0})v_0 = 0. \tag{24}$$

The finite-dimensional system $\Sigma(\check{A}(\mathrm{e}^{\mathrm{j}\theta_0}), \check{B}(\mathrm{e}^{\mathrm{j}\theta_0}), \check{C}(\mathrm{e}^{\mathrm{j}\theta_0}), \check{D}(\mathrm{e}^{\mathrm{j}\theta_0}))$ is not observable. We show that this will imply that $\check{Q}$, the solution of the Lyapunov equation (23), is not coercive.

For every $\theta \in [0, 2\pi]$, consider pointwise correspondent of the Lyapunov equation (23)

$$\check{A}(\mathrm{e}^{\mathrm{j}\theta})^*\check{Q}(\mathrm{e}^{\mathrm{j}\theta}) + \check{Q}(\mathrm{e}^{\mathrm{j}\theta})\check{A}(\mathrm{e}^{\mathrm{j}\theta}) = -\check{C}(\mathrm{e}^{\mathrm{j}\theta})^*\check{C}(\mathrm{e}^{\mathrm{j}\theta}). \tag{25}$$

Since $\check{A}(\mathrm{e}^{\mathrm{j}\theta})$ and $\check{C}(\mathrm{e}^{\mathrm{j}\theta})$ are continuous in θ on the finite interval $[0, 2\pi]$, we conclude that $\check{Q}(\mathrm{e}^{\mathrm{j}\theta})$ inherits this property (see Lancaster and Rodman (1995, Theorem 11.2.1)). For $\theta = \theta_0$, taking in (25) inner product with v_0 and using (24) gives the equality

$$\mathrm{Re}\,\lambda\langle v_0, \check{Q}(\mathrm{e}^{\mathrm{j}\theta_0})v_0\rangle = 0.$$

Using the exponential stability of $\check{A}$ we must have $\langle v_0, \check{Q}(\mathrm{e}^{\mathrm{j}\theta_0})v_0\rangle = 0$. This implies that $\det \check{Q}(\mathrm{e}^{\mathrm{j}\theta_0}) = 0$. Using this together with the continuity of $\check{Q}(\mathrm{e}^{\mathrm{j}\theta})$ on $[0, 2\pi]$ and Lemma A.3 one can conclude that $\check{Q}$ is not a coercive operator. Consequently, the system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is not exactly observable in infinite time, and so $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is not exactly observable on $[0, T]$. ∎

Since exact observability and exact controllability are dual concepts, one can also claim the following equivalence related to exact controllability.

Theorem 3.4. *Suppose that $\check{A}(\mathrm{e}^{\mathrm{j}\theta})$ and $\check{B}(\mathrm{e}^{\mathrm{j}\theta})$ are continuous in θ on $[0, 2\pi]$. Then the following statements are equivalent:*

(1) *There exists a $T > 0$ such that the system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly controllable on $[0, T]$.*
(2) *The system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly controllable on $[0, T]$ for every $T > 0$.*
(3) *The systems $\Sigma(\check{A}(\mathrm{e}^{\mathrm{j}\theta}), \check{B}(\mathrm{e}^{\mathrm{j}\theta}), \check{C}(\mathrm{e}^{\mathrm{j}\theta}), \check{D}(\mathrm{e}^{\mathrm{j}\theta}))$ are controllable for all $\theta \in [0, 2\pi]$ t, i.e.,*

$$\mathrm{rank}\left[(\lambda I - \check{A}(\mathrm{e}^{\mathrm{j}\theta})) : \check{B}(\mathrm{e}^{\mathrm{j}\theta})\right] = n \tag{26}$$

for all $\theta \in [0, 2\pi]$ and for all $\lambda \in \mathbb{C}$.

As a consequence of the necessity proof of Theorem 3.3 one can obtain the following results on exact controllability and observability in infinite time.

Corollary 3.5. *Suppose that $\check{A}(\mathrm{e}^{\mathrm{j}\theta})$, $\check{B}(\mathrm{e}^{\mathrm{j}\theta})$ and $\check{C}(\mathrm{e}^{\mathrm{j}\theta})$ are continuous in θ on $[0, 2\pi]$.*

If condition (16) is satisfied and (26) holds for all $\theta \in [0, 2\pi]$ and for all $\lambda \in \mathbb{C}$ then $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly controllable in infinite time.

