Componentwise ultimate bound and invariant set computation for switched linear systems

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Abstract

We present a novel ultimate bound and invariant set computation method for continuous-time switched linear systems with disturbances and arbitrary switching. The proposed method relies on the existence of a transformation that takes all matrices of the switched linear system into a convenient form satisfying certain properties. The method provides ultimate bounds and invariant sets in the form of polyhedral and/or mixed ellipsoidal/polyhedral sets, is completely systematic once the aforementioned transformation is obtained, and provides a new sufficient condition for practical stability. We show that the transformation required by our method can easily be found in the well-known case where the subsystem matrices generate a solvable Lie algebra, and we provide an algorithm to seek such transformation in the general case. An example comparing the bounds obtained by the proposed method with those obtained from a common quadratic Lyapunov function computed via linear matrix inequalities shows a clear advantage of the proposed method in some cases.

Key words: Ultimate bounds, invariant sets, switched systems, componentwise methods, solvable Lie algebras

1 INTRODUCTION

Switched systems are a special type of dynamical systems that combine a finite number of subsystems by means of a switching rule [13,11]. Switched systems constitute a convenient description for many systems of practical importance, including many industrial processes, aircraft control, control of mechanical systems in general, and power systems. The stability and stabilizability of switched systems is an area where considerable research effort has been spent in recent years [12,2,16,13]. Different stability problems for switched systems arise depending on whether stability should hold for every admissible switching signal (arbitrary switching), for every switching signal within some class (constrained switching) or for a specific switching signal (switching stabilization). This paper focuses on the arbitrary switching case.

In general, most attention has been devoted to analysing or ensuring the asymptotic stability of an equilibrium point for the switched system [12,2,16,13]. However, there exist numerous reasons why asymptotic stability may be prevented in a realistic setting. One such reason is that switching may be employed to drive the state of the switched system close to a point that is not an equilibrium point of all subsystems [18]. Another reason is that nonvanishing perturbations (also named persistent disturbances) may act on the system [8, Ch. 9]. When asymptotic stability is not possible, ensuring some type of practical stability such as the ultimate boundedness of the state trajectories becomes important.

Some results have been reported on the practical stability of switched systems. In [17], a switched discrete-time system is considered where switching is state-dependent and the problem is that of finding controls that steer the state from a set of initial states to a set of "safe" states. References [21] and [20] address control design to ensure uniform ultimate boundedness for switched linear systems with parametric uncertainties under arbitrary switching by means of a common Lyapunov function approach. In [22], the authors address the design of both the control and switching strategy to achieve uniform ultimate boundedness of the system state. Most existing ultimate bound computation methods either make use of level sets of a Lyapunov function or employ some norm of the system state to compute the ultimate bound

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set. For switched linear systems, a quadratic Lyapunov function common to all subsystems can be computed via linear matrix inequalities (LMIs) in case it exists (see, for example, Section 4.3 of [16] and the references therein).

In this paper, we address the computation of ultimate bounds and invariant sets for switched continuous-time linear systems. We derive a novel computation method that is based on componentwise analysis and extends previous results presented by the authors in [9], [5]. The proposed method provides a new sufficient condition for practical stability and relies on the existence of a transformation that takes all matrices of the switched linear system into a form satisfying certain properties. These properties relate to the concept of Metzler matrices and an associated matrix operation [see (1) in the *Notation* subsection below]. The use of these new tools and comparison-type results based on these tools distinguish the present paper from our previous results for non-switched continuous-time systems [9,5] and, thus, constitute one of this paper's novel aspects. We show that the transformation required by the proposed method can be found in the well-known case where the subsystem matrices of the switched linear system generate a solvable Lie algebra. More importantly, another contribution of the present paper is to provide an algorithm to seek the desired transformation that is not restricted to the solvable Lie algebra case. Note that obtaining the required transformation in the switched-linear case is a much more difficult task than in the non-switched case treated in [9], [5], where the transformation was simply a change of coordinates to the Jordan canonical form.

Advantages of the proposed method include its complete systematicity and that it requires neither the computation of a Lyapunov function nor the use of a norm for the system state. An interesting feature of the method is that the ultimate bounds obtained are polyhedral if the required transformation is real, and of a mixed polyhedral/ellipsoidal form if the transformation is complex. To illustrate the results, we provide an example where the matrices of the switched linear system do not generate a solvable Lie algebra. We show that the algorithm is able to find the transformation required by our method, which yields ultimate bounds that are tighter than those obtained by means of a common quadratic Lyapunov function computed via LMIs. A preliminary conference version of parts of the results presented here, as well as parallel results for discrete-time switched linear systems, was published in [6].

