Stabilization of Sets with Application to Multi-Vehicle Coordinated Motion ⋆

Sergey G. Nersesov [a], Parham Ghorbanian [a], Amir G. Aghdam [b]

[a] *Department of Mechanical Engineering, Villanova University, Villanova, PA 19085-1681*

[b] *Department of Electrical and Computer Engineering, Concordia University, Montreal, QC, Canada H3G 1M8*

Abstract

In this paper, we develop stability and control design framework for time-varying and time-invariant sets of nonlinear dynamical systems using vector Lyapunov functions. Several Lyapunov functions arise naturally in multi-agent systems, where each agent can be associated with a generalized energy function which further becomes a component of a vector Lyapunov function. We apply the developed control framework to the problem of multi-vehicle coordinated motion to design distributed controllers for individual vehicles moving in a specified formation. The main idea of our approach is that a moving formation of vehicles can be characterized by a time-varying set in the state space, and hence, the problem of distributed control design for multi-vehicle coordinated motion is equivalent to design of stabilizing controllers for time-varying sets of nonlinear dynamical systems. The control framework is shown to ensure global exponential stabilization of multi-vehicle formations. Finally, we implement the feedback stabilizing controllers for time-invariant sets to achieve global exponential stabilization of static formations of multiple vehicles.

Key words: Stabilization of sets, vector Lyapunov functions, multi-vehicle systems, coordinated motion, cooperative control.

1 Introduction

In the analysis and control design of complex interconnected dynamical systems it is often desirable to treat the overall system as a collection of interconnected subsystems. The behavior of the aggregate or interconnected system can then be predicted from the behaviors of the individual subsystems and their interconnections. The use of vector Lyapunov functions [1,22,29,16,25] in interconnected systems analysis offers a very flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function. In particular, each component of a vector Lyapunov function need not be positive definite with a negative or even negative-semidefinite derivative. Alternatively, the time derivative of the vector Lyapunov function need only satisfy an element-by-element inequality involving a vector field of a certain comparison system.

Multi-agent systems present a class of interconnected dynamical systems where agents are often coupled through the common task that they need to accomplish, but otherwise dynamically decoupled, meaning that the motion of one does not directly affect the others. The complexity of cooperative manoeuvres that multi-agent systems need to perform as well as environmental conditions often necessitate the design of feedback control algorithms that use information about current position and velocity of each vehicle to steer them while maintaining a specified formation. For example, for mobile agents operating in foggy environment or located far from each other, open-loop visual control for coordinated motion becomes impractical. In this case, feedback control algorithms are required for individual vehicle steering which determine how a given vehicle maneuvers based on positions and velocities of nearby vehicles and/or on those of a formation leader. The leader could be real, that is, one of the vehicles in a formation leads the others or it could be virtual, that is, vehicles synthesize a leader and the motions of the vehicles in a formation are defined with respect to a virtual agent whose positions and velocities are known at each instant of time.

⋆ This paper was not presented at any IFAC meeting. Corresponding author: S.G. Nersesov. Tel.: +1 610-519-8977. Fax: +1 610-519-7312.

Email addresses: sergey.nersesov@villanova.edu (Sergey G. Nersesov), parham.ghorbanian@villanova.edu (Parham Ghorbanian), aghdam@ece.concordia.ca (Amir G. Aghdam).

Analysis and control design for networks of mobile agents has received considerable attention in the literature. Some of the common manoeuvres that a group of mobile agents may perform are flocking [26,32], cyclic pursuit [20], (virtual) leader following [9,30], rendezvous [4,28], etc. Graph-theoretic notions [8] are essential in the analysis and control design for a system of mobile agents performing a common task [14,10]. A number of recent papers propose rigorous mathematical techniques for the analysis of networks of agents. Specifically, authors in [5,29] use the graph theory to model interconnected systems, while [10,7,23] involve graph theoretic notions for stability analysis of formations of large number of agents. Authors in [19,27,26] use potential functions to analyze flocking and [6] resorts to control Lyapunov functions to design feedback controllers for coordinated motion of multi-robot platforms. Distributed control of robotic networks has been extensively studied in [3,28] where the authors develop a variety of control algorithms for network consensus. Furthermore, distributed nonlinear static and dynamic control architectures for multi-agent coordination using thermodynamic principles was presented in [12]. A survey of recent research results in cooperative control of multi-vehicle systems was performed in [24].

Stability of time-varying sets for nonlinear dynamical systems have not been widely studied in the literature. Notable exceptions include [18,17,21] where stability analysis for conditionally invariant sets was developed. In this paper, we develop stability analysis and control design framework for time-varying and time-invariant sets of nonlinear dynamical systems using vector Lyapunov functions. In multi-agent systems, several Lyapunov functions arise naturally where each agent can be associated with a generalized energy function which further becomes a component of a vector Lyapunov function. Furthermore, since a specified formation of multiple vehicles can be characterized by a time-varying set in the state space, the problem of control design for multi-vehicle coordinated motion is equivalent to design of stabilizing controllers for time-varying sets of nonlinear dynamical systems. Thus, using the stability and control results developed for time-varying sets based on vector Lyapunov functions, we design distributed control algorithms for stabilization of multi-vehicle formations. These distributed control algorithms use only local information about individual vehicle relative position and velocity with respect to the leader in order to maintain a specified formation for a system of multiple vehicles. Finally, we specialize the results obtained for time-varying sets to address stabilization of time-invariant sets and to further develop stabilizing control algorithms for static formations (rendezvous) of multiple vehicles. The developed cooperative control algorithms are shown to globally exponentially stabilize both moving and static formations.

2 Stability and Stabilization of Time-Varying Sets

In this section, we present the results on stability and stabilization of time-varying sets for time-varying nonlinear dynamical systems using vector Lyapunov functions [1,22,29,16,25]. To elucidate this, consider the time-varying nonlinear dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, is the solution to (1), $t_0 \in [0, \infty)$, $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f(t, 0) = 0$, $t \geq t_0$, and $f(\cdot, \cdot)$ is Lipschitz continuous on $[0, \infty) \times \mathcal{D}$.

The following definition introduces several types of stability for time-varying sets of nonlinear time-varying dynamical systems. For this definition, $\mathcal{D}_0^t \triangleq \mathcal{D}_0(t)$, $t \geq t_0$, is a time-varying set such that, at each instant of time $t \geq t_0$, $\mathcal{D}_0(t)$ is a compact set and $\mathcal{O}_\varepsilon(\mathcal{D}_0(t)) \triangleq \{x \in \mathcal{D} : \operatorname{dist}(x, \mathcal{D}_0(t)) < \varepsilon\}$, $t \geq t_0$, defines the ε-neighborhood of $\mathcal{D}_0(t)$ at each instant of time $t \geq t_0$, where $\operatorname{dist}(x, \mathcal{D}_0(t)) \triangleq \inf_{y \in \mathcal{D}_0(t)} \|y - x\|$, $t \geq t_0$.

Definition 2.1 *Consider the nonlinear time-varying dynamical system (1). Let $\mathcal{D}_0^t$ be a time-varying set such that $\mathcal{D}_0^t$ is positively invariant with respect to (1) and at each instant of time $t \in [t_0, \infty)$, $\mathcal{D}_0(t)$ is a compact set.*

i) *$\mathcal{D}_0^t$ is Lyapunov stable if for every $\varepsilon > 0$ and $t_0 \in [0, \infty)$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $x(t) \in \mathcal{O}_\varepsilon(\mathcal{D}_0(t))$ for all $t \geq t_0$.*
ii) *$\mathcal{D}_0^t$ is uniformly Lyapunov stable if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $x(t) \in \mathcal{O}_\varepsilon(\mathcal{D}_0(t))$ for all $t \geq t_0$ and for all $t_0 \in [0, \infty)$.*
iii) *$\mathcal{D}_0^t$ is asymptotically stable if it is Lyapunov stable and for every $t_0 \in [0, \infty)$, there exists $\delta = \delta(t_0) > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{D}_0(t)) = 0$.*
iv) *$\mathcal{D}_0^t$ is uniformly asymptotically stable if it is uniformly Lyapunov stable and there exists $\delta > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{D}_0(t)) = 0$ uniformly in t_0 and x_0 for all $t_0 \in [0, \infty)$.*
v) *$\mathcal{D}_0^t$ is globally asymptotically stable if it is Lyapunov stable and $\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{D}_0(t)) = 0$ for all $x_0 \in \mathbb{R}^n$ and $t_0 \in [0, \infty)$.*
vi) *$\mathcal{D}_0^t$ is globally uniformly asymptotically stable if it is uniformly Lyapunov stable and $\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{D}_0(t)) = 0$ uniformly in t_0 and x_0 for all $x_0 \in \mathbb{R}^n$ and $t_0 \in [0, \infty)$.*
vii) *$\mathcal{D}_0^t$ is uniformly exponentially stable if there exist scalars $\alpha > 0$, $\beta > 0$, $\delta > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $\operatorname{dist}(x(t), \mathcal{D}_0(t)) \leq \alpha \operatorname{dist}(x_0, \mathcal{D}_0(t_0)) e^{-\beta(t-t_0)}$, $t \geq t_0$, for all $t_0 \in [0, \infty)$.*
viii) *$\mathcal{D}_0^t$ is globally uniformly exponentially stable if there exist scalars $\alpha > 0$, $\beta > 0$ such that $\operatorname{dist}(x(t), \mathcal{D}_0(t)) \leq$*

$\alpha \operatorname{dist}(x_0, \mathcal{D}_0(t_0)) e^{-\beta(t-t_0)}$, $t \geq t_0$, *for all* $x_0 \in \mathbb{R}^n$ *and* $t_0 \in [0, \infty)$.

The following definition introduces the notion of class $\mathcal{W}$ functions involving *quasimonotone increasing* functions.