If condition (18) is satisfied and (19) holds for all $\theta \in [0, 2\pi]$ and for all $\lambda \in \mathbb{C}$ then $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is exactly observable in infinite time.

That the converse to Corollary 3.5 is not true is illustrated by the following example. However, if $\mathrm{e}^{\check{A}t}$ is exponentially stable the converse to Corollary 3.5 holds (see Russell and Weiss (1994)).

Example 3.6. Consider the following system

$$.\check{A}(\mathrm{e}^{\mathrm{j}\theta}) = -2\sin^2\theta, \qquad \check{C}(\theta) = 2\sin\theta. \tag{27}$$

It is easy to see that these are continuous functions of θ on $[0, 2\pi]$. Furthermore, for $\theta = \pi$ the finite-dimensional system $\Sigma(\check{A}(\pi), \check{C}(\pi))$ is not observable, so condition (19) does not hold for $\theta = \pi$, and so $\Sigma(\check{A}, \check{C})$ is not exactly observable on $[0, T]$.

We claim that the system $\Sigma(\check{A}, \check{C})$ is exactly observable in infinite time. It is easy to see that $\check{Q} = I$ is a solution of the Lyapunov equation (23) corresponding to our system, which implies that the system is exactly observable in infinite time.

4. Exponential stabilizability and detectability

We recall that $\Sigma(A, B, C)$ is exponentially stabilizable if there exists an $F \in \mathcal{L}(Z, U)$ such that $A + BF$ is exponentially stable and it is exponentially detectable if there exists an $L \in \mathcal{L}(Y, Z)$ such that $A + LC$ is exponentially stable. We derive simple conditions for exponential stabilizability and exponential detectability by analyzing its Fourier transformed system.

In the case of continuous symbols, to show that $\Sigma(\check{A}, \check{B}, \check{C})$ is exponentially stabilizable we just need to show that $(\check{A}(e^{j\theta}), \check{B}(e^{j\theta}))$ is stabilizable for each $\theta \in [0, 2\pi]$. This was claimed earlier in Bamieh et al. (2002, Corollary 3), but without an adequate proof.

Theorem 4.1. *Suppose that $\check{A}(e^{j\theta})$ and $\check{B}(e^{j\theta})$ are continuous in θ on $[0, 2\pi]$.*

$\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially stabilizable if and only if $(\check{A}(e^{j\theta}), \check{B}(e^{j\theta}))$ is exponentially stabilizable for each $\theta \in [0, 2\pi]$, i.e.,

$$\operatorname{rank}\left[(\lambda I - \check{A}(e^{j\theta})) : \check{B}(e^{j\theta})\right] = n \tag{28}$$

for all $\theta \in [0, 2\pi]$ and for all $\lambda \in \overline{\mathbb{C}_0^+}$.

The dual statement corresponding to Theorem 4.1 is provided in what follows.

Theorem 4.2. *Suppose that $\check{A}(e^{j\theta})$ and $\check{C}(e^{j\theta})$ are continuous in θ on $[0, 2\pi]$.*

$\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially detectable if and only if $(\check{A}(e^{j\theta}), \check{C}(e^{j\theta}))$ is exponentially detectable for each $\theta \in [0, 2\pi]$, i.e.,

$$\operatorname{rank}\begin{bmatrix}\lambda I - \check{A}(e^{j\theta}) \\ \check{C}(e^{j\theta})\end{bmatrix} = n \tag{29}$$

for all $\theta \in [0, 2\pi]$ and for all $\lambda \in \overline{\mathbb{C}_0^+}$.