The componentwise ultimate-bound computation method of [9], [5] has been successfully applied to the analysis of sampled-data systems with quantisation [4] and to the development of new controller design methods [10]. Moreover, a novel application in fault tolerant control systems has been recently reported in, e.g., [15,14] and [19]. In these papers, the method of [9] has been employed to obtain invariant sets where the system behaviour under "healthy" and "faulty" operation can be confined; fault tolerance can be achieved whenever those sets are "separated" in some sense. Thus, the results presented in the current paper have relevance in fault tolerant control systems and we envisage their application in the analysis and design of improved strategies with fault tolerance guarantees.

Notation. $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}_0$ and $\mathbb{C}$ denote the reals, nonnegative reals, nonnegative integers and complex numbers, respectively, and j the imaginary unit ($j^2 = -1$). If $x(t)$ is a vector-valued function, then $\limsup_{t\to\infty} x(t)$ denotes the vector obtained by taking $\limsup_{t\to\infty}$ of each component of $x(t)$, and similarly for 'max'. $|M|$, $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ denote the *elementwise* magnitude, real part, and imaginary part, respectively, of a matrix or vector M. The (i,k)-th entry of M is denoted $M_{i,k}$ and its k-th column $(M)_{:,k}$. If $X, Y \in \mathbb{R}^{n\times m}$, the expression '$X \preceq Y$' denotes the set of componentwise inequalities $X_{i,k} \le Y_{i,k}$, $i = 1,\dots,n$, $k = 1,\dots,m$, and similarly for $X \succeq Y$. Given matrices M_{ℓ_1}, M_{ℓ_2}, $\dots$, M_{ℓ_n}, the notation $\left(\prod_{r=\ell_n}^{r=\ell_1} M_r\right)$ denotes the product $M_{\ell_1}M_{\ell_2}\dots M_{\ell_n}$. Given a matrix $M \in \mathbb{C}^{n\times n}$, $\boldsymbol{\rho}(M)$ denotes its spectral radius, that is, the maximum magnitude of its eigenvalues. A matrix $M \in \mathbb{R}^{n\times n}$ is *Metzler* if $M_{i,k} \ge 0$ for all $i \ne k$. M is Metzler if and only if $e^{Mt} \succeq 0$ for all $t \ge 0$. Given an arbitrary matrix $N \in \mathbb{C}^{n\times n}$, we define $\mathcal{M}(N) \in \mathbb{R}^{n\times n}$ as the matrix whose entries satisfy

$$[\mathcal{M}(N)]_{i,k} = \begin{cases} \mathrm{Re}\{N_{i,k}\} & \text{if } i = k, \\ |N_{i,k}| & \text{if } i \ne k. \end{cases} \tag{1}$$

Note that $\mathcal{M}(N)$ is Metzler for every $N \in \mathbb{C}^{n\times n}$.

2 Main results

Consider the continuous-time switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + E_{\sigma(t)}w(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^p$ is a perturbation, and

$$\sigma : \mathbb{R}_+ \to \{1, 2, \dots, \mathrm{N}\} \tag{3}$$

is the piecewise constant switching function, assumed to have a finite number of discontinuities in every bounded interval. The evolution of the perturbation w is unknown but assumed to have a componentwise bound

$$|w(t)| \preceq \mathbf{w}, \quad \text{for all } t \ge 0, \tag{4}$$

where $\mathbf{w} \in \mathbb{R}^p_+$ is a known constant vector.

Theorem 1 below derives transient and ultimate bounds on the switched continuous-time system state that are

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valid for any realization of the switching function σ and, in addition, can take the componentwise form of the perturbation bound (4) into account. The proof of Theorem 1 is a minor modification of that of Theorem 2 in [6] and is omitted for the sake of conciseness.