Definition 2.2 ([29,25]) *A function* $w = [w_1, ..., w_q]^{\mathrm{T}} : [0, \infty) \times \mathbb{R}^q \to \mathbb{R}^q$ *is of* class $\mathcal{W}$ *if, for every fixed* $t \in [0, \infty)$*, each component* $w_i(\cdot, \cdot)$, $i = 1, ..., q$*, of* $w(\cdot, \cdot)$ *satisfies* $w_i(t, z') \leq w_i(t, z'')$ *for all* $z', z'' \in \mathbb{R}^q$ *such that* $z'_j \leq z''_j$, $j = 1, ..., q$, $j \neq i$, *and* $z'_i = z''_i$*, where* z_i *denotes the* i*th component of* z.

Note that if $w(t, \cdot) \in \mathcal{W}$, it follows that $w(\cdot, \cdot)$ is essentially nonnegative [11] which implies that a time-varying dynamical system whose dynamics are represented by $w(\cdot, \cdot)$ exhibits solutions that belong to the nonnegative orthant for all nonnegative initial conditions [11]. Throughout this paper, we use notation $x \leq\leq y$ (respectively, $x << y$), where $x, y \in \mathbb{R}^q$, to denote that each component of x and y satisfies inequality $x_i \leq y_i$ (respectively, $x_i < y_i$), $i = 1, \ldots, q$. Furthermore, we use $\mathbf{e}$ to denote the vector given by $\mathbf{e} \triangleq [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^q$. The following result presents sufficient conditions for several types of stability of time-varying sets with respect to nonlinear time-varying dynamical systems using vector Lyapunov functions.

Theorem 2.1 *Consider the nonlinear time-varying dynamical system (1). Assume there exists a continuously differentiable vector function* $V(t, x) = [V_1, \ldots, V_q]^{\mathrm{T}} : [0, \infty) \times \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}^q_+$*, where* $\mathcal{Q} \subset \mathbb{R}^q$ *and* $0 \in \mathcal{Q}$*; a continuous function* $w = [w_1, \ldots, w_q]^{\mathrm{T}} : [0, \infty) \times \mathcal{Q} \to \mathbb{R}^q$*; and class* $\mathcal{K}$ *functions [15]* $\alpha(\cdot)$, $\beta(\cdot)$ *such that* $V_i(t, x) = 0$, $x \in \mathcal{D}_i(t)$, $t \geq t_0$*, where* $\mathcal{D}_i(t) \subset \mathcal{D}$, $t \geq t_0$*;* $V_i(t, x) > 0$, $x \in \mathcal{D} \backslash \mathcal{D}_i(t)$, $t \geq t_0$, $i = 1, \ldots, q$*;* $\mathcal{D}_0^t = \mathcal{D}_0(t) \triangleq \cap_{i=1}^q \mathcal{D}_i(t) \neq \emptyset$ *is a positively invariant time-varying set with respect to (1) which is compact at each instant of time* $t \geq t_0$*;* $w(t, \cdot) \in \mathcal{W}$*;* $w(t, 0) = 0$, $t \geq 0$*;*

$$\alpha(\operatorname{dist}(x, \mathcal{D}_0(t))) \leq \mathbf{e}^{\mathrm{T}} V(t, x) \leq \beta(\operatorname{dist}(x, \mathcal{D}_0(t))), \quad (x, t) \in \mathcal{D} \times [0, \infty), \quad (2)$$

and

$$\frac{\partial V_i(t, x)}{\partial t} + V'_i(t, x) f(t, x) \leq w_i(t, V(t, x)), \quad (x, t) \in \mathcal{D} \times [0, \infty), \quad i = 1, \ldots, q. \quad (3)$$

In addition, assume that the vector comparison system

$$\dot{z}(t) = w(t, z(t)), \quad z(0) = z_0, \quad t \geq t_0, \quad (4)$$

has a unique solution $z(t)$, $t \geq t_0$*, forward in time. Then the following statements hold:*

i) If the zero solution to (4) is uniformly Lyapunov stable, then $\mathcal{D}_0^t$ *is uniformly Lyapunov stable with respect to (1).*

ii) If the zero solution to (4) is uniformly asymptotically stable, then $\mathcal{D}_0^t$ *is uniformly asymptotically stable with respect to (1).*

iii) If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, $\alpha(\cdot)$ *and* $\beta(\cdot)$ *are class* $\mathcal{K}_\infty$ *functions, and the zero solution to (4) is globally uniformly asymptotically stable, then* $\mathcal{D}_0^t$ *is globally uniformly asymptotically stable with respect to (1).*

iv) If there exist constants $\nu \geq 1$, $\alpha > 0$, $\beta > 0$ *such that, instead of (2), the following inequality holds*

$$\alpha \left[\operatorname{dist}(x, \mathcal{D}_0(t))\right]^\nu \leq \mathbf{e}^{\mathrm{T}} V(t, x) \leq \beta \left[\operatorname{dist}(x, \mathcal{D}_0(t))\right]^\nu, \quad (x, t) \in \mathcal{D} \times [0, \infty), \quad (5)$$

and the zero solution to (4) is uniformly exponentially stable, then $\mathcal{D}_0^t$ *is uniformly exponentially stable with respect to (1).*

v) If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$ *and there exist constants* $\nu \geq 1$, $\alpha > 0$, $\beta > 0$ *such that (5) holds and the zero solution to (4) is globally uniformly exponentially stable, then* $\mathcal{D}_0^t$ *is globally uniformly exponentially stable with respect to (1).*

Proof. i) Let $\varepsilon > 0$. It follows from uniform Lyapunov stability of the nonlinear dynamical system (4) that there exists $\mu = \mu(\varepsilon) > 0$ such that if $\|z_0\|_1 < \mu$ and $z_0 \in \overline{\mathbb{R}}^q_+$, where $\|\cdot\|_1$ denotes the absolute sum norm, then $\|z(t)\|_1 < \alpha(\varepsilon)$, $t \geq t_0$, and $z(t) \in \overline{\mathbb{R}}^q_+$, $t \geq t_0$. Now, choose $z_0 = V(t_0, x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$, $t_0 \in [0, \infty)$, and, for $\mu = \mu(\varepsilon) > 0$, choose $\delta = \delta(\mu(\varepsilon)) = \delta(\varepsilon) > 0$ such that $\beta(\delta) = \mu$. Then, for $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$, it follows from (2) that

$$\|z_0\|_1 = \mathbf{e}^{\mathrm{T}} z_0 = \mathbf{e}^{\mathrm{T}} V(t_0, x_0) \leq \beta(\operatorname{dist}(x_0, \mathcal{D}_0(t_0))) < \beta(\delta) = \mu, \quad (6)$$

which implies that $\mathbf{e}^{\mathrm{T}} z(t) = \|z(t)\|_1 < \alpha(\varepsilon)$, $t \geq t_0$. Now, with $z_0 = V(t_0, x_0) \geq\geq 0$ and the assumption that $w(t, \cdot) \in \mathcal{W}$ it follows from (3) and the vector comparison principle [25] that $0 \leq\leq V(t, x(t)) \leq\leq z(t)$, $t \geq t_0$. Thus, using (2), it follows that if $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$, then

$$\alpha(\operatorname{dist}(x(t), \mathcal{D}_0(t))) \leq \mathbf{e}^{\mathrm{T}} V(t, x(t)) \leq \mathbf{e}^{\mathrm{T}} z(t) < \alpha(\varepsilon), \quad t \geq t_0, \quad (7)$$

which implies that $x(t) \in \mathcal{O}_\varepsilon(\mathcal{D}_0(t))$, $t \geq t_0$. This proves uniform Lyapunov stability of the time-varying set $\mathcal{D}_0^t$ with respect to (1).

ii) Uniform Lyapunov stability of $\mathcal{D}_0^t$ with respect to (1) follows from i). Furthermore, since the zero solution to (4) is uniformly asymptotically stable, there exists $\mu > 0$ such that if $\|z_0\|_1 < \mu$, then $\lim_{t \to \infty} z(t) = 0$. As in i), let $z_0 = V(t_0, x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$, $t_0 \in [0, \infty)$, and choose $\delta = \delta(\mu) > 0$ such that $\beta(\delta) = \mu$. In this case, if $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$, it follows from (2) that

$$\|z_0\|_1 = \mathbf{e}^{\mathrm{T}} z_0 = \mathbf{e}^{\mathrm{T}} V(t_0, x_0) \leq \beta(\operatorname{dist}(x_0, \mathcal{D}_0(t_0))) < \beta(\delta) = \mu, \quad (8)$$

which implies that $\lim_{t\to\infty} z(t) = 0$. Since $w(t,\cdot) \in \mathcal{W}$ and $z_0 = V(t_0, x_0)$, it follows from (3) and the vector comparison principle that $0 \leq\leq V(t, x(t)) \leq\leq z(t)$, $t \geq t_0$. Now, using (2), it follows that, for $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$,

$$\alpha(\mathrm{dist}(x(t), \mathcal{D}_0(t))) \leq \mathbf{e}^{\mathrm{T}} V(t, x(t)) \leq \mathbf{e}^{\mathrm{T}} z(t), \tag{9}$$

for all $t \geq t_0$. Since $\lim_{t\to\infty} z(t) = 0$, it follows from (9) that $\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{D}_0(t)) = 0$ which proves uniform asymptotic stability of $\mathcal{D}_0^t$ with respect to (1).

iii) Uniform Lyapunov stability of $\mathcal{D}_0^t$ follows from *i*). Next, for any $x_0 \in \mathbb{R}^n$, $t_0 \in [0,\infty)$, and $z_0 = V(t_0, x_0)$, identical arguments as in *ii*) can be used to show that $\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{D}_0(t)) = 0$, which, along with the uniform Lyapunov stability, implies global uniform asymptotic stability of $\mathcal{D}_0^t$ with respect to (1).

iv) It follows from the uniform exponential stability of the nonlinear dynamical system (4) that there exist positive constants γ, μ, and η such that if $\|z_0\|_1 < \mu$, then

$$\|z(t)\|_1 \leq \gamma \|z_0\|_1 e^{-\eta(t-t_0)}, \quad t \geq t_0. \tag{10}$$

