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Journal Article

Author(s):
Subbotin, Maksim V.; Smith, Roy

Publication date:
2009-11

Permanent link:
https://doi.org/10.3929/ethz-b-000639811

Originally published in:
Automatica 45(11), https://doi.org/10.1016/j.automatica.2009.07.005

Design of Distributed Decentralized Estimators for Formations with Fixed and Stochastic Communication Topologies ⋆

Maksim V. Subbotin [a], Roy S. Smith [b]

[a] *LaunchPoint Technologies, Inc., 5735 Hollister Avenue, Suite B, Goleta, CA 93117, USA*

[b] *Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560, USA*

Published: M.V. Subbotin & R.S. Smith, "Design of Distributed Decentralized Estimators for Formations with Fixed and Stochastic Communication Topologies," *Automatica,* Vol. 45, No. 11, pp. 2491–2501, 2009. DOI: https://doi.org/10.1016/j.automatica.2009.07.005,

Abstract

This paper proposes a solution to the problem of synthesizing distributed decentralized estimators for a formation of agents. The collected dynamics of the formation are modeled by a discrete LTI system. In the considered estimation structure each agent of the formation carries an estimate of the entire formation state. Agents of the formation can communicate information between each other through unidirectional links modeled with a fixed or a stochastic communication topology. The design procedures are based on a set of convex optimization problems with linear matrix inequalities and result in the suboptimal choice of estimator gains which stabilize the estimation error dynamics and minimize a norm of the estimation error correlation matrix.

Key words: Estimation; Co-operative control; Decentralized systems; Markov models.

1 Introduction

Decentralized and distributed estimation and control problems have been a research focus for many years due to the variety of applications which require the use of decentralized control architectures with distributed measurements. Amongst the most interesting and challenging applications are formation control problems, sensor networks, distributed power systems, and vehicle platoons. Starting from the Witsenhausen (1968) example these problems have been shown to be challenging and have demanded the development of new techniques. A wide variety of papers devoted to the subject considered different problem formulations, performance requirements and constraints. Recent papers have addressed the issues of communication constraints and their influence on the system stability and performance. Fax and Murray (2004) explicitly related the communication topology to the formation stability. Tatikonda and Mitter (2004) considered the best achievable performance of a formation under communication constraints. Smith and Hadaegh (2007) analyzed the stability properties of a decentralized estimator and derived the relationship between the eigenvalues of the estimation error dynamics and the communication topology of a formation. Yan, Kang, and Bitmead (2005) and Yan and Bitmead (2003) considered a coordinated control problem for a formation of vehicles and offered an estimator design procedure for a class of decoupled linear systems with a particular communication architecture. Gupta, Hassibi, and Murray (2005) considered multi-sensor fusion with packet drops and developed an algorithm for the choice of transmitted data which optimizes the state estimate. Xu and Hespanha (2004, 2005) investigated the decentralized estimation problem with multiple smart sensors which optimally process local measurements and send data through a network to a remote estimator. Jiang,

⋆ This paper was not presented at any IFAC meeting. Corresponding author R. S. Smith. Tel. +1–805–893–2967.

Email addresses: `subbotin@engineering.ucsb.edu` (Maksim V. Subbotin), `roy@ece.ucsb.edu` (Roy S. Smith).

Voulgaris, and Neogi (2007) and Yadav, Voulgaris, and Salapaka (2005) developed techniques for the synthesis of distributed controllers for decoupled and triangular systems.

In this paper we focus on the estimation part of the problem and develop synthesis tools for the design of distributed estimators able to provide state information to the decentralized control system. We propose estimator synthesis procedures for the type of systems represented by a collection of agents composing a formation and coupled by a common objective function. This type of system was considered by Smith and Hadaegh (2007), where the authors solved the stability analysis part of the problem. The proposed formulation does not impose any assumptions on the structure of the formation system dynamics and hence allows us to consider general LTI systems with arbitrary dynamic coupling between agents. We consider decentralized and distributed estimation structures in which each agent of the formation carries an estimate of the entire formation state. This estimation architecture requires more computational power from each agent than other proposed schemes, for example the one in (Fax and Murray, 2004), but is essential for decentralized control problems with high-performance formation-wide objective functions. These include precise formation-keeping, formation reconfiguration and collision avoidance without a centralized controller. In addition, the overall robustness of the formation to faults and uncertainties may be improved as a result of the redundancy in state estimates. Our work was particularly motivated by a problem of precisely controlled coordination for spacecraft formation flying applications. This type of problem has been described for several formation flying interferometers (Bik, Visser and Jennrich, 2007), (Shipley et al., 2002). For these systems the control objective is specified in terms of the entire formation state in relative coordinates and requires each agent of the formation to adjust its actions and monitor the behavior of other agents through the local agent's formation state estimate.

Communication was shown to play a central role in defining stability properties and performance limitations in formations (Fax and Murray, 2004), (Tatikonda and Mitter, 2004). In the presence of distributed and parallel computation and control it allows agents to reach some level of consensus and agree on cooperative action. In this paper we consider formations with arbitrary, but specified, communication topologies, and hence explicitly consider systems with limited communication. From a graph theoretical point of view each agent's estimator is viewed as a node of a directed graph specified by a Laplacian.

Throughout the paper we consider two models for inter-agent communication links, and develop in parallel synthesis tools for both models. The first model resembles an analog communication link where the transmitted information is corrupted with zero-mean Gaussian noise. The second model describes a discrete channel which can deliver transmitted information without any corruption, but information may be lost. To capture the stochastic nature of the data loss in the discrete channel we model it with a two-state Markov chain. This results in a Markov communication topology that can be studied via the results of Costa and Fragoso (1993) and Costa and Guerra (2002).

The main results of this paper are two design procedures for the synthesis of the suboptimal gains of the distributed decentralized estimator. The methods developed in this paper have been presented in our preliminary work, (Subbotin and Smith, 2007(a)), (Subbotin and Smith, 2007(b)). As the performance measure in the design procedures we use a norm of an estimation error correlation matrix. Utilizing the recent results of de Oliveira, Bernussou, and Geromel (1999) and de Oliveira, Geromel, and Bernussou (2002), we propose linear matrix inequality (LMI) based synthesis methods for the estimator design. Using the results of de Oliveira et al. makes it possible to impose the structural constraints induced by the limited communication on the design variables without imposing conservative structural constraints on the performance measure matrices. For formations without noise in the communication signals and no restrictions on the communication signals' dimensions we can formulate a convex optimization problem. When noise is present in the communication signals, the design variables appear in the equations in a way which leads to a bilinear optimization problem. In both situations the intermediate variables used to find the estimator gains introduce some conservatism in the design and result in the suboptimal choice of the gains even though the individual problems appear to be convex.

The rest of the paper consists of three main parts. In Section two we describe the class of systems we consider throughout the paper, introduce the notation and variables we use, and describe the two communication models. In Section three we develop the tools necessary for the estimator design and describe the synthesis procedures. In the fourth section we present experimental results for a formation with three agents. Our experiments show how to apply the developed techniques for a practical system.

Throughout this paper we use the following notation. Letters i and j are used primarily for indexing vectors and matrices. The symbol $\otimes$ is used to denote the Kronecker product for matrices. The identity matrix with dimension $n \times n$ is defined as I_n and a column vector with the dimension n and all elements equal to 1 is defined as 1_n. A block diagonal matrix B with submatrices $B_i,\ i = 1, ..., n$ on the diagonal is denoted by $B \equiv \mathrm{diag}(B_1, ..., B_n)$ or $B \equiv \mathrm{diag}_i(B_i)$. The matrix B' is the transpose of B.

2 Problem formulation

We consider discrete LTI systems described by,

$$x(k+1) = Ax(k) + B_u u(k) + B_v v(k), \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the system state, $u(k) \in \mathbb{R}^{m_u}$ is the actuation input, and $v(k) \in \mathbb{R}^{m_v}$ is a zero-mean, Gaussian process noise with covariance Q_v. The state dynamics (1) represent the collected formation dynamics of N vehicles; the agents of the formation. The control input is composed of individual control inputs of each agent, $u(k) = \sum_{i=1}^{N} u_i(k)$, and the i^{th} agent's control signal, which corresponds to the control of local actuators, is defined by $u_i(k) = \Pi_i u(k) \in \mathbb{R}^{m_u}$, where Π_i is the projection matrix and $\sum_{i=1}^{N} \Pi_i = I$. Each agent is able to measure the signal, $y_i(k) = C_i x(k) + n_i(k)$, where $y_i(k) \in \mathbb{R}^{k_{y_i}}$ is the system output available to agent i, and $n_i(k) \in \mathbb{R}^{k_{y_i}}$ is a zero-mean Gaussian measurement noise with covariance Q_{n_i}.