Theorem 1 *Consider the switched system* (2) *with switching function* (3) *and componentwise perturbation bound* (4)*. Let $V \in \mathbb{C}^{n\times n}$ be invertible and define*

$$\Lambda_i \triangleq V^{-1}A_iV, \quad \Lambda \triangleq \max_{i=1,\dots,\mathrm{N}} \mathcal{M}(\Lambda_i), \tag{5}$$

$$\mathbf{z} \triangleq \max_{i=1,2,\dots,\mathrm{N}} \left[\max_{w:|w|\preceq \mathbf{w}} |V^{-1}E_iw| \right] \tag{6}$$

where $\mathcal{M}(\cdot)$ is the operation defined in (1)*. Suppose that Λ is Hurwitz and define*

$$\phi \triangleq \max\{|V^{-1}x(0)|, -\Lambda^{-1}\mathbf{z}\}, \text{ and } \eta \triangleq \phi + \Lambda^{-1}\mathbf{z}. \tag{7}$$

Then, the states of system (2)–(4) *are bounded as*

$$|V^{-1}x(t)| \preceq -\Lambda^{-1}\mathbf{z} + e^{\Lambda t}\eta, \tag{8}$$

for all $t \geq 0$, and ultimately bounded as

$$\limsup_{t\to\infty} |V^{-1}x(t)| \preceq -\Lambda^{-1}\mathbf{z}. \tag{9}$$

We next present two corollaries which provide, respectively, componentwise bounds and an invariant set for the states of the linear switched system (2)–(4).

Corollary 2 *Under the conditions of Theorem 1, the states of the linear switched continuous-time system* (2)–(4) *are componentwise bounded as $|x(t)| \preceq |V|(-\Lambda^{-1}\mathbf{z} + e^{\Lambda t}\eta)$ for all $t \geq 0$, and componentwise ultimately bounded as $\limsup_{t\to\infty} |x(t)| \preceq |V|(-\Lambda^{-1}\mathbf{z})$.*

PROOF. Immediate from the bounds (8) and (9) and the inequality $|x(t)| \preceq |V||V^{-1}x(t)|$. □

Corollary 3 *Under the conditions of Theorem 1, the set $S_c \triangleq \{x \in \mathbb{R}^n : |V^{-1}x| \preceq -\Lambda^{-1}\mathbf{z}\}$ is invariant for the state trajectories of the linear switched continuous-time system* (2)–(4)*.*

PROOF. Suppose $x(0) \in S_c$. Then $|V^{-1}x(0)| \preceq -\Lambda^{-1}\mathbf{z}$ and ϕ and η defined in (7) satisfy $\phi = -\Lambda^{-1}\mathbf{z}$ and $\eta = 0$. Substituting the latter into (8) yields $|V^{-1}x(t)| \preceq -\Lambda^{-1}\mathbf{z}$ for all $t \geq 0$; that is, $x(t) \in S_c$ for all $t \geq 0$ and the result then follows. □

Theorem 1 presents a systematic method to compute transient and ultimate bounds for the continuous-time switched system (2)–(4). These bounds were used to obtain componentwise bounds (Corollary 2) and an invariant set (Corollary 3) for the switched system. The method relies on a transformation V such that a function [given by the operation $\mathcal{M}(\cdot)$ defined in (1)] of the transformed matrices $V^{-1}A_iV$, for $i = 1, \dots, \mathrm{N}$, is bounded by a Metzler Hurwitz matrix. Note that the latter also consitutes a novel sufficient condition for practical stability. Such a transformation can be found in some special cases of interest; for example, when the matrices $A_1, \dots, A_\mathrm{N}$ of system (2) generate a solvable Lie algebra [1]. The simplest such case is when $\mathrm{N} = 1$, i.e., when switching becomes immaterial because the system is comprised of only one subsystem. In this simple case, it is straightforward to show that the current results are consistent with those in [9] and [5] for non-switching systems, and that the hypotheses of Theorem 1 incur no loss of generality. In the more general solvable Lie algebra case, let $V \in \mathbb{C}^{n\times n}$ be the transformation that renders $\Lambda_i = V^{-1}A_iV$ upper triangular for $i = 1, \dots, \mathrm{N}$ and consider Λ in (5). Since the Λ_i are all upper triangular, then the eigenvalues of Λ_i are its main-diagonal entries, and those of $\mathcal{M}(\Lambda_i)$ are the real parts of those of Λ_i. Note then that Λ is Hurwitz if and only if the Λ_i are all Hurwitz. Therefore, note that also in this case, the hypotheses of Theorem 1 incur no loss of generality.