As in *i*), let $z_0 = V(t_0, x_0) \geq\geq 0, x_0 \in \mathcal{D}, t_0 \in [0,\infty)$ and choose $\delta = \delta(\mu) = \left(\frac{\mu}{\beta}\right)^{\frac{1}{\nu}} > 0$. In this case, if $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$, it follows from (5) that

$$\begin{aligned}\|z_0\|_1 = \mathbf{e}^{\mathrm{T}} z_0 = \mathbf{e}^{\mathrm{T}} V(t_0, x_0) &\leq \beta \left[\mathrm{dist}(x_0, \mathcal{D}_0(t_0))\right]^\nu \\ &< \beta\delta^\nu = \mu.\end{aligned} \tag{11}$$

Since $w(t,\cdot) \in \mathcal{W}$ and $z_0 = V(t_0, x_0)$, it follows from (3) and the vector comparison principle that $0 \leq\leq V(t, x(t)) \leq\leq z(t)$, $t \geq t_0$. Now, using (5) and (10), it follows that, for $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$,

$$\begin{aligned}\alpha \left[\mathrm{dist}(x(t), \mathcal{D}_0(t))\right]^\nu &\leq \mathbf{e}^{\mathrm{T}} V(t, x(t)) \\ &\leq \mathbf{e}^{\mathrm{T}} z(t) \\ &\leq \gamma \|z_0\|_1 e^{-\eta(t-t_0)} \\ &\leq \gamma\, \beta \left[\mathrm{dist}(x_0, \mathcal{D}_0(t_0))\right]^\nu e^{-\eta(t-t_0)}, \\ &\qquad t \geq t_0,\end{aligned} \tag{12}$$

which implies that

$$\mathrm{dist}(x(t), \mathcal{D}_0(t)) \leq \left(\frac{\gamma\beta}{\alpha}\right)^{\frac{1}{\nu}} \mathrm{dist}(x_0, \mathcal{D}_0(t_0)) e^{-\frac{\eta}{\nu}(t-t_0)}, \quad t \geq t_0, \tag{13}$$

establishing uniform exponential stability of $\mathcal{D}_0^t$ with respect to (1).

v) The proof is identical to the proof of *iv*). □

Remark 2.1 *If $w(t, z) \equiv w(z)$, then uniform stability of (4) is equivalent to the regular notion of stability for autonomous systems.*

Remark 2.2 *Note that, if in Theorem 2.1, $\mathcal{D}_i(t) \triangleq \{x \in \mathbb{R}^n : \mathcal{X}_i(t, x) = 0\}$, $t \geq t_0$, where $\mathcal{X}_i : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^{s_i}$ are continuous functions for all $i = 1, \ldots, q$, then the result of the theorem still holds for the definition of a distance given by $\mathrm{dist}(x, \mathcal{D}_0(t)) \triangleq \left[\mathcal{X}^{\mathrm{T}}(t, x)\mathcal{X}(t, x)\right]^{\frac{1}{2}}$, where $\mathcal{X}(t, x) \triangleq [\mathcal{X}_1^{\mathrm{T}}(t, x), \ldots, \mathcal{X}_q^{\mathrm{T}}(t, x)]^{\mathrm{T}}$.*

Next, we use the result of Theorem 2.1 to design stabilizing controllers for time-varying sets of multi-agent dynamical systems composed of q agents whose dynamics are given by

$$\dot{x}_i(t) = f_i(t, x(t)) + G_i(t, x(t))u_i(t), \quad t \geq t_0, \quad i = 1, \ldots, q, \tag{14}$$

where $x(t) = [x_1^{\mathrm{T}}(t) \ldots, x_q^{\mathrm{T}}(t)]^{\mathrm{T}}$, $x_i(t) \in \mathbb{R}^{n_i}$, $t \geq 0$, $f_i : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^{n_i}$ and $G_i : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^{n_i \times m_i}$ are continuous functions for all $i = 1, \ldots, q$. Consider the time-varying sets given by $\mathcal{D}_i(t) \triangleq \{x \in \mathbb{R}^n : \mathcal{X}_i(t, x_i) = 0\}$, $t \geq t_0$, where $\mathcal{X}_i : [0,\infty) \times \mathbb{R}^{n_i} \to \mathbb{R}^{s_i}$ are continuous functions for all $i = 1, \ldots, q$. Define the motion of the ith agent on the set $\mathcal{D}_i(\cdot)$ as $x_{\mathrm{e}i}(t)$, $t \geq t_0$, and note that $\mathcal{X}_i(t, x_{\mathrm{e}i}(t)) \equiv 0$. Assume there exist vector functions $u_{\mathrm{e}i}(t)$, $t \geq t_0$, $i = 1, \ldots, q$, such that

$$G_i(t, x_{\mathrm{e}}(t))u_{\mathrm{e}i}(t) = \dot{x}_{\mathrm{e}i}(t) - f_i(t, x_{\mathrm{e}}(t)), \quad t \geq t_0, \quad i = 1, \ldots, q, \tag{15}$$

where $x_{\mathrm{e}}(t) \triangleq [x_{\mathrm{e}1}^{\mathrm{T}}(t), \ldots, x_{\mathrm{e}q}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$.

The next result presents a controller design that guarantees stabilization of a time-varying set for the time-varying nonlinear dynamical system (14) using vector Lyapunov functions.

Theorem 2.2 *Consider the multi-agent dynamical system given by (14). Assume there exist a continuously differentiable, component decoupled vector function $V : [0,\infty) \times \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, that is, $V(t, x) = [V_1(t, x_1), \ldots, V_q(t, x_q)]^{\mathrm{T}}$, $(x, t) \in \mathbb{R}^n \times [0,\infty)$; a continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : [0,\infty) \times \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, and class $\mathcal{K}$ functions $\alpha : [0,\infty) \to [0,\infty)$ and $\beta : [0,\infty) \to [0,\infty)$ such that $V_i(t, x_i) = 0$, $x \in \mathcal{D}_i(t) \subset \mathbb{R}^n$, $t \geq t_0$, $i = 1, \ldots, q$; $V_i(t, x_i) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{D}_i(t)$, $t \geq t_0$, $i = 1, \ldots, q$; $\mathcal{D}_0^t = \mathcal{D}_0(t) \triangleq \cap_{i=1}^q \mathcal{D}_i(t) \neq \emptyset$, $t \geq t_0$, is a compact set at each $t \geq t_0$; $w(t,\cdot) \in \mathcal{W}$; $w(t, 0) = 0$, $t \geq 0$;*

$$\alpha(\mathrm{dist}(x, \mathcal{D}_0(t))) \leq \mathbf{e}^{\mathrm{T}} V(t, x) \leq \beta(\mathrm{dist}(x, \mathcal{D}_0(t))), \quad (x, t) \in \mathbb{R}^n \times [0,\infty), \tag{16}$$

and, for all $i = 1, \ldots, q$,

$$\frac{\partial V_i(t, x_i)}{\partial t} + V_i'(t, x_i) f_i(t, x) \leq w_i(t, V(t, x)), \quad (x, t) \in \mathcal{R}_i, \tag{17}$$

where $\mathcal{R}_i \triangleq \{(x,t) \in \mathbb{R}^n \times [0,\infty) : V_i'(t,x_i)G_i(t,x) = 0\}$, $i = 1,\ldots,q$. In addition, assume that the zero solution $z(t) \equiv 0$ to

$$\dot{z}(t) = w(t,z(t)), \quad z(0) = z_0, \quad t \geq t_0, \tag{18}$$

is uniformly asymptotically stable. Then $\mathcal{D}_0^t$ is uniformly asymptotically stable with respect to the nonlinear dynamical system (14) with the feedback control law $u = \phi(t,x) = [\phi_1^{\mathrm{T}}(t,x),\ldots,\phi_q^{\mathrm{T}}(t,x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, $t \in [0,\infty)$, given by

$$\phi_i(t,x) = \begin{cases} u_{\mathrm{e}i}(t) - \left(c_{0i} + \frac{\mu_i(t,x)+\sqrt{\lambda_i(t,x)}}{\sigma_i^{\mathrm{T}}(t,x)\sigma_i(t,x)}\right)\sigma_i(t,x), \\ \qquad\qquad \text{if } \sigma_i(t,x) \neq 0; \\ u_{\mathrm{e}i}(t), \ \text{if } \sigma_i(t,x) = 0, \end{cases} \tag{19}$$

where $u_{\mathrm{e}i}(t)$, $t \geq t_0$, satisfies (15), $\lambda_i(t,x) \triangleq \mu_i^2(t,x) + (\sigma_i^{\mathrm{T}}(t,x)\sigma_i(t,x))^2$, $\mu_i(t,x) \triangleq \rho_i(t,x) - w_i(t,V(t,x)) + \frac{\partial V_i(t,x_i)}{\partial t} + \sigma_i^{\mathrm{T}}(t,x)u_{\mathrm{e}i}(t)$, $\rho_i(t,x) \triangleq V_i'(t,x_i)f_i(t,x)$, $\sigma_i(t,x) \triangleq G_i^{\mathrm{T}}(t,x)V_i'^{\mathrm{T}}(t,x_i)$, and $c_{0i} > 0$, $i = 1,\ldots,q$. If, in addition, $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions and the zero solution $z(t) \equiv 0$ to (18) is globally uniformly asymptotically stable, then $\mathcal{D}_0^t$ is globally uniformly asymptotically stable with respect to (14) with the feedback control law $u = \phi(t,x)$ given by (19). Furthermore, if there exist constants $\nu \geq 1$, $\alpha > 0$ and $\beta > 0$ such that, instead of (16), the following inequality holds

$$\alpha\left[\mathrm{dist}(x,\mathcal{D}_0(t))\right]^\nu \leq \mathbf{e}^{\mathrm{T}}V(t,x) \leq \beta\left[\mathrm{dist}(x,\mathcal{D}_0(t))\right]^\nu, \quad (x,t) \in \mathbb{R}^n \times [0,\infty), \tag{20}$$

and the zero solution to (18) is uniformly exponentially stable, then $\mathcal{D}_0^t$ is uniformly exponentially stable with respect to (14) with the feedback control law (19). Finally, if (20) holds and the zero solution to (18) is globally uniformly exponentially stable, then $\mathcal{D}_0^t$ is globally uniformly exponentially stable with respect to (14) with the feedback control law (19).