We assume that a stabilizing state feedback, $u(k) = -Kx(k)$, which satisfies a formation-wide objective function, is given and specifies the desired closed-loop dynamics of the formation through the matrix $A_{clp} = A - B_u K$. Since we focus on the estimation part of the problem, we do not consider a particular method for the choice of K. The formation control law is calculated and implemented by each agent individually using available measurements and information transmitted from other agents, resulting in a decentralized and distributed architecture. Each agent's control system is a combination of a full-order formation state estimator which provides $\hat{x}_i(k) \in \mathbb{R}^{n_x}$ and state feedback for the calculation of $u_i(k)$. As a result, the i^{th} agent's contribution to the control input is given by, $u_i(k) = -\Pi_i K \hat{x}_i(k)$. Each agent's estimate of the system state contains errors and also differs from every other agent's estimate. Furthermore each agent updates its estimate of the system state using, in part, an estimate of every other agent's control action. The estimated control actions also contain errors and this results in a coupling of the estimation error dynamics of the agents. This coupling prevents the use of more standard estimator design methods. See (Smith and Hadaegh, 2007) for a detailed analysis of this error dynamics coupling.

If we define each agent's estimation error as, $e_i(k) \equiv x(k) - \hat{x}_i(k)$, then the closed-loop plant dynamics can be written as,

$$\begin{aligned} x(k+1) &= Ax(k) - B_u \sum_{i=1}^{N} \Pi_i K \hat{x}_i(k) + B_v v(k) \\ &= A_{clp} x(k) + B_u \sum_{i=1}^{N} \Pi_i K e_i(k) + B_v v(k). \end{aligned} \tag{2}$$

2.1 Fixed communication topology with noise

In this section we consider an analog communication model, which is described with a fixed communication topology and additive noise in the signals transmitted between agents. Here we assume that the information transferred through an individual unidirectional communication link can be represented by,

$$t_{ij}(k) = H_{ij} \hat{x}_j(k) + w_{ij}(k), \tag{3}$$

where $t_{ij}(k) \in \mathbb{R}^{k_{ij}}$ is the signal received by estimator i from estimator j and $w_{ij}(k) \in \mathbb{R}^{k_{ij}}$ is a zero-mean, Gaussian communication noise with variance $Q_{w_{ij}}$. We assume that the transmitter gain matrix, H_{ij}, is to be designed, while its row dimension specified by k_{ij} is given. This formulation is motivated by the fact that the number of channels in an individual communication link may be limited and the limitation is reflected in k_{ij}. Specifying the transmission gains, H_{ij}, as design variables gives a potentially higher performance network as the transmission gains are chosen with respect to both the communication noise properties and the overall estimation performance objective. It is easy to modify this formulation to consider more specific communicated information. For example: full estimate transmission ($H_{ij} = I$); measurement transmission ($t_{ij}(k) = y_j(k-1)$); or actuation transmission ($H_{ij} = -\Pi_j K$); giving sub-optimal but simpler estimator design problems.

The i^{th} agent's full order formation state estimator updates its estimate according to the following model,

$$\begin{aligned} \hat{x}_i(k+1) &= A_{clp} \hat{x}_i(k) + L_i(y_i(k) - C_i \hat{x}_i(k)) \\ &\quad + \sum_j F_{ij}(t_{ij}(k) - H_{ij} \hat{x}_i(k)), \end{aligned} \tag{4}$$

where L_i is an estimator gain related to the agent's local measurements, and F_{ij} is a receiver gain matrix which corresponds to the transmitter gain matrix H_{ij} and the sum is taken over all received signals. We consider both F_{ij} and H_{ij} as design variables in the estimator synthesis problem. This structure for applying the information communicated from other estimators preserves the separation between the collected error dynamics and the closed-loop plant dynamics.

We can now consider the dynamics of this estimation error. From (2) and (4) we get,

$$\begin{aligned} e_i(k+1) &= (A_{clp} - L_i C_i) e_i(k) \; + \; B_u \sum_{l=1}^{N} \Pi_l K e_l(k) \\ &\quad - \sum_j F_{ij} H_{ij}(e_i(k) - e_j(k)) + B_v v(k) \\ &\quad - \; L_i n_i(k) \; - \; \sum_j F_{ij} w_{ij}(k), \end{aligned} \tag{5}$$

where the summations with index j are again taken over all received signals. We now introduce a graph-theoretic based notation to simply these summations when considering the collected estimators' errors.

To specify the communication links each agent of the formation is considered to be a node of a graph in a fixed communication topology. We specify the topology using Laplacian matrices, $\mathcal{L}_j \in I\!R^{N\times N}$, and transmission indicator matrices, $\mathcal{T}_j \in I\!R^{\sum_{i=1}^{N} k_{ij} \times \sum_{i=1}^{N} k_{ij}}$, $j = 1, ..., N$. Each Laplacian, $\mathcal{L}_j$, and transmission indicator matrix, $\mathcal{T}_j$, specifies communication links between the j^{th} agent and all other agents of the formation. The Laplacian is defined as follows: elements l_{sk} of the Laplacian, $\mathcal{L}_j$, satisfy, $l_{sj} = -1$, $l_{ss} = 1$ if there is a communication link from agent j to agent s and $l_{sk} = 0$ otherwise. Similarly the transmission indicator matrix is defined to be $\mathcal{T}_j = \mathrm{diag}(\delta_{1j} I_{k_{1j}},\ \delta_{2j} I_{k_{2j}},\ ...,\ \delta_{Nj} I_{k_{Nj}})$ and $\delta_{ij} = 1$ if there is communication from agent j to agent i, and $\delta_{ij} = 0$ otherwise.

To specify the communication topology of the whole formation, we introduce collected matrix variables: the collected Laplacian,

$$\mathcal{L}_f = \begin{bmatrix} \mathcal{L}_1 \otimes I_{n_x} \\ \mathcal{L}_2 \otimes I_{n_x} \\ \vdots \\ \mathcal{L}_N \otimes I_{n_x} \end{bmatrix} \in I\!R^{N^2 n_x \times N n_x}, \qquad (6)$$

and the collected transmission indicator matrix,

$$\mathcal{T}_f = \mathrm{diag}(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_N) \\ \in I\!R^{\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij} \times \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij}}. \qquad (7)$$

We also introduce the collected receiver gain matrix,

$$F_f = [F_1 \;\; F_2 \; ... \; F_N] \in I\!R^{N n_x \times \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij}}, \qquad (8)$$

where $F_j = \mathrm{diag}(F_{1j}, F_{2j}, ..., F_{Nj})$ contains the receiver gains of agents receiving signals from the j^{th} agent, $F_{ij} \in I\!R^{n_x \times k_{ij}}$. The final definition required is the collected transmitter gain matrix,

$$H_f = \mathrm{diag}(H_1, H_2, ..., H_N) \in I\!R^{\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij} \times N^2 n_x}, \qquad (9)$$

where $H_j = \mathrm{diag}(H_{1j}, H_{2j}, ..., H_{Nj})$ contains the transmitter gains of the j^{th} agent, and $H_{ij} \in I\!R^{k_{ij} \times n_x}$.

If we collect estimation errors from all estimators—with each specified by (5)—in one vector we get, $e(k) = [e_1(k)' \; e_2(k)' \; ... \; e_N(k)']' \in I\!R^{N n_x}$, and we can express the collected estimation error dynamics as,

$$e(k+1) = (A_B - L_f C_f - F_f H_f \mathcal{L}_f) e(k) \\ + \Gamma v(k) - L_f n(k) - F_f \mathcal{T}_f w(k), \qquad (10)$$

where $A_B = I_N \otimes A_{clp} + S_N$, $S_N = 1_N \otimes S_1$, with $S_1 = [B_u \Pi_1 K \; ... \; B_u \Pi_N K]$. The collected estimator measurement update gains have been defined as,

$$L_f = \mathrm{diag}(L_1, ..., L_N), \qquad (11)$$

and the collected measurement matrices are $C_f = \mathrm{diag}(C_1, ..., C_N)$. We also collect the noise inputs via $\Gamma = 1_N \otimes B_v$, $n(k) = [n_1(k)' \; ... \; n_N(k)']' \in I\!R^{\sum_{i=1}^{N} k_{y_i}}$, and $w(k) \in I\!R^{\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij}}$ is a collected communication noise with the corresponding covariance, $Q_w = \mathrm{cov}(w(k))$.

Equation (10) gives the estimation error dynamics for all agents in the formation and allows us to specify separate transmitter and receiver gain pairs, H_{ij}, F_{ij}, and channel dimensions for each possible link. To maintain the simplicity of the equations, if there is no communication link from agent j to agent i, we assign that link to have a channel dimension, $k_{ij} = 1$. To ensure that this fictitious link does not influence the estimation error dynamics, we also impose the constraints, $F_{ij} = 0 \in I\!R^{n_x \times 1}$ and $H_{ij} = 0 \in I\!R^{1 \times n_x}$. The noise terms associated with fictional links have no influence on the design or analysis problems as the corresponding Q_w will enter the following equations as $\mathcal{T}_f Q_w \mathcal{T}_f'$ where it is eliminated via multiplications with zero in $\mathcal{T}_f$.