3 Systematic Bound Computation

The results of Section 2 require a matrix V so that some conditions be satisfied. As we have shown, the existence of such matrix V is ensured whenever the subsystem matrices $A_1, \dots, A_\mathrm{N}$ generate a solvable Lie algebra. However, in the general case such Lie algebra is most likely to not be solvable, and no simple procedure for finding the required V exists. In this section, we give a systematic method to seek a matrix V that satisfies the hypotheses of Theorem 1, i.e., such that Λ in (5) is Hurwitz.

We begin by explaining the rationale of the proposed method. Suppose that $A_1, \dots, A_\mathrm{N}$ are Hurwitz and generate a solvable Lie algebra. Since (5) is continuous on the entries of A_i for a fixed V, and since the eigenvalues of a matrix are continuous on its entries, then if Λ is Hurwitz, it will also be Hurwitz for small perturbations of the entries of $A_1, \dots, A_\mathrm{N}$, even if the Lie algebra generated by the latter matrices will no longer be solvable, i.e., even if $A_1, \dots, A_\mathrm{N}$ will no longer be simultaneously triangularizable. Given $A_1, \dots, A_\mathrm{N}$, for $V \in \mathbb{C}^{n\times n}$, decompose $V^{-1}A_iV$ uniquely as follows

$$V^{-1}A_iV = T_i^{\mathrm{U}}(V) + T_i^{\mathrm{L}}(V), \tag{10}$$

where $T_i^{\mathrm{U}}(V)$ is upper triangular and $T_i^{\mathrm{L}}(V)$ is strictly lower triangular. Recalling (5), (1), and employing (10), we have

$$\Lambda = \max_i \mathcal{M}(T_i^{\mathrm{U}}(V)) + \max_i |T_i^{\mathrm{L}}(V)|. \tag{11}$$

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If $A_1, \ldots, A_{\rm N}$ generate a solvable Lie algebra and V achieves simultaneous triangularization, then $T_i^{\rm L}(V) = 0$ for all i and, as we have established in Section 2, Λ Hurwitz $\Leftrightarrow$ A_i Hurwitz for all i. Hence the condition of Theorem 1 is ensured because the subsystem matrices A_i are necessarily stable. In the case where $A_1, \ldots, A_{\rm N}$ are Hurwitz but do not generate a solvable Lie algebra, no V can achieve simultaneous triangularization. However, V can be said to achieve "approximate" simultaneous triangularization if the entries of $T_i^{\rm L}(V)$ are small enough, since in such case (10) shows that $V^{-1}A_iV$ will be close to $T_i^{\rm U}(V)$ and (11) that Λ will be close to $\max_i \mathcal{M}(T_i^{\rm U})$, so that Λ will be Hurwitz.

We next develop a method to seek V such that the entries of $T_i^{\rm L}(V)$ are minimized in some appropriate way. In the exact triangularization case there is no loss of generality in choosing V unitary, i.e. such that $V^*V = \mathrm{I}$. In the approximate triangularization case, we will restrict our search to a unitary V due to several numerical advantages that will become clear in the sequel. We next introduce the method and subsequently show precisely in what sense the entries of $T_i^{\rm L}(V)$ are minimized. Our method is shown in pseudocode as Algorithm 1.

Algorithm 1: Iterative approximate triangularisation

Data: $A_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, \rm N$

begin Initialisation

$A_i^1 \triangleq A_i$, $U_1 \triangleq \mathrm{I}$, $U = [\,]$ (empty), $\ell \leftarrow 0$

repeat

$\ell \leftarrow \ell + 1$

$$v_1^\ell \leftarrow \underset{\substack{v\in\mathbb{C}^{n-\ell+1}\\ v^*v=1}}{\operatorname{argmin}} \sum_{i=1}^{\rm N} \left(v^*(A_i^\ell)^*A_i^\ell v - |v^*A_i^\ell v|^2\right) \quad (12)$$

$$(U)_{:,\ell} \leftarrow \left(\prod_{r=\ell}^{r=1} U_r\right) v_1^\ell = U_1U_2\cdots U_\ell v_1^\ell \quad (13)$$

if $\ell < n$ **then**

Construct an orthonormal basis for $\mathbb{C}^{n-\ell+1}$:

$$\{v_1^\ell, \ldots, v_{n-\ell+1}^\ell\} \quad (14)$$

Assign
$$U_{\ell+1} \leftarrow [v_2^\ell|\cdots|v_{n-\ell+1}^\ell], \quad (15)$$
$$A_i^{\ell+1} \leftarrow U_{\ell+1}^* A_i^\ell U_{\ell+1}. \quad (16)$$

until $\ell = n$;

Algorithm returns U.