Proof. The vector Lyapunov derivative components $\dot{V}_i(\cdot,\cdot)$, $i = 1,\ldots,q$, along the trajectories of the closed-loop system (14), with $u = \phi(t,x)$, $(x,t) \in \mathbb{R}^n \times [0,\infty)$, given by (19), satisfy

$$\begin{aligned} \dot{V}_i(t,x_i) &= \frac{\partial V_i(t,x_i)}{\partial t} + \rho_i(t,x) + \sigma_i^{\mathrm{T}}(t,x)\phi_i(t,x) \\ &= \begin{cases} \frac{\partial V_i(t,x_i)}{\partial t} + \rho_i(t,x) + \sigma_i^{\mathrm{T}}(t,x)u_{\mathrm{e}i}(t) \\ \quad -c_{0i}\sigma_i^{\mathrm{T}}(t,x)\sigma_i(t,x) - \mu_i(t,x) \\ \quad -\sqrt{\lambda_i(t,x)}, \ \text{if } \sigma_i(t,x) \neq 0; \\ \frac{\partial V_i(t,x_i)}{\partial t} + \rho_i(t,x), \ \text{if } \sigma_i(t,x) = 0, \end{cases} \\ &\leq w_i(t,V(t,x)), \quad (x,t) \in \mathbb{R}^n \times [0,\infty). \end{aligned}$$

Now, the result is a direct consequence of Theorem 2.1. □

Remark 2.3 *Note that if, in Theorem 2.2, $q = 1$, $\mathcal{D}_0^t \equiv \{0\}$, and (14) is a time-invariant system, then we can set $x_{\mathrm{e}}(t) \equiv 0$, $u_{\mathrm{e}}(t) \equiv 0$, $w(t,z) \equiv 0$, and $V(t,x) \equiv V(x)$. In this case, the feedback control law (19) specializes to Sontag's universal formula [31].*

Remark 2.4 *If $\mathcal{R}_i = \emptyset$, $i = 1,\ldots,q$, then $w(\cdot,\cdot)$ in (17) can be chosen arbitrarily so that the comparison system (18) is (globally) uniformly asymptotically (respectively, exponentially) stable. In addition, since $\mathcal{D}_i(t) = \{x \in \mathbb{R}^n : \mathcal{X}_i(t,x_i) = 0\}$, $t \geq t_0$, then $V_i(\cdot,\cdot)$, $i = 1,\ldots,q$, can be chosen arbitrarily provided that $V_i(t,x_i) = 0$, $x \in \mathcal{D}_i(t)$, $t \geq t_0$, $V_i(t,x_i) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{D}_i(t)$, $t \geq t_0$, $i = 1,\ldots,q$, and (16) (respectively, (20)) holds. For example, $V_i(\cdot,\cdot)$ can be taken as $V_i(t,x_i) = \mathcal{X}_i^{\mathrm{T}}(t,x_i)P_i\mathcal{X}_i(t,x_i)$, $x_i \in \mathbb{R}^{n_i}$, where $P_i \in \mathbb{R}^{s_i \times s_i}$ is such that $P_i > 0$, $i = 1,\ldots,q$. In this case, $\alpha(\mathrm{dist}(x,\mathcal{D}_0(t))) = \alpha\left[\mathrm{dist}(x,\mathcal{D}_0(t))\right]^2$, where $0 < \alpha \leq \min_{i=1,\ldots,q}\{\lambda_{\min}(P_i)\}$, $\beta(\mathrm{dist}(x,\mathcal{D}_0(t))) = \beta\left[\mathrm{dist}(x,\mathcal{D}_0(t))\right]^2$, where $\beta \geq \max_{i=1,\ldots,q}\{\lambda_{\max}(P_i)\}$, and $\mathrm{dist}(x,\mathcal{D}_0(t)) \triangleq \left[\mathcal{X}^{\mathrm{T}}(t,x)\mathcal{X}(t,x)\right]^{\frac{1}{2}}$, where $\mathcal{X}(t,x) \triangleq [\mathcal{X}_1^{\mathrm{T}}(t,x_1),\ldots,\mathcal{X}_q^{\mathrm{T}}(t,x_q)]^{\mathrm{T}}$. In this case, it follows from Remark 2.2 and Theorem 2.2 that for the closed-loop system (14), (19) the time-varying set $\mathcal{D}_0^t$ is (globally) uniformly asymptotically (respectively, exponentially) stable.*

3 Control Design for Multi-Vehicle Coordinated Motion

In this section, we apply the results of Section 2 to a problem of coordinated motion of multiple vehicles in pursuit of a (virtual) leader. Specifically, we design a distributed feedback control law that drives individual vehicles to a configuration with specified distance and orientation with respect to a leader while maintaining this configuration throughout the motion of the leader. The leader can be either real or virtual. In the latter case, the agents synthesize a motion with respect to which they need to maintain a specified formation. To elucidate the control design, consider planar motion of q agents with the individual agent dynamics given by

$$\ddot{x}_i(t) = u_{xi}(t), \quad x_i(0) = x_{i0}, \quad \dot{x}_i(0) = \dot{x}_{i0}, \quad t \geq 0, \tag{21}$$

$$\ddot{y}_i(t) = u_{yi}(t), \quad y_i(0) = y_{i0}, \quad \dot{y}_i(0) = \dot{y}_{i0}, \tag{22}$$

where $x_i : [0,\infty) \to \mathbb{R}$ and $y_i : [0,\infty) \to \mathbb{R}$, $i = 1,\ldots,q$, are the displacements of the ith agent in the horizontal and vertical directions, respectively, and u_{xi} and u_{yi} are the control forces acting on the ith agent in the horizontal and vertical directions, respectively. Next, define $\eta_i \triangleq [x_i, y_i, \dot{x}_i, \dot{y}_i]^{\mathrm{T}}$, $i = 1,\ldots,q$, and $\eta \triangleq [\eta_1^{\mathrm{T}},\ldots,\eta_q^{\mathrm{T}}]^{\mathrm{T}}$.

Then the generalized dynamics (21), (22) for q agents can be written in the state space form as

$$\dot{\eta}(t) = (I_q \otimes A)\eta(t) + (I_q \otimes B)u(t), \quad \eta(0) = \eta_0, \quad t \geq 0, \tag{23}$$

where $\eta_0 = [\eta_{10}^{\mathrm{T}}, \ldots, \eta_{q0}^{\mathrm{T}}]^{\mathrm{T}}$, $\eta_{i0} = [x_{i0}, y_{i0}, \dot{x}_{i0}, \dot{y}_{i0}]^{\mathrm{T}}$, $u \triangleq [u_1^{\mathrm{T}}, \ldots, u_q^{\mathrm{T}}]^{\mathrm{T}}$, $u_i \triangleq [u_{xi}, u_{yi}]^{\mathrm{T}}$, "$\otimes$" is the Kronecker product [2], $I_q \in \mathbb{R}^{q\times q}$ is the identity matrix, and A, B are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{24}$$

Furthermore, we define the time-varying sets

$$\mathcal{D}_i(t) \triangleq \{\eta \in \mathbb{R}^{4q} : \eta_i - p_i(t) = 0\}, \quad t \geq 0, \quad i = 1, \ldots, q, \tag{25}$$

where

$$p_i(t) \triangleq \begin{bmatrix} x_{\mathrm{L}}(t) + l_{xi\mathrm{L}} \\ y_{\mathrm{L}}(t) + l_{yi\mathrm{L}} \\ \dot{x}_{\mathrm{L}}(t) \\ \dot{y}_{\mathrm{L}}(t) \end{bmatrix}, \quad t \geq 0, \quad i = 1, \ldots, q, \tag{26}$$

$x_{\mathrm{L}}(t), y_{\mathrm{L}}(t), t \geq 0$, are, respectively, horizontal and vertical positions of the leader, $\dot{x}_{\mathrm{L}}(t), \dot{y}_{\mathrm{L}}(t), t \geq 0$, are, respectively, horizontal and vertical velocities of the leader, and $l_{xi\mathrm{L}}, l_{yi\mathrm{L}} \in \mathbb{R}$ are, respectively, desired horizontal and vertical distances between the ith agent and the leader. Note that each set $\mathcal{D}_i(t)$, $t \geq 0$, $i = 1, \ldots, q$, defines relative position and velocity of the ith agent with respect to the leader. In order to construct the set $\mathcal{D}_i(t)$, $t \geq 0$, $i = 1, \ldots, q$, only the local information about the ith agent position and velocity is needed. The position and velocity of the leader at each instant of time are assumed to be known. Furthermore, the intersection of the sets (25) given by

$$\mathcal{D}_0^t = \mathcal{D}_0(t) \triangleq \bigcap_{i=1,\ldots,q} \mathcal{D}_i(t), \quad t \geq 0, \tag{27}$$

characterizes the desired formation of agents with respect to the leader where all agents maintain specified distances and velocities with respect to the leader.