The collected estimation error dynamics (10) together with (2) completely specify the closed-loop dynamics for the formation with N parallel distributed estimators exchanging the information between each other according to a fixed topology defined by $\mathcal{L}_f$ and the signal model represented in (3). From (2) and (10) we can see that the estimation error dynamics are decoupled from the closed-loop plant dynamics, while the latter are driven by the estimation error. While the plant dynamics are assigned by a particular choice of K, the estimation error dynamics determined by K and the gains L_f, H_f, and F_f.

In the sequel we will pose distributed estimator design problems with L_f, H_f, and F_f (defined in (11), (9) and (8)) as the design variables. By definition, these variables are constrained to have a sparse structure. Additional structural constraints are imposed in the case where a link is not present in the communication topology. It will be seen that these structural constraints can introduce some conservatism in the design.

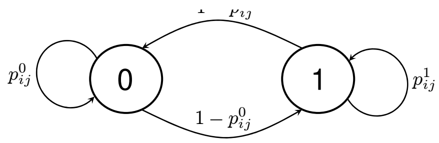

Fig. 1. Two-state Markov chain modeling the communication link packet loss probability.

2.2 Markov communication topology

In this section we consider an alternative model for the communication links. Each of the communication link is able to carry vectors of real numbers without any corruption, but the information transmitted through a link can—with a certain probability—be lost. This type of model is often used in the literature (Hespanha, Naghshtabrizi, and Xu, 2005) to describe lossy digital packet communication channel. With this motivation in mind, we consider the following model for a single unidirectional communication link used to transmit information from agent j to agent i,

$$t_{ij}(k) = \mu_{ij}(k) H_{ij} \hat{x}_j(k), \tag{12}$$

where $t_{ij}(k) \in I\!R^{k_{ij}}$ is the signal received by the estimator of the i^{th} agent from the estimator of the j^{th} agent. $H_{ij} \in I\!R^{k_{ij} \times n_x}$ is the transmitter gain matrix, and $\mu_{ij}(k)$ is a binary parameter describing the success of transmission: $\mu_{ij}(k) = 1$ if the data is successfully received and $\mu_{ij}(k) = 0$ if not. The binary parameter $\mu_{ij}(k)$ modeling success or failure of the transmission can be defined to be either a stochastic or deterministic variable. The most common and widely accepted method for specifying $\mu_{ij}(k)$ is modeling it as a stochastic variable described by either a Bernoulli process or a finite-state Markov chain (see Gilbert (1960), Elliott (1963) and Hespanha, Naghshtabrizi, and Xu (2005) and the references therein). In this section we consider the Gilbert model, where $\mu_{ij}(k)$ is described by a two-state Markov chain represented graphically in Figure 1. State 0 corresponds to a failure in a link and state 1 corresponds to a successful transmission. Transition probabilities p_{ij}^0 and p_{ij}^1 describe probabilities of staying in state 0 and state 1 correspondingly. Modeling $\mu_{ij}(k)$ with a Bernoulli process results in a simpler design problem than the one considered here.

Assume that there are N_l unidirectional links in the topology describing communication between agents' estimators in the formation. The fact that each of N_l agents received or did not receive its corresponding information at step k can be described by a vector $\Theta(k) = [\Theta_1(k)\ \Theta_2(k)\ ...\ \Theta_{N_l}(k)] \in I\!R^{N_l}$, where each element of the vector is equal to the state of the Markov chain $\mu_{ij}(k)$ for the corresponding link in the communication topology. If the transitions between the states of the Markov chains for individual links are independent, then $\Theta(k)$ is itself an element of a finite-state Markov chain with $M = 2^{N_l}$ states, since each element of the $\Theta(k)$ vector can take one of two values, 0 or 1, independently of other elements of the vector. To define the M-state Markov chain modeling communication, we introduce a state $\theta(k)$ which takes values in $\{1, ..., M\}$ and corresponds to one of the M possible states of $\Theta(k)$. We also define $\pi_s(k) = P\{\theta(k) = s\}$, $s = 1, ..., M$, a probability of being in state s at time k, and $\mathcal{P} \in I\!R^{M \times M}$, a transition probability matrix. The elements of matrix $\mathcal{P}$, p_{st}, $s = 1, ..., M$, $t = 1, ..., M$ can be calculated using the transition probabilities of Markov chains for individual links, p_{ij}^0, p_{ij}^1, by simply taking the products of M probabilities describing transition from state $\theta(k)$ to $\theta(k+1)$. For later derivations we define a row vector of the probability distribution for the states of the chain, $\pi(k) = [\pi_1(k)\ \pi_2(k)\ ...\ \pi_M(k)] \in I\!R^M$, and its evolution is described by,

$$\pi(k+1) = \pi(k)\mathcal{P}. \tag{13}$$

Using the proposed communication model we derive new equations for the collected estimation error dynamics. Each agent of the formation receiving information from other agents' estimators updates its formation state estimate according to the following model. For the ith agent the update equation is:

$$\begin{aligned} \hat{x}_i(k+1) = {} & A_{clp}\hat{x}_i(k) + L_i(y_i(k) - C_i\hat{x}_i(k)) \\ & + \sum_j \mu_{ij}(k) F_{ij}(t_{ij}(k) - H_{ij}\hat{x}_i(k)), \end{aligned} \tag{14}$$

where F_{ij} is the receiver gain matrix which corresponds to the transmitter gain matrix H_{ij} and the sum is taken over all received signals. After the time interval allocated for the transmission, each i^{th} agent of the formation knows if it received or did not receive the information from the other agents. This fact is reflected in binary variable $\mu_{ij}(k)$ present in the update equation, so if agent i did not receive information from agent j, then $\mu_{ij}(k) = 0$ and $\mu_{ij}(k) = 1$ otherwise. Note that no acknowledgment signals are sent, so the transmitting agent j has no information about $\mu_{ij}(k)$.

The Laplacians $\mathcal{L}_j$ and the collected Laplacian $\mathcal{L}_f$ are now stochastic variables described by the states of the Markov chain, $\theta(k)$. To emphasize this fact we further use the notation $\mathcal{L}_{j\{\theta(k)\}}$ and $\mathcal{L}_{f\{\theta(k)\}}$ for this type of communication model. So for any given choice of Markov state, $\theta(k) = i$, $i = 1, ..., M$, the corresponding collected Laplacian $\mathcal{L}_{f\{\theta(k)=i\}}$ carries the information about successful or failed links in the specified communication topology.

Using this notation, the collected estimation error dynamics equation can be written compactly as,

$$
\begin{aligned}
e(k+1) &= (A_B - L_f C_f - F_f H_f \mathcal{L}_{f\{\theta(k)\}})e(k) \\
&\quad + \Gamma v(k) - L_f n(k) = \bar{A}_{\{\theta(k)\}} e(k) + \bar{B}\bar{u}(k),
\end{aligned} \tag{15}
$$

where

$$
\bar{A}_{\{\theta(k)\}} = A_B - L_f C_f - F_f H_f \mathcal{L}_{f\{\theta(k)\}}, \tag{16}
$$

$\bar{B} = [\Gamma \ -L_f]$, and $\bar{u}(k) = [v(k)' \ n(k)']'$. The collected estimation error dynamics (15) together with (2) describe the complete closed-loop dynamics of the formation with N agents and estimators exchanging the information between each other through links modeled with the 2-state Markov chains. In this formulation the estimation error dynamics are determined by the choice of the design variables L_f, F_f, H_f, and the Laplacian $\mathcal{L}_{f\{\theta(k)\}}$.

3 Estimator Design

In this section we develop the tools and procedures for the synthesis of decentralized distributed estimators. As a performance measure in the design procedures we use an estimation error correlation matrix.