Proof of Theorem 4.1. Sufficiency: Suppose that (28) holds. For each $\theta \in [0, 2\pi]$ the matrix pair $\check{A}(e^{j\theta}), \check{B}(e^{j\theta})$ is stabilizable if and only if

$$\operatorname{rank}\left[(\lambda I - \check{A}(e^{j\theta})) : \check{B}(e^{j\theta})\right] = n \quad \text{for all } \lambda \in \overline{\mathbb{C}_0^+}$$

if and only if there exists a stabilizing solution to the Riccati equation

$$\check{A}(e^{j\theta})^*\check{Q}(e^{j\theta}) + \check{Q}(e^{j\theta})\check{A}(e^{j\theta}) - \check{Q}(e^{j\theta})\check{B}(e^{j\theta})\check{B}(e^{j\theta})^*\check{Q}(e^{j\theta}) + I = 0. \tag{30}$$

Moreover, it is the unique nonnegative solution. Hence we have a unique nonnegative solution for all $\theta \in [0, 2\pi]$ and since $\check{A}(e^{j\theta})$, $\check{B}(e^{j\theta})$ and $\check{C}(e^{j\theta})$ are continuous in θ on the finite interval $[0, 2\pi]$, we conclude that $\check{Q}(e^{j\theta})$ inherits this property (see Lancaster and Rodman (1995, Theorem 11.2.1)). This provides a nonnegative self-adjoint solution $\check{Q} \in \mathcal{L}(\mathbf{W}(\partial\mathbb{D}; \mathbb{C}^n))$ to the corresponding operator Riccati equation, since

$$\|\check{Q}\|_\infty = \operatorname{ess\,sup}_{0\le\theta\le2\pi}\|\check{Q}(e^{j\theta})\| = \max_{0\le\theta\le2\pi}\|\check{Q}(e^{j\theta})\| < \infty.$$

Now

$$\check{A}^{cl}(e^{j\theta}) := \check{A}(e^{j\theta}) - \check{B}(e^{j\theta})\check{B}(e^{j\theta})^*\check{Q}(e^{j\theta})$$

is continuous in θ and it is stable for all $\theta \in [0, 2\pi]$ and so

$$\{\operatorname{Re}(\lambda) | \exists\theta \in [0, 2\pi] \text{ s.t. } \lambda \in \sigma(\check{A}^{cl}(e^{j\theta}))\} < 0.$$

Comparing this with (9), we see that $\check{A} - \check{B}\check{B}^*\check{Q}$ generates an exponentially stable semigroup.

Necessity: Suppose now that the system is exponentially stabilizable, but (28) fails at $\theta = \theta_0$. Then there exists a nonzero $v_0 \in \mathbb{C}^n$ and $\lambda \in \overline{\mathbb{C}_0^+}$ such that

$$v_0^*\check{A}(e^{j\theta_0}) = \lambda v_0^*, \qquad v_0^*\check{B}(e^{j\theta_0}) = 0. \tag{31}$$

Choose $\check{z}_0(e^{j\theta}) = \check{f}(e^{j\theta})v_0$, where $\check{f}(e^{j\theta})$ is $1/\sqrt{2\epsilon}$ on $[\theta_0 - \epsilon, \theta_0 + \epsilon]$ and zero outside this interval. Since $\check{A}$ and $\check{B}$ are continuous we have

$$\check{A}^*\check{z}_0 = \overline{\lambda}\check{f}(v_0 + a(\epsilon)), \qquad \check{B}^*\check{z}_0 = 0 + \check{f}b(\epsilon),$$

where $a(\epsilon)$ and $b(\epsilon)$ are continuous vectors with norms of the order of ε.

Since the system is exponentially stabilizable, there exists a unique nonnegative solution $\check{Q} \in L_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times m})$ to Eq. (30). We rewrite (30) as

$$(\check{A} + \check{B}\check{F})\check{Q} + \check{Q}(\check{A} + \check{B}\check{F})^* = -I, \tag{32}$$

where $\check{F} = -\check{B}^*\check{Q}$. We take inner products of the first term in A with respect to z_0 to obtain

$$\langle\check{z}_0, \check{A}\check{Q}\check{z}_0\rangle = \overline{\lambda}\langle v_0, \check{Q}v_0\rangle + \langle\check{f}a(\epsilon), \check{f}\check{Q}v_0\rangle,$$

where the last term is of the order ϵ. Next we take inner products with the first term in B to obtain

$$\langle\check{B}^*\check{z}_0, \check{F}\check{Q}\check{z}_0\rangle = \langle b(\epsilon)\check{f}, \check{F}\check{Q}\check{f}v_0\rangle,$$

which is of the order of ϵ. So taking inner products of (32) yields up to first order terms in ϵ

$$\operatorname{Re}\lambda\langle\check{z}_0, \check{Q}\check{z}_0\rangle = -\langle\check{z}_0, \check{z}_0\rangle,$$

and this is in contradiction to the assumption that $\lambda \in \overline{\mathbb{C}_0^+}$. Hence the system is not exponentially stabilizable. ■