Remark 3.1 *Note that this approach of characterizing multi-vehicle formations via time-varying sets also captures formations where only neighbor-to-neighbor interactions are permitted [14,26,28,12,3]. In this case, as long as the connectivity graph describing the entire multi-vehicle formation is strongly connected [10,12], the formation is uniquely defined by a time-varying set characterizing neighbor-to-neighbor relative positions and velocities.*

Next, we define the component decoupled vector function $V : [0, \infty) \times \mathbb{R}^{4q} \to \mathbb{R}^q$ such that $V(t, \eta) = [V_1(t, \eta_1), \ldots, V_q(t, \eta_q)]^{\mathrm{T}}$, where

$$V_i(t, \eta_i) = (\eta_i - p_i(t))^{\mathrm{T}} P(\eta_i - p_i(t)), \quad \eta_i \in \mathbb{R}^4, \quad t \geq 0, \quad i = 1, \ldots, q, \tag{28}$$

where

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0. \tag{29}$$

Note that $V_i(t, \eta_i) = 0$, $\eta \in \mathcal{D}_i(t)$, $t \geq 0$, and $V_i(t, \eta_i) > 0$, $\eta \in \mathbb{R}^{4q} \setminus \mathcal{D}_i(t)$, $t \geq 0$, $i = 1, \ldots, q$. In addition, since $\lambda_{\min}(P) = \lambda_{\max}(P) = 1$, condition (20) is satisfied with $\alpha = \frac{1}{2}$, $\beta = 2$, $\nu = 2$, $\mathrm{dist}(\eta, \mathcal{D}_0(t)) \triangleq \left[(\eta - p(t))^{\mathrm{T}}(\eta - p(t))\right]^{\frac{1}{2}}$, $\eta \in \mathbb{R}^{4q}$, $t \geq 0$, where $p(t) \triangleq [p_1^{\mathrm{T}}(t), \ldots, p_q^{\mathrm{T}}(t)]^{\mathrm{T}}$. Furthermore, it can be shown that, for $\mathcal{R}_i \triangleq \{(\eta, t) \in \mathbb{R}^{4q} \times [0, \infty) : V_i'(t, \eta_i)B = 0\}$, $i = 1, \ldots, q$, condition (17) is satisfied with

$$\frac{\partial V_i(t, \eta_i(t))}{\partial t} + V_i'(t, \eta_i(t))A\eta_i(t) \leq -\gamma_i V_i(t, \eta_i(t)), \quad (\eta, t) \in \mathcal{R}_i, \quad i = 1, \ldots, q, \tag{30}$$

for

$$\gamma_i \in (0, 1], \quad i = 1, \ldots, q. \tag{31}$$

In this case, the zero solution to (18) is globally exponentially stable with

$$w(z) = [-\gamma_1 z_1, \ldots, -\gamma_q z_q]^{\mathrm{T}}. \tag{32}$$

Hence, it follows from Theorem 2.2 that the time-varying set $\mathcal{D}_0^t$ defined by (27) is globally uniformly exponentially stable with respect to (23) with the feedback control law $u_i = \phi_i(t, \eta_i)$, $i = 1, \ldots, q$, given by (19) with $\mu_i(t, \eta_i) \triangleq \rho_i(t, \eta_i) - w_i(V_i(t, \eta_i)) + \frac{\partial V_i(t, \eta_i)}{\partial t} + \sigma_i^{\mathrm{T}}(t, \eta_i)u_{\mathrm{e}i}(t)$, $\rho_i(t, \eta_i) \triangleq V_i'(t, \eta_i)A\eta_i$, $\sigma_i(t, \eta_i) \triangleq B^{\mathrm{T}}V_i'^{\mathrm{T}}(t, \eta_i)$, $u_{\mathrm{e}i}(t) = [\ddot{x}_{\mathrm{L}}(t), \ddot{y}_{\mathrm{L}}(t)]^{\mathrm{T}}$, and $w(V(t, \eta))$ given by (32). Note that the feedback control law $u_i = \phi_i(t, \eta_i)$, $i = 1, \ldots, q$, is a distributed control algorithm [28,3] which uses only local information about relative position and velocity of the ith agent with respect to the leader. This allows to reproduce this controller without

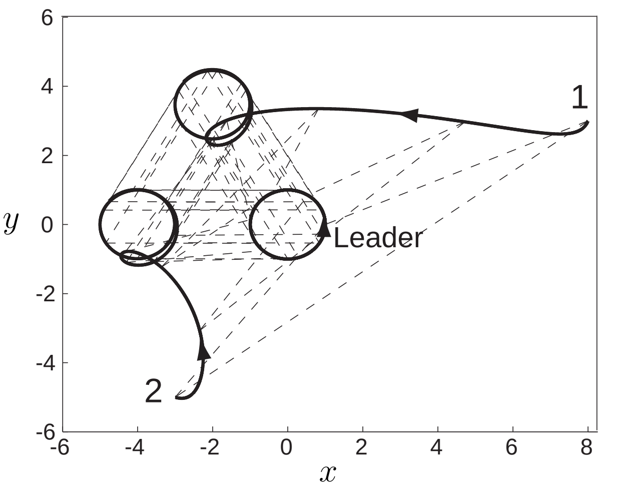

Fig. 1. Position phase portrait of two agents following the leader.

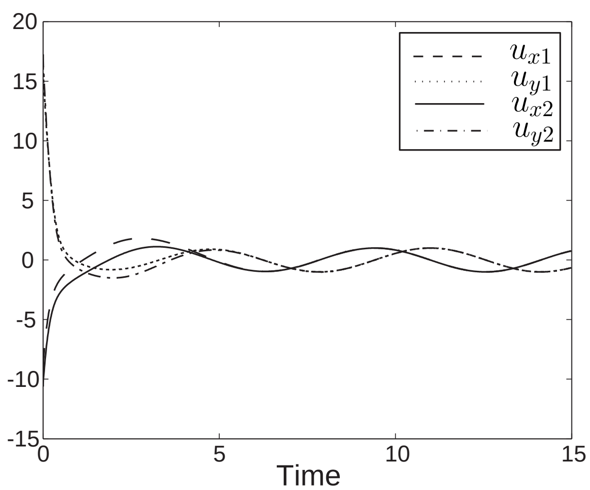

Fig. 2. Control forces in horizontal and vertical directions versus time.

changing its structure as many times as the number of agents in order to steer individual agent while maintaining a specified formation with respect to the leader.

In the following simulation, we consider two agents pursuing a leader in a triangular formation. For this, we set $l_{x1\mathrm{L}} = -2$, $l_{y1\mathrm{L}} = 2\sqrt{3}$, $l_{x2\mathrm{L}} = -4$, $l_{y2\mathrm{L}} = 0$, $c_{0i} = 0.5$, $i = 1, 2$, $\gamma_i = \frac{1}{5}$, $i = 1, 2$, $\eta_{10} = [8, 3, -1, -3]^{\mathrm{T}}$, and $\eta_{20} = [-3, -5, 3, -1]^{\mathrm{T}}$. With this choice of the parameters $l_{xi\mathrm{L}}$ and $l_{yi\mathrm{L}}$, $i = 1, 2$, the agents will form a configuration of an equilateral triangle with respect to the leader. Furthermore, the leader is set to be moving counter-clockwise around a circle of radius 1 according to $x_{\mathrm{L}}(t) = \cos t$, $y_{\mathrm{L}}(t) = \sin t$, $t \geq 0$. For the feedback controller (19), Figure 1 shows position phase portrait of two agents following the leader and Figure 2 shows the time history of the control forces acting on each agent.

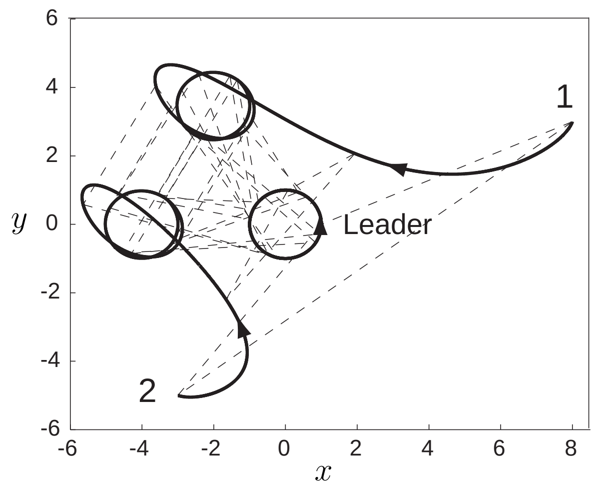

Fig. 3. Position phase portrait of two agents following the leader.

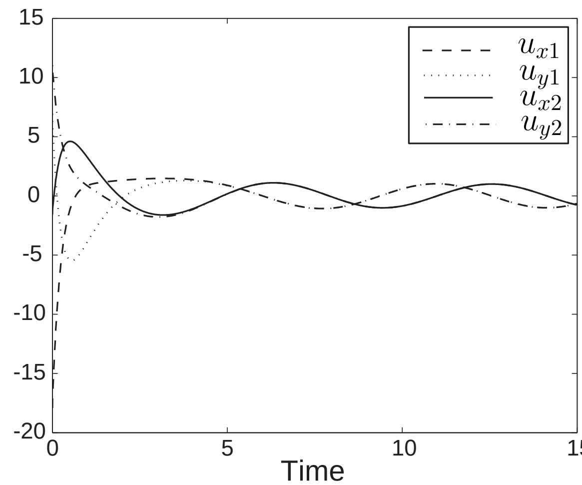

Fig. 4. Control forces in horizontal and vertical directions versus time.

Next, we compare the performance of the control law (19) with the performance of two other cooperative control algorithms developed in [28] for the same formation control problem as above with the same data. Specifically, the first control law that we consider for the system (21), (22) is given by

$$
\begin{aligned}
[u_{xi}(t), u_{yi}(t)]^{\mathrm{T}} = & -K_g \tilde{h}_i(t) - D_g \dot{h}_i(t) \\
& -K_f(\tilde{h}_i(t) - \tilde{h}_{i-1}(t)) \\
& -D_f(\dot{h}_i(t) - \dot{h}_{i-1}(t)) \\
& -K_f(\tilde{h}_i(t) - \tilde{h}_{i+1}(t)) \\
& -D_f(\dot{h}_i(t) - \dot{h}_{i+1}(t)), \quad t \geq 0, \qquad (33)
\end{aligned}
$$

where $i = 1, 2$, $\tilde{h}_i(t) \triangleq h_i(t) - h_{i\mathrm{d}}(t)$, $h_i(t) \triangleq [x_i(t), y_i(t)]^{\mathrm{T}}$, $h_{i\mathrm{d}}(t) \triangleq [x_{\mathrm{L}}(t) + l_{xi\mathrm{L}}, y_{\mathrm{L}}(t) + l_{yi\mathrm{L}}]^{\mathrm{T}}$, $\tilde{h}_3(t) \triangleq \tilde{h}_1(t)$, $\tilde{h}_0(t) \triangleq \tilde{h}_2(t)$, $h_3(t) \triangleq h_1(t)$, and $h_0(t) \triangleq h_2(t)$. The control gains $K_g \in \mathbb{R}^{2\times 2}$, $D_g \in \mathbb{R}^{2\times 2}$ are

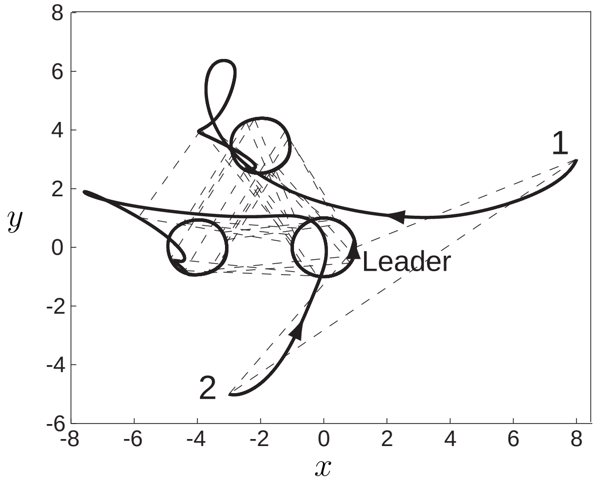

Fig. 5. Position phase portrait of two agents following the leader.