3.1 Fixed communication topology with noise

First, we present our results on the synthesis of decentralized estimator for a formation of agents with the communication model described in Section 2.1. With the assumption that estimation error dynamics (10) are stable we can write the propagation equation for the estimation error correlation as,

$$
\begin{aligned}
&P(k+1) \equiv \operatorname{cov}(e(k+1)) = E(e(k+1)e(k+1)') = \\
&(A_B - L_f C_f - F_f H_f \mathcal{L}_f)P(k)(A_B - L_f C_f - F_f H_f \mathcal{L}_f)' \\
&\quad + \Gamma Q_v \Gamma' + L_f Q_n L_f' + F_f \mathcal{T}_f Q_w \mathcal{T}_f' F_f'.
\end{aligned} \tag{17}
$$

The stability assumption on the estimation error dynamics is natural, because in our design problem we are looking for the stabilizing combination of the estimator gains, L_f, H_f, F_f. We consider the time-invariant case, which corresponds to the steady-state solution of (17), $P \equiv P(k) = P(k+1)$, and should satisfy,

$$
\begin{aligned}
&P - (A_B - L_f C_f - F_f H_f \mathcal{L}_f)P(A_B - L_f C_f - F_f H_f \mathcal{L}_f)' \\
&\quad - \Gamma Q_v \Gamma' - L_f Q_n L_f' - F_f \mathcal{T}_f Q_w \mathcal{T}_f' F_f' = 0.
\end{aligned} \tag{18}
$$

At this point we can formulate an optimization problem to find the structured estimator gain matrix, L_f, receiver and transmitter gain matrices, F_f and H_f, which stabilize the estimation error dynamics (10) and minimize an upper bound, X, of the estimation error correlation, $X \geq P$. This problem is formulated as follows:

$$
\begin{aligned}
&\underset{X, L_f, F_f, H_f}{\text{minimize}} \ \|X\|, \quad \text{subject to:} \\
&\qquad X = X' > 0, \\
&X - (A_B - L_f C_f - F_f H_f \mathcal{L}_f)X(A_B - L_f C_f - F_f H_f \mathcal{L}_f)' \\
&\quad - \Gamma Q_v \Gamma' - L_f Q_n L_f' - F_f \mathcal{T}_f Q_w \mathcal{T}_f' F_f' > 0,
\end{aligned} \tag{19}
$$

where L_f satisfies a block-diagonal structural constraint (given in (11)), and F_f, H_f satisfy the corresponding structural constraints (given in (8) and (9)).

It is important to mention that some limitation should be imposed on the available choices of the transmitter gain, H_f, in the design procedure. The inequality (19) does not adequately constrain H_f as it only enters as a product with the F_f design variable. The $F_f \mathcal{T}_f Q_w \mathcal{T}_f' F_f'$ can be made arbitrarily small while H_f is chosen to make the product $H_f F_f$ constant. In any practical problem the transmitter should have a limited power. It is appealing to apply a direct norm bound to H_f but this does not allow us to both maintain convexity and use the structured design results given below. So in order to incorporate some form of H_f constraint in the design procedure we introduce an implicit weight for the signals transmitted between agents.

To this end, we consider the problem of minimizing the $\mathcal{H}_2$ norm of the transfer matrix $T_{z\bar{u}}(\cdot)$ for the following system,

$$
\begin{aligned}
e(k+1) &= \bar{A}e(k) + \bar{B}\bar{u}(k), \\
z(k) &= \bar{C}e(k),
\end{aligned} \tag{20}
$$

where $\bar{A} = A_B - L_f C_f - F_f H_f \mathcal{L}_f$, $\bar{B} = [\Gamma \ -L_f \ -F_f \mathcal{T}_f]$, $\bar{u}(k) = [v(k)' \ n(k)' \ \omega(k)']'$,

$$
\bar{C} = \begin{bmatrix} I \\ \gamma \bar{H}_f \end{bmatrix},
$$

$\bar{H}_f = [H_1' \ H_2' \ldots H_N']'$, and $\gamma > 0$ is a positive constant which can be used to add weight on the corresponding part of the output vector. Clearly, (20) represents a system with the required estimation error dynamics (10). The output, $z(k)$, contains the estimation error, $e(k)$, and is augmented with $\gamma \bar{H}_f e(k)$ in order to weight the transmitter gains, H_j. With this notation the transfer matrix from $\bar{u}(k)$ to $z(k)$ is defined as $T_{z\bar{u}}(\zeta) = \bar{C}(I\zeta - \bar{A})^{-1}\bar{B}$. Minimizing the $\mathcal{H}_2$ norm of $T_{z\bar{u}}$, weighted by the covariances on the noise and disturbance inputs, has the required effect of minimizing a combination of $\|X\|$ and the H_f gains.

An important feature of inequality (19)—and standard optimization approaches for minimizing $\|T_{z\bar{u}}\|_2$—is that the unknown design variables, F_f and H_f, enter both as a product and individually. This prevents us from formulating this as a standard LMI design for $\mathcal{H}_2$ minimization. Direct application of the matrix inequality constraints in this case leads to a biaffine matrix inequality (BMI). We instead propose an alternative iterative formulation. This will involve iterating between an *H-problem* (minimizing over H_f when F_f and L_f are considered fixed) and an *F-problem* (minimizing over F_f and L_f when H_f is considered fixed).

We now outline the results required to establish the LMIs to be used in the iterative procedure. The following result, due to de Oliveira, Geromel, and Bernussou (2002), allows the formulation of the $\mathcal{H}_2$ norm minimization problem as an LMI.

Lemma 1 *The inequality* $\|T_{z\bar{u}}\bar{Q}^{1/2}\|_2^2 < \mu$ *holds if, and only if, there exists a matrix* G *and symmetric matrices* X *and* W *such that,* $\text{trace}(W) < \mu$ *and*

$$\begin{bmatrix} X & \bar{A}G & \bar{B} \\ G'\bar{A}' & G+G'-X & 0 \\ \bar{B}' & 0 & \bar{Q}^{-1} \end{bmatrix} > 0, \tag{21}$$

$$\begin{bmatrix} W & \bar{C}G \\ G'\bar{C}' & G+G'-X \end{bmatrix} > 0, \tag{22}$$

where $\bar{Q} = \text{diag}(Q_v,\ Q_n,\ Q_w)$ *is the input weighting matrix.*

To motivate the application of Lemma 1 to the design procedure, we observe that the inequality (19) is equivalent to the LMI (21) by Theorem 1 in de Oliveira, Bernussou, and Geromel (1999), and hence, the variable X in both inequalities represents the upper bound on the estimation error correlation matrix, P. As a consequence, minimizing the $\mathcal{H}_2$ norm of $T_{z\bar{u}}(\cdot)\bar{Q}^{1/2}$ (achieved by minimizing its bound, μ in Lemma 1), minimizes the trace of W and, as a consequence, minimizes a norm of X, the upper bound on P. The presence of the slack variable G, allows us to impose structural constraints on the design variables, L_f, F_f, and H_f, without imposing any structural constraints on X.

To use the LMIs (21) and (22) for the design of H_f, we define a structured variable,

$$G \equiv \text{diag}(\bar{G}, ..., \bar{G}) \in I\!R^{Nn_x \times Nn_x}, \tag{23}$$

and observe that, due to the matching sparsity of $\mathcal{L}_f$ and G, we have $H_f\mathcal{L}_f G = E_d\mathcal{L}_f$, where,

$$\begin{aligned} &E_d \equiv \text{diag}(E_1, ..., E_N) \\ &= \text{diag}(H_1 G, ..., H_N G) \in I\!R^{\sum_{i=1}^N \sum_{j=1}^N k_{ij} \times N^2 n_x}. \end{aligned} \tag{24}$$

Note that the communication topology constraints of the form $H_{ij} = 0$ are are now reflected as constraints on blocks of E_d. Define a new matrix variable,

$$E_c \equiv [E_1' \ ...\ E_N']' \in I\!R^{\sum_{i=1}^N \sum_{j=1}^N k_{ij} \times Nn_x}, \tag{25}$$

where each E_j has same dimensions and structure as the corresponding H_j, $E_j \in I\!R^{\sum_{i=1}^N k_{ij} \times Nn_x}$. With these variables we rewrite (21) as,

$$\begin{bmatrix} X & \bar{A}_G & \bar{B} \\ \bar{A}_G' & G+G'-X & 0 \\ \bar{B}' & 0 & \bar{Q}^{-1} \end{bmatrix} > 0, \tag{26}$$

where $\bar{A}_G = A_B G - L_f C_f G - F_f E_d \mathcal{L}_f$, and (22) as,

$$\begin{bmatrix} W & \begin{bmatrix} G \\ \gamma E_c \end{bmatrix} \\ \begin{bmatrix} G' & \gamma E_c' \end{bmatrix} & G+G'-X \end{bmatrix} > 0. \tag{27}$$

The inequalities (26) and (27) are linear in the variables X, W, G, and E_j, $j = 1, ..., N$. If we consider L_f, F_f, and γ to be fixed, we can pose the design of the transmitter gains, H_{ij}, as the following convex optimization problem.

H-problem:

$$\min_{X,W,G,E_j} \mu, \quad \text{subject to:}$$

$$\text{trace}(W) < \mu,$$

Inequalities (26) and (27),and the structural constraints on $G, E_d,$ and E_c (given in (23), (24) and (25)).

The transmitter gains are then calculated via $H_j = E_j G^{-1}$.

For the *F-problem* part of the design procedure we need LMIs dual to those used in Lemma 1. For this purpose we state the following lemma.