The matrix U returned by Algorithm 1 is a candidate V for Theorem 1. We next establish the relationship between U and the minimization of the entries of $T_i^{\rm L}(U)$. We need a preliminary lemma, whose proof follows from previous definitions and properties of unitary matrices.

Lemma 4 *Let $V_1, V_2 \in \mathbb{C}^{n\times n}$ satisfy $V_1^*V_1 = V_2^*V_2 = \mathrm{I}$ and suppose that $(V_1)_{:,k} = (V_2)_{:,k}$ for $k = 1, \ldots, \ell$. Then, for $i = 1, \ldots, \rm N$ and $k = 1, \ldots, \ell$,*

$$\|\,(T_i^{\rm L}(V_1))_{:,k}\,\| = \|\,(T_i^{\rm L}(V_2))_{:,k}\,\|. \quad (17)$$

Lemma 4 shows that, for any unitary $V \in \mathbb{C}^{n\times n}$, the norm of the k-th column of $T_i^{\rm L}(V)$, namely $\|\,(T_i^{\rm L}(V))_{:,k}\,\|$, depends only on the first k columns of V. Consequently, if v_k denotes the k-th column of V, we can define, for $k = 1, \ldots, n$, the function

$$F_i^k(v_1, \ldots, v_k) \triangleq \|\,(T_i^{\rm L}(V))_{:,k}\,\|^2. \quad (18)$$

We are ready to state the main result of this section.

Theorem 5 *Consider the matrix U returned by Algorithm 1 and let u_k denote the k-th column of U. Then,*

i) $U^*U = \mathrm{I}$.
ii) *For $k = 1, \ldots, n$, u_k satisfies*

$$\sum_{i=1}^{\rm N} F_i^k(u_1, \ldots, u_k) = \min_z \sum_{i=1}^{\rm N} F_i^k(u_1, \ldots, u_{k-1}, z), \quad (19)$$

where the minimum above is taken over all $z \in \mathbb{C}^n$ such that $z^*z = 1$ and, if $k > 1$, also

$$z^*u_r = 0, \ \textit{for}\ r = 0, 1, \ldots, k-1. \quad (20)$$

PROOF. i) Straightforward from previous definitions and properties of unitary matrices. ii) From the definition of $T_i^{\rm L}$ in (10), we can write

$$(T_i^{\rm L}(U))_{:,k} = \begin{bmatrix} \mathbf{0}_{k\times k} & 0 \\ 0 & \mathrm{I}_{n-k} \end{bmatrix} U^*A_iu_k = \begin{bmatrix} \mathbf{0}_{k\times n} \\ \hline u_{k+1}^* \\ \hline \vdots \\ \hline u_n^* \end{bmatrix} A_iu_k. \quad (21)$$

From (13),

$$u_k = \left(\prod_{r=k}^{r=1} U_r\right) v_1^k. \quad (22)$$

Using (22), (15), and (16),

$$(T_i^{\rm L}(U))_{:,k} = \begin{bmatrix} \mathbf{0}_{k\times(n-k+1)} \\ \hline w_2^* \\ \hline \vdots \\ \hline w_{n-k+1}^* \end{bmatrix} A_i^kv_1^k, \quad (23)$$

where $w_r = \left(\prod_{s=r+k-1}^{s=k+1} U_s\right) v_1^{r+k-1}$, for $r = 2, \ldots, n-k+1$. Define $w_1 \triangleq v_1^k$ and let W be the matrix whose columns are $w_1, \ldots, w_{n-k+1}$. Note that

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$W^*W = WW^* = \mathrm{I}_{n-k+1}$ by i). Rewrite (23) as

$$
\begin{aligned}
(T_i^{\mathrm{L}}(U))_{:,k} &= \begin{bmatrix} \mathbf{0}_{k\times 1} & 0 \\ 0 & \mathrm{I}_{n-k} \end{bmatrix} W^* A_i^k v_1^k \\
&= \begin{bmatrix} \mathbf{0}_{(k-1)\times(n-k+1)} \\ \mathrm{I}_{n-k+1} - \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{0}_{(n-k)\times(n-k)} \end{bmatrix} \end{bmatrix} W^* A_i^k v_1^k
\end{aligned}
$$