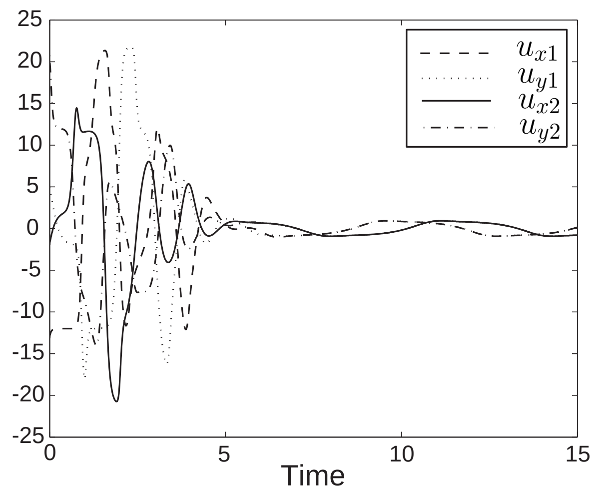

Fig. 6. Control forces in horizontal and vertical directions versus time.

positive-definite matrices and control gains $K_f \in \mathbb{R}^{2\times 2}$, $D_f \in \mathbb{R}^{2\times 2}$ are positive-semidefinite matrices. The second control law for the system (21), (22) accounts for the actuator saturation and is given by

$$\begin{aligned}[u_{xi}(t), u_{yi}(t)]^{\mathrm{T}} = &-k_g \tanh(k\tilde{h}_i(t)) - d_g \tanh(k\dot{\tilde{h}}_i(t)) \\ &-k_f \tanh[k(\tilde{h}_i(t) - \tilde{h}_{i-1}(t))] \\ &-k_f \tanh[k(\tilde{h}_i(t) - \tilde{h}_{i+1}(t))], \qquad (34)\end{aligned}$$

where $k_g > 0$, $k > 0$, $d_g > 0$, $k_f \geq 0$, and $\tanh(\cdot)$ is the hyperbolic tangent defined componentwise.

Figures 3 and 4 show the performance of the controller (33) with $K_g = D_g = K_f = D_f = I_2$. Furthermore, Figures 5 and 6 show the performance of the controller (34) with $k_g = 7$, $d_g = 5$, $k_f = 5$, and $k = 1$. In both cases, the values of the control gains were slightly altered from the ones in [28] to yield the best compromise between the convergence time and the control effort. It was observed that both controllers, (33) and (34), retain a steady-state error between the desired and actual positions of each agent. This corresponds to a triangular steady state formation of two agents with respect to the leader that oscillates around a desired equilateral triangle but never converges to it. Alternatively, controller (19) ensures exponential stabilization of the desired formation. In addition, the rate of change for the controller (34) is significantly higher than that of (19) and (33).

4 Stability and Stabilization of Time-Invariant Sets

In this section, we present results on stabilization of time-invariant sets for time-invariant nonlinear dynamical systems using vector Lyapunov functions. Consider the time-invariant nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \qquad (35)$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq 0$, is the system state vector, $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f(0) = 0$, and $f(\cdot)$ is Lipschitz continuous on $\mathcal{D}$.

Definition 4.1 *For the nonlinear dynamical system (35), let $\mathcal{D}_0 \subset \mathcal{D}$ be a compact positively invariant set with respect to (35). $\mathcal{D}_0$ is Lyapunov stable if, for every open neighborhood $\mathcal{O}_1 \subseteq \mathcal{D}$ of $\mathcal{D}_0$, there exists an open neighborhood $\mathcal{O}_2 \subseteq \mathcal{O}_1$ of $\mathcal{D}_0$ such that $x(t) \in \mathcal{O}_1$, $t \geq 0$, for all $x_0 \in \mathcal{O}_2$. $\mathcal{D}_0$ is* asymptotically stable *if it is Lyapunov stable and there exists a neighborhood $\mathcal{O}_1$ of $\mathcal{D}_0$ such that* $\mathrm{dist}(x(t), \mathcal{D}_0) \to 0$ *as $t \to \infty$ for all $x_0 \in \mathcal{O}_1$. $\mathcal{D}_0$ is* globally asymptotically stable *if it is Lyapunov stable and* $\mathrm{dist}(x(t), \mathcal{D}_0) \to 0$ *as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. $\mathcal{D}_0$ is exponentially stable if there exist $\alpha > 0$, $\beta > 0$, and a neighborhood $\mathcal{O}_1$ of $\mathcal{D}_0$ such that* $\mathrm{dist}(x(t), \mathcal{D}_0) \leq \alpha\, \mathrm{dist}(x_0, \mathcal{D}_0) e^{-\beta t}$, *$t \geq 0$, for all $x_0 \in \mathcal{O}_1$. Finally, $\mathcal{D}_0$ is globally exponentially stable if there exist $\alpha > 0$ and $\beta > 0$ such that* $\mathrm{dist}(x(t), \mathcal{D}_0) \leq \alpha\, \mathrm{dist}(x_0, \mathcal{D}_0) e^{-\beta t}$, *$t \geq 0$, for all $x_0 \in \mathbb{R}^n$.*

Theorem 4.1 *Consider the nonlinear dynamical system (35). Assume there exists a continuously differentiable vector function $V = [V_1, \ldots, V_q]^{\mathrm{T}} : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$, where $\mathcal{Q} \subset \mathbb{R}^q$ and $0 \in \mathcal{Q}$, such that $V_i(x) = 0$, $x \in \mathcal{D}_i$, where $\mathcal{D}_i \subset \mathcal{D}$, $i = 1, \ldots, q$; $V_i(x) > 0$, $x \in \mathcal{D}\backslash\mathcal{D}_i$, $i = 1, \ldots, q$; $\mathcal{D}_0 \triangleq \cap_{i=1}^q \mathcal{D}_i \neq \emptyset$ is a compact positively invariant set with respect to (35), and*

$$V'(x)f(x) \leq\leq w(V(x)), \quad x \in \mathcal{D}, \qquad (36)$$

where $w : \mathcal{Q} \to \mathbb{R}^q$ is continuous, $w(\cdot) \in \mathcal{W}$, and $w(0) = 0$. In addition, assume that the vector comparison system

$$\dot{z}(t) = w(z(t)), \quad z(0) = z_0, \quad t \geq 0, \qquad (37)$$

has a unique solution in forward time $z(t)$, $t \geq 0$. Then the following statements hold:

i) *If the zero solution $z(t) \equiv 0$ to (37) is Lyapunov stable, then $\mathcal{D}_0$ is Lyapunov stable with respect to (35).*

ii) *If the zero solution $z(t) \equiv 0$ to (37) is asymptotically stable, then $\mathcal{D}_0$ is asymptotically stable with respect to (35).*

iii) *If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, and $v(x) \triangleq \mathbf{e}^{\mathrm{T}}V(x)$, $x \in \mathbb{R}^n$, is such that $v(x) \to \infty$ as $\mathrm{dist}(x, \mathcal{D}_0) \to \infty$, and the zero solution $z(t) \equiv 0$ to (37) is globally asymptotically stable, then $\mathcal{D}_0$ is globally asymptotically stable with respect to (35).*

iv) *If there exist constants $\nu \geq 1$, $\alpha > 0$, $\beta > 0$ such that*

$$\alpha[\mathrm{dist}(x, \mathcal{D}_0)]^\nu \leq \mathbf{e}^{\mathrm{T}}V(x) \leq \beta[\mathrm{dist}(x, \mathcal{D}_0)]^\nu, \quad x \in \mathcal{D}, \tag{38}$$

and the zero solution $z(t) \equiv 0$ to (37) is exponentially stable, then $\mathcal{D}_0$ is exponentially stable with respect to (35).

v) *If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, (38) holds for all $x_0 \in \mathbb{R}^n$, and the zero solution $z(t) \equiv 0$ to (37) is globally exponentially stable, then $\mathcal{D}_0$ is globally exponentially stable with respect to (35).*

Proof. The proof is similar to the proof of Theorem 2.1, and hence, is omitted. □

Next, we use the result of Theorem 4.1 to design controllers to stabilize time-invariant sets for multi-agent dynamical systems composed of q agents whose dynamics are given by

$$\dot{x}_i(t) = f_i(x(t)) + G_i(x(t))u_i(t), \quad t \geq t_0, \quad i = 1, \ldots, q, \tag{39}$$

where $x(t) = [x_1^{\mathrm{T}}(t) \ldots, x_q^{\mathrm{T}}(t)]^{\mathrm{T}}$, $x_i(t) \in \mathbb{R}^{n_i}$, $t \geq 0$, $f_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ satisfying $f_i(0) = 0$ and $G_i : \mathbb{R}^n \to \mathbb{R}^{n_i \times m_i}$ are continuous functions for all $i = 1, \ldots, q$, and $u_i(\cdot)$, $i = 1, \ldots, q$, satisfy sufficient regularity conditions such that the nonlinear dynamical system (39) has a unique solution forward in time.