Lemma 2 *There exists a matrix* G*, and symmetric matrices* X *and* W*, such that,* $\text{trace}(W) < \mu$ *and (21), (22) hold, if and only if, there exists a matrix* D*, and a symmetric matrix* Y*, such that* $\text{trace}(W) < \mu$ *and*

$$\begin{bmatrix} Y & \bar{A}'D' & 0 \\ D\bar{A} & D+D'-Y & D\bar{B} \\ 0 & \bar{B}'D' & \bar{Q}^{-1} \end{bmatrix} > 0, \tag{28}$$

$$\begin{bmatrix} W & \bar{C} \\ \bar{C}' & Y \end{bmatrix} > 0. \tag{29}$$

Proof. We prove the necessity part of the lemma; the sufficiency part can be proven in a similar way. Assume that (21) and (22) are feasible with the matrices $X = X'$, $W = W'$, G and trace$(W) < \mu$. Since (21) is satisfied, $X = X' > 0$ and $G + G' > X > 0$. Hence G is nonsingular and we can define a nonsingular matrix

$$T_G = \begin{bmatrix} 0 & G^{-1} & 0 \\ G^{-1} & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

where the identity matrix, I, has the same dimension as $\bar{Q}$. Premultiplying (21) by T_G' and postmultiplying by T_G, we arrive at,

$$\begin{bmatrix} G^{-1}+G'^{-1}-G'^{-1}XG^{-1} & \bar{A}'G^{-1} & 0 \\ G'^{-1}\bar{A} & G'^{-1}XG^{-1} & G'^{-1}\bar{B} \\ 0 & \bar{B}'G^{-1} & \bar{Q}^{-1} \end{bmatrix} > 0.$$

With new matrix variables $Y = Y' = G^{-1} + G'^{-1} - G'^{-1}XG^{-1}$ and $D = G'^{-1}$, we observe that the above inequality is equal to (28). If we define a nonsingular matrix

$$T_{GW} = \begin{bmatrix} I & 0 \\ 0 & G^{-1} \end{bmatrix},$$

and premultiply (22) by T_{GW}' and postmultiply it by T_{GW}, then we arrive at (29) with $Y = G^{-1} + G'^{-1} - G'^{-1}XG^{-1}$. Hence, the necessity part of the lemma is proven. □

Now, we use the LMIs of Lemma 2 to design the estimator gain, L_f, along with the receiver gain, F_f, for the case when H_f is considered to be fixed. We define the variables,

$$D \equiv \operatorname{diag}(\bar{D}, ..., \bar{D}) \in I\!R^{Nn_x \times Nn_x}, \tag{30}$$

$$R \equiv DL_f = \operatorname{diag}(R_1, ..., R_N) \in I\!R^{Nn_x \times \sum_{i=1}^{N} k_{y_i}} \tag{31}$$

and

$$\begin{aligned} E &\equiv DF_f \\ &= [E_1 \;\; E_2 \; ... \; E_N] \in I\!R^{Nn_x \times \sum_{i=1}^{N} \sum_{j=1}^{N} k_{ij}}. \end{aligned} \tag{32}$$

Note that communication topology constraints of the form $F_{ij} = 0$ are now reflected as block structural constraints on E. Using these definitions, rewrite (28) as,

$$\begin{bmatrix} Y & \bar{A}_D & 0 \\ \bar{A}_D' & D + D' - Y & \bar{B}_D \\ 0 & \bar{B}_D' & \bar{Q}^{-1} \end{bmatrix} > 0, \tag{33}$$

where $\bar{A}_D = A_B'D' - C_f'R' - \mathcal{L}_f'H_f'E'$ and $\bar{B}_D = [D\Gamma - R \; - E\mathcal{T}_f]$. We also rewrite (29) as,

$$\begin{bmatrix} W & \begin{bmatrix} I \\ \gamma\bar{H}_f \end{bmatrix} \\ \begin{bmatrix} I & \gamma\bar{H}_f' \end{bmatrix} & Y \end{bmatrix} > 0. \tag{34}$$

In this notation it is clear that (33) and (34) are linear in the variables Y, W, D, R, and E. If γ and H_f, correspondingly $\bar{H}_f$, are considered to be fixed then we can pose the following a convex optimization problem for the design of estimator gains, L_f, and the receiver gains, F_{ij}.

F-problem:

$$\underset{Y,W,D,R,E}{\text{minimize}} \; \mu, \quad \text{subject to:}$$

$$\operatorname{trace}(W) < \mu,$$

Inequalities (33) and (34) and the structural constraints on $D, R,$ and E (given in (30), (31) and (32)).

The unknown gains are then calculated via $L_f = D^{-1}R$ and $F_f = D^{-1}E$.

The *H-problem* and the *F-problem* can be combined in an iteration to give suboptimal choice of L_f, H_f, and F_f. To initialize the iteration, we choose a constant weight $\gamma > 0$ and make an initial guess for the matrix containing the transmitter gains, H_f. The initial choice of H_f should satisfy the previously defined structural constraints given in (9), in addition to the $H_{ij} = 0$ constraints imposed if there is no communication from agent j to agent i. At each step we solve two optimization problems. First, for the fixed transmitter gain matrix H_f, we solve the *F-problem* and find F_f and L_f. Second, by holding fixed these values of F_f and L_f, we solve the *H-problem* and find a new H_f to be used at the next step of iteration. This iteration can be repeated a fixed number of times or until the difference between the costs μ at the consecutive steps is small.

To guarantee the convergence of the proposed iterative procedure, we have to show that the cost function does not increase at each step and feasibility of each constraint in *F-problem* implies feasibility of each constraint in *H-problem* and vice versa. The following theorem establishes the desired result.

Theorem 1 *F-problem is feasible with the value of cost equal to μ, if and only if, H-problem is feasible with the value of cost equal to μ.*

Theorem 1 follows directly from Lemma 2 which establishes the equivalence of the LMIs used in the formulations of *H-problem* and *F-problem*. Note that if G satisfies the structural constraints given in (23) then the construction used in the proof, $D = G'^{-1}$, gives a D satisfying the required structural constraints in (30). The converse also holds. The result of the iterative procedure outlined above is a suboptimal choice of the estimator gain, L_f, the transmitter gain, H_f, and the receiver gain, F_f, which guarantee that the estimation error dynamics are stable and the estimation error correlation matrix, P, is bounded from above by X. Once the iteration is over, we can check if H_f satisfies an additional practical limitation, a bound on the transmitter power, for example, and if not, increase the γ weight accordingly and repeat the iteration. If we are satisfied with the choice of H_f, we complete the design by solving the following optimization problem,

$$\underset{X, L_f, F_f}{\text{minimize}} \ \|X\|, \quad \text{subject to:} \quad \text{Inequality (33)},$$

where $Y = X^{-1}$, $R = DL_f$, $E = DF_f$, and the new slack matrix variable $D = \mathrm{diag}(D_1, ..., D_N)$, $D_j \in I\!R^{n_x \times n_x}$, $j = 1, ..., N$, does not have identical blocks on the diagonal. This problem formulation differs from *F-problem* in that we use a less conservative slack matrix, D, and minimize the upper bound on the estimation error correlation, P, directly. Once the problem is solved, we calculate the unknown gains from $L_f = D^{-1}R$ and $F_f = D^{-1}E$.

3.2 Markov communication topology

In this section we describe the design procedure for the system with the Markov communication topology described in Section 2.2. Observe that the estimation error dynamics (15) is a description of a discrete-time Markov jump linear system. To be able to approach our design problem we use the results of Costa and Fragoso (1993) and Costa and Guerra (2002), which allow us to formulate the design problem as a set of LMIs with their feasibility guaranteeing the mean square stability (MSS) of system (15).

