Analysis and synthesis of attractive quantum Markovian dynamics

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Abstract

We propose a general framework for investigating a large class of stabilization problems in Markovian quantum systems. Building on the notions of invariant and attractive quantum subsystem, we characterize attractive subspaces by exploring the structure of the invariant sets for the dynamics. Our general analysis results are exploited to assess the ability of open-loop Hamiltonian and output-feedback control strategies to synthesize Markovian generators which stabilize a target subsystem, subspace, or pure-state. In particular, we provide an algebraic characterization of the manifold of stabilizable pure states in arbitrary finite-dimensional Markovian systems, that leads to a constructive strategy for designing the relevant controllers. Implications for stabilization of entangled pure states are addressed by example.

Keywords: Quantum control; Quantum dynamical semigroups; Quantum subsystems.

1 Introduction

Stabilization problems have a growing significance for a variety of quantum control applications, ranging from state preparation of optical, atomic, and nanomechanical systems to the generation of noise-protected realizations of quantum information in realistic devices [1, 2]. Dynamical systems undergoing Markovian evolution [3, 4] are relevant from the standpoint of typifying irreversible quantum dynamics and present distinctive control challenges [5]. In particular, open-loop quantum-engineering and (approximate) stabilization methods based on dynamical decoupling cease to be viable in the Markovian regime [6, 7]. It is our goal in this work to show how a wide class of Markovian stabilization

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problems can nevertheless be effectively treated within a general framework, provided by *invariant and attractive quantum subsystems*.

After providing the relevant technical background, we proceed establishing a first analysis result that fully characterizes the attractive subspaces for a given generator. This is done by analyzing the structure induced by the generator in the system's Hilbert space, and by invoking Krasowskii-LaSalle's invariance principle. We next explore the application of the result to stabilization problems for Markovian Hamiltonian and output-feedback control. Our approach leads to a complete characterization of the stabilizable pure states, subspaces, and subsystems as well as to constructive design strategies for the control parameters. Some partial results in this sense have been presented in [8], and in the conference paper [9]. We also refer to the journal article [8] for a more detailed discussion of the connection between invariant, attractive and noiseless subsystems, along with a thorough analysis of model robustness issues which shall not be our focus here.

2 Preliminaries and background

2.1 Quantum Markov processes

Consider a separable Hilbert space \mathcal{H} over the complex field \mathbb{C} . Let $\mathfrak{B}(\mathcal{H})$ represent the set of linear bounded operators on \mathcal{H} , with $\mathfrak{H}(\mathcal{H})$ denoting the real subspace of Hermitian operators, and \mathbb{I}, \mathbb{O} being the identity and the zero operator, respectively. Throughout our analysis, we consider a *finite-dimensional* quantum system Q: following the standard quantum statistical mechanics formalism [10], we associate to \mathcal{Q} a complex, finite-dimensional \mathcal{H} . Our (possibly uncertain) knowledge of the state of Q is condensed in a *density operator* ρ on \mathcal{H} , with $\rho \geq 0$ and trace $(\rho) = 1$. Density operators form a convex set $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{H}(\mathcal{H})$, with one-dimensional projectors corresponding to extreme points (*pure states*, $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$). Observable quantities are represented by Hermitian operators in $\mathfrak{H}(\mathcal{H})$, and expectation values are computed by using the trace functional: $\mathbb{E}_{\rho}(X) = \operatorname{trace}(\rho X)$. If \mathcal{Q} is the composite system obtained from two distinguishable quantum systems Q_1 , Q_2 , the corresponding mathematical description is carried out in the tensor product space, $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, observables and density operators being associated with Hermitian and positive-semidefinite, normalized operators on \mathcal{H}_{12} , respectively. The *partial trace* over \mathcal{H}_2 is the unique linear operator trace₂(·) : $\mathfrak{B}(\mathcal{H}_{12}) \to \mathfrak{B}(\mathcal{H}_1)$, ensuring that for every $X_1 \in \mathfrak{B}(\mathcal{H}_1), X_2 \in \mathfrak{B}(\mathcal{H}_2), \operatorname{trace}_2(X_1 \otimes X_2) = X_1 \operatorname{trace}(X_2).$ Partial trace is used to compute marginal states and partial expectations on multipartite systems.

In the presence of either intended or unwanted couplings (such as with a measurement apparatus, or with a surrounding quantum environment), the evolution of a subsystem of interest is no longer unitary and reversible, and the general formalism of *open quantum systems* is required [11, 3, 4]. A wide class of open quantum systems obeys Markovian dynamics [3, 12, 13, 4]. Let \mathcal{I}

denote the physical quantum system of interest, with associated Hilbert space \mathcal{H}_I , dim $(\mathcal{H}_I) = d$. Assume that we have no access or control over the state of the system's environment, and that the dynamics in $\mathfrak{D}(\mathcal{H}_I)$ is continuous in time and described at each instant $t \geq 0$ by a Trace-Preserving Completely Positive (TPCP) linear map $\mathcal{T}_t(\cdot)$ [14]. If a forward composition law is also assumed, we obtain a quantum Markov process, or Quantum Dynamical Semigroup (QDS):

Definition 1 (QDS) A quantum dynamical semigroup is a one-parameter family of TPCP maps $\{\mathcal{T}_t(\cdot), t \geq 0\}$ that satisfies:

- $\mathcal{T}_0 = \mathbb{I}$,
- $\mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s}, \ \forall t, s > 0,$
- trace($\mathcal{T}_t(\rho)X$) is a continuous function of $t, \forall \rho \in \mathfrak{D}(\mathcal{H}_I), \forall X \in \mathcal{B}(\mathcal{H}_I)$.

Due to the trace- and positivity-preserving assumptions, a QDS is a semigroup of contractions. As proven in [12, 15], the Hille-Yoshida generator for the semigroup exists and can be cast in the following canonical form:

$$\dot{\rho}(t) = \mathcal{L}(\rho(t)) = -\frac{i}{\hbar} [H, \rho(t)] + \sum_{k=1}^{p} \gamma_k \mathcal{D}(L_k, \rho(t))$$

$$= -\frac{i}{\hbar} [H, \rho(t)] + \sum_{k=1}^{p} \gamma_k \Big(L_k \rho(t) L_k^{\dagger} - \frac{1}{2} \{ L_k^{\dagger} L_k, \rho(t) \} \Big),$$
(1)

with $\{\gamma_k\}$ denoting the spectrum of A. The *effective Hamiltonian* H and the noise operators L_k (also known as "Lindblad operators") completely specify the dynamics, including the effect of the Markovian environment. In general, H is equal to the Hamiltonian for the isolated, free evolution of the system, H_0 , plus a correction, H_L , induced by the coupling to the environment (aka "Lamb shift"). The non-Hamiltonian terms $\mathcal{D}(L_k, \rho(t))$ in (1) account for the non-unitary character of the dynamics, specified by noise operators $\{L_k\}$.