Theorem 4.2 *Consider the multi-agent dynamical system given by (39). Assume there exist a continuously differentiable, component decoupled vector function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, that is, $V(x) = [V_1(x_1), \ldots, V_q(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, and continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, such that $V_i(x_i) = 0$, $x \in \mathcal{D}_i \subset \mathbb{R}^n$, $i = 1, \ldots, q$; $V_i(x_i) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{D}_i$, $i = 1, \ldots, q$; $\mathcal{D}_0 \triangleq \cap_{i=1}^q \mathcal{D}_i \neq \emptyset$ is a compact set; $w(\cdot) \in \mathcal{W}$; $w(0) = 0$, and, for all $i = 1, \ldots, q$,*

$$V_i'(x_i)f_i(x) \leq w_i(V(x)), \quad x \in \mathcal{R}_i, \tag{40}$$

where $\mathcal{R}_i \triangleq \{x \in \mathbb{R}^n : V_i'(x_i)G_i(x) = 0\}$, $i = 1, \ldots, q$. In addition, assume that the zero solution $z(t) \equiv 0$ to (37) is asymptotically stable. Then $\mathcal{D}_0$ is asymptotically stable with respect to the nonlinear dynamical system (39) with the feedback control law $u = \phi(x) = [\phi_1^{\mathrm{T}}(x), \ldots, \phi_q^{\mathrm{T}}(x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, given by

$$\phi_i(x) = \begin{cases} -\left(c_{0i} + \frac{(\rho_i(x) - w_i(V(x))) + \sqrt{\lambda_i(x)}}{\sigma_i^{\mathrm{T}}(x)\sigma_i(x)}\right)\sigma_i(x), \\ \qquad\qquad \text{if } \ \sigma_i(x) \neq 0; \\ 0, \qquad\qquad \text{if } \ \sigma_i(x) = 0, \end{cases} \tag{41}$$

where $\lambda_i(x) \triangleq (\rho_i(x) - w_i(V(x)))^2 + (\sigma_i^{\mathrm{T}}(x)\sigma_i(x))^2$, $\rho_i(x) \triangleq V_i'(x_i)f_i(x)$, $x \in \mathbb{R}^n$, $\sigma_i(x) \triangleq G_i^{\mathrm{T}}(x)V_i'^{\mathrm{T}}(x_i)$, $x \in \mathbb{R}^n$, and $c_{0i} > 0$, $i = 1, \ldots, q$. If, in addition, $v(x) \triangleq \mathbf{e}^{\mathrm{T}}V(x)$, $x \in \mathbb{R}^n$, is such that $v(x) \to \infty$ as $\mathrm{dist}(x, \mathcal{D}_0) \to \infty$, and the zero solution $z(t) \equiv 0$ to (37) is globally asymptotically stable, then $\mathcal{D}_0$ is globally asymptotically stable with respect to (39) with the feedback control law (41). Furthermore, if there exist constants $\nu \geq 1$, $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\,[\mathrm{dist}(x, \mathcal{D}_0)]^\nu \leq \mathbf{e}^{\mathrm{T}}V(x) \leq \beta\,[\mathrm{dist}(x, \mathcal{D}_0)]^\nu\,, \quad x \in \mathbb{R}^n, \tag{42}$$

and the zero solution to (37) is exponentially stable, then $\mathcal{D}_0$ is exponentially stable with respect to (39) with the feedback control law (41). Finally, if (42) holds and the zero solution to (37) is globally exponentially stable, then $\mathcal{D}_0$ is globally exponentially stable with respect to (39) with the feedback control law (41).

Proof. The vector Lyapunov derivative components $\dot{V}_i(\cdot)$, $i = 1, \ldots, q$, along the trajectories of the closed-loop dynamical system (39), with $u = \phi(x)$, $x \in \mathbb{R}^n$, given by (41), are given by

$$\begin{aligned} \dot{V}_i(x_i) &= V_i'(x_i)[f_i(x) + G_i(x)\phi_i(x)] \\ &= \rho_i(x) + \sigma_i^{\mathrm{T}}(x)\phi_i(x) \\ &= \begin{cases} -c_{0i}\sigma_i^{\mathrm{T}}(x)\sigma_i(x) - \sqrt{\lambda_i(x)} \\ \qquad\qquad + w_i(V(x)), & \text{if } \ \sigma_i(x) \neq 0; \\ \rho_i(x), & \text{if } \ \sigma_i(x) = 0, \end{cases} \\ &\leq w_i(V(x)), \quad x \in \mathbb{R}^n. \end{aligned} \tag{43}$$

Now, the result is a direct consequence of Theorem 4.1. □

Remark 4.1 *As in Remark 2.4, if $\mathcal{R}_i = \emptyset$, $i = 1, \ldots, q$, then $w(\cdot)$ in (40) and (41) can be chosen arbitrarily so that the comparison system (37) is (globally) asymptotically (respectively, exponentially) stable. In addition, if $\mathcal{D}_i = \{x \in \mathbb{R}^n : \mathcal{X}_i(x_i) = 0\}$, where $\mathcal{X}_i : \mathbb{R}^{n_i} \to \mathbb{R}^{s_i}$ are continuous functions for all $i = 1, \ldots, q$, then $V_i(\cdot)$, $i = 1, \ldots, q$, can be chosen arbitrarily provided that $V_i(x_i) = 0$, $x \in \mathcal{D}_i$, and $V_i(x_i) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{D}_i$, $i = 1, \ldots, q$. For example, $V_i(\cdot)$ can be taken as $V_i(x_i) = \mathcal{X}_i^{\mathrm{T}}(x_i)P_i\mathcal{X}_i(x_i)$,*

$x_i \in \mathbb{R}^{n_i}$, *where* $P_i \in \mathbb{R}^{s_i \times s_i}$ *is such that* $P_i > 0$, $i = 1, \ldots, q$.

5 Control Design for Static Formations

In this section, we apply the results of Section 4 to stabilize static formations of multiple vehicles. Specifically, we design a feedback control law that drives two agents to a configuration with specified distance between the agents and orientation with respect to the horizontal. For this, consider the dynamics of two agents given by (21), (22) and rewrite them in the state space form as

$$\dot{\xi}_1(t) = A\xi_1(t) + B\tilde{u}_1(t), \quad \xi_1(0) = \xi_{10}, \tag{44}$$
$$\dot{\xi}_2(t) = A\xi_2(t) + B\tilde{u}_2(t), \quad \xi_2(0) = \xi_{20}, \tag{45}$$

where $\xi_1 \triangleq [x_1, x_2, \dot{x}_1, \dot{x}_2]^{\mathrm{T}}$, $\xi_2 \triangleq [y_1, y_2, \dot{y}_1, \dot{y}_2]^{\mathrm{T}}$, $\xi \triangleq [\xi_1^{\mathrm{T}}, \xi_2^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{u}_1 \triangleq [u_{x1}, u_{x2}]^{\mathrm{T}}$, $\tilde{u}_2 \triangleq [u_{y1}, u_{y2}]^{\mathrm{T}}$, and A, B are given by (24). Next, define the sets $\mathcal{D}_1 \triangleq \{\xi \in \mathbb{R}^8 : E\xi_1 - p_x = 0\}$ and $\mathcal{D}_2 \triangleq \{\xi \in \mathbb{R}^8 : E\xi_2 - p_y = 0\}$, where

$$E = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad p_x \triangleq \begin{bmatrix} l_x \\ 0 \\ 0 \end{bmatrix}, \quad p_y \triangleq \begin{bmatrix} l_y \\ 0 \\ 0 \end{bmatrix}, \tag{46}$$

and $l_x, l_y \in \mathbb{R}$. Note that $\mathcal{D}_0 \triangleq \mathcal{D}_1 \cap \mathcal{D}_2$ determines a family of formations for two agents where both agents are at the equilibrium and the distance between the agents and the angle with respect to the horizontal, respectively, are given by

$$L = (l_x^2 + l_y^2)^{\frac{1}{2}} \quad \text{and} \quad \theta = \begin{cases} \arctan\left(\frac{l_y}{l_x}\right), & l_x \geq 0, \\ \pi + \arctan\left(\frac{l_y}{l_x}\right), & l_x < 0. \end{cases} \tag{47}$$

Furthermore, note that for any pair of $L > 0$ and $\theta \in [-\frac{\pi}{2}, \frac{3}{2}\pi]$, there exist unique $l_x \in \mathbb{R}$ and $l_y \in \mathbb{R}$ such that (47) is satisfied. Next, define a component decoupled vector function $V : \mathbb{R}^8 \to \mathbb{R}^2$ such that $V(\xi) = [V_1(\xi_1), V_2(\xi_2)]^{\mathrm{T}}$, $\xi = [\xi_1^{\mathrm{T}}, \xi_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^8$, where

$$V_1(\xi_1) = \frac{1}{2}(E\xi_1 - p_x)^{\mathrm{T}} P (E\xi_1 - p_x), \quad \xi_1 \in \mathbb{R}^4, \tag{48}$$
$$V_2(\xi_2) = \frac{1}{2}(E\xi_2 - p_y)^{\mathrm{T}} P (E\xi_2 - p_y), \quad \xi_2 \in \mathbb{R}^4, \tag{49}$$

with

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} > 0. \tag{50}$$

Note that $V_i(\xi_i) = 0, \xi \in \mathcal{D}_i$, and $V_i(\xi_i) > 0, \xi \in \mathbb{R}^8 \backslash \mathcal{D}_i$, $i = 1, 2$. It can be shown that, for $\mathcal{R}_i \triangleq \{\xi \in \mathbb{R}^8 : V_i'(\xi_i)B = 0\}$, condition (40) is satisfied with

$$V_i'(\xi_i)A\xi_i \leq -\gamma_i V_i(\xi_i), \quad \xi \in \mathcal{R}_i, \quad i = 1, 2, \tag{51}$$

for

$$\gamma_i \in (0, 1], \quad i = 1, 2. \tag{52}$$

In this case, the zero solution to (37) is globally exponentially stable with

$$w(V) = \begin{bmatrix} -\gamma_1 V_1 \\ -\gamma_2 V_2 \end{bmatrix}. \tag{53}$$

Furthermore, since $\lambda_{\min}(P) = 1$ and $\lambda_{\max}(P) = 3$, condition (42) is satisfied with $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\nu = 2$, and

$$\mathrm{dist}(\xi, \mathcal{D}_0) \triangleq \left([(E\xi_1 - p_x)^{\mathrm{T}}, (E\xi_2 - p_y)^{\mathrm{T}}] \begin{bmatrix} E\xi_1 - p_x \\ E\xi_2 - p_y \end{bmatrix} \right)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}^8. \tag{54}$$

Thus, it follows from Theorem 4.2 that $\mathcal{D}_0$ is globally exponentially stable with respect to (44), (45) with the feedback control law $\tilde{u}_i = \phi_i(\xi_i)$ given by (41) where $\rho_i(\xi_i) = V_i'(\xi_i)A\xi_i$, $\sigma_i(\xi_i) = B^{\mathrm{T}} V_i'^{\mathrm{T}}(\xi_i)$, $i = 1, 2$, and $w(V)$ is given by (53).