Following the notational conventions in Costa and Guerra (2002), we define new vector variables $z_j(k) \equiv E\{e(k)\ 1_{\theta(k)=j}\} \in I\!R^{Nn_x}$, $j = 1, ..., M$, where $1_{\theta(k)=j}$ is the Dirac measure. Hence $z_j(k)$ is the estimation error expectation depending on the state j of the Markov chain at time step k. We also define the collected vector, $z(k) \equiv [z_1(k)'\ z_2(k)'\ ...\ z_M(k)']' \in I\!R^{Nn_xM}$, and matrix, $Z_j(k) \equiv E\{z_j(k)z_j(k)'\} \in I\!R^{Nn_x \times Nn_x}$, $j = 1, ..., M$. As shown in Costa and Fragoso (1993),

$$Z(k) \equiv E\{z(k)z(k)'\} = \mathrm{diag}(Z_1(k), ..., Z_M(k)), \quad (35)$$

$$Z(k+1) = \underset{j}{\mathrm{diag}} \left(\sum_{i=1}^{M} p_{ij} \bar{A}_i Z_i(k) \bar{A}_i' + \bar{B} Q \bar{B}' \sum_{i=1}^{M} \pi_i(k) p_{ij} \right), \quad (36)$$

where $\bar{A}_i$ is given in (16) by specifying $\theta(k) = i$, the collected covariances are $Q = \mathrm{diag}(Q_{\underline{v}}, Q_n)$, and we have used the fact that the input matrix $\bar{B}$ is independent of states of the Markov chain. Equation (36) is the update equation for the augmented estimation error correlation matrix, $Z(k)$. The estimation error correlation matrix is then,

$$\begin{aligned} P(k) \equiv E\{e(k)e(k)'\} &= E\{\sum_{j=1}^{M} z_j(k) \sum_{j=1}^{M} z_j(k)'\} \\ &= [I\ ...\ I] Z(k) [I\ ...\ I]' = \sum_{j=1}^{M} Z_j(k). \end{aligned}$$

We would like to design the distributed estimator with constant gains and for that purpose consider a time-invariant formulation which corresponds to a steady-state solution, $Z \equiv Z(k) = Z(k+1)$, or a long-run average solution of (36). In equation (36) the probability distribution, $\pi(k) \in I\!R^M$, of the states of the Markov chain is a dynamic variable with an evolution described by (13). To be able to consider the time-invariant case we make several observations about the properties of the Markov chain. First, observe that according to our definition of the transition probability matrix, $\mathcal{P}$, the Markov chain describing communication topology can exhibit both aperiodic and periodic behavior. For the aperiodic case there exists a steady-state solution of (13), $\pi = \lim_{k\to\infty} \pi(k)$, which generally depends on the initial value of distribution $\pi(0)$. In a more specific situation the Markov chain can be ergodic, then the steady-state value of $\pi(k)$ is independent of $\pi(0)$. In any of these two situations we can find the steady-state distribution, π, and consider the steady-state solution of (36) with $Z = Z(k) = Z(k+1)$.

In the periodic case when $\lim_{k\to\infty} \pi_i(k)$ does not exist, we can consider a Cesaro limit, and the long-run average solution of (36) is given by,

$$\lim_{k\to\infty}\frac{1}{k}\sum_{l=0}^{k-1}Z(l+1)= \lim_{k\to\infty}\frac{1}{k}\sum_{l=0}^{k-1}\Big[\operatorname*{diag}_j\Big(\sum_{i=1}^{M}p_{ij}\bar{A}_iZ_i(l)\bar{A}_i' + \bar{B}Q\bar{B}'\sum_{i=1}^{M}\pi_i(l)p_{ij}\Big)\Big]. \quad (37)$$

From standard results in Markov chain theory we know that for a periodic case there exists a limit $\pi \equiv \lim_{k\to\infty}(\pi(0)+\pi(1)+...+\pi(k-1))/k$ and hence the limit for the last term of the right-hand side of (37) is defined. For the left-hand side of (37), $\lim_{k\to\infty}\frac{1}{k}\sum_{l=0}^{k-1}Z(l+1) = \lim_{k\to\infty}\frac{1}{k}\left(\sum_{l=0}^{k-1}[Z(l)]-Z(0)+Z(k)\right)$, and if $Z(0)$ and $Z(k)$ are bounded, then $\lim_{k\to\infty}\frac{1}{k}\sum_{l=0}^{k-1}Z_i(l+1) = \lim_{k\to\infty}\frac{1}{k}\sum_{l=0}^{k-1}Z_i(l)$, $i = 1,...,M$. We assume that this limit exits, which is true if the system is stable, and define $Z_i \equiv \lim_{k\to\infty}\frac{1}{k}\sum_{l=0}^{k-1}Z_i(l)$.

With these assumptions the steady state or the long-run average solution of (36) satisfies,

$$Z = \operatorname*{diag}_j(Z_j),$$

$$Z_j = \sum_{i=1}^{M}p_{ij}\bar{A}_iZ_i\bar{A}_i' + \bar{B}Q\bar{B}'\sum_{i=1}^{M}\pi_ip_{ij}, \quad j=1,...,M.$$

At this point we can consider the problem of designing a distributed parallel estimator which stabilizes the collected estimation error dynamics (15) and minimizes a steady-state or a long-run average estimation error correlation matrix, $P \equiv \sum_{j=1}^{M}Z_j$. We state this optimization problem as follows:

$$\min_{L_f,F_f,H_f,X_j} \|\bar{P}\|, \quad \text{subject to}$$

$$\bar{P}=\bar{P}'=\sum_{j=1}^{M}X_j>0, \quad \text{and}$$

$$X_j-\sum_{i=1}^{M}p_{ij}\bar{A}_iX_i\bar{A}_i'-\bar{B}Q\bar{B}'\sum_{i=1}^{M}\pi_ip_{ij}>0, \quad \text{for all } j=1,...,M, \quad (38)$$

where L_f, F_f, and H_f satisfy the structural constraints in (6), (8) and (9) and those of the form $H_{ij}=0$ and $F_{ij}=0$ arising from the communication topology.

In the case of an aperiodic Markov chain, the feasibility of the matrix inequalities (38) is equivalent to the MSS of the Markov jump linear system described by (15), due to the result of Costa and Fragoso (1993). The MSS of the system (15) implies the existence of $e \in I\!R^{Nn_x}$ and $P \in I\!R^{Nn_x\times Nn_x}$ independent of $e(0)$ such that, $\|E\{e(k)\}-e\| \to 0$ and $\|E\{e(k)e(k)'\}-P\| \to 0$ as $k\to\infty$.

To be able to use inequalities (38) in the design procedure, we state the following lemma.

Lemma 3 *If there exists a matrix $G \in I\!R^{Nn_x\times Nn_x}$ and $Y = \mathrm{diag}(Y_1,...,Y_M)$ with $Y_j = Y_j' > 0$, such that for all $j=1,...,M$,*

$$\begin{bmatrix} Y & \hat{A}_j'G' & 0 \\ G\hat{A}_j & G+G'-Y_j & \sqrt{\sigma_j}G\bar{B} \\ 0 & \bar{B}'G'\sqrt{\sigma_j} & Q^{-1}\end{bmatrix} > 0, \quad (39)$$

with $\sigma_j = \sum_{i=1}^{M}\pi_ip_{ij}$ and

$$\hat{A}_j = \left[\sqrt{p_{1j}}\bar{A}_1 \ \sqrt{p_{2j}}\bar{A}_2 \ ... \ \sqrt{p_{Mj}}\bar{A}_M\right],$$

then $X_j = Y_j^{-1}$ satisfies $X_j = X_j' > 0$, $j=1,...,M$ and the M LMI conditions in (38).

The line of argument in the proof of this lemma is similar to the proof of Theorem 1 in de Oliveira, Bernussou, and Geromel (1999) and we omit it here due to space limitations. To use the LMIs in (39) for the design of the gains L_f, F_f, and H_f, we define a block-diagonal structured variable,

$$G \equiv \mathrm{diag}(G_1,...,G_N), \quad (40)$$

with $G_i \in I\!R^{n_x\times n_x}$, $i=1,...,N$. We also define

$$R \equiv GL_f = \mathrm{diag}(R_1,...,R_N) \in I\!R^{Nn_x\times\sum_{i=1}^{N}k_{y_i}}, \quad (41)$$

with $R_i = G_iL_i$, and

$$E \equiv GF_fH_f = [E_1 \ \ E_2 \ ... \ E_N] \in I\!R^{Nn_x\times N^2n_x}, \quad (42)$$

where $E_j = \mathrm{diag}(E_{1j},...,E_{Nj})$ and $E_{ij} = G_iF_{ij}H_{ij}$. The G, R and E variables in the Markov communication problem play a similar role to those defined for the Gaussian noise problem although they are not defined identically. They do however play an identical role in expressing structural constraints, including those due the absence of links in the communication topology.

Note that each $G\bar{A}_i = GA_B - RC_f - E\mathcal{L}_{f\{i\}}$ and as a consequence all $G\hat{A}_j$ are linear in new variables G, R, and E. The product $G\bar{B} = [G\Gamma \ -R]$ is also linear in G and R, and hence all inequalities in (39) are linear in the matrix variables Y_j, $j=1,...,M$, G, R, and E. Now we can redefine the design problem as a convex optimization problem with LMIs:

$$\begin{aligned}
&\underset{G,R,E,Y_j}{\text{maximize}}\,\gamma, \quad \text{subject to:}\\
&\qquad 0 < \gamma I \leq \sum_{j=1}^{M} Y_j,\\
&\text{the } M \text{ LMI constraints in (39), and the structural}\\
&\text{constraints on } G, R, \text{and } E \text{ (given in (40), (41)}\\
&\text{and (42)).}
\end{aligned} \tag{43}$$

When the feasible solution which minimizes the upper bound on the estimation error correlation matrix P is found, we can calculate L_f and $F_f H_f$ from $L_f = G^{-1}R$ and $F_f H_f = G^{-1}E$ since G is nonsingular. It is important to mention that, even though the above optimization problem is convex in the design variables, the structural constraints on G, R, and E introduce potential conservatism into design of the estimator gains.