In principle, the exact form of the generator of a QDS may be rigorously derived from a Hamiltonian model for the joint system-environment dynamics under appropriate limiting conditions (the so-called "singular coupling limit" or the "weak coupling limit," respectively [3, 4]). In most physical situations, however, an analytical derivation is unfeasible, since the full microscopic Hamiltonian describing the system-environment interaction is unavailable. A Markovian generator of the form (1) is then postulated on a phenomenological basis. In practice, it is often the case that knowledge of the noise effect may be assumed, allowing to specify the Markovian generator by directly assigning a set of noise operators L_k (not necessarily orthogonal or complete) in (1), and the corresponding noise strengths γ_k . Each of the noise operators L_k may be associated to a distinct noise channel $\mathcal{D}(L_k, \rho(t))$, by which information irreversibly leaks from the system to the environment.

2.2 Quantum subsystems: Invariance and attractivity

Quantum subsystems are the basic building block for describing composite systems in quantum mechanics [10], and provide a general framework for scalable quantum information engineering in physical systems. In fact, the so-called *subsystem principle* [2, 1, 16] states that any "faithful" representation of information in a quantum system requires to specify a subsystem desired information. Many of the control tasks considered in this paper are motivated by the need for strategies to create and maintain quantum information in open quantum systems. A definition of quantum subsystem suitable to our scopes is the following:

Definition 2 (Quantum subsystem) A quantum subsystem S of a system \mathcal{I} defined on \mathcal{H}_I is a quantum system whose state space is a tensor factor \mathcal{H}_S of a subspace \mathcal{H}_{SF} of \mathcal{H}_I ,

$$\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R, \tag{2}$$

for some co-factor \mathcal{H}_F and remainder space \mathcal{H}_R . The set of linear operators on $\mathcal{S}, \mathcal{B}(\mathcal{H}_S)$, has the same statistical properties and is isomorphic to the (associative) subalgebra of $\mathcal{B}(\mathcal{H}_I)$ of operators of the form $X_I = X_S \otimes \mathbb{I}_F \oplus \mathbb{O}_R$.

Let $n = \dim(\mathcal{H}_S)$, $f = \dim(\mathcal{H}_F)$, $r = \dim(\mathcal{H}_R)$, and let $\{|\phi_j^S\rangle\}_{j=1}^n$, $\{|\phi_k^F\rangle\}_{k=1}^f$, $\{|\phi_l^R\rangle\}_{l=1}^r$ be orthonormal bases for \mathcal{H}_S , \mathcal{H}_F , \mathcal{H}_R , respectively. The decomposition (2) is then naturally associated with the following basis for \mathcal{H}_I :

$$\{|\varphi_m\rangle\} = \{|\phi_j^S\rangle \otimes |\phi_k^F\rangle\}_{j,k=1}^{n,f} \cup \{|\phi_l^R\rangle\}_{l=1}^r.$$

This induces a block structure for matrices acting on \mathcal{H}_I :

$$X = \begin{pmatrix} X_{SF} & X_P \\ \hline X_Q & X_R \end{pmatrix}, \tag{3}$$

where, in general, $X_{SF} \neq X_S \otimes X_F$. We denote by Π_{SF} the projector onto \mathcal{H}_{SF} , that is, $\Pi_{SF} = (\mathbb{I}_{SF} | 0)$.

In this paper, we study Markov dynamics of a quantum system \mathcal{I} with a given decomposition of the associated Hilbert space of the form (2), with respect to the quantum subsystem \mathcal{S} associated to \mathcal{H}_S . By describing the dynamics in the Schrödinger picture, *i.e.*, with evolving states and time-invariant observables, the first step is to specify whether the system \mathcal{I} has been properly initialized in a state which faithfully represents a state of the subsystem \mathcal{S} , and what is the structure of such states.

Definition 3 (State initialization) The system \mathcal{I} with state $\rho \in \mathfrak{D}(\mathcal{H}_I)$ is initialized in \mathcal{H}_S with state $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$ if the blocks of ρ satisfy:

(i) $\rho_{SF} = \rho_S \otimes \rho_F$ for some $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$;

(ii) $\rho_P = 0, \rho_R = 0.$

We denote by $\mathfrak{I}_S(\mathcal{H}_I)$ the set of states that satisfy (i)-(ii) for some ρ_S .

Condition (ii) guarantees that $\bar{\rho}_S = \text{trace}_F(\Pi_{SF}\rho\Pi_{SF}^{\dagger})$ is a valid (normalized) state of S, while condition (i) ensures that measurements or dynamics affecting the factor \mathcal{H}_F have no effect on the state in \mathcal{H}_S .

We now proceed to characterize in which sense, and under which conditions, a quantum subsystem may be defined as invariant. Recall that a set \mathcal{W} is said to be *invariant* for a dynamical system if the trajectories that start in \mathcal{W} remain in \mathcal{W} for all $t \geq 0.^1$ In view of Definition 3, the natural definition considering dynamics in the state space may be phrased as follows:

Definition 4 (Invariance) Let \mathcal{I} evolve under a family of TPCP maps $\{\mathcal{T}_t, t \geq 0\}$. \mathcal{S} is an invariant subsystem if $\mathfrak{I}_S(\mathcal{H}_I)$ is an invariant subset of $\mathfrak{D}(\mathcal{H}_I)$.

In explicit form, as given in [8], this means that $\forall \rho_S \in \mathfrak{D}(\mathcal{H}_S), \rho_F \in \mathfrak{D}(\mathcal{H}_F)$, the state of \mathcal{I} obeys

$$\mathcal{T}_t\left(\begin{array}{c|c} \rho_S \otimes \rho_F & 0\\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathcal{T}_t^S(\rho_S) \otimes \mathcal{T}_t^F(\rho_F) & 0\\ \hline 0 & 0 \end{array}\right), t \ge 0, \tag{4}$$

where, for every $t \geq 0$, $\mathcal{T}_t^S(\cdot)$ and $\mathcal{T}_t^F(\cdot)$ are TPCP maps on \mathcal{H}_S and \mathcal{H}_F , respectively, not depending on the initial state. For Markovian evolution of \mathcal{I} , $\{\mathcal{T}_t^F\}$ and $\{\mathcal{T}_t^F\}$ are required to be QDSs on their respective domain.

We next recall a characterization of dynamical models able to ensure invariance for a fixed subsystem, based on appropriately constraining the blockstructure of the matrix representation of the operators specifying the Markovian generator. We refer to [8] for the proofs.

Lemma 1 (Markovian invariance) Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, and let H, $\{L_k\}$ be the Hamiltonian and the error generators of a Markovian QDS as in (1). Then \mathcal{H}_S supports an invariant subsystem iff $\forall k$:

$$L_{k} = \left(\begin{array}{c|c} L_{S,k} \otimes L_{F,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array} \right),$$

$$iH_{P} - \frac{1}{2} \sum_{k} (L_{S,k}^{\dagger} \otimes L_{F,k}^{\dagger}) L_{P,k} = 0,$$

$$H_{SF} = H_{S} \otimes \mathbb{I}_{F} + \mathbb{I}_{S} \otimes H_{F},$$

(5)

where for each k either $L_{S,k} = \mathbb{I}_S$ or $L_{F,k} = \mathbb{I}_F$ (or both).