Operating on the result above yields

$$
\| (T_i^{\mathrm{L}}(U))_{:,k} \|^2 = (v_1^k)^*(A_i^k)^* A_i^k v_1^k - |(v_1^k)^* A_i^k v_1^k|^2, \quad (24)
$$

where we have used the fact that $WW^* = \mathrm{I}_{n-k+1}$ and $w_1 = v_1^k$. By (18), we also know that

$$
\| (T_i^{\mathrm{L}}(U))_{:,k} \|^2 = F_i^k(u_1, \ldots, u_{k-1}, u_k). \quad (25)
$$

Comparing (24) with (25), we note that the minimization (12) is equivalent to minimizing the right-hand side of (25) with respect to v_1^k. Noting that $u_1, \ldots, u_{k-1}$ do not depend on v_1^k, and recalling that u_k and v_1^k are related by (22), we then have

$$
v_1^k = \operatorname*{argmin}_{v^*v=1} \sum_{i=1}^{\mathrm{N}} F_i^k \left(u_1, \ldots, u_{k-1}, \left(\prod_{r=k}^{r=1} U_r \right) v \right). \quad (26)
$$

By i), all vectors u_k that satisfy (20) and $u_k^* u_k = 1$ are parameterised as $\left(\prod_{r=k}^{r=1} U_r\right) v$ with $v^*v = 1, v \in \mathbb{C}^{n-k+1}$. Again, note that U_r, for $r = 1, \ldots, k$ do not depend on u_k. Therefore, (26) leads to $u_k = \operatorname{argmin}_z \sum_{i=1}^{\mathrm{N}} F_i^k(u_1, \ldots, u_{k-1}, z)$, where the minimization is performed over z satisfying (20) and $z^*z = 1$, whence (19) follows. □

Theorem 5 i) shows that the matrix U returned by Algorithm 1 is indeed unitary and Theorem 5 ii) establishes the relationship between such matrix and the minimization of the entries of $T_i^{\mathrm{L}}(U)$, the strictly lower triangular part of $U^{-1}A_iU$. This relationship can be expressed as follows. Each column of U is selected so that the sum of the squared norm of the corresponding column of $T_i^{\mathrm{L}}(U)$ is minimized, given the previous columns of U. Note that in the solvable Lie algebra case Algorithm 1 is guaranteed to return a matrix U satisfying $T_i^{\mathrm{L}}(U) = 0$.

Combining the results of the previous sections, the proposed systematic method for ultimate bound and invariant set computation can be summarized as follows: given the matrices $A_1, \ldots, A_{\mathrm{N}}$ of the switched linear system, apply Algorithm 1 to seek the transformation U that achieves simultaneous "approximate" triangularization; letting $V = U$, check if V satisfies the conditions required by Theorem 1 and, if so, compute the ultimate bounds and invariant sets using the explicit formulas given in such theorem and its corollaries.

4 EXAMPLE

Consider the open-loop unstable system

$$
\dot{x}(t) = Ax(t) + Bu(t), \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 2798 & 0 & -19.6 \\ 0 & 0 & -24.39 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 2.439 \end{bmatrix},
$$

which models a magnetic ball levitation system, with same parameters and states as in [7, Ch. 4.7], and linearized around the same operating point. The state components represent the ball vertical position, vertical speed, and coil current, respectively. Assuming that the full state is available for measurement, two different stabilising feedback matrices are designed according to different performance objectives: $K_1 = [\,10316 \,|195\, |-49\,]$ and $K_2 = [\,9048.8\, |171.1\, |-43.4\,]$. Noise is assumed to affect the measured state $\hat{x} = x + w$, with componentwise bound $|w(t)| \preceq \mathbf{w} = [0.1\ |1\ |1]^T \cdot 10^{-3}$, corresponding to uncertainty of ± 0.1 mm in position, ± 1 mm/s in speed and ± 1 mA in coil current. Application of the feedback control $u = -K_i\hat{x}$ yields, for $i = 1, 2$,

$$
\dot{x}(t) = A_i x(t) + E_i w(t), \quad A_i = A + BK_i, \quad E_i = BK_i,
$$

Knowing whether the controller can switch between controllers without affecting stability is desired, jointly with an ultimate bound for the system state. Computation of the unitary matrix U as in Section 3 yields