Note that the solution to the proposed optimization problem gives the transmitter and the receiver matrix gains as a product, $F_f H_f$, which must be factored to complete the design. Another issue which should be addressed is the rank constraint on each individual block, $F_{ij}H_{ij}$, of the product $F_f H_f$ and consequently the variable E. Since the dimension of a transmitter gain matrix, H_{ij}, is $k_{ij} \times n_x$ and $F_{ij}H_{ij} \in I\!R^{n_x \times n_x}$, the ranks of all N_l, where N_l is the number of links in the topology, nonzero products $F_{ij}H_{ij}$ should be less or equal than the corresponding k_{ij}. To tackle this issue we can use results of Fazel, Hindi, and Boyd (2003) or Orsi, Helmke, and Moore (2006) and impose additional LMI constraints on N_l nonzero blocks of the matrix variable E, $E_{ij} = G_i F_{ij} H_{ij}$. Hence, imposing the rank constraints introduces an additional conservatism into the formulation of the design problem.

Once all the nonzero blocks of E_{ij} in the solution to the optimization problem satisfy the rank constraints, we can find F_{ij} and H_{ij} by taking a singular value decomposition (SVD) of each product $F_{ij}H_{ij}$,

$$F_{ij}H_{ij} = \begin{bmatrix} U_{ij}^1 & U_{ij}^2 \end{bmatrix} \begin{bmatrix} D_{ij} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{ij}^1 \\ V_{ij}^2 \end{bmatrix}, \tag{44}$$

where $D_{ij} \in I\!R^{k_{ij} \times k_{ij}}$ is a diagonal matrix with possibly some zeros on the diagonal. Then a possible choice of transmitter and receiver gains is $H_{ij} = V_{ij}^1$ and $F_{ij} = U_{ij}^1 D_{ij}$. To guarantee that each H_{ij} satisfies a possible power limitation we simply scale the matrices in the SVD product.

The absence of communication noise allowed us to formulate the optimization problem for the system with Markov topology as a convex optimization. At the same time dealing with the constraints on the dimensions of the communication signals requires the introduction of rank constraints which are usually hard to handle. As an alternative to introducing rank constraints we can adapt the iterative tools developed in the previous section for the design of the gains of the estimator with Markov topology. The synthesis tools from this section and Section 3.1 can also be combined in a straightforward manner to result in the synthesis procedure for the Markov communication topology with noise. If the design tools from the both sections are combined, the resulting mathematical formulation can be considered as the dual to the state feedback design problem presented in (do Val, Geromel, and Goncalves, 2002).

4 Experimental results

Now we illustrate the design procedure for the system with the Markov communication topology on an experimental example. The experimental configuration is a formation with three agents, where each agent is a motor cart able to move along a track. We use the term "agent" when referring to the estimator/controller and the term "cart" when describing the physical system; Agent 1 corresponds to Cart 1, etc.. The control input applied to each cart is the motor voltage, while the outputs available for measurement are the positions of the carts on the tracks. This experimental setup was implemented with three Quanser motor-cart modules and one computer station with an acquisition and control board. All three controllers, each consisting of a full formation state estimator and state feedback, were implemented in MATLAB Simulink in a single diagram. The communication links between estimators—including either Gaussian noise or Markov model packet loss—were also implemented by links on the same Simulink diagram. The cart position measurements were taken via encoders and the acquisition and contain sensor noise.

The dynamics of each motor cart can be described by,

$$\ddot{p}_i = 3.78u_i - 16.88\dot{p}_i, \quad i = 1, 2, 3,$$

where p_i is the position of a cart on the track and u_i is the control input: the voltage applied to the cart motor. We rewrite this system in the state-space form and find its discrete zero-order-hold equivalent with sampling period $T_s = 0.0005$ seconds,

$$\bar{x}_i(k+1) = A_i \bar{x}_i(k) + B_i u_i(k).$$

The full formation dynamics are then described by,

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}_u u(k) + \bar{B}_v v(k), \tag{45}$$

where $\bar{x}(k) = [\bar{x}_1(k)'\ \bar{x}_2(k)'\ \bar{x}_3(k)']' \in I\!R^6$, $u(k) = [u_1(k)'\ u_2(k)'\ u_3(k)']' \in I\!R^3$, $\bar{A} = \mathrm{diag}(A_1,\ A_2,\ A_3)$, $\bar{B}_u = \mathrm{diag}(B_1,\ B_2,\ B_3)$, and we augment the original system dynamics with the zero-mean Gaussian process noise $v(k) \in I\!R^3$ with covariance $Q_v = 10^{-6}I$ entering the system through $\bar{B}_v = \mathrm{diag}(\bar{b},\ \bar{b},\ \bar{b})$, $\bar{b} = [0\ 1]'$.

We define the formation by specifying the relative distances between agents. The control objective is to guarantee that the agents converge to and keep relative distances specified with a vector $d = [d_{12}\ d_{23}\ d_{13}]$, where d_{ij} is the distance between Agent i and Agent j. In the experiment shown we have chosen $d_{12} = 0.2\ m$, $d_{23} = 0.2\ m$, and $d_{13} = 0.4\ m$ as the reference formation. We assume that the measurements available for the agents are their relative distances to one of the other agents. In particular, Agent 1 is able to measure its distance to Agent 2, Agent 2 measures the distance to Agent 3, and Agent 3 its distance to Agent 1. With these measurements the three output matrices for each of the agents in the formation are: $\bar{C}_1 = [-1\ 0\ 1\ 0\ 0\ 0]$, $\bar{C}_2 = [0\ 0\ 1\ 0\ \ -1\ 0]$, $\bar{C}_3 = [1\ 0\ 0\ 0\ \ -1\ 0]$. System (45) with the output matrix $\bar{C} = [\bar{C}_1'\ \bar{C}_2'\ \bar{C}_3']'$ is not observable, due to the fact that no absolute cart positions can be determined from the measured variables. We can reduce the dimension of the system (45) by removing the unobservable part without influencing the relative position performance of the formation. To do this we apply a similarity transformation $x(k) = T\bar{x}(k)$ and truncate unobservable states of the system to arrive at,

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) + B_v v(k), \\ y_i(k) &= C_i x(k) + n_i(k),\ i = 1, 2, 3, \end{aligned} \tag{46}$$

where $x(k) \in I\!R^4$, and $n_i(k) \in I\!R$ is the zero-mean Gaussian measurement noise with covariance $Q_{n_i} = 10^{-6}$.

First, we design a stabilizing state feedback, $u(k) = -Kx(k)$, which renders the closed-loop formation system matrix, $A - B_u K$, Hurwitz and guarantees that the formation control objective is satisfied if the estimation error dynamics are designed to be stable. To find K, we use a standard LQR design method and first specify weights $Q = \mathrm{diag}(100,\ 10,\ 100,\ 10,\ 100,\ 10)$ and $R = \mathrm{diag}(0.1,\ 0.1,\ 0.1)$ for the standard quadratic cost function, $J = \sum_{k=1}^{\infty}(\bar{x}(k)'Q\bar{x}(k) + u(k)'Ru(k))$, in the original formation coordinates $\bar{x}(k)$. These are then transformed to the reduced system coordinates using the defined similarity transformation, T, and state truncation. The individual control inputs for each cart are defined by, $u_i(k) = -\Pi_i K\hat{x}_i(k)$. Here $\hat{x}_i(k)$ is the estimate of state $x(k)$ at i^{th} agent's estimator and Π_i is the corresponding projection matrix, $\Pi_1 = \mathrm{diag}(1,\ 0,\ 0)$, $\Pi_2 = \mathrm{diag}(0,\ 1,\ 0)$, $\Pi_3 = \mathrm{diag}(0,\ 0,\ 1)$.

We assume that the agents of the formation communicate according to the Markov communication topology model described in Section 2.2. For our experiment we allow Agent 1 to communicate a function of its estimates to Agent 2, and Agent 2 to communicate a function of its estimates to both Agents 1 and 3. We assume that each agent can transmit signals with four variables, so $k_{21} = 4$, $k_{12} = 4$, and $k_{32} = 4$, and,

$$\mathcal{L}_1 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathcal{L}_3 = 0_N.$$

This communication topology has $N_l = 3$ links and we model each link with the two-state Markov chain as shown in Figure 1, with the same transition probabilities for all links, $p^0_{21} = p^0_{12} = p^0_{32} = p^0 = 0.1$ and $p^1_{21} = p^1_{12} = p^1_{32} = p^1 = 0.95$. Then according to our definition the result of communication at step k can be described by the state of the Markov chain with $M = 2^3 = 8$ states: $\theta(k) = 1 :\ \Theta(k) = [0\ 0\ 0]$, $\theta(k) = 2 :\ \Theta(k) = [0\ 1\ 0]$, $\theta(k) = 3 :\ \Theta(k) = [1\ 0\ 0]$, $\theta(k) = 4 :\ \Theta(k) = [1\ 1\ 0]$, $\theta(k) = 5 :\ \Theta(k) = [0\ 0\ 1]$, $\theta(k) = 6 :\ \Theta(k) = [0\ 1\ 1]$, $\theta(k) = 7 :\ \Theta(k) = [1\ 0\ 1]$, $\theta(k) = 8 :\ \Theta(k) = [1\ 1\ 1]$.