One may require S to have dynamics independent from evolution affecting \mathcal{H}_F and \mathcal{H}_R also in the case where \mathcal{I} is not initialized in the sense of Definition

¹For clarity, let us also recall other standard dynamical systems notions relevant in our context. Given $\dot{\rho} = \mathcal{L}(\rho)$ and a suitable norm for the state manifold, we call *invariant*, stationary, or equilibrium state any ρ such that $\dot{\rho} = 0$. An equilibrium state ρ is said to be stable if for every $\epsilon \geq 0$ there exists δ such that if $\|\rho_0 - \rho\| \leq \delta$, then any trajectory starting from ρ_0 does not leave the ball of radius ϵ centered in ρ . A state ρ is said to be (globally) attractive if the trajectories from any initial condition converge to it.

3. If neither conditions (i)-(ii) are satisfied, one may still define an unnormalized reduced state for the subsystem:

$$\tilde{\rho}_S = \operatorname{trace}_F(\Pi_{SF}\rho\Pi_{SF}^{\dagger}), \quad \operatorname{trace}(\tilde{\rho}_S) \le 1.$$

This allows for entangled mixed states to be supported on \mathcal{H}_{SF} , as well as for blocks ρ_P , ρ_R to differ from zero. Similar to the case of so-called "initializationfree" subsystems considered in [17, 8], an additional constraint on the Lindblad operators is required to ensure independent reduced dynamics in this case. That is, with respect to the matrix block-decomposition above, it must be $L_{P,k} = 0$ for every k. Such a constraint decouples the evolution of the SF-block of the state from the rest, rendering both $\mathfrak{I}_S(\mathcal{H}_I)$ and $\mathfrak{I}_R(\mathcal{H}_I)$ separately invariant.

This imposes tighter conditions on the noise operators, which may be hard to ensure in reality and, from a control perspective, leave less room for Hamiltonian compensation as examined in Section 4.1. In order to address situations where such extra constraints cannot be met, as well as a question which is interesting on its own, we explore conditions for a subsystem to be *attractive*:

Definition 5 (Attractive subsystem) Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$. Then \mathcal{H}_S supports an attractive subsystem with respect to a family $\{\mathcal{T}_t\}_{t\geq 0}$ of TPCP maps if $\forall \rho \in \mathfrak{D}(\mathcal{H}_I)$ the following condition is asymptotically obeyed:

$$\lim_{t \to \infty} \left(\mathcal{T}_t(\rho) - \left(\begin{array}{c|c} \bar{\rho}_S(t) \otimes \bar{\rho}_F(t) & 0 \\ \hline 0 & 0 \end{array} \right) \right) = 0, \tag{6}$$

where

$$\bar{\rho}_S(t) = \operatorname{trace}_F[\Pi_{SF}\mathcal{T}_t(\rho)\Pi_{SF}^{\dagger}],$$
$$\bar{\rho}_F(t) = \operatorname{trace}_S[\Pi_{SF}\mathcal{T}_t(\rho)\Pi_{SF}^{\dagger}].$$

This implies that every trajectory in $\mathfrak{D}(\mathcal{H}_I)$ converges to $\mathfrak{I}_S(\mathcal{H}_I)$. Thus, an attractive subsystem may be thought of as a subsystem that "self-initializes" in the long-time limit, by reabsorbing initialization errors. Although such a desirable behavior only emerges asymptotically, for QDSs one can see that convergence is exponential, as long as the relevant eigenvalues of \mathcal{L} have strictly negative real part.

We conclude this section by recalling two partial results on attractive subsystems which we established in [8]. The first is a negative result, which shows, in particular, how the possibility of "initialization-free" and attractive behavior are mutually exclusive.

Proposition 1 Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, $\mathcal{H}_R \neq 0$, and let H, $\{L_k\}$ be the Hamiltonian and the error generators as in (1), respectively. Let \mathcal{H}_S support an invariant subsystem. If $L_{P,k} = L_{Q,k}^{\dagger} = 0$ for every k, then \mathcal{H}_S is not attractive.

Note that he conditions of the above Proposition are obeyed, in particular, if $L_k = L_P^{\dagger}$, $\forall k$. As a consequence, attractivity is never possible for the class of

unital (I-preserving) Markovian QDSs with purely self-adjoint L_k 's. Still, even if the condition $L_{P,k} = L_{Q,k}^{\dagger} = 0$ condition holds, attractive subsystems may exist in the pure-factor case, where $\mathcal{H}_R = 0$. Sufficient conditions are provided by the following:

Proposition 2 Assume that $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$ ($\mathcal{H}_R = 0$), and let \mathcal{H}_S be invariant under a QDS with generator of the form

$$\mathcal{L} = \mathcal{L}_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes \mathcal{L}_F.$$

If $\mathcal{L}_F(\cdot)$ has a unique attractive state $\hat{\rho}_F$, then \mathcal{H}_S is attractive.

Interesting linear-algebraic conditions for determining whether a generator $\mathcal{L}_F(\cdot)$ has a unique attractive state (though not necessarily pure) are presented in [18, 19].

3 Characterizing attractive Markovian dynamics

We begin by presenting new necessary and sufficient conditions for attractivity of a subspace, which will provide the basis for the synthesis results in the next sections. Notice that if \mathcal{H}_{SF} supports an attractive subsystem, the entire set of states with support on \mathcal{H}_{SF} , $\mathfrak{I}_{SF}(\mathcal{H}_I)$, is attractive. Once this is verified, the dynamics confined to the invariant subspace (that supports a pure subsystem) may be studied with the aid of the results recalled in the previous section. The following Lemma will be used in the proof of the main result, but is also interesting on its own. We denote with $\operatorname{supp}(X)$ the support of $X \in \mathfrak{B}(\mathcal{H})$, *i.e.*, the orthogonal complement of its kernel.

Lemma 2 Let \mathcal{W} be an invariant subset of $\mathfrak{D}(\mathcal{H}_I)$ for the QDS dynamics generated by $\dot{\rho} = \mathcal{L}(\rho)$, and define:

$$\mathcal{H}_{\mathcal{W}} = \operatorname{supp}(\mathcal{W}) = \bigcup_{\rho \in \mathcal{W}} \operatorname{supp}(\rho).$$

Then $\mathfrak{I}_{\mathcal{W}}(\mathcal{H}_I)$ is invariant.