$$
U = \begin{bmatrix} 0.0147 & -0.0023 & -0.9998 \\ -0.7802 & 0.0021 & -0.0189 \\ -0.0046 & -0.6253 & 0.0036 \end{bmatrix} + j \begin{bmatrix} -0.0118 & 0.0029 & 0 \\ 0.6252 & -0.0046 & 0 \\ 0.0022 & 0.7803 & 0 \end{bmatrix},
$$

for which $\max_{i=1,2} \mathcal{M}(U^{-1}A_iU)$ is Hurwitz [see (5)]. This implies that the switched system $\dot{x} = A_ix$ is stable for arbitrary switching. Application of Theorem 1 and Corollary 2 with $V = U$ yields

$$
\begin{aligned}
\limsup_{t\to\infty} |V^{-1}x(t)| &\preceq \mathbf{b} \triangleq \begin{bmatrix} 0.106 \\ 0.385 \\ 0.001 \end{bmatrix}, \quad \text{and} \\
\limsup_{t\to\infty} |x(t)| &\preceq \begin{bmatrix} 4.1 \\ 107.9 \\ 385.2 \end{bmatrix} \cdot 10^{-3}. \quad (27)
\end{aligned}
$$

Applying Corollary 3 yields that the set $S_d = \{x \in \mathbb{R}^3 : |U^{-1}x| \preceq \mathbf{b}\}$ is invariant for the state trajectories of the linear switched system under arbitrary switching.

We next compare the bound (27) with that obtained via a common quadratic Lyapunov function (CQLF). We follow a bounding procedure similar to that in [8, Ch. 9] but adapted to switched systems. To compute a CQLF for this system, we seek $P = P^T > 0$ and $Q = Q^T > 0$ so that $A_i^TP + PA_i + Q < 0$, for $i = 1, 2$. Solving such LMIs via Matlab's `feasp` function yields

$$
P = \begin{bmatrix} 2.698 & 0.0669 & -0.0087 \\ * & 0.0028 & -0.0003 \\ * & * & 4.94\cdot 10^{-5} \end{bmatrix}, \quad Q = \begin{bmatrix} 4.3231 & -0.018 & -0.0264 \\ * & 0.0417 & 0.0007 \\ * & * & 0.0004 \end{bmatrix}
$$

where the corresponding CQLF is $V(x) = x^TPx$. Straightforward calculations show that an attractive

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invariant set for this system is given by

$$\left\{x \in \mathbb{R}^3 : x^T P x \leq \frac{4\lambda_{\max}(P)\mathbf{p}}{\lambda_{\min}^2(Q)} \approx 117\right\}, \qquad (28)$$

where $\mathbf{p} = \max_{i=1,2} \max_{|w| \preceq \mathbf{w}} \|PE_i w\|^2 = 7.34 \cdot 10^{-4}$. From (28) we can derive the following ultimate bound on the state norm

$$\limsup_{t\to\infty} \|x(t)\| \leq 2.86 \cdot 10^3. \qquad (29)$$

Note that a componentwise bound several times larger than (27) can still fit in the regions given by (29) or (28).

5 CONCLUSIONS

We have presented a novel componentwise ultimate bound and invariant set computation method for continuous-time switched linear systems with disturbances under arbitrary switching. The method requires a transformation matrix satisfying certain properties. We have also provided an algorithm that seeks such transformation matrix. The matrix provided by the algorithm is guaranteed to allow the application of our method when the evolution matrices of the switching subsystems either generate or are close, in some appropriate sense, to generating a solvable Lie algebra. In addition, we have presented a practical example for which the bounds obtained by our method are tighter than those obtained via a common quadratic Lyapunov function approach. One interesting question for future research is the analysis of the applicability of our method, e.g., determining whether our method can be applied to a larger, smaller, or different class of switching linear systems than those which admit a CQLF. Another future research question is determining in what cases our method yields tighter bounds than those obtained by other existing methods such as CQLF. As discussed, our results apply to the arbitrary switching case and, as such, they can be interpreted as worst-case results over all possible switching sequences. In addition, the switched system is assumed to be autonomous, that is, without a control input. An important topic for future work is then the design of suitable switching sequences and/or control inputs to assign ultimate bounds or obtain bounds that are minimal in some sense. Some initial work along a related line has been reported in [3].

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