This description captures all possible outcomes of communication at step k, where each individual link is transmitting or failing to transmit information independently of all other links. With $\pi(k) \in I\!R^8$ and the elements of transition probability matrix calculated from individual probabilities of each link, for example $p_{24} = (1 - p^0)p^1p^0$, the Markov chain modeled with (13) is aperiodic. With $\pi(0) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 1]$ we found the steady-state solution of (13) to be $\pi = [0.0001\ 0.0026\ 0.0026\ 0.0472\ 0.0026\ 0.0472\ 0.0472\ 0.8503]$ and used this π in the design calculations. The binary signals implementing success or failure of the links were generated by a Markov chain model from Matlab and are shown on Figure 2 for the first second of the experiment.

Using the algorithm described in detail in Section 3.2, we designed the distributed estimator gains L_i, $i = 1, 2, 3$, transmitter gains H_{21}, H_{12}, H_{32}, and receiver gains F_{21}, F_{12}, F_{32}. To find these gains we implemented the proposed synthesis procedure using *yalmip* (Löfberg, 2004) in MATLAB. The achieved upper bound on the estimation error correlation matrix was $\|P\|_2 = 1.22$. It is worth mentioning, that we designed and tested two other estimators with different communication topologies. Increasing the number of communication links in the topology, consistently resulted in a lower bound on the estimation error correlation matrix, $\|P\|_2$, and higher estimator performance during the experiment.

The experimental results for the proposed system architecture and gains calculated with the transmitter constraints $\|H_{ij}\|_F \leq 1$ are shown in Figures 3 and 4. Figure 3 shows the positions of carts along the tracks versus time. At $t = 0$ seconds the system was initialized

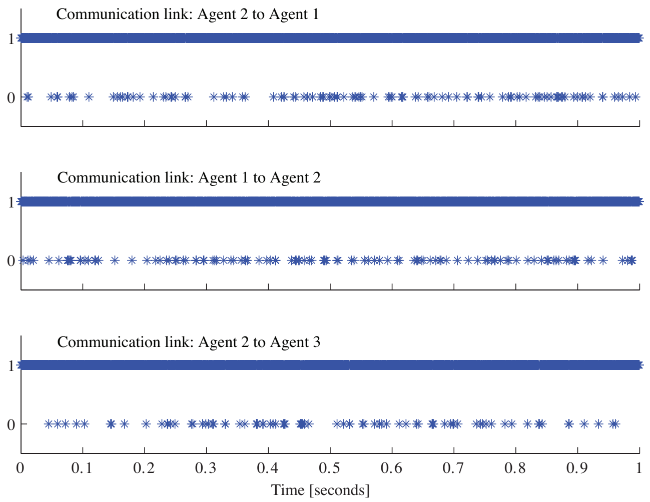

Fig. 2. Markov model generated binary signals implementing success, 1, or failure, 0, of each of the communication links.

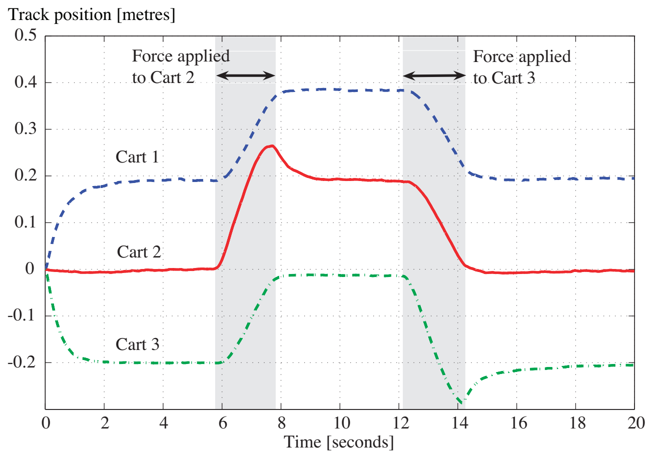

Fig. 3. Positions of carts on the tracks for an experiment with a Markov communication topology. External forces are applied to Carts 2 and 3 at the times shown and move the entire formation.

with carts located at the same point on their respective tracks. Once the controller was on, they started moving into formation and by the time $t = 4$ seconds the formation was in order. From time $t = 5.9$ seconds for about 2 seconds, we applied a force to Cart 2, causing the formation to drift in the direction of applied force. At time $t = 11$ the formation was arranged again and at time $t = 12$ seconds we applied a force to Cart 3 and maintained it for about 2 seconds. As a result the formation drifted in the opposite direction and by the time $t = 19$ seconds it was in the nominal formation again.

Figure 4 shows the error in the estimates of relative distances for all three agents during the experiment. The estimation errors for d_{12} are shown in the uppermost sub-plot. Agent 1 (on Cart 1) has access to a direct measurement of d_{12} and Agent 1's estimation error decays

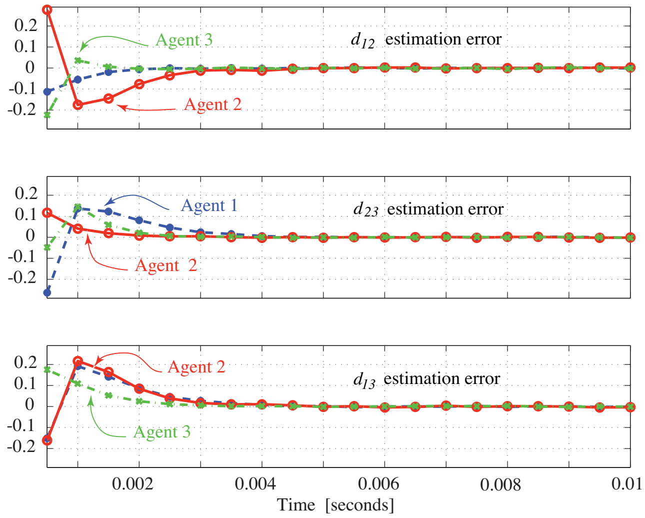

Fig. 4. Estimation errors corresponding to each of the three measured state variables. Each agent maintains an estimate of each variable. The time scale corresponds to the first 20 time steps of the experiment.

exponentially due to the dominant contribution of the measurement. Agents 2 and 3 do not have access to this measurement and receive information about d_{12} via the intermittent communication links and the propagation of their system models. Analogous comments apply to the estimate of d_{23} by Agent 2 and the estimate of d_{13} by Agent 3. Note that the estimation error dynamics are several orders of magnitude faster than the control dynamics in this system. It is the control actuation saturation levels that limit the overall system performance.

5 Conclusion

Decentralized control structures based on distributed estimation—in which each agent maintains an estimate of the state of the entire formation—are beneficial for control problems with formation-wide objective functions. While keeping the estimate of the formation state is computationally demanding, it offers the formation a higher level of autonomy in decision making and higher robustness with respect to faults and uncertainties. Communication between the agents of the formation can resolve the stability issues arising from using agents' local estimates for global state feedback and can improve the performance of overall closed-loop system.

We have presented results on the design of distributed decentralized estimators for a formation of agents. A time-invariant formulation and two communication models, describing analog and digital communication between the agents of the formation, were considered. For each model we proposed procedures for the synthesis of the suboptimal gains of the distributed estimator formulated with LMIs. Classical estimator gains as well as the terms introduced by the communication between agents were used as the design variables. For formations with analog communication with additive noise the optimization problem was bilinear, while for formations with digital communication it was convex, but in the case of channel limitations requires additional rank constraints on the design variables. Structural constraints on the design variables were handled effectively through the introduction of structured slack variables. To evaluate the validity of the proposed methods, they were applied to the design of distributed estimators for an experimental formation with three agents. Experimental results illustrated the performance of estimators and showed that the design procedures are applicable to practical systems.

The proposed methods provide tools for the design of decentralized distributed estimators under the assumption that the formulated design problems are feasible. This assumption may not hold for a formation described with a general LTI system and having any specified topology. The system may have unstable fixed modes (Tarokh, 1985) or eigenvalues of estimation error dynamics (10), (15) may depend on each other in a restrictive fashion (Karcanias, Laios, and Giannakopoulos, 1988), which can prevent from stabilizing the estimation error through the design variables L_f, H_f, F_f. These issues are to be considered in our future work.

Acknowledgements

The authors would like to thank João P. Hespanha and Payam Naghshtabrizi for helpful discussions.

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