Proof. Let $\hat{\mathcal{W}}$ be the convex hull of \mathcal{W} . Thus, every element $\hat{\rho}$ of $\hat{\mathcal{W}}$ may be expressed as $\hat{\rho} = \sum_k p_k \rho_k$, where $p_k \ge 0$, $\sum_k p_k = 1$, and $\rho_1, \ldots, \rho_k \in \mathcal{W}$. By using linearity of the dynamics,

$$\mathcal{T}_t(\hat{\rho}) = \sum_k p_k \mathcal{T}_t(\rho_k) = \sum_k p_k \rho'_k, \quad \forall t \ge 0.$$

with $\rho'_1, \ldots, \rho'_k \in \mathcal{W}$. Hence $\hat{\mathcal{W}}$ is invariant. Furthermore, from the definition of $\hat{\mathcal{W}}$, there exist a $\bar{\rho} \in \hat{\mathcal{W}}$ such that $\operatorname{supp}(\bar{\rho}) = \operatorname{supp}(\hat{\mathcal{W}}) = \mathcal{H}_{\mathcal{W}}$. Consider $\mathcal{H}_I = \mathcal{H}_{\mathcal{W}} \oplus \mathcal{H}_{\mathcal{W}}^{\perp}$, and the corresponding matrix partitioning:

$$X = \left(\begin{array}{c|c} X_{\mathcal{W}} & X_P \\ \hline X_Q & X_R \end{array}\right).$$

With respect to this partition, the block $\bar{\rho}_{\mathcal{W}}$ of $\bar{\rho}$ is full-rank, while $\bar{\rho}_{P,Q,R}$ are zero-blocks. The trajectory $\{\mathcal{T}_t(\bar{\rho}), t \geq 0\}$ is contained in $\hat{\mathcal{W}}$ only if:

$$\frac{d}{dt}\bar{\rho} = \begin{pmatrix} \mathcal{L}_{\mathcal{W}}(\bar{\rho}_{\mathcal{W}}) & 0\\ 0 & 0 \end{pmatrix},$$

so that, upon computing explicitly the generator blocks, we must impose:

$$\begin{cases} \bar{\rho}_{\mathcal{W}} \left(iH_P - \frac{1}{2} \sum_k L_{\mathcal{W},k}^{\dagger} L_{P,k} \right) = 0, \\ -\frac{1}{2} \sum_k \{ L_{Q,k}^{\dagger} L_{Q,k}, \bar{\rho}_{\mathcal{W}} \} = 0. \end{cases}$$

Since $\bar{\rho}_{\mathcal{W}}$ is full-rank and positive, it must be:

$$\begin{cases} iH_P - \frac{1}{2} \sum_k L_{\mathcal{W},k}^{\dagger} L_{P,k} = 0, \\ L_{Q,k} = 0, \quad \forall k. \end{cases}$$

Comparing with the conditions given in Corollary 1, we infer that $\mathfrak{I}_{\mathcal{W}}(\mathcal{H}_I)$ is invariant, hence we conclude.

We are now in a position to prove our main result:

Theorem 1 (Subspace attractivity) Let $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, and assume that \mathcal{H}_S is an invariant subspace for the QDS dynamics generated by (1). Define:

$$\mathcal{H}_{R'} = \bigcap_{k=1}^{p} \ker(L_{P,k}),\tag{7}$$

with the matrix blocks $L_{P,k}$ representing linear operators from \mathcal{H}_R to \mathcal{H}_S . Then \mathcal{H}_S is an attractive subspace iff $\mathcal{H}_{R'}$ does not support any invariant subsystem.

Proof. Clearly, if $\mathcal{H}_{R'}$ supports an invariant set \mathcal{W}_R , then \mathcal{H}_S cannot be attractive, since for every $\bar{\rho} \in \mathcal{W}_R$, the dynamics is confined to \mathcal{W}_R . To prove the other implication, we shall prove that if $\mathcal{H}_{R'}$ does not support an invariant *set*, then \mathcal{H}_S is attractive. Consider the non-negative, linear functional $V(\rho) = \operatorname{trace}(\Pi_R \rho)$. It is zero iff $\rho_R = 0$, *i.e.*, for perfectly initialized states. By LaSalle's invariance principle (see *e.g.* [20]), every trajectory will converge to the largest invariant subset \mathcal{W} contained in the set:

$$\mathcal{Z} = \{ \rho \in \mathfrak{D}(\mathcal{H}_I) | \dot{V}(\rho) = 0 \}.$$

Explicit calculation of the blocks of the generator (see also [8]) yields:

$$\dot{V}(\rho) = \operatorname{trace}(\Pi_R \mathcal{L}(\rho)) = -\operatorname{trace}\left(\sum_k L_{P,k}^{\dagger} L_{P,k} \rho_R\right).$$

By the cyclic property of the trace, the last term is equivalent to the trace of $\sum_{k} L_{P,k} \rho_R L_{P,k}^{\dagger}$, which is a sum of positive operators, and thus can be zero iff each term is zero. Being the $L_{P,k}$'s fixed, this can hold iff ρ_R has support

contained in $\mathcal{H}_{R'}$, defined as above. Thus, the support of \mathcal{Z} is $\mathcal{H}_S \oplus \mathcal{H}_{R'}$. Call $\mathcal{H}_{\mathcal{W}}$ the support of the maximal invariant set \mathcal{W} in \mathcal{Z} : By Lemma 2, $\mathfrak{I}_{\mathcal{W}}(\mathcal{H}_I)$ is invariant. But \mathcal{W} is defined as the maximal invariant set in \mathcal{Z} , so it must be $\mathcal{W} = \mathfrak{I}_{\mathcal{W}}(\mathcal{H}_I)$. Recalling that by hypothesis $\mathfrak{I}_S(\mathcal{H}_I)$ is itself an invariant subset contained in \mathcal{Z} , it must be $\mathcal{H}_{\mathcal{W}} = \mathcal{H}_S \oplus \mathcal{H}_{R''}$, with $\mathcal{H}_{R''} \subset \mathcal{H}_{R'}$. We next prove that $\mathcal{H}_{R'}$ supports an invariant set iff $\mathcal{H}_{\mathcal{W}} \neq \mathcal{H}_S$, *i.e.* $\mathcal{H}_{R'} \neq 0^2$. Consider a $\hat{\rho} \in \mathcal{W}$ such that $\hat{\rho}$ has non-trivial support on \mathcal{H}_S^1 . If no such state exists, \mathcal{W} has support only on \mathcal{H}_S , so clearly $\mathcal{H}_{R'} = 0$. If such a state exists, let

$$\hat{\rho}' = \frac{\Pi_{R'}\hat{\rho}\Pi_{R'}}{\operatorname{trace}(\Pi_{R'}\hat{\rho})},$$

where $\Pi_{R'}$ is the orthogonal projector on $\mathcal{H}_{R'}$. Since $\hat{\rho}'$ has support only on $\mathcal{H}_{R'} \subset \mathcal{H}_{W}$, its trajectory $\{\hat{\rho}'(t) = \mathcal{T}_t(\hat{\rho}'), t \geq 0\}$ is confined to \mathcal{W} . On the other hand, $\mathcal{W} \subset \mathcal{Z}$, hence it must be $\dot{V}(\hat{\rho}(t)) = 0$ for all $t \geq 0$. By observing that $V(\hat{\rho}') = 1$ and that $V(\rho), \dot{V}(\rho)$ are continuous, we can conclude that the trajectory $\{\hat{\rho}'(t)\}$ must have support only on some $\mathcal{H}_{R''} \subset \mathcal{H}_{R'}$, endowing $\mathcal{H}_{R'}$ with an invariant set, and by the Lemma above, the invariant subsystem associated to $\mathfrak{I}_{R''}(\mathcal{H}_I)$. We conclude by observing that if $\mathcal{H}_{R'}$ does not support an invariant set, then $\mathcal{H}_{W} = \mathcal{H}_S$, hence \mathcal{H}_S is attractive.

In spite of its non-constructive nature, the power of this characterization will be apparent in the proofs of the results concerning *active stabilization* of states and subspaces by Hamiltonian control in the next section. We observe here that Lemma 2 lends itself to the following useful specialization:

Proposition 3 If $\bar{\rho}$ is an invariant state for the QDS dynamics generated by $\dot{\rho} = \mathcal{L}(\rho)$, and $\mathcal{H}_B = supp(\bar{\rho})$, then $\mathfrak{I}_B(\mathcal{H}_I)$ is invariant. Conversely, if $\bar{\mathcal{H}}$ supports an invariant subset, it contains at least an invariant state.

Proof. The first implication follows from Lemma 2 above. If $\overline{\mathcal{H}}$ supports an invariant subset, then by the same Lemma it supports an invariant subsystem, and the density operators with support on $\overline{\mathcal{H}}$ form a convex, compact set that evolves accordingly to a (reduced) QDS. Hence, it must admit at least an invariant state [3].

This result provides us with an explicit criterion for verifying whether $\mathcal{H}_{R'}$ contains an invariant subset: It will suffice to check if $\mathcal{H}_{R'}$ supports an invariant state. Invariant (or "fixed") states may be found by analyzing the structure of $\ker(\mathcal{L}(\cdot))$. An efficient algorithm for generic TPCP maps has been recently presented in [16].

4 Engineering attractive Markovian dynamics

In this section, we illustrate the relevance of the theoretical framework developed thus far to a wide class of Markovian stabilization problem associated with

 $^{^{2}}$ To the scope of this proof, the "if" implication would suffice, but since the converse arises naturally, we prove both.

the task of making a desired (fixed) quantum subsystem invariant or attractive. Interestingly, these problems may be regarded as instances of *Markovian reservoir engineering*, which has long been investigated on a phenomenological basis by the physics community in the context of both decoherence mitigation and the quantum-classical transition, see *e.g.* [21, 22, 23].

In the special yet relevant case of sinthesizing attractive dynamics with respect to an intended *pure* state, our results fully characterize the manifold of pure states that may be "prepared" given a reference dissipative dynamics using either open-loop Hamiltonian or feedback control resources. As discussed in [8], provided a sufficient level of accuracy in tuning the relevant control parameters may be ensured, the "direct" Markovian feedback considered here has the important advantage of substantially relaxing implementation constraints in comparison with "Bayesian" feedback techniques requiring real-time state estimation update [24, 25].

4.1 Open-loop Hamiltonian control

We begin by exploring what can be achieved by considering only open-loop Hamiltonian control, specifically, the application of *time-independent* Hamiltonians to the dynamical generator. This allows us to consider generators involving, in general, multiple L_k , and yields interesting characterizations of the possibilities offered by this class of controls for stabilization problems, complementing previous work from a controllability perspective [26, 27]. The results established below will also be of key importance in the proofs of the theorems on closed-loop stabilization. Lastly, a separate presentation will serve to clarify the different scopes and limitations of the two class of control strategies. As a direct consequence of the Markovian invariance theorem, we have the following:

Corollary 1 (Open-loop invariant subspaces) Let $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Then $\mathfrak{I}_S(\mathcal{H}_I)$ can be made invariant by open-loop Hamiltonian control iff $L_{Q,k} = 0$ for every k.

Proof. By specializing Corollary 1, \mathcal{H}_S supports an invariant subsystem iff:

$$L_{Q,k} = 0, \quad \forall k \tag{8}$$

$$iH_P - \frac{1}{2}\sum_k L_{S,k}^{\dagger} L_{P,k} = 0.$$
(9)

The only condition that is affected by a change of Hamiltonian is (9), which however can always be satisfied by an appropriate choice of control Hamiltonian. This leaves us with condition (8) alone.

The above result makes it possible to enforce invariant subspaces for the controlled dynamics by solely using Hamiltonian resources, *without directly modifying the non-unitary part.* The ability of open-loop Hamiltonian control to induce stronger attractivity properties is characterized in the following: **Theorem 2 (Open-loop attractive subspaces)** Let $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ and assume that \mathcal{H}_S supports an invariant subsystem. Then $\mathfrak{I}_S(\mathcal{H}_I)$ can be made attractive by open-loop Hamiltonian control iff $\mathfrak{I}_R(\mathcal{H}_I)$ is not invariant.

Proof. If \mathcal{H}_R supports an invariant subsystem, then by Corollary 1 it must be $L_{P,k} = 0$ for every k. Since \mathcal{H}_S invariant, this implies $H_P = 0$. Any Hamiltonian control perturbation that preserves invariance on \mathcal{H}_S must satisfy this condition, hence preserve invariance on \mathcal{H}_R too, thus \mathcal{H}_S cannot be rendered attractive. If the whole \mathcal{H}_R does not support an invariant subsystem, we can devise an iterative procedure that builds up a control Hamiltonian H_c such that \mathcal{H}_S becomes attractive. Theorem 1 states that if there is no invariant subsystem supported in $\mathcal{H}_{R'}$ (defined in (7)), then \mathcal{H}_S is attractive. If there is an invariant subsystem with support $\mathcal{H}_T \subset \mathcal{H}_{R'}$, let us consider the following Hilbert space decomposition:

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_T \oplus \mathcal{H}_Z.$$

After imposing the invariance conditions on \mathcal{H}_S and \mathcal{H}_T , the associated blockdecomposition of the Lindblad operators and Hamiltonian turns out to be of the form:

$$L_{k} = \begin{pmatrix} L_{S,k} & 0 & L_{P',k} \\ 0 & L_{T,k} & L_{P'',k} \\ \hline 0 & 0 & L_{Z,k} \end{pmatrix},$$
$$H = \begin{pmatrix} H_{S} & 0 & H_{P'} \\ \hline 0 & H_{T} & H_{P''} \\ \hline H_{P'}^{\dagger} & H_{P''}^{\dagger} & H_{Z} \end{pmatrix},$$

subject to the conditions:

$$iH_{P'} - \frac{1}{2} \sum_{k} L^{\dagger}_{S,k} L_{P',k} = 0,$$

$$iH_{P''} - \frac{1}{2} \sum_{k} L^{\dagger}_{T,k} L_{P'',k} = 0.$$

One sees that the most general Hamiltonian perturbation that preserves the invariance of \mathcal{H}_S has the form:

$$H_c = \begin{pmatrix} H_1 & 0 & 0\\ 0 & H_2 & H_M \\ 0 & H_M^{\dagger} & H_3 \end{pmatrix}.$$