Corollary 4.3. *If $\check{A}$ and $\check{B}$ are in the Wiener class, then there exists an $\check{F}$ in the Wiener class such that $\check{A} + \check{B}\check{F}$ generates an exponentially stable semigroup if and only if (28) holds.*

Proof. Since the $\|\cdot\|_1$-norm is strictly larger than the infinity norm, we need only prove sufficiency.

First we show the existence of an $\check{F}$ in the Wiener class. From Theorem 4.1 we have a stabilizing $\check{F} = -\check{B}^*\check{Q}$ so that there exist positive constants M, α such that $\|e^{(\check{A}+\check{B}\check{F})t}\|_\infty \le Me^{-\alpha t}$. Since $\check{F}(\theta)$ is continuous and periodic on $[0, 2\pi]$, it is approximable by $\check{F}^N(\theta) = \sum_{l=-N}^{N} F_l e^{-jl\theta}$ in the $\|\cdot\|_\infty$-norm (Young, 1980, Proposition 1, p. 113). Clearly $\check{F}^N$ is in the Wiener class. Choose N sufficiently large so that

$$\|\check{F} - \check{F}^N\|_\infty < \epsilon = \frac{\alpha}{2M\|\check{B}\|_\infty}.$$

Then using the perturbation result from Curtain and Zwart (1995, Theorem 3.2.1) we have

$$\|e^{(\check{A}+\check{B}\check{F}^N)t}\|_\infty \le Me^{-\alpha t}e^{M\|\check{B}\|_\infty\epsilon t} = Me^{-\frac{\alpha}{2}t},$$

which shows exponential stability in the $\|\cdot\|_\infty$-norm. ■

We remark that in Theorem 4.1 it is essential that $\check{A}(e^{j\theta})$, $\check{B}(e^{j\theta})$ and $\check{C}(e^{j\theta})$ be continuous in θ on $[0, 2\pi]$. Unlike the approximate controllability condition, the rank condition for exponential stabilizability should hold for all $\theta \in [0, 2\pi]$ as the following example illustrates.

Example 4.4. Consider Example 3.2 where

$$\check{A}(e^{j\theta}) = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}, \qquad \check{B}(e^{j\theta}) = \begin{bmatrix}0 \\ 1 - e^{-j\theta}\end{bmatrix}.$$

The Riccati equation (30) has the unique positive solution

$$\check{Q}(e^{j\theta}) = \begin{bmatrix}\sqrt{1 + 1/\sin(\theta/2)} & 1/(2\sin(\theta/2)) \\ 1/(2\sin(\theta/2)) & \dfrac{\sqrt{1+\sin(\theta/2)}}{2(\sin(\theta/2))^{3/2}}\end{bmatrix}$$

for each $\theta \in [0, 2\pi]$, but $\check{Q}(\theta)$ is not bounded on $[0, 2\pi]$. So the operator Riccati equation

$$\check{A}^*\check{Q} + \check{Q}\check{A} - \check{Q}\check{B}\check{B}^*\check{Q} + I = 0$$

does not have a nonnegative self-adjoint solution in $L_\infty(\partial\mathbb{D})$. From Theorem 6.2.4 in Curtain and Zwart (1995) we conclude that $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ is not exponentially stabilizable. This checks with the observation that the condition (28) fails for $\theta = 0$.

So for this class of systems approximate controllability does not imply exponential stabilizability.

5. Conclusions

We have developed simple tests for system theoretic properties such as stability, stabilizability, controllability for a class of spatially distributed systems. In the case of continuous symbols, the main system theoretic properties such as exponential stability, exact controllability and observability and exponential stabilizability and detectability, can be verified by using the known finite-dimensional tests for each theta. We illustrated the theory with several simple examples and counterexamples.