In the following simulation we set $l_x = \frac{1}{\sqrt{2}}$, $l_y = \frac{1}{\sqrt{2}}$, $c_{0i} = 0.2$, $\gamma_i = \frac{1}{2}$, $i = 1, 2$, $\xi_{10} = [2, 5, -3, 2]^{\mathrm{T}}$, and $\xi_{20} = [3, 4, 4, 1]^{\mathrm{T}}$. With this choice of the parameters l_x and l_y, the steady state distance between agents is 1 with the angle with respect to the horizontal being $\frac{\pi}{4}$. Figure 7 shows position phase portrait of two agents and Figure 8 shows the time history of the control forces acting on each agent.

6 Conclusion

In this paper, we developed stability analysis and control design framework for time-varying and time-invariant sets of nonlinear time-varying dynamical systems. Specifically, we presented sufficient conditions guaranteeing several types of stability of time-varying sets using vector Lyapunov functions. Based on the stability results, we developed a distributed control framework for time-varying sets and applied it to stabilization of moving formations of multiple agents in pursuit of a (virtual) leader. It was shown that, for a system of planar double integrators, the developed distributed control algorithm globally exponentially stabilizes a specified

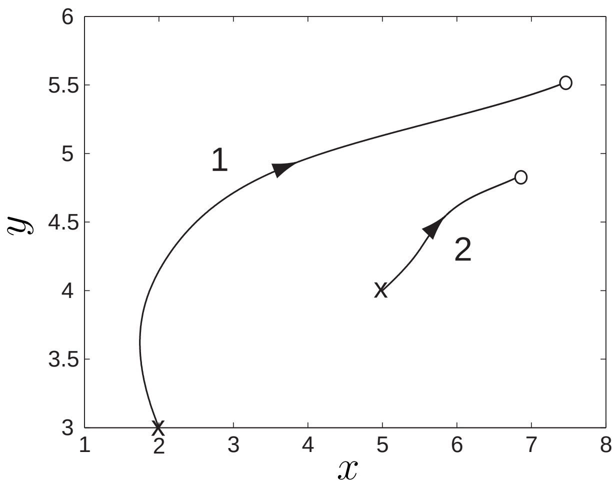

Fig. 7. Position phase portrait of two agents.

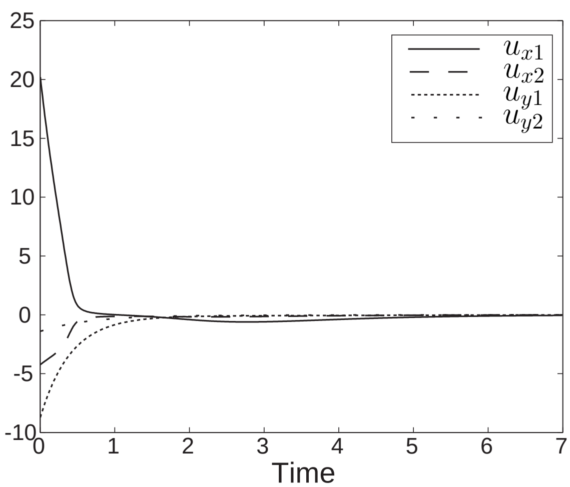

Fig. 8. Control forces in horizontal and vertical directions versus time.

moving formation with respect to the leader. Furthermore, we developed stabilizing control framework for time-invariant sets and applied it to globally exponentially stabilize static formations of multiple vehicles. Finally, it should be noted that the stability results for time-varying and time-invariant sets of nonlinear dynamical systems developed in this paper can be used to design various other control algorithms to achieve stable coordinated motion of multi-vehicle systems. Furthermore, the results presented here for systems of double integrators can be extended to more general models using the feedback linearization technique [13].

Acknowledgements

This research was supported in part by the Office of Naval Research under Grant N00014-09-1-1195.

References

[1] R. Bellman. Vector Lyapunov functions. *SIAM Journal of Control*, 1:32–34, 1962.

[2] D. S. Bernstein. *Matrix Mathematics.* Princeton University Press, Princeton, NJ, 2005.

[3] F. Bullo, J. Cortéz, and S. Martinéz. *Distributed Control of Robotic Networks.* Princeton University Press, Princeton, NJ, 2009.

[4] P. R. Chandler, M. Pachter, and S. Rasmussen. UAV cooperative control. In *Proc. Amer. Contr. Conf.*, pages 50–55, Arlington, VA, 2001.

[5] J. Corfmat and A. Morse. Decentralized control of linear multivariable systems. *Automatica*, 12:476–495, 1976.

[6] P. Örgen, M. Egerstedt, and X. Hu. A control Lyapunov function approach to multiagent coordination. *IEEE Trans. Robot. Automat.*, 18(5):847–851, 2002.

[7] J. P. Desai, J. P. Ostrowski, and V. Kumar. Modeling and control of formations of nonholonomic mobile robots. *IEEE Trans. Robot. Automat.*, 17:905–908, 2001.

[8] R. Diestel. *Graph Theory.* Springer-Verlag, New York, NY, 1997.

[9] M. Egerstedt, X. Hu, and A. Stotsky. Control of mobile platforms using a virtual vehicle approach. *IEEE Trans. Autom. Contr.*, 46:1777–1782, 2001.

[10] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Trans. Autom. Contr.*, 49:1465–1476, 2004.

[11] W. M. Haddad and V. Chellaboina. Stability and dissipativity theory for nonnegative dynamical systems: A unified analysis framework for biological and physiological systems. *Nonlinear Analysis: Real World Applications*, 6:35–65, 2005.

[12] Q. Hui and W. M. Haddad. Distributed nonlinear control algorithms for network consensus. *Automatica*, 44:2375–2381, 2008.

[13] A. Isidori. *Nonlinear Control Systems.* Springer-Verlag, Berlin, 1995.

[14] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Autom. Contr.*, 48:988–1001, 2003.

[15] H. K. Khalil. *Nonlinear Systems.* Prentice-Hall, Upper Saddle River, NJ, 3rd edition, 2002.

[16] V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram. *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems.* Kluwer Academic Publishers, Dordrecht, Netherlands, 1991.

[17] V. Lakshmikantham and S. Sivasundaram. Stability of moving invariant sets and uncertain dynamic systems on time scales. *Computers and Mathematics with Applications*, 36:339–346, 1998.

[18] S. Leela and N. Shahzad. On stability of moving conditionally invariant sets. *Nonlinear Analysis: Theory, Methods, and Applications*, 27(7):797–800, 1996.

[19] N. E. Leonard and E. Fiorelli. Virtual leaders, artificial potentials, and coordinated control of groups. In *Proc. IEEE Conf. Dec. Contr.*, pages 2968–2873, Orlando, FL, 2001.

[20] J. A. Marshall, M. E. Broucke, and B. A. Francis. Formations of vehicles in cyclic pursuit. *IEEE Trans. Autom. Contr.*, 49:1963–1974, 2004.

[21] Yu. A. Martynyuk-Chernienko. On the stability of motion of uncertain systems. *International Applied Mechanics*, 35(2):212–216, 1999.

[22] V. M. Matrosov. Method of vector Liapunov functions of interconnected systems with distributed parameters (Survey). *Avtomatika i Telemekhanika*, 33:63–75, 1972. (in Russian).

[23] M. Mesbahi. On state-dependent dynamic graphs and their controllability properties. *IEEE Trans. Autom. Contr.*, 50:387–392, 2005.

[24] R. M. Murray. Recent research in cooperative control of multi-vehicle systems. *J. Dyn. Syst. Meas. Contr.*, 129:571–583, 2007.

[25] S. G. Nersesov and W. M. Haddad. On the stability and control of nonlinear dynamical systems via vector Lyapunov functions. *IEEE Trans. Autom. Contr.*, 51(2):203–215, 2006.

[26] R. Olfati-Saber. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Trans. Autom. Contr.*, 51:401–420, 2006.

[27] R. Olfati-Saber and R. M. Murray. Distributed cooperative control of multiple vehicle formations using structural potential functions. In *IFAC World Congr.*, Barcelona, Spain, 2002.

[28] W. Ren and R. W. Beard. *Distributed Consensus in Multi-Vehicle Cooperative Control*. Springer-Verlag, London, UK, 2008.

[29] D. D. Šiljak. *Large-Scale Dynamic Systems: Stability and Structure*. Elsevier North-Holland Inc., New York, NY, 1978.

[30] T. R. Smith, H. Hanssmann, and N. E. Leonard. Orientation control of multiple underwater vehicles with symmetry-breaking potentials. In *Proc. IEEE Conf. Dec. Contr.*, pages 4598–4603, Orlando, FL, 2001.

[31] E. D. Sontag. A universal construction of Artstein's theorem on nonlinear stabilization. *Sys. Contr. Lett.*, 13:117–123, 1989.

[32] H. G. Tanner, A. Jadbabaie, and G. J. Pappas. Flocking in fixed and switching networks. *IEEE Trans. Autom. Contr.*, 52:863–868, 2007.