Consider a control Hamiltonian H_c such that the block H_M has full columnrank, while H_1, H_3 are arbitrary and H_2 is still to be determined. If $\dim(\mathcal{H}_T) \leq \frac{1}{2}\dim(\mathcal{H}_R)$, then $i\rho_T H_M \neq 0$ for every ρ_T , hence \mathcal{H}_T cannot support any invariant subsystem, since conditions in Corollary 1 cannot be satisfied for any subspace of \mathcal{H}_T . Conversely, if $\dim(\mathcal{H}_T) > \frac{1}{2}\dim(\mathcal{H}_R)$, choosing an H_M as above, by dimension comparison H_M must have a non-trivial left kernel \mathcal{K} , $\mathcal{K}H_M = 0$, and thus there exists a $\mathcal{H}_{T'} \subset \mathcal{K}$ that supports an invariant $\mathfrak{I}_{T'}(\mathcal{H}_I)$, whose dimension is strictly lesser than dimension of $\dim(\mathcal{H}_T)$. We can iterate the reasoning with a new, refined decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_{T'} \oplus \mathcal{H}_{Z'}$, with $\mathcal{H}_{Z'} = \mathcal{H}_Z \oplus (\mathcal{H}_T \oplus \mathcal{H}_{T'})$. With this decomposition, the generator matrices exhibits the same block structure as above, with $\dim(\mathcal{H}_{T'}) < \dim(\mathcal{H}_T)$. Thus, we can exploit the freedom of choice on the block \mathcal{H}_2 to further reduce the dimension of the invariant set. At each iteration, the procedure either stops rendering \mathcal{H}_S attractive, if $\dim(\mathcal{H}_T) \leq \frac{1}{2}\dim(\mathcal{H}_R)$, or decrease the dimension of the invariant set by at least 1. The procedure thus ends in at most $\dim(\mathcal{H}_{R'})$ steps.

Remarkably, the proof of the above theorem, combined with a strategy to find invariant subspaces, provides a constructive procedure to build a constant Hamiltonian that makes the desired invariant subspace attractive whenever the Theorem's hypothesis are satisfied.

4.2 Markovian feedback control

The potential of Hamiltonian compensation for controlling Markovian evolutions is clearly limited by the impossibility to directly modify the noise action. To our scopes, open-loop control is then mostly devoted to connect subspaces in \mathcal{H} that are already invariant, and to adjust the generator parameters so that the interplay between Hamiltonian and dissipative contributions (as in Eq. (5)) can stabilize the desired subspace or subsystem.

A way to overcome these limitations is offered by closed-loop control strategies. Measurement-based feedback control requires the ability to both effectively monitor the environment, and condition the target evolution upon the measurement record. Feedback strategies have been considered since the beginning of the quantum control field [28], and successfully employed in a wide variety of settings (see *e.g.* [29, 30, 24, 31, 32]).

We focus on a measurement scheme which mimics optical homo-dyne detection for field-quadrature measurements, whereby the target system (e.g. an atomic cloud trapped in an optical cavity) is indirectly monitored via measurements of the outgoing laser field quadrature [29, 33]. The conditional dynamics of the state is stochastic, driven by the fluctuation one observes in the measurement. Considering a suitable infinitesimal feedback operator determined by a *feedback Hamiltonian* F, and taking the expectation with respect to the noise trajectories, this leads to the Wiseman-Milburn Markovian *Feedback Master equation* (FME) [29, 30]:

$$\dot{\rho}_t = -i\hbar \Big[H + \frac{1}{2} (FM + M^{\dagger}F), \, \rho_t \Big] + \mathcal{D}(M - iF, \rho_t). \tag{10}$$

The feedback state-stabilization problem for Markovian dynamics has been extensively studied for a single two-level system (*qubit*) [34, 35]. The standard approach is to to design a Markovian feedback loop by assigning both the measurement and feedback operators M, F, and to treat the measurement strength and the feedback gain as the relevant control parameters accordingly.

Throughout the following section, we will assume to have more freedom, by considering, for a fixed measurement operator M, both F and H as tunable control Hamiltonians

Definition 6 (CHC) A controlled FME of the form (10) supports complete Hamiltonian control (CHC) if (i) arbitrary feedback Hamiltonians $F \in \mathfrak{H}(\mathcal{H}_I)$ may be enacted; (ii) arbitrary constant control perturbations $H_c \in \mathfrak{H}(\mathcal{H}_I)$ may be added to the free Hamiltonian H.

This leads to both new insights and constructive control protocols for systems where the noise operator is a generalized angular momentum-type observable, for generic finite-dimensional systems. Physically, the CHC assumption must be carefully scrutinized on a case by case basis, since constraints on the form of the Hamiltonian with respect to the Lindblad operator may emerge, notably in the abovementioned weak-coupling limit derivations of Markovian models [3].

We now address the general subspace-stabilization problem for controlled Markovian dynamics described by FMEs. A characterization of the subspaces supporting stabilizable subsystems is provided by the following:

Theorem 3 (Feedback attractive subspaces) Let $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, with Π_S being the orthogonal projection on \mathcal{H}_S . Assume CHC capabilities. Then, for any measurement operator M, there exist a feedback Hamiltonian F and a Hamiltonian compensation \mathcal{H}_c that make the subsystem supported by \mathcal{H}_S attractive for the FME (10) iff

$$[\Pi_S, (M+M^{\dagger})] \neq 0. \tag{11}$$

Proof. Write $M = M^H + iM^A$, with both M^H and M^A being Hermitian, thus $L = M^H + i(M^A - F)$. Condition (11) holds iff M^H is not block-diagonal when partitioned according to the chosen decomposition. If M^H is block-diagonal, then, by Corollary 1, enforcing invariance of the subsystem supported by \mathcal{H}_S requires that $L_Q = 0$. But then it must also be $L_P = 0$, so that \mathcal{H}_R supports an invariant subsystem. Since the choice of L_S and L_R does not affect invariance, by Theorem 2 it follows that \mathcal{H}_S cannot be made attractive by Hamiltonian control. On the other hand, if M^H is not block-diagonal, we can always find F in such a way that L is upper diagonal, $L_P \neq 0$, by choosing $F_P = iM_P^H + M_P^A$. With L as the new noise operator, we now have to devise a control Hamiltonian H_c with a block $H_{c,P}$ that makes \mathcal{H}_S invariant (this is always possible by Corollary 1, since $L_Q=0$), and a block $H_{c,R}$ constructed following the procedure in the proof of Theorem 2.

The following specialization to pure states, $\it i.e.$ one-dimensional subspaces, is immediate:

Corollary 2 Assume CHC. For any measurement operator M, there exist a feedback Hamiltonian F and a Hamiltonian compensation H_c able to stabilize an arbitrary desired pure state ρ_d for the FME (10) iff

$$[\rho_d, (M+M^{\dagger})] \neq 0. \tag{12}$$

The proof of Theorem 3 provides a constructive algorithm for designing the feedback and correction Hamiltonians needed for the stabilization task. In particular, our analysis recovers the qubit stabilization results of [34] recalled before. For example, the states that are not stabilizable within the control assumptions of [34] are the ones commuting with the Hermitian part of $M = \sigma_+$, that is, $M^H = \sigma_x$. In the *xz*-plane of the Bloch's representation, the latter correspond precisely to the equatorial points.

As a corollary of Theorem 3 and Proposition 2, we present sufficient and necessary conditions for engineering a generic attractive quantum subsystem (with a non-trivial co-factor). We start with a Lemma, which is a straightforward specialization of Proposition 5 in [8]:

Lemma 3 Assume that $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$, $(\mathcal{H}_R = 0)$, and a QDS of the form $\mathcal{L} = \mathbb{I}_S \otimes \mathcal{L}_F$. If $\mathcal{L}_F(\cdot)$ admits at least two invariant states, then \mathcal{H}_S is not attractive.

Theorem 4 (Feedback attractive subsystems) Let $\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = \mathcal{H}_S \otimes \mathcal{H}_F \oplus \mathcal{H}_R$, with dim (\mathcal{H}_S) , dim $(\mathcal{H}_F) \geq 2$, and assume CHC capabilities. Then for any M, with Hermitian part M^H , there exist a feedback Hamiltonian F and a Hamiltonian compensation \mathcal{H}_c that make the subsystem S attractive for the FME (10) iff the following conditions hold:

$$[\Pi_{SF}, M^H] \neq 0, \tag{13}$$

ii)

i)

$$\Pi_{SF} M^{H} \Pi_{SF}^{\dagger} = \begin{cases} \mathbb{I}_{S} \otimes C_{F}, \text{ or} \\ C_{S} \otimes \mathbb{I}_{F}, \end{cases}$$
(14)

iii)

$$\Pi_{SF} M^H \Pi_{SF}^{\dagger} \neq \lambda \mathbb{I}_{SF}, \ \forall \lambda \in \mathbb{C}.$$
(15)

Proof. By Theorem 3, condition (13) is necessary and sufficient to render \mathcal{H}_{SF} attractive, which is a necessary condition for attractivity of \mathcal{H}_S . In fact, if this is not the case, by Theorem 1 there would exist an invariant subsystem whose support is contained \mathcal{H}_R . To ensure invariance of $\mathfrak{I}_S(\mathcal{H}_I)$, by Corollary 1, the block L_{SF} of L = M - iF has to satisfy $L_{SF} = L_S \otimes L_F$, with $L_S = \mathbb{I}_S$ or $L_F = \mathbb{I}_F$ (or both). Thus, both the Hermitian and anti-Hermitian parts of L_{SF} must have the same structure. The Hermitian part of L is equal the Hermitian part of M, whereby it follows that (14) is necessary for invariance of $\mathfrak{I}_S(\mathcal{H}_I)$. Assume $C_F \neq \mathbb{I}_F$ (the other case may be treated in a similar way, by interchanging the roles of \mathcal{H}_F and \mathcal{H}_S in what follows). If (15) is not satisfied, then L_{SF} must be unitarily similar to a diagonal matrix for any choice of F that ensures invariance of $\mathfrak{I}_S(\mathcal{H}_I)$. Hence, the dynamics restricted to \mathcal{H}_F admits at least two different stationary states (dim(\mathcal{H}_F) ≥ 2 by hypothesis). By Lemma 3, we conclude that $\mathfrak{I}_{\mathcal{S}}(\mathcal{H}_{I})$ cannot be attractive. Conversely, if i) holds, following the proof of Theorem 3, we can devise a Hamiltonian correction H_c and a feedback Hamiltonian F for which \mathcal{H}_{SF} is attractive. Since the SFblock is irrelevant to this stage, H_c and F may be further chosen to render a pure state of \mathcal{H}_F attractive for the reduced dynamics. Assume ii) and iii), with C_F different from a scalar matrix (again, to treat the other case, $C_S \neq \lambda \mathbb{I}_S$, it suffices to switch the appropriate subscripts in what follows). Thus, there exists a one-dimensional projector ρ_1 such that $[\rho_1, C_F] \neq 0$. By Corollary 2, we can find F_F and H_F that render it attractive. By choosing an Hamiltonian control so that $H_{SF} = \mathbb{I}_S \otimes H_F$, and $F_{SF} = \mathbb{I}_S \otimes F_F$, the stated conditions are also sufficient for the existence of attractivity-ensuring controls.

5 Applications

The following examples will serve to exemplify the application of our stabilization results to prototypical finite-dimensional control systems, which are also of direct relevance to quantum information devices. Different scenarios may arise depending on whether the target system is (or is regarded as) indecomposable, or explicit reference to a decomposition into subsystems is made.

5.1 Single systems

Example 1: Consider a single qubit on $\mathcal{H} \simeq \mathbb{C}^2$, with uncontrolled dynamics specified by $H = n_0 \mathbb{I}_2 + n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$, with $n_0, n_x, n_y, n_z \in \mathbb{R}$ and $M = \frac{\hbar}{2} \sigma_x$. Assume we wish to stabilize $\rho_d = \text{diag}(1,0)$. Since $[\rho_d, \sigma_x] \neq 0$, this is possible. Following the procedure in the above proof, consider $F = -\frac{\hbar}{2} \sigma_y$, so that

$$L = \hbar \frac{\sigma_x + i\sigma_y}{2} = \hbar \sigma_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and $H_c = -n_x \sigma_x - n_y \sigma_y$. Substituting in the FME (10), one obtains the desired result, as it can also be directly verified by using Proposition 7 in [8].

Assume, more generally, that it is possible to continuously monitor an arbitrary single-spin observable, $\vec{\sigma} \cdot \vec{n}$. Since the choice of the reference frame for the spin axis is conventional, by suitably adjusting the relative orientation of the measurement apparatus and the sample, it is then in principle possible to prepare and stabilize any desired pure state with a similar control strategy.

Example 2: Consider a three-level system (a *qutrit*), whose Hilbert space $\mathcal{H} \simeq \mathbb{C}^2$ carries a spin-1 representation of spin angular momentum observables J_{α} , $\alpha = x, y, z$. Without loss of generality, we may choose a basis in \mathcal{H} such that the desired pure state to be stabilized is $\rho_d = \text{diag}(1,0,0)$, and by CHC we may also ensure that H = 0. In analogy with Example 1, a natural strategy is to continuously monitor a non-diagonal spin observable, for instance:

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}.$$

Since $[\rho_d, J_x] \neq 0$, the state is stabilizable. Choosing the feedback Hamiltonian

$$F = -J_y = -\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0 \end{pmatrix},$$

yields

$$L = J_x + iJ_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix}.$$

Unlike the qubit case, $H' = \frac{1}{2}(FM + M^{\dagger}F) \neq 0$, thus a Hamiltonian compensation H_c is needed to ensure that $i(H' + H_c)_P - \frac{1}{2}L_S^{\dagger}L_P = 0$. With these choices, it is easy to see that $\mathcal{H}_{R'}$ does not support any invariant subsystem, hence ρ_d is attractive.

Provided that a similar structure of the observables is ensured, the previous examples naturally extend to generic d-level systems, as formally established in [8] by using Lyapunov techniques.

5.2 Bipartite systems

If a multipartite structure is specified on \mathcal{H} , it is both conceptually and practically important to understand whether stabilization of physically relevant class of states (including non-classical *entangled states*) is achievable with control resources which respect appropriate operational constraints, such as locality. We focus here on the simplest setting offered by *bipartite qubit systems*, with emphasis on Markovian-feedback preparation of entangled states, which has also been recently analyzed within a quantum filtering approach in [36].

Example 3: Consider a two-qubit system defined on a Hilbert space $\mathcal{H} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$, with a preferred basis $\mathcal{C} = \{ |ab\rangle = |a\rangle \otimes |b\rangle | a, b = 0, 1 \}$ (*e.g.*, \mathcal{C} defines the *computational basis* in quantum information applications). The control task is to engineer a QDS generator that stabilizes the maximally entangled "cat state":

$$\rho_d = \frac{1}{2} (|00\rangle + |11\rangle) (\langle 00| + \langle 11|).$$

In order to employ the synthesis techniques developed above, we consider a change of basis such that in the new representation $\rho_d = \text{diag}(1,0,0,0)$. A particularly natural choice is to consider the so-called *Bell basis*:

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \\ \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle), \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \right\}$$

Let U be the unitary matrix realizing the change of basis. In the Bell basis, which we use to build our controller, we consider a Hilbert space decomposition $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, where $\mathcal{H}_S = \text{span}\{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\}$, and $\mathcal{H}_R = \mathcal{H}_S^{\perp}$, and the associated matrix block decomposition.

as

Let us consider $M = \sigma_z \otimes \mathbb{I}$ in the canonical basis. It is easy to verify that $[M, \rho_d] \neq 0$. In the Bell basis, $M^B = UMU^{\dagger} = \mathbb{I} \otimes \sigma_x$, and $M_P = (0, 1, 0, 0)$. If, in this basis, we are able to implement the feedback Hamiltonian $F^B = \mathbb{I} \otimes \sigma_y$ (where now the tensor product should simply be meant as a matrix operation), we render ρ_d invariant, yet obtaining $L^B = M^B - iF^B$ with $L_P^B \neq 0$. Direct computation yields $F = U^{\dagger}(\mathbb{I} \otimes \sigma_y)U = \sigma_y \otimes \sigma_x$, back in the computational basis. With this choice, using the definitions in the proof of Theorem 2, we have:

$$\mathcal{H}_{R'} = \operatorname{span}\left\{\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\right\},\,$$

and $\mathcal{H}_{R'}$ is itself invariant. Hence, we need to produce a control Hamiltonian H_c to "destabilize" $\mathcal{H}_{R'}$. By inspection, we find that $\mathcal{H}_{R'}$ contains a proper subspace $\mathcal{H}_T = \operatorname{span}\{\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\}$ that supports an invariant and attractive state for the dynamics reduced to $\mathcal{H}_{R'}$. To "connect" this state to the attractive domain of ρ_d , we need a non-trivial Hamiltonian coupling between $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ and $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This may be obtained by a control Hamiltonian $H_c = \sigma_y \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_y$ in the standard basis – which completes the specification of the control strategy that renders ρ_d the unique attractive state for the dynamics. Notice that both the measurement and Hamiltonian compensation can be implemented locally, which may be advantageous in practice.

This example suggests how our results, obtained under CHC assumptions, may be interesting to explore the compatibility with existing control constraints. A further illustration comes from the following example.

Example 4: Consider again the above two-qubit system, but now imagine that we can only implement "non-selective" measurement and control Hamiltonians, *i.e.*, M, F, H must commute with the operation that swaps the qubit states. It is then natural to restrict attention to the dynamics in the three-dimensional subspace generated by the *triplet* states, which correspond to eigenvalue $\hbar^2 J(J+1), J = 1$, of the total spin angular momentum $J_{\alpha} = \frac{\hbar}{2} (\sigma_{\alpha} \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_{\alpha}), \alpha = x, y, z$ [10]:

$$\mathcal{H}_{J=1} = \operatorname{span}\left\{|00\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |11\rangle\right\}.$$

Notice that $\mathcal{H}_{J=1}$ corresponds to the fixed subspace with respect to the swap operation.

Our goal is to engineer a FME such that the maximally entangled state $\rho_d = \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)$ is attractive for the dynamics restricted to $\mathcal{H}_{J=1}$. Consider a collective measurement of spin along the *x*-axis, described by J_x . Upon reordering the triplet vectors so that in the new (primed) basis the *z*-projection ranges over 0, 1, -1 and $\rho_d = \text{diag}(1, 0, 0)$, we have:

$$J'_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix}.$$

Thus, in this basis we are looking for a feedback Hamiltonian of the form:

$$F' = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & i & i \\ -i & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

that is, F' corresponds to the non-selective operator $F = \frac{1}{\hbar}(J_z J_y + J_y J_z)$. Hence, by choosing $M' = J_x$ and F' as above we get:

$$L' = M' - iF' = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

This, with a choice of $H'_c = J_z$ suffices to make ρ_d attractive. In fact, considering the Hilbert space decomposition $\mathcal{H}_{J=1} = \mathcal{H}_S \oplus \mathcal{H}_R$, where $\mathcal{H}_S = \operatorname{span}\{\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\}$, and $\mathcal{H}_R = \mathcal{H}_S^{\perp}$, we find that the largest invariant subset in \mathcal{H}_R has support in $\mathcal{H}_{R'} = \operatorname{span}\{\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\}$. By observing that $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ is not an eigenstate of J_z , we infer that the chosen control parameters make ρ_d attractive in $\mathcal{H}_{J=1}$.

6 Conclusion

We have revisited the fundamental concepts of invariance and attractivity for quantum Markovian subsystems from a system-theoretic viewpoint. Building on the characterization of invariant subsystems and some partial results presented in [8], a linear-algebraic approach and Lyapunov's stability theory methods have provided us with an explicit characterization of attractive subspaces, along with an explicit attractivity test. In the special case of a single pure state, our results directly characterize the semigroup generators which support *state-preparation via dissipative Markovian dynamics*.

In the second part of the work, the conditions identified for subsystem invariance and attractivity have been exploited for designing Hamiltonian and output-feedback Markovian control strategies which actively achieve the intended quantum stabilization. In addition to a complete characterization of subspaces and subsystems that can be rendered attractive, our results include constructive recipes for synthesizing the required control parameters, which have been illustrated in simple yet paradigmatic examples. While our present analysis assumes perfect detection efficiency, a perturbative argument confirms that unique attractive states depend in a continuous fashion on the model parameters [8].

Further work is needed in order to establish feedback stabilization results which include finite bandwidth and detection efficiency, as well as simultaneous monitoring of multiple observables. In addition, the analysis of Markovian stabilization problems in the presence of control resources different and/or more constrained than assumed here appears especially well worth pursuing. For instance, as illustrated in the last examples, one may want to limit possible operations to local or collective observables of multipartite systems in various settings of physical relevance. The latter may include opto-mechanical systems, for which feedback control strategies based on homodyne detection have been considered before [37], or non-equilibrium many-body systems, for which preparation of a class of entangled states using "quasi-local" Markovian dissipation has recently being investigated in a physically motivated setting [38]. Additional investigation is also required to establish the full potential of Hamiltonian control and Markovian feedback in synthesizing not only invariant and attractive, but also *noiseless* structures [8]. This may point to yet new venues for producing protected realizations of quantum information in physical systems described by quantum Markovian semigroups.

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