Appendix

In this section we recall the definitions and introduce the notations for various frequency domain spaces with respect to the unit disc. Denote by $\mathbb{D}$ the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ and by $\partial\mathbb{D}$ its boundary, the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.

Definition A.1. We define the following frequency-domain spaces:

$$\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q) = \Big\{ f : \partial\mathbb{D} \to \mathbb{C}^q \mid f \text{ is measurable and } \|f\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{j\theta})|^2 d\theta \right)^{\frac{1}{2}} < \infty \Big\}$$

$$\mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times q}) = \Big\{ F : \partial\mathbb{D} \to \mathbb{C}^{k\times q} \mid F \text{ is measurable and } \|F\|_\infty = \operatorname{ess\,sup}_{0<\theta\le 2\pi} \|F(e^{j\theta})\| < \infty \Big\} \quad \blacksquare$$

$\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$ is a Hilbert space under the inner product

$$\langle f_1, f_2 \rangle_2 = \frac{1}{2\pi} \int_0^{2\pi} \langle f_1(e^{j\theta}), f_2(e^{j\theta}) \rangle_{\mathbb{C}^q} d\theta.$$

$\mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times q})$ is a Banach space under the $\|\cdot\|_\infty$-norm. Its elements induce a bounded operator from $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$ to $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^k)$.

Property A.2. *If $F \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times q})$ and $u \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$, then $Fu \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^k)$. The multiplication map $\Lambda_F : u \mapsto Fu$ defines a bounded linear operator from $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$ to $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^k)$ (often called a Laurent operator) and*

$$\|\Lambda_F\| = \sup_{u\neq 0} \frac{\|\Lambda_F u\|_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^k)}}{\|u\|_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^q)}} = \|F\|_\infty. \tag{A.1}$$

If $k = q$, we obtain the Banach algebra $\mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times k})$. We quote the following result from Gohberg, Goldberg, and Kaashoek (1993, Theorem 2.4, p. 569) on the existence of an inverse.

Lemma A.3. *$\mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times k})$ is a Banach algebra and $F \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times k})$ is boundedly invertible if and only if $\exists$ a $\gamma > 0$ such that $\{\theta \mid |\det(F(e^{j\theta}))| < \gamma\}$ has measure zero. If F is continuous, then $F \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{k\times k})$ is boundedly invertible if and only if $\det F(e^{j\theta}) \neq 0$ for all $\theta \in [0, 2\pi]$.*

Elements of $\mathbf{L}_2(\partial\mathbb{D}, \mathbb{C}^q)$ arise naturally as Fourier transforms of elements in $\ell_2(\mathbb{C}^q) = \{z = (z_r)_{r=-\infty}^{\infty}, z_r \in \mathbb{C}^q \mid \sum_{r=-\infty}^{\infty} \|z_r\|_{\mathbb{C}^q}^2 < \infty\}$.

Definition A.4. The Fourier transform $\mathfrak{F}$ of an element of $z \in \ell_2(\mathbb{C}^q)$ is defined by

$$\check{z}(e^{j\theta}) = \sum_{r=-\infty}^{\infty} z_r e^{-jr\theta}, \quad \theta \in [0, 2\pi], \tag{A.2}$$

which is precisely the Fourier series representation of an element $\check{z} \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$ with the Fourier coefficients

$$z_r = \frac{1}{2\pi} \int_0^{2\pi} \check{z}(e^{j\theta}) e^{jr\theta} d\theta.$$

Note that an element $\check{z} \in \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$ has the inverse Fourier transform $z = (z_r)_{r=-\infty}^{\infty}, z_r \in \mathbb{C}^q$, where z_r are the Fourier coefficients of $\check{z}$. Moreover, $\check{}: \ell_2(\mathbb{C}^q) \to \mathbf{L}_2(\partial\mathbb{D}; \mathbb{C}^q)$ is an isometric isometry with $\|\check{z}\|_{\mathbf{L}_2(\partial\mathbb{D};\mathbb{C}^q)} = \|z\|_{\ell_2(\mathbb{C}^q)}